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Extensions of Valuations

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Abstract. Continuous valuations have been proposed by several authors as a way of modeling probabilistic non-determinism in programming language semantics. Let $(X, \mathcal{O})$ be a topological space. A quasi-simple valuation on X is the sup of a directed family of simple valuations. We show that quasi-simple valuations are exactly those valuations that extend to continuous valuations to the Alexandroff topology on the specialization preordering of the topology $\mathcal{O}$. A number of applications are presented. In particular, we retrieve Jones' result that every continuous valuation is quasi-simple if X is a continuous dcpo—in this case, there is a least extension to the Alexandroff topology. We show that this can be refined if X is algebraic, where every continuous valuation is the sup of a directed family of simple valuations based on finite elements. We exhibit another class of spaces where every continuous valuation is quasi-simple, the so-called finitarily coherent spaces—in this case, there is a largest extension to the Alexandroff topology. In general, the extension to the Alexandroff topology is not unique, unless the original valuation is bicontinuous for example. We also show that other natural spaces of valuations, namely those of discrete valuations and point-continuous valuations, can be characterized by similar extension theorems.

1 Introduction

Giving a faithful account of probabilistic non-determinism in denotational semantics has attracted quite some research since the pioneering work of Saheb-Djahromi, and of Jones and Plotkin. Mixing domain theory with probabilities seems to require one to use *continuous valuations* rather than measures, as justified and studied in Claire Jones' PhD thesis [8]. Amongst all continuous valuations, those that can be built as sups of directed families of simple valuations are the most natural, and can be handled most conveniently. (We give definitions shortly.) For example, if ν is a continuous valuation that can be written as the sup of the directed family $\left(\sum_{j=1}^{n_i} a_{ij}\delta_{x_{ij}}\right)_{i\in I}$, then the integral $\int_{x\in X} f(x)d\nu$ (as defined by Tix [11] for example, extending Jones [8]) can be *computed*, not just characterized, as the sup of all finite sums $\sum_{j=1}^{n_i} a_{ij}f(x_{ij})$. This is one of the ingredients in [4], notably.

For short, call *quasi-simple* those valuations that are sups of directed families of simple valuations. The initial purpose of this paper is to refine our understanding of quasi-simple valuations. This will be achieved by proving Theorem 1 below, and exploring some of its applications. This will lead us to realize that various classes of continuous valuations can be characterized similarly.

Let us give a few definitions. Let $(X, \mathcal{O})$ be a topological space. A *valuation* on X is a map $\nu : \mathcal{O} \to (\mathbb{R}_+ \cup \{+\infty\})$ such that $\nu(\emptyset) = 0$ (*strictness*); $U \subseteq V$ implies $\nu(U) \leq \nu(V)$ for every opens $U, V \in \mathcal{O}$ (*monotonicity*); and $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for every opens $U, V \in \mathcal{O}$ (*modularity*). A *continuous* valuation in addition satisfies $\nu(\bigcup_i U_i) = \sup_i \nu(U_i)$ for every directed family $(U_i)_{i\in I}$ of opens—such a family is *directed* if and only if $I \neq \emptyset$ and for every $i, j \in I$, there is $k \in I$ such that $U_i, U_j \subseteq U_k$. Continuous valuations are a concept close to that of measure; while measures are defined on σ-algebras, valuations are naturally defined on topological spaces.

We can build continuous valuations by a number of different means. First, the *Dirac valuation* δ_x maps every open containing x to 1, and every open not containing x to 0. Second, any finite linear combination $\sum_{i=1}^{n} a_i\nu_i$ of continuous valuations $(\nu_i)_{1\leq i\leq n}$ ($a_i \in \mathbb{R}_+ \cup \{+\infty\}$) is again a continuous valuation; a linear combination $\sum_{i=1}^{n} a_i\delta_{x_i}$ of Dirac valuations is called a *simple valuation*. Third, ordering valuations pointwise, any sup of directed families of continuous valuations is a continuous valuation again. We shall

call directed sups of simple valuations *quasi-simple valuations*. It is in general not the case that every bounded continuous valuation is quasi-simple, however Jones [8] showed that every bounded continuous valuation is indeed quasi-simple when X is the topological space underlying a continuous dcpo.

The main point of this paper is to show an equivalent characterization of quasi-simple valuation:

Theorem 1. *Let $(X, \mathcal{O})$ be any topological space. A bounded function $\nu : \mathcal{O} \to \mathbb{R}_+$ is a quasi-simple valuation if and only if it extends to a continuous valuation on the Alexandroff extension of the topology $\mathcal{O}$.*

Here ν is *bounded* just means that the range of ν is included in $\mathbb{R}_+$, in particular $\nu(X) < +\infty$. The major ingredient of the theorem is the Alexandroff extension of a topology. First, recall that on every topological space $(X, \mathcal{O})$ we may define a *specialization preordering* $\leq_{\mathrm{s}}$ by $x \leq_{\mathrm{s}} y$ if and only if x is in the closure of y; equivalently, if every closed set containing y contains x; equivalently, if every open set containing x contains y. This is an ordering exactly when X is T_0. Conversely, given any preordering $\leq$, the set of sets that are upward closed wrt. $\leq$ is a topology, called the *Alexandroff topology* over $\leq$. We call the Alexandroff topology over the specialization preordering $\leq_{\mathrm{s}}$ the *Alexandroff extension* of the topology $\mathcal{O}$. Notice that every $U \in \mathcal{O}$ is upward closed in $\leq_{\mathrm{s}}$, so all opens are opens of the Alexandroff extension. In the sequel, we shall call any extension of a continuous valuation ν to the Alexandroff extension of the topology an *A-extension* of ν.

Outline. We prove Theorem 1 in Section 2, then explore a number of applications in Section 3. First, we shall show in Section 3.1 that Jones' Theorem that every bounded continuous valuation ν on a continuous dcpo is quasi-simple is an easy corollary of this theorem. In fact, ν not only has an A-extension, but also a *least* one in this case. We shall then refine this in Section 3.2 to show that every bounded continuous valuation on an algebraic dcpo is the sup of a directed family of simple valuations *based on finite elements*, filling a gap in Sünderhauf's argument [10]. We notice that the space of A-extensions in general has no remarkable structure, in particular A-extensions need not be unique. A case where A-extensions are necessary unique is that of bicontinuous valuations à la Bukatin and Shorina, which we briefly touch upon in Section 3.3. Dually to the continuous dcpo case, we exhibit in Section 3.4 a new class of topological spaces where every continuous valuation is quasi-simple, this time because there is a *largest* A-extension: the *finitarily coherent spaces*. We derive other simpler extensions result for other classes of continuous valuations in Section 4, the discrete and the point-continuous valuations respectively. We sprinkle the text liberally with remarks and examples. We conclude by reviewing related work in Section 5.

2 Proof of Theorem 1

2.1 Only If

The only if direction of the Theorem is relatively easy. First, any simple valuation $\nu = \sum_{i=1}^{n} a_i \delta_{x_i}$ extends to the Alexandroff valuation of $\mathcal{O}$, namely the valuation $\hat{\nu}$ that is just $\sum_{i=1}^{n} a_i \delta_{x_i}$ again, only defined on all $\leq_{\mathrm{s}}$-upward closed subsets of X.

Second, we notice the well-known fact that any sup ν of a directed family of continuous valuations ν_i, $i \in I$, on any topology, is a continuous valuation on the same topology. The main difficulty here is to show that ν is indeed modular. To this end, we first prove that: $(*)$ for any two opens U, V, $\nu(U) + \nu(V) = \sup_{i \in I}(\nu_i(U) + \nu_i(V))$: the $\geq$ direction is clear; conversely for any $\epsilon > 0$ there is $j \in I$ such that $\nu(U) \leq \nu_j(U) - \epsilon/2$, and there is $k \in I$ such that $\nu(V) \leq \nu_k(V) - \epsilon_2$; since $(\nu_i)_{i \in I}$ is directed, for some $i \in I$ $\nu_j, \nu_k \leq \nu_i$, so $\nu(U) + \nu(V) \leq \nu_i(U) + \nu_i(V) - \epsilon$. Having shown $(*)$, it follows:

$$\begin{aligned}
\nu(U \cup V) + \nu(U \cap V) &= \sup_{i \in I}(\nu_i(U \cup V) + \nu_i(U \cap V)) \qquad \text{(by } (*)\text{)} \\
&= \sup_{i \in I}(\nu_i(U) + \nu_i(V)) \quad \text{(since } \nu_i \text{ is modular)} \quad = \nu(U) + \nu(V) \qquad \text{(by } (*)\text{)}
\end{aligned}$$

We then need the following auxiliary lemma, which extends slightly Jones' Splitting Lemma:

Lemma 1 ([10], Theorem 2.4, Corollary 2.6). *Let $(X, \mathcal{O})$ be a topological space, and $\leq_{\mathrm{s}}$ its specialization preorder. Define the following binary relations on the set of simple valuations on X:*

1. $\nu \leq_1 \nu'$ *if and only if* $\nu' = \nu + a\delta_x$ *for some* $a \in \mathbb{R}_+$*, and* $x \in X$*;*
2. $\nu \leq_2 \nu'$ *if and only if* ν *is of the form* $\nu_0 + a\delta_x$*, and* $\nu' = \nu_0 + a\delta_y$ *for some simple valuation* ν_0*, some* $a \in \mathbb{R}_+$*, and some pair* x, y *of elements of* x *such that* $x \leq_{\mathrm{s}} y$*.*

The preordering $\leq$ *on simple valuations is exactly the smallest preordering containing* $\leq_1 \cup \leq_2$*.*

An equivalent, but more synthetic characterization, is to say that $\sum_{i=1}^n a_i\delta_{x_i} \leq \sum_{j=1}^n b_j\delta_{x_j}$ if and only if there is a matrix $(t_{ij})_{1\leq i,j\leq n}$ of non-negative reals such that $a_i = \sum_{j=1}^n t_{ij}$ for all i, $b_j = \sum_{i=1}^n t_{ij}$ for all j, and whenever $t_{ij} \neq 0$ then $x_i \leq_{\mathrm{s}} x_j$.

Lemma 1 in particular implies that the property $\sum_{i=1} a_i\delta_{x_i} \leq \sum_{j=1}^n b_j\delta_{x_j}$ between simple valuations does not depend on the topology $\mathcal{O}$ but only on its specialization preordering $\leq_{\mathrm{s}}$. It follows that, given any directed family $(\nu_i)_{i\in I}$ of simple valuations on $(X, \mathcal{O})$, the family $(\overline{\nu}_i)_{i\in I}$ of simple valuations on the Alexandroff extension of $\mathcal{O}$ is also directed. Indeed, $\mathcal{O}$ and its Alexandroff extension have the same specialization preordering. Then the sup of the family $(\overline{\nu}_i)_{i\in I}$ is a continuous valuation on the Alexandroff extension of $\mathcal{O}$, as a sup of a directed family of continuous valuations on the same Alexandroff extension. Furthermore, $\sup_{i\in I} \overline{\nu}_i$ clearly extends $\sup_{i\in I} \nu_i$. □

2.2 If

The if direction is more complex, but proceeds along lines very similar to [8], Section 5.1. We mimic the latter to help the reader notice the similarity.

Fix a set X, a preordering $\leq$ on X. Write $\uparrow A$ for $\{y | \exists x \in A \cdot x \leq y\}$, for each $A \subseteq X$. Write also $\uparrow x$ for $\uparrow \{x\}$. Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$. (For example, this is the case if $X_0 = X$, where we retrieve the Alexandroff topology.) We shall show that every bounded continuous valuation on $(X, \mathcal{O})$ is a sup of a directed family of simple valuations $\sum_{i=1}^n a_i\delta_{x_i}$, where $x_i \in X_0$ for every i.

A *crescent* C is any difference $U \setminus V$ of (Alexandroff) opens U, V. A *field* of sets is any family of sets such that every finite union and every finite intersection of elements of the field is again in the field. This includes $\emptyset$ and the whole space X. The field generated by the open sets is the smallest field containing the opens; this is exactly the set of all finite disjoint unions of crescents. Any bounded valuation ν extends to a unique additive function on the field of all crescents, defined by $\nu(U \setminus V) = \nu(U) - \nu(U \cap V) = \nu(U \cup V) - \nu(V)$. (An *additive function* ν maps any finite disjoint union $\bigcup_i C_i$ to $\sum_i \nu(C_i)$.) This is the Smiley-Horn-Tarski Theorem, often attributed to Pettis (see [2], Section 1.11). We shall equate continuous valuations on $(X, \mathcal{O})$ with their unique extension to the field of disjoint unions of crescents.

Notice that, for every continuous valuation ν, for every family $(O_i)_{i\in I}$ of opens, $\nu(\bigcup_{i\in I} O_i) = \sup_{J \text{ finite} \subseteq I} \nu(\bigcup_{j\in J} O_j)$. This is because the family of finite unions of O_i is directed.

Lemma 2. *Let* ν *be a continuous valuation,* O *and* O_i*,* $i \in I$*, be open. Then*

$$\nu\left(\bigcup_{i\in I} O_i \setminus O\right) = \sup_{J \text{ finite} \subseteq I} \nu\left(\bigcup_{j\in J} O_j \setminus O\right) \tag{1}$$

Proof. This is exactly Lemma 5.1 of [8], although on a different topology. For completeness, here is the proof. The left-hand side of (1) is:

$$\begin{aligned}\nu\left(\bigcup_{i\in I} O_i \setminus O\right) &= \nu\left(\bigcup_{i\in I} O_i\right) - \nu\left(\bigcup_{i\in I} O_i \cap O\right) \\ &= \sup_{J \text{ finite} \subseteq I} \nu\left(\bigcup_{j\in J} O_j\right) - \sup_{J \text{ finite} \subseteq I} \nu\left(\bigcup_{j\in J} O_j \cap O\right)\end{aligned}$$

by continuity of ν. For any $\epsilon > 0$, there is a finite subset J_ϵ that makes this greater than or equal to:

$$\sup_{J \text{ finite} \subseteq I} \nu\left(\bigcup_{j\in J} O_j\right) - \nu\left(\bigcup_{j\in J_\epsilon} O_j \cap O\right) - \epsilon \geq \nu\left(\bigcup_{j\in J_\epsilon} O_j\right) - \nu\left(\bigcup_{j\in J_\epsilon} O_j \cap O\right) - \epsilon$$
$$= \nu\left(\bigcup_{j\in J_\epsilon} O_j \setminus O\right) - \epsilon$$

As this holds for every $\epsilon > 0$, the left-hand side of (1) is greater than or equal to the right-hand side. Conversely, for every $\epsilon > 0$, there is a finite subset J_ϵ that makes the left-hand side less than or equal to:

$$\nu\left(\bigcup_{j\in J_\epsilon} O_j\right) + \epsilon - \sup_{J \text{ finite} \subseteq I} \nu\left(\bigcup_{j\in J} O_j \cap O\right) \leq \nu\left(\bigcup_{j\in J_\epsilon} O_j\right) + \epsilon - \nu\left(\bigcup_{j\in J_\epsilon} O_j \cap O\right)$$
$$= \nu\left(\bigcup_{j\in J_\epsilon} O_j \setminus O\right) + \epsilon$$

So the left-hand side of (1) is less than or equal to the right-hand side. □

Call a *pre-dissection* D any finite set of pairs (x_i, U_i), $1 \leq i \leq n$, where $x_i \in X$ and U_i is open in $\mathcal{O}$—that is, U_i is of the form $\uparrow A_i$ for $A_i \subseteq X_0$. By convention, write $C_i = \uparrow x_i \setminus U_i$. D is a *dissection* if additionally $C_1, \ldots, C_n$ are pairwise disjoint.

Given a dissection D and a continuous valuation ν, define

$$\nu[D] = \sum_{i=1}^{n} \nu(C_i)\delta_{x_i} \tag{2}$$

It is clear that $\nu[D] \leq \nu$, since for every open U, $\nu[D](U) = \sum_{x_i\in U} \nu(C_i) = \nu(\bigcup_{i/x_i\in U} C_i)$ (since ν is additive and the C_is are pairwise disjoint), and $\bigcup_{i/x_i\in U} C_i \subseteq \bigcup_{i/x_i\in U} \uparrow x_i \subseteq U$ (because U is upward-closed). In fact:

Lemma 3. *Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$. Every bounded continuous valuation ν on $\mathcal{O}$ is the sup of $(\nu[D])_{D \text{ dissection}}$.*

Proof. Let U be any open of the form $\uparrow\{x_1, \ldots, x_n\}$—call this a *finitary* open. Let U_i be the open $\uparrow\{x_1, \ldots, x_{i-1}\}$ for each i, $1 \leq i \leq n$. The set D of all pairs (x_i, U_i) is a pre-dissection. We claim that it is a dissection. Indeed, for any $i < j$, C_j does not intersect U_j by construction; but U_j contains $\uparrow x_i$ (since $i < j$), which contains C_i, so C_j cannot intersect C_i.

On the other hand, $\bigcup_{i=1}^n C_i = U$. Indeed, by induction on k, $\bigcup_{i=1}^k C_i = \uparrow\{x_1, \ldots, x_k\}$. So $\nu[D](U) = \sum_{i/x_i\in U} \nu(C_i) = \sum_{i=1}^n \nu(C_i) = \nu(\bigcup_{i=1}^n C_i) = \nu(U)$.

Fix an arbitrary open U, and an arbitrary $\epsilon > 0$. Since every open is the union of the directed family of all finitary opens contained in it, and since ν is continuous, there is a finitary open $U_0 \subseteq U$ such that $\nu(U) \leq \nu(U_0) + \epsilon$. Now by the above there is a dissection D such that $\nu(U_0) = \nu[D](U_0)$. Since $U_0 \subseteq U$, $\nu[D](U_0) \leq \nu[D](U)$, so $\nu(U) \leq \nu[D](U) + \epsilon$. As $\epsilon > 0$ is arbitrary, $\nu(U) \leq \sup_D \nu[D](U)$. Since U is arbitrary, $\nu \leq \sup_D \nu[D]$. We conclude since $\nu[D] \leq \nu$ for all dissections D. □

However the family of all $\nu[D]$, where D ranges over all dissections, is not necessarily directed. Lemma 3 clearly entails:

Corollary 1. *Under the assumptions of Lemma 3, ν is the sup of all valuations $r.\nu[D]$, where D ranges over all dissections, and $0 < r < 1$.*

We shall show that the family $(r.\nu[D])_{D \text{ dissection}, 0<r<1}$ is directed. The key observation is the following lemma, which is roughly similar to Jones' Lemma 5.3:

Lemma 4. *Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$.*

For any open sets $O_1, \ldots, O_n$ in $\mathcal{O}$, for every $0 \leq s < 1$, there is a dissection D such that, for any $B \subseteq \{1, \ldots, n\}$,

$$s.\nu\left(\bigcup_{i\in B} O_i\right) \leq \nu[D]\left(\bigcup_{i\in B} O_i\right) \tag{3}$$

Proof. For every $J \subseteq \{1, \ldots, n\}$, let C_J be the crescent $\bigcap_{j\in J} O_j \setminus \bigcup_{j\notin J} O_j$; empty intersections are meant to denote the whole set X. Clearly the C_Js are pairwise disjoint, and their union is $\bigcup_{i=1}^n O_i$. In fact, for every $B \subseteq \{1, \ldots, n\}$, $\bigcup_{i\in B} O_i = \bigcup_{J\supseteq B} C_J$, where the union on the right is a disjoint union. In particular,

$$\nu\left(\bigcup_{i\in B} O_i\right) = \sum_{J\supseteq B} \nu(C_J) \tag{4}$$

We now extract a dissection from each C_J, whose measure approximates that of C_J, using the continuity Lemma 2. Observe that since $\bigcap_{i\in J} O_i$ is open, it is a union of basic opens $\uparrow x$ with $x \in X_0$, hence it is the union of all $\uparrow x$ with $x \in \bigcap_{i\in J} O_i \cap X_0$. We claim that

$$C_J = \left(\bigcup_{x\in C_J\cap X_0} \uparrow x\right) \setminus \bigcup_{j\notin J} O_j \tag{5}$$

for every J. Indeed, let y be an arbitrary element of C_J. Then $y \in \bigcap_{i\in J} O_i = \bigcup_{x\in\bigcap_{i\in J} O_i\cap X_0} \uparrow x$, so let x be some element of $\bigcap_{i\in J} O_i \cap X_0$ such that $y \geq x$. Since $y \in C_J$, $y \notin \bigcup_{j\notin J} O_j$, and since $\bigcup_{j\notin J} O_j$ is open, hence upward closed, x is not in $\bigcup_{j\notin J} O_j$ either. So x is in $\left(\bigcap_{i\in J} O_i \cap X_0\right) \setminus \bigcup_{j\notin J} O_j = C_J \cap X_0$. Since $y \geq x$, in particular y is in $\bigcup_{x\in C_J\cap X_0} \uparrow x$. So y is in the right-hand side of (5). Conversely, let y be arbitrary in the right-hand side of (5). In particular, $y \geq x$ for some $x \in C_J \cap X_0$. Since $x \in C_J$, in particular $x \in \bigcap_{j\in J} O_j$. Since the open $\bigcap_{j\in J} O_j$ is upward closed, $y \in \bigcap_{j\in J} O_j$. Since on the other hand $y \notin \bigcup_{j\notin J} O_j$, $y \in C_J$.

Let $\epsilon_J = (1-s)\nu(C_J)$. From (5) and Lemma 2 we infer that, when $\epsilon_J > 0$, there is a finite set of points $x_{J1}, \ldots, x_{Jm_J}$ in $C_J \cap X_0$ such that

$$\nu(C_J) \leq \nu\left(\bigcup_{i=1}^{m_J} \uparrow x_{Ji} \setminus \bigcup_{j\notin J} O_j\right) + \epsilon_J \tag{6}$$

This also holds when $\epsilon_J = 0$, since in this case $\nu(C_J) = 0$ and therefore (6) is trivial.

Let D_J be the dissection $\{(x_{Ji}, U_{Ji}) | 1 \leq i \leq m_J\}$, with $U_{Ji} = \uparrow\{x_{J1}, \ldots, x_{J(i-1)}\} \cup \bigcup_{j\notin J} O_j$ (compare with the proof of Lemma 3). Notice in particular that since every x_{Ji} is in X_0, U_{Ji} is indeed open. Let C_{Ji} be the pairwise disjoint sets $\uparrow x_{Ji} \setminus U_{Ji}$, $1 \leq i \leq m_J$. Then $\bigcup_{i=1}^{m_J} C_{Ji} = \bigcup_{i=1}^{m_J} \uparrow x_{Ji} \setminus \bigcup_{j\notin J} O_j$, so by (6)

$$\nu(C_J) \leq \sum_{i=1}^{m_J} \nu(C_{Ji}) + \epsilon_J \tag{7}$$

Let D be the union of all D_J, $J \subseteq \{1, \ldots, n\}$. This is a pre-dissection. To show that it is a dissection, it remains to show that C_{Ji} does not intersect $C_{J'i'}$ when $J \neq J'$. This is straightforward, since $C_{Ji} \subseteq \bigcup_{i=1}^{m_J} C_{Ji} \subseteq C_J$, similarly $C_{J'i'} \subseteq C_{J'}$, and C_J and $C_{J'}$ do not intersect.

For every $J, J' \subseteq \{1, \ldots, n\}$, $x_{J'i} \in C_J$ if and only if $J = J'$. Indeed, remember that $x_{J'i}$ is in $C_{J'}$, and the C_Js are pairwise disjoint. So

$$\nu[D](C_J) = \sum_{J'\subseteq\{1,\ldots,n\}} \sum_{i=1}^{m_{J'}} \nu(C_{J'i})\delta_{x_{J'i}}(C_J) = \sum_{1\leq i\leq m_J} \nu(C_{Ji}) \geq \nu(C_J) - \epsilon_J \qquad \text{using (7)}$$

Summing over $J \supseteq B$, we obtain

$$\nu[D]\left(\bigcup_{i\in B} O_i\right) \geq \nu\left(\bigcup_{i\in B} O_i\right) - \sum_{J\supseteq B} \epsilon_J = \nu\left(\bigcup_{i\in B} O_i\right) - (1-s)\sum_{J\supseteq B}\nu(C_J)$$
$$= \nu\left(\bigcup_{i\in B} O_i\right) - (1-s)\nu\left(\bigcup_{i\in B} O_i\right) = s.\nu\left(\bigcup_{i\in B} O_i\right)$$

□

We claim that the family $(r.\nu[D])_{D \text{ dissection}, 0<r<1}$ is directed. Consider indeed $r_1.\nu[D_1]$ and $r_2.\nu[D_2]$, where $0 < r_1, r_2 < 1$ and D_1 and D_2 are two dissections. Let D_i be written as the set of all (x_{ij}, U_{ij}), $1 \leq j \leq n_i$, $i \in \{1,2\}$. Write C_{ij} for $\uparrow x_{ij} \setminus U_{ij}$. Let $O_1 = \uparrow x_{11}, \ldots, O_{n_1} = \uparrow x_{1n_1}, O_{n_1+1} = \uparrow x_{21}, \ldots, O_{n_2} = \uparrow x_{2n_2}$, and $n = n_1 + n_2$. Let $0 \leq s < 1$. By Lemma 4 there is a dissection D such that, for any $B \subseteq \{1, \ldots, n\}$,

$$s.\nu\left(\bigcup_{i\in B} O_i\right) \leq \nu[D]\left(\bigcup_{i\in B} O_i\right)$$

Let O be any open. Let B_1 be the set of indices j such that $x_{1j} \in O$. In particular O contains $\bigcup_{j\in B_1} \uparrow x_j$, so

$$s.\nu\left(\bigcup_{i\in B_1} O_i\right) \leq \nu[D](O)$$

On the other hand,

$$\nu[D_1](O) = \sum_{i=1}^{n_1} \nu(C_{1i})\delta_{x_{1i}}(O) = \sum_{i\in B_1}\nu(C_{1i}) = \nu\left(\bigcup_{i\in B_1} C_{1i}\right)$$
$$\leq \nu\left(\bigcup_{i\in B_1} \uparrow x_{1i}\right) = \nu\left(\bigcup_{i\in B_1} O_i\right)$$

So $s\nu[D_1](O) \leq \nu[D](O)$. Similarly we prove that $s\nu[D_2] \leq \nu[D](O)$. Since O is arbitrary, $s\nu[D_1] \leq \nu[D]$ and $s\nu[D_2] \leq \nu[D]$. Recall that s is arbitrary. We may choose s and some r such that $0 < s, r < 1$ and $r_1, r_2 \leq rs$. It follows that $r_1\nu[D_1] \leq r\nu[D]$ and $r_2\nu[D_2] \leq r\nu[D]$. Therefore $(r\nu[D])_{D \text{ dissection}, 0<r<1}$ is indeed directed. We conclude by Corollary 1. We have proved:

Proposition 1. *Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$. Then every continuous valuation on $(X, \mathcal{O})$ is a sup of a directed family of simple valuations $\sum_{i=1}^n a_i\delta_{x_i}$, where $x_i \in X_0$ for every i.*

The if direction of Theorem 1 follows by taking $X_0 = X$.

3 Applications

Given any continuous valuation ν on a topological space $(X, \mathcal{O})$, there are two extreme ways of attempting to extend ν to the Alexandroff extension of $\mathcal{O}$, i.e., to build an A-extension of ν. None of these yields a continuous valuation in general, but in particular cases they will.

First, we may approximate ν from below: define ν_* by

$$\nu_*(U) = \nu\left(\overset{\circ}{U}\right)$$

for each upward closed set U, where $\overset{\circ}{U}$ is the interior of U for the topology $\mathcal{O}$. We may also approximate ν from above: define ν^* by

$$\nu^*(U) = \inf_{O\in\mathcal{O}, O\supseteq U} \nu(O)$$

Both ν^* and ν_* are strict and monotonic. Clearly ν_* and ν^* bound all A-extensions of ν:

Lemma 5. *Let $(X, \mathcal{O})$ be a topological space. For every valuation ν on $(X, \mathcal{O})$, for every A-extension $\overline{\nu}$ of ν,*

$$\nu_* \leq \overline{\nu} \leq \nu^*$$

Proof. For every Alexandroff open U, $\overline{\nu}(U) \leq \overline{\nu}(O) = \nu(O)$ for every open $O \in \mathcal{O}$ such that $U \subseteq O$. So $\overline{\nu}(U) \leq \inf_{O \in \mathcal{O}, O \supseteq U} \nu(O) = \nu^*(U)$. On the other hand, $\overset{\circ}{U} \subseteq U$, so $\nu_*(U) = \nu(\overset{\circ}{U}) = \overline{\nu}(\overset{\circ}{U}) \leq \overline{\nu}(U)$. □

We shall refine the upper bound in Section 3.4, Fact 2.

We shall be particularly interested in topological spaces arising from directed complete partial orders. Recall that a *directed complete partial order*, or *dcpo*, is a partial order $(X, \leq)$ such that every directed subset of X has a least upper bound $\sup X$. A subset A of X is *directed* provided A is not empty and any two elements in A have an upper bound in A. The *Scott topology* on X has as opens all upward closed subsets O such that, whenever $\sup D \in O$ for some directed subset D of X, then D contains some element of X. The specialization ordering of the Scott topology is $\leq$, just as for the Alexandroff topology.

Remark 1. There is no reason why ν^* or ν_* should be modular. Let X be $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with the ordering $(j, k) \leq (m, n)$ if and only if either $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. This is a dcpo, which we shall study more closely in Remark 8. Its Scott opens contain all but finitely many points of the form (m, ω). In particular, every finite intersection of non-empty Scott opens is non-empty. This implies that the function ν mapping $\emptyset$ to 0 and each non-empty Scott open to 1 is modular, hence a continuous valuation. But ν^* maps every non-empty upward closed set to 1: let $U = \{(2i, \omega) | i \in \mathbb{N}\}$, $V = \{(2i+1, \omega) | i \in \mathbb{N}\}$, then $\nu^*(U \cup V) + \nu^*(U \cap V) = 1 + 0 = 1$ while $\nu^*(U) + \nu^*(V) = 1 + 1 = 2$. Similarly, $\nu_*(U \cup V) + \nu_*(U \cap V) = \nu_*\{(i, \omega) | i \in \mathbb{N}\} + \nu_*(\emptyset) = \nu\{(i, \omega) | i \in \mathbb{N}\} = 1$, while $\nu_*(U) + \nu_*(V) = 0 + 0 = 0$. □

Remark 2. There is no reason why ν_* should be continuous either. Take again ν and X as in Remark 1. Let $U_n = \{(i, \omega) | 0 \leq i \leq n\}$. Then $\nu_*(U_n) = \nu(\emptyset) = 0$, so $\sup_{n \in \mathbb{N}} \nu_*(U_n) = 0$, while $\nu_*(\sup_{n \in \mathbb{N}} U_n) = \nu_*\{(i, \omega) | i \in \mathbb{N}\} = \nu\{(i, \omega) | i \in \mathbb{N}\} = 1$. □

Remark 3. In general, ν^* is not continuous either. Let X be $[0, 1]$ with its usual, metric topology. Its specialization preordering is just equality, since X is Hausdorff, hence T_1. Let ν be the Lebesgue measure on the Borel σ-algebra built on top of this topology. By restriction to the open sets, ν yields a strict, monotonic and modular function. It is also continuous, because ν is a Radon measure, hence is inner regular, and X is locally compact. (More concretely: let O be the union $\bigcup_{i \in I} O_i$ of a directed family of open sets O_i. Since X is locally compact, for each $i \in I$, there is a compact K_i such that $O_i \subseteq \overset{\circ}{K}_i \subseteq K_i \subseteq O$. So $\nu(O) = \sup_{i \in I} \nu(K_i) = \nu(O)$ since $O = \bigcup_{i \in I} K_i$ and ν is inner-regular.) So ν defines a continuous valuation on X.

However, ν^* is not continuous. Indeed, let O be any open subset of X. O is the directed union of its finite subsets $\mathbb{K} \subseteq O$. But clearly $\nu^*(\mathbb{K}) = 0$. If ν^* was continuous, therefore, ν^* would be identically zero, which is absurd; e.g., $\nu^*(X) = \nu(X) = 1$. □

3.1 Continuous Dcpos and Other Spaces Where Interior Distributes Over Unions

We first reprove Jones' result (Corollary 2 below). Let $(X, \leq)$ be a dcpo. The relation $\ll$ defined by $x \ll y$ if and only if for every directed subset D of X such that $\sup D \geq y$ then $z \geq x$ for some $z \in D$ is called the *way-below* relation, and we write $\Uparrow x = \{y \in X | x \ll y\}$. $(X, \leq)$ is a *continuous* dcpo if and only if, for every $y \in X$, the set $D = \{x \in X | x \ll y\}$ is directed and $y = \sup D$. If X is a continuous dcpo, in particular $\Uparrow x$ is Scott open for every x. In fact the sets $\Uparrow x$, $x \in X$, forms a basis of the Scott topology: that is, every Scott open O is the union of all $\Uparrow x$, $x \in O$ (see e.g., [8], Section 2.1.3).

In any topological space, the interior operator distributes over intersection: the interior of $A \cap B$ is always $\overset{\circ}{A} \cap \overset{\circ}{B}$. In general the interior does not distribute over union: take X be $\mathbb{R}$ with its usual topology, A be the set of rational numbers, $B = \mathbb{R} \setminus A$, then the interior of $A \cup B$ is $\mathbb{R}$, while both $\overset{\circ}{A}$ and $\overset{\circ}{B}$ are empty. However,

Lemma 6. *Let $(X, \leq)$ be a continuous dcpo. Then Scott interior distributes over unions of upward closed subsets, i.e., for any family $(U_i)_{i \in I}$ of upward closed subsets of X,*

$$\overbrace{\bigcup_{i \in I} U_i}^{\circ} = \bigcup_{i \in I} \overset{\circ}{U_i}$$

Proof. First note that the interior of any upward closed subset U is $\bigcup_{x \in U} \uparrow x$. Indeed, on the one hand, for every $x \in U$, $\uparrow x$ is open since X is continuous. Every $y \in \uparrow x$ is such that $x \ll y$, in particular $x \leq y$, so $y \in U$ since U is upward closed. Therefore, for every $x \in U$, $\uparrow x$ is a Scott open contained in U. In particular $\overset{\circ}{U} \supseteq \bigcup_{x \in U} \uparrow x$. Conversely, $\overset{\circ}{U} = \bigcup_{O \text{ Scott open} \subseteq U} O = \bigcup_{O \text{ Scott open} \subseteq U} \bigcup_{x \in O} \uparrow x \subseteq \bigcup_{x \in U} \uparrow x$, since every $x \in O \subseteq U$ is in U.

Now let U and V be upward closed in X. Since $U \cup V$ is upward closed, the interior of $U \cup V$ is the union of all $\uparrow x$, $x \in U \cup V$, which is the union of $\bigcup_{x \in U} \uparrow x = U$ and $\bigcup_{x \in V} \uparrow x = V$. A similar argument shows that interior distributes over infinite unions as well. □

Lemma 7. *Let $(X, \mathcal{O})$ be a topological space, and assume that interior distributes over unions of upward closed sets. Then every bounded continuous valuation on X is quasi-simple.*

Proof. By Theorem 1 it suffices to show that every bounded valuation ν extends to the Alexandroff topology over $\leq$. Define the extension $\overline{\nu}$ as ν_*; in other words, $\overline{\nu}(U) = \nu(\overset{\circ}{U})$ for every upward closed subset U. Clearly $\overline{\nu}$ is strict and monotonic. It is also modular:

$$\begin{aligned}
\overline{\nu}(U \cup V) + \overline{\nu}(U \cap V) &= \nu\left(\overbrace{U \cup V}^{\circ}\right) + \nu\left(\overbrace{U \cap V}^{\circ}\right) \\
&= \nu(\overset{\circ}{U} \cup \overset{\circ}{V}) + \nu(\overset{\circ}{U} \cap \overset{\circ}{V}) \quad \text{(since interior distributes over } \cup, \cap) \\
&= \nu(\overset{\circ}{U}) + \nu(\overset{\circ}{V}) \qquad \text{(since } \nu \text{ is modular)} \qquad = \overline{\nu}(U) + \overline{\nu}(V)
\end{aligned}$$

Finally $\overline{\nu}$ is continuous. Indeed, if $(U_i)_{i \in I}$ is a directed family of subsets,

$$\begin{aligned}
\overline{\nu}\left(\bigcup_{i \in I} U_i\right) &= \nu\left(\overbrace{\bigcup_{i \in I} U_i}^{\circ}\right) = \nu\left(\bigcup_{i \in I} \overset{\circ}{U_i}\right) \qquad \text{(since interior distributes over directed unions)} \\
&= \sup_{i \in I} \nu(\overset{\circ}{U_i}) \qquad \text{(since } \nu \text{ is continuous)} \qquad = \sup_{i \in I} \overline{\nu}(U_i)
\end{aligned}$$

□

The proof above says something more:

Fact 1 *Under the assumption of Lemma 7, every bounded continuous valuation ν has a* least *A-extension.*

Namely $\overline{\nu} = \nu_*$: see Lemma 5. This is in particular the case for continuous dcpos, by Lemma 6. Also, the following is an immediate consequence:

Corollary 2 (Jones). *Every bounded continuous valuation on a continuous dcpo (with the Scott topology) is quasi-simple.*

Remark 4 (Non-uniqueness of extensions). A natural question is whether every bounded continuous valuation extends to a *unique* continuous valuation on the Alexandroff topology, when it extends at all. This is wrong, even when X is a continuous dcpo. Take indeed $X = \mathbb{N} \cup \{\omega\}$, with the natural ordering on $\mathbb{N}$ and $n < \omega$ for all $n \in \mathbb{N}$. This is a continuous dcpo, and the non-empty Scott opens are all sets $[k, \omega] = \{k, k+1, k+2, \ldots, \omega\}$, $k \in \mathbb{N}$. Note that $\{\omega\}$ is upward closed but not Scott open. Define

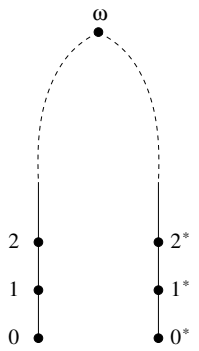
$\nu([k,\omega]) = (ak+1)/(k+1)$, where a is some real number between 0 and 1. This is a continuous probability valuation on X, which can be extended by letting $\nu\{\omega\}$ be any real number b between 0 and a. If $a > 0$, we therefore get uncountably many distinct A-extensions of ν. Correspondingly, there are uncountably many distinct ways of writing ν as sups of directed families of simple valuations, e.g., as $\sup_{n\in\mathbb{N}} \nu_n$, where ν_n is the increasing (hence directed) sequence of valuations

$$\nu_n = \sum_{k=0}^{n-1} \frac{1-a}{(k+1)(k+2)}\delta_k + \left(\frac{an+1}{n+1} - b\right)\delta_n + b\delta_\omega$$

for all $n \in \mathbb{N}$. Note that ν_n is just $\nu[D]$ where D is the dissection obtained from the finitary open $\uparrow\{\omega, n, n-1, \ldots, 1, 0\}$ as we did in the proof of Lemma 3. □

Remark 5 (On extensions to measures). Recall that the Borel σ-field $\sigma(\mathcal{O})$ generated by a given topology $\mathcal{O}$ on X is the least collection of sets containing $\mathcal{O}$ and such that every complement, every countable union, every countable intersection of sets in $\sigma(\mathcal{O})$ is again in $\sigma(\mathcal{O})$. The sets in $\sigma(\mathcal{O})$ are called the *Borel subsets* of X when $\mathcal{O}$ is understood. Theorem 2.16 of [2] states that on monotone convergence spaces, every quasi-simple valuation extends to a unique measure on the Borel subsets. (A *monotone convergence space* is a T_0 topological space $(X, \mathcal{O})$ whose specialization ordering $\leq_s$ induces a dcpo structure whose Scott topology is finer than $\mathcal{O}$. Every dcpo is a monotone convergence space.) On the example of Remark 4, every subset of $\mathbb{N} \cup \{\omega\}$ is a Borel subset, and the unique measure μ that extends ν maps each subset A of $\mathbb{N} \cup \{\omega\}$ to $\sum_{k\in A} \frac{1-a}{(k+1)(k+2)}$, to which we must add a if $\omega \in A$. It follows that among the A-extensions of ν, only the one that is obtained by letting b be exactly a extends to a measure. In particular the assumption that X be a monotone convergence space in Theorem 2.16 of op.cit. is essential, in that the example above illustrates the non-existence of extensions of continuous valuations to measures when X is equipped with an Alexandroff topology; a space with the Alexandroff topology is a monotone convergence space only when it coincides with the Scott topology. □

Remark 6 (Extensions on non-continuous dcpos). Another natural question is whether a form of converse to Corollary 2 holds: assume $(X, \leq)$ is a dcpo on which every bounded continuous valuation is quasi-simple; is X a *continuous* dcpo? Again, the answer is no. Consider the dcpo X shown as the Hasse diagram below.



X is $\mathbb{N} \cup \mathbb{N}^* \cup \{\omega\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{0^*, 1^*, 2^*, \ldots\}$, with the obvious ordering. The upward-closed subsets are the empty set $\emptyset$, the singleton $\{\omega\}$, the sets $[n, \omega] = \{n, n+1, n+2, \ldots, \omega\}$ and $[n^*, \omega] = \{n^*, (n+1)^*, (n+2)^*, \ldots, \omega\}$, $n \in \mathbb{N}$, and the sets $A_{m,n} = [m, \omega] \cup [n^*, \omega]$, $m, n \in \mathbb{N}$.

Amongst these, only $\emptyset$ and $A_{m,n}$ are Scott open. In particular, $[n, \omega]$ is not Scott open since ω can be reached as the sup of all m^*, $m \in \mathbb{N}$, but no m^* is in $[n, \omega]$.

Observe that X is *not* continuous. Indeed, note that there is no element way-below ω, so the set of points way-below ω is not directed, and its sup is not ω. (Check that $n \not\ll \omega$, since ω is the sup of all m^*, but $m^* \geq n$ for no m ; similarly for the other cases.)

Now consider any bounded continuous valuation ν on the Scott topology; let $\nu(X) = 1$, without loss of generality. Then ν is entirely determined by an infinite matrix $(\nu_{ij})_{i,j\in\mathbb{N}}$, where $\nu_{ij} = \nu(A_{ij})$. In particular $\nu_{ij} \leq \nu_{i'j'}$ as soon as $i \geq i'$ and $j \geq j'$. Observe that $A_{ij} \cup A_{i'j'} = A_{\min(i,i')\ \min(j,j')}$, and $A_{ij} \cap A_{i'j'} = A_{\max(i,i')\ \max(j,j')}$. In particular $A_{i(j+1)} \cup A_{(i+1)j} = A_{ij}$ and $A_{i(j+1)} \cap A_{(i+1)j} = A_{(i+1)(j+1)}$, so modularity of ν entails

$$\nu_{ij} + \nu_{(i+1)(j+1)} = \nu_{(i+1)j} + \nu_{i(j+1)} \tag{8}$$

for all $i, j \in \mathbb{N}$. Rearranging summands, it obtains $\nu_{(i+1)j} - \nu_{ij} = \nu_{(i+1)(j+1)} - \nu_{i(j+1)}$. In other words, $\nu_{(i+1)j} - \nu_{ij}$ is independent of j, therefore $\nu_{0j} - \nu_{ij}$ is also independent of j. Name this quantity α_i, and note that $\alpha_i \geq 0$. Paraphrasing the definition of α_i, we obtain

$$\nu_{ij} = \nu_{0j} - \alpha_i \tag{9}$$

Similarly, (8) also entails $\nu_{(i+1)(j+1)} - \nu_{(i+1)j} = \nu_{i(j+1)} - \nu_{ij}$. So $\nu_{i(j+1)} - \nu_{ij}$ and therefore also $\alpha^*_j = \nu_{i0} - \nu_{ij}$ is independent of i. In other words,

$$\nu_{ij} = \nu_{i0} - \alpha^*_j \tag{10}$$

Putting (9) and (10) together, we obtain $\nu_{ij} = 1 - \alpha_i - \alpha^*_j$ (remember that $\nu_{00} = \nu(A_{00}) = \nu(X) = 1$ by assumption). Extend ν to $\overline{\nu}$, defined on all upward closed sets, by letting $\overline{\nu}\{\omega\} = 0$, $\overline{\nu}([i,\omega]) = a - \alpha_i$ and $\overline{\nu}([j,\omega]) = a^* - \alpha^*_j$, where a and a^* are arbitrary non-negative reals such that $a + a^* = 1$, $a \geq \sup_{i\in\mathbb{N}} \alpha_i$, $a^* \geq \sup_{j\in\mathbb{N}} \alpha^*_j$. This extension is clearly strict, monotonic and continuous. It is also modular: there are many cases to consider, let us look at a few representative ones. If $U = [i,\omega]$ and $V = [j^*,\omega]$, then

$$\begin{aligned}\overline{\nu}(U \cup V) + \overline{\nu}(U \cap V) &= \overline{\nu}(A_{ij}) + \overline{\nu}\{\omega\} = \overline{\nu}(A_{ij}) = \nu_{ij} \\ &= 1 - \alpha_i - \alpha^*_j = a - \alpha_i + a^* - \alpha^*_j = \nu(U) + \nu(V)\end{aligned}$$

Consider the case $U = [n,\omega]$ and $V = A_{ij}$, then

$$\begin{aligned}\overline{\nu}(U \cup V) + \overline{\nu}(U \cap V) &= \overline{\nu}(A_{\min(n,i)j}) + \overline{\nu}([\max(n,i),\omega]) \\ &= 1 - \alpha_{\min(n,i)} - \alpha^*_j + a - \alpha_{\max(n,i)} \\ &= 1 - \alpha_i - \alpha^*_j + a - \alpha_n = \overline{\nu}(A_{ij}) + \overline{\nu}([n,\omega]) = \overline{\nu}(V) + \overline{\nu}(U)\end{aligned}$$

So X is a non-continuous dcpo on which every bounded continuous valuation ν has an A-extension $\overline{\nu}$, and is therefore quasi-simple by Theorem 1. Notice also that there are in general uncountably many A-extensions $\overline{\nu}$, at least as many as the number of possible pairs (a, a'). (There are others, where $\overline{\nu}\{\omega\}$ is not restricted to be 0.) We shall see in Section 3.4 that X falls into another natural class of spaces where continuous valuations are quasi-simple. □

Remark 7 (Non-existence of extensions). We might then ask whether there are any cases at all of a continuous valuation on a topological space that is not quasi-simple. Indeed there are. Take $X = \mathbb{N}$ with the *cofinite* topology, i.e., where the closed sets are $\mathbb{N}$ and all finite subsets of $\mathbb{N}$. Since every finite intersection of non-empty open sets is again non-empty, the function ν mapping $\emptyset$ to 0 and every non-empty open to 1 is modular. It is also clearly strict, monotonic, and continuous, hence ν is a continuous valuation. The specialization preordering $\leq_s$ is just equality, so the Alexandroff extension of the cofinite topology is the discrete topology. Take any continuous valuation $\overline{\nu}$ extending ν to the discrete topology. For any $n \in \mathbb{N}$, $\overline{\nu}[0,n]$ must be 0 since $\overline{\nu}[0,n] = \nu(\mathbb{N}) - \nu[n,+\infty)$. Since $\overline{\nu}$ is continuous, we must have $\nu(\mathbb{N}) = \overline{\nu}(\mathbb{N}) = \sup_{n\in\mathbb{N}} \overline{\nu}[0,n] = 0$, a contradiction. By Theorem 1, ν is therefore not quasi-simple. □

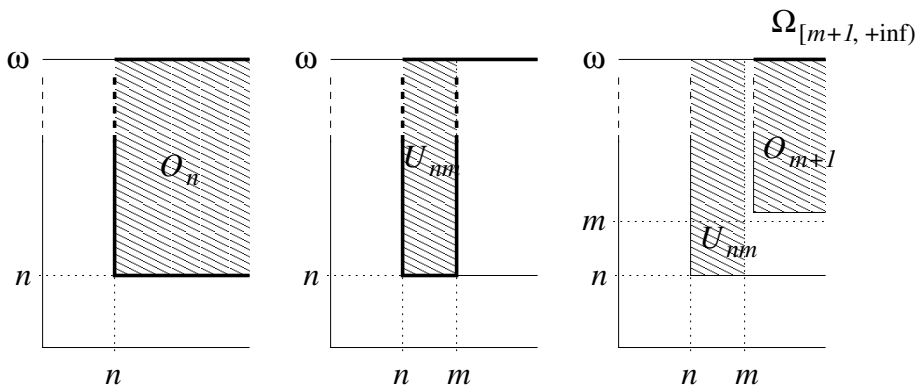


Fig. 1. The various sets used in Remark 8

Remark 8 (Non-existence of extensions on dcpos). We may refine the observation that there are continuous, non-quasi-simple valuations, even when X is a dcpo. The simplest known example uses Johnstone's famous example of a non-sober dcpo; see Alvarez-Manilla [2], Section 2.7, p.73. At this point, we should warn the reader that non-sobriety is a red herring here; we shall return to this example in Remark 16.

The space X is the example we used in Remark 1: let X be $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with the ordering $(j,k) \leq (m,n)$ if and only if either $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. Recall that this is a dcpo, and that the function ν mapping $\emptyset$ to 0 and each non-empty Scott open to 1 is modular, hence a continuous valuation. However, it cannot be extended to all upward closed sets; Alvarez-Manilla relies on two theorems to show this, one saying that any quasi-simple valuation extends uniquely to a measure on the Borel subsets of the topology (Theorem 2.16 in op.cit.), and the other stating that for every measure μ, for every decreasing sequence of measurable sets A_n, $n \in \mathbb{N}$, whose intersection is empty, then $\lim_{n \to +\infty} \mu(A_n) = 0$ (Proposition 1.14 in op.cit.) We rely on Theorem 1 instead, avoiding the need to introduce measures.

Assume ν has an A-extension $\overline{\nu}$. Fix $\epsilon \geq 0$ such that $\epsilon < 1/4$. For each subset S of $\mathbb{N}$, let Ω_S be the upward closed, but not Scott open in general, set $\{(i,\omega) | i \in S\}$. Let O_n be the Scott open $\{(i,j) | i,j \geq n\} \cup \Omega_{[n,+\infty)}$. For every $m \geq n$, let U_{nm} the upward closed set $\{(i,j) | i,j \geq n, i \leq m\} \cup \Omega_{[n,+\infty)}$; this is not Scott open. (See Figure 1 to see what these subsets look like in the plane.) However O_n is the directed union of all U_{nm}, $m \geq n$. Since $\overline{\nu}$ is continuous and $\overline{\nu}(O_n) = \nu(O_n) = 1$, there is an $m \geq n$ such that $\overline{\nu}(U_{nm}) \geq 1 - \epsilon$. By modularity $\overline{\nu}(U_{nm}) + \overline{\nu}(O_{m+1}) = \overline{\nu}(U_{nm} \cup O_{m+1}) + \overline{\nu}(U_{nm} \cap O_{m+1})$. The left-hand side is greater than or equal to $2 - \epsilon$, while $\overline{\nu}(U_{nm} \cup O_{m+1}) \leq \overline{\nu}(O_n) = 1$ by monotonicity. Since $U_{nm} \cap O_{m+1} = \Omega_{[m+1,+\infty)}$, it follows that $\overline{\nu}(\Omega_{[m+1,+\infty)}) \geq 1 - \epsilon$. Since $\Omega_{[m+1,+\infty)}$ is the directed union of all $\Omega_{[m+1,p]}$, $p \geq m+1$, by continuity again, there is $p \geq m+1$ such that $\overline{\nu}(\Omega_{[m+1,p]}) \geq 1 - 2\epsilon$. Remember that n was arbitrary, so we have proved:

$$\text{For every } n \in \mathbb{N}, \text{ there are } p > m \geq n \text{ such that } \overline{\nu}(\Omega_{[m+1,p]}) \geq 1 - 2\epsilon \qquad (*)$$

Now use $(*)$ twice. First take $n = 0$, giving $p > m$ such that $\overline{\nu}(\Omega_{[m+1,p]}) \geq 1 - 2\epsilon$. Then take $n = p+1$, yielding $p' > m' \geq p+1$ such that $\overline{\nu}(\Omega_{[m'+1,p']}) \geq 1 - 2\epsilon$. By modularity, it then follows that $\overline{\nu}(\Omega_{[m,p] \cup [m'+1,p']}) \geq 2 - 4\epsilon > 1$ since $\epsilon < 1/4$. But this is impossible, since $\overline{\nu}(\Omega_{[m,p] \cup [m'+1,p']}) \leq \overline{\nu}(X) = \nu(X) = 1$. □

Remark 9 (Sups of undirected families). Defining quasi-simple valuations as sups *of directed families* of simple valuations is important. First and foremost, the sup of an undirected family of (simple) valuations needs not be a valuation, because it needs not be modular: take ν as the sup of δ_a and δ_b on the two-element set $\{a,b\}$ with $=$ as ordering, then $\nu(\{a\} \cup \{b\}) + \nu(\{a\} \cap \{b\}) = 1$ but $\nu(\{a\}) + \nu(\{b\}) = 2$. More subtly, there are continuous valuations that are sups of simple valuations but are not quasi-simple: e.g., the valuation ν of Remark 1 and Remark 8 is the sup of all δ_x, $x \in X$, but is not quasi-simple. Clearly, the family $(\delta_x)_{x \in X}$ is not directed. □

3.2 Algebraic Dcpos

An element x of a dcpo $(X, \leq)$ is *finite* if and only if $x \ll x$. Let X_0 be the set of finite elements in X. X itself is *algebraic* if and only if, for every $y \in X$, the set $D = \{x \in X_0 | x \leq y\}$ is directed and $y = \sup D$. Every algebraic dcpo is continuous. Furthermore, if X is algebraic, then $\uparrow x$ is Scott open for every $x \in X_0$, and in fact every Scott open O is $\bigcup_{x \in O \cap X_0} \uparrow x$. Proposition 1 immediately entails:

Corollary 3. *Let $(X, \leq)$ be an algebraic dcpo, and X_0 its subset of finite elements. Then every bounded continuous valuation on X, equipped with its Scott topology, is the sup of a directed family of* simple valuations based on finite elements*, i.e., of simple valuations $\sum_{i=1}^{n} a_i \delta_{x_i}$ such that $x_1, \ldots, x_n \in X_0$.*

Remark 10. This was first claimed by Sünderhauf [10], whose Theorem 3.4 states the corollary above, in particular, although the fact that we can choose the family of simple valuations to be directed is stated without justification. Theorem 3.4 of [10] also states that each simple valuation based on finite elements is finite, but this is wrong: as already observed by Jones ([8], p.84), $\delta_x \ll \delta_y$ holds of no pair of elements x and y such that $x \ll y$; in particular, δ_x is never finite, even when x is finite. In fact, it is easy to see that the only finite valuation ν (in the sense that $\nu \ll \nu$) is the everywhere zero valuation: if $\nu \neq 0$, the directed family $(r\nu)_{0 \leq r < 1}$ has ν as supremum, but $r\nu \geq \nu$ is false for all $r < 1$. □

Corollaries 2 and 3 apply to bounded continuous valuations, in particular to *sub-probability* valuations ν, such that $\nu(X) \leq 1$. However the construction of Section 2.1 requires valuations $r\nu[D]$ such that $r\nu[D](X)$ may be strictly less than 1. These results extend to *probability* valuations ν (such that $\nu(X) = 1$) in the case where X has a least element $\bot$:

Corollary 4. *Let $(X, \leq)$ be a continuous dcpo with bottom $\bot$.*

Every bounded continuous probability valuation ν on X, with its Scott topology, is the sup of a directed family of simple probability valuations $\sum_{i=1}^{n} a_i \delta_{x_i}$ on X.

If X is algebraic, we may additionally require that x_i is finite, $1 \leq i \leq n$.

Proof. Use Edalat's trick [4]: apply Corollaries 2 and 3 to the restriction ν' of ν to the dcpo $X' = X \setminus \{\bot\}$. Then ν' is the sup of a directed family of simple sub-probability valuations $(\nu'_i)_{i \in I}$ on X', where $\nu'_i = \sum_j a_{ij} \delta_{x_{ij}}$, and $\sum_j a_{ij} \leq 1$. Let ν_i be $\sum_j a_{ij} \delta_{x_{ij}} + (\sum_j a_{ij}) \delta_\bot$—i.e., add the missing mass at $\bot$. Then it is easy to check that $(\nu_i)_{i \in I}$ is directed and $\sup_{i \in I} \nu_i = \nu$. □

3.3 Bicontinuous valuations

Before we turn to other kinds of spaces in Section 3.4, let us have a quick glance at bicontinuous valuations, as introduced by Bukatin and Shorina [3]. Let $(X, \mathcal{O})$ be a topological space, $\leq_{\mathrm{s}}$ its specialization preordering, and $\downarrow x$ be $\{y \in X | y \leq_{\mathrm{s}} x\}$. Note that $\downarrow x$ is always closed. In fact, $\downarrow x$ is just the closure of x.

Lemma 8. *Let $(X, \mathcal{O})$ be a topological space, $\leq_{\mathrm{s}}$ its specialization preordering. For every upward closed subset U, $U = \bigcap_{O \in \mathcal{O}, O \supseteq U} O$.*

Proof. The argument is a slight generalization of one due to Jones ([8], p.81). The $\subseteq$ direction is clear. Conversely, assume $x \notin U$. In particular $\downarrow x$ does not intersect U, otherwise there would be $y \in U$ with $y \leq_{\mathrm{s}} x$, i.e., such that x is in every open containing y: so x would be in U, a contradiction. Since $\downarrow x$ does not intersect U, U is contained inside the open $X \setminus \downarrow x$. Clearly x is not in the open $O = X \setminus \downarrow x$, hence not in $U = \bigcap_{O \in \mathcal{O}, O \supseteq U} O$. □

A valuation ν is *cocontinuous* ([3]; [2], Section 1.10, p.41) if and only if, for every filtered family F of opens, $\nu\left(\overset{\circ}{\bigcap_{O \in F} O}\right) = \inf_{O \in F} \nu(O)$. F is *filtered* if and only if F is not empty, and for every $O_1, O_2 \in F$, there is $O \in F$ such that $O \subseteq O_1, O_2$. The valuation ν is *bicontinuous* [9] if and only if it is both continuous and cocontinuous.

Recall that we have defined two approximations of continuous valuations (Lemma 5).

Lemma 9. *For every cocontinuous valuation ν, $\nu_* = \nu^*$.*

Proof. Assume that ν is cocontinuous. Let U be any Alexandroff open. By Lemma 8, $U = \bigcap_{O \in \mathcal{O}, O \supseteq U} O$. The family of opens O containing U is clearly filtered. Since ν is cocontinuous, $\nu(\overset{\circ}{U}) = \inf_{O \in \mathcal{O}, O \supseteq U} \nu(O)$. That is, $\nu_* = \nu^*$. □

It follows that every bicontinuous valuation has at most one A-extension.

The sup of any directed family of continuous valuations is again a continuous valuation ([8], Theorem 4.1, p.65). It is not clear whether the sup of a directed family of quasi-simple valuations should be again a quasi-simple valuation (although it is a continuous valuation). However, due to Lemma 9, we obtain:

Lemma 10. *On any topological space $(X, \mathcal{O})$, the sup of any directed family of quasi-simple bicontinuous valuations is quasi-simple.*

Proof. Let $(\nu_i)_{i \in I}$ be a directed family of quasi-simple bicontinuous valuations, with sup ν. By Theorem 1, Lemma 5 and Lemma 9, ν_i has a unique A-extension $\overline{\nu}_i$. Moreover, since the map $\nu \mapsto \nu_*$ (or $\nu \mapsto \nu^*$) is monotonic, so is $\nu_i \mapsto \overline{\nu}_i$. It follows that the family $(\overline{\nu}_i)_{i \in I}$ is a directed family of continuous valuations on the Alexandroff extension of $\mathcal{O}$. So it has a sup $\overline{\nu}$, which is indeed an extension of ν to the Alexandroff extension of $\mathcal{O}$. By Theorem 1, ν is quasi-simple. □

3.4 Finitarily Coherent Spaces

Let $(X, \mathcal{O})$ be a topological space. Recall that $K \subseteq X$ is *compact* if and only if for every family of opens $O_i \in \mathcal{O}$, $i \in I$, such that $K \subseteq \bigcup_{i \in I} O_i$, there is a finite sub-family $I_{fin} \subseteq I$ such that $K \subseteq \bigcup_{i \in I_{fin}} O_i$. A subset A of X is *saturated* if and only if $A = \bigcap_{O \in \mathcal{O} \supseteq A} O$. (By Lemma 8, the saturated sets are exactly those that are upward closed in the specialization preordering.) Write $\mathcal{Q}$ the set of compact saturated subsets of X.

A topological space is *locally compact* if and only if every neighborhood of any point contains a compact neighborhood. In other words, for every open O, for every $x \in O$, there is a compact K such that $x \in \overset{\circ}{K} \subseteq K \subseteq O$.

A topological space $(X, \mathcal{O})$ is *sober* if and only if every irreducible closed subset is the closure of a unique point x. A closed set F is *irreducible* if and only if it is non-empty and, for any two closed subsets F_1 and F_2 such that $F \subseteq F_1 \cup F_2$, either $F \subseteq F_1$ or $F \subseteq F_2$. Given the specialization preordering $\leq_{\mathrm{s}}$, the closure of a point x is $\downarrow x$. That x should be unique means that $\leq_{\mathrm{s}}$ should be an ordering, i.e., X should be T_0. The other point is that every irreducible closed subset should be of the form $\downarrow x$ for some x. In the example of Remark 1 and Remark 8, X is not sober: X itself is irreducible, but is not of the form $\downarrow x$.

The most important property of sober spaces is the Hofmann-Mislove theorem ([1], Proposition 7.2.9, p.74): the sets $\mathcal{N}_Q = \{O \in \mathcal{O} | O \supseteq Q\}$, where Q is compact saturated, are exactly the Scott open filters of opens in $\mathcal{O}$. It follows ([2], Proposition 3.9, p.78), that, if $(X, \mathcal{O})$ is sober, then $(X, \leq_{\mathrm{s}})$ is a dcpo, and $\mathcal{O}$ is *order-consistent*, meaning that it is finer than the *upper topology*, generated by all complements of sets of the form $\downarrow x$, $x \in X$, and it is coarser than the Scott topology on $\leq_{\mathrm{s}}$. Also, every filtered intersection of compact saturated sets is again compact saturated and if $(Q_i)_{i \in I}$ is a filtered family of compact saturated sets, and $\bigcap_{i \in I} Q_i \subseteq O$ for some open $O \in \mathcal{O}$, then $Q_i \in O$ for some i. (A family is *filtered* for $\subseteq$ if and only if it is directed for $\supseteq$.)

While the union of compact saturated subsets is again compact saturated, their intersection needs not be. A topological space that is sober, locally compact, and such that the intersection of any two compact saturated sets is again compact saturated is called a *coherent space* ([11], Definition 3.1, p.17; or [1], Definition 7.2.17, p.75). Some authors use different definitions. For example, Alvarez-Manilla ([2], Definition 1.11, p.31) drops local compactness from the definition of coherent spaces. By [2], Proposition 1.12, p.32, coherent spaces are also exactly the so-called *stably locally compact* spaces, as well as the locally compact supersober spaces.

A result by Tix, also discovered by Norberg and Vervaat ([11], Satz 3.4, p.19; [2], Proposition 3.33, p.88) is that, on a coherent space $(X, \mathcal{O})$, given any continuous valuation ν on $\mathcal{O}$, the restriction of ν^* to compact saturated subsets is strict, monotonic, modular, *inf-continuous*, i.e., for every filtered family of compact saturated subsets $(Q_i)_{i \in I}$, $\nu^*(\bigcap_{i \in I} Q_i) = \inf_{i \in I} \nu^*(Q_i)$; moreover, for every open $O \in \mathcal{O}$, $\nu(O) = \sup_{Q \text{ compact saturated } \subseteq O} \nu^*(Q)$.

Earlier, we have called subsets of the form $\uparrow \{x_1, \ldots, x_n\}$ the *finitary opens* of the Alexandroff topology of a preorder $(X, \leq)$. When $\leq$ is the specialization preordering $\leq_{\mathrm{s}}$ of a topological space, such subsets are compact saturated. Indeed, by the remark above, they are saturated. Furthermore, if $\uparrow \{x_1, \ldots, x_n\} \subseteq \bigcup_{i \in I} O_i$, where each O_i is open, choose O_{i_1} containing $x_1, \ldots, O_{i_n}$ containing x_n, then $\uparrow \{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} O_{i_k}$, so $\uparrow \{x_1, \ldots, x_n\}$ is compact. Note that this is true whatever the topology $\mathcal{O}$, as soon as it has $\leq$ as specialization preordering. Call such subsets $\uparrow \{x_1, \ldots, x_n\}$ *finitary compacts*.

While there may be more compact saturated subsets than just the finitary ones in general, the compact saturated subsets of an Alexandroff topology are exactly the finitary compacts. (And therefore every compact saturated subset is also open.) Indeed, given any compact saturated set Q, $Q = \bigcup_{x \in Q} \uparrow x$ since Q is saturated, hence upward closed; since Q is compact, Q is in fact equal to a finite union of sets of the form $\uparrow x$, $x \in Q$, so Q is a finitary compact.

Following Heckmann [7], call a topological space *locally finitary* if and only if, for every open O, for every $x \in O$, there is a *finitary* compact $\mathbb{K}$ such that $x \in \overset{\circ}{\mathbb{K}} \subseteq \mathbb{K} \subseteq O$. In other words, if each point has a fundamental system of finitary compact neighborhoods. In particular, every locally finitary space is locally compact.

In the sequel, the right kind of topological space we need looks like a coherent space. The main role will however be played not by compact saturated subsets but by finitary compacts: instead of it being locally

compact, we require it to be locally finitary; instead of requiring that the intersection of two compacts saturated subsets be compact, we require that the intersection of two finitary compacts is finitary compact; finally, we replace sobriety by a weaker property:

Definition 1. *A topological space $(X, \mathcal{O})$ is called* finitarily sober *if and only if, for every open O and every filtered family $(\mathbb{K}_i)_{i\in I}$ of finitary compacts such that $\bigcap_{i\in I} \mathbb{K}_i \subseteq O$, then $\mathbb{K}_i \subseteq O$ for some $i \in I$.*

$(X, \mathcal{O})$ is finitarily coherent *if and only if it is finitarily sober, locally finitary, and the intersection of any two finitary compacts is finitary compact.*

Remark 11. Every sober space is clearly finitarily sober. The converse does not hold. For one, a finitarily sober space needs not even be T_0. On the other hand, a finitarily sober topology is coarser than the Scott topology with respect to its specialization preorder $\leq_s$: if $(x_i)_{i\in I}$ is a directed family in $(X, \leq_s)$, then $(\uparrow x_i)_{i\in I}$ is a filtered family of finitary compacts, so for any open O, if $(x_i)_{i\in I}$ has a sup x in O, then $\bigcap_{i\in I} \uparrow x_i = \uparrow x \subseteq O$, so $\uparrow x_i \in O$, therefore $x_i \in O$ for some $i \in I$. (This discussion requires one to generalize slightly the definition of Scott topology to non-dcpos, see e.g. [2], Section 1.6, p.30.)

Conversely, if $(X, \leq)$ is a dcpo, then the Scott topology on X is always finitarily sober: this is Lemma 3.9.4 of Heckmann's PhD thesis [6]. In particular, if $(X, \mathcal{O})$ is a monotone convergence space (see Remark 5 for the definition), then it is finitarily sober. □

Remark 12. That the intersection of any two finitary compacts is finitary compact is a purely-order theoretic property: it is equivalent to requiring that for any two points x, y of X, there is a finite set $\{x_1, \ldots, x_n\}$ such that $z \geq x$ and $z \geq y$ is equivalent to $z \geq x_i$ for some i. In particular, this is so if x and y have no common upper bound, or when x and y have a least upper bound. □

For readability, write $\mathbb{K}$, possibly subscripted or primed, for finitary compacts, and O for opens. Tix's Satz 3.4 can be adapted to the finitarily coherent case:

Proposition 2. *Let $(X, \mathcal{O})$ be a finitarily coherent space. For every continuous valuation ν on $\mathcal{O}$,*

1. *for every open O, $\nu(O) = \sup_{\mathbb{K}\subseteq O} \nu^*(\mathbb{K})$;*
2. *ν^* is strict on the finitary compacts, i.e., $\nu^*(\emptyset) = 0$;*
3. *ν^* is monotonic on the finitary compacts, i.e., $\nu^*(\mathbb{K}_1) \leq \nu^*(\mathbb{K}_2)$ as soon as $\mathbb{K}_1 \subseteq \mathbb{K}_2$;*
4. *ν^* is modular on the finitary compacts, i.e., $\nu^*(\mathbb{K}_1 \cup \mathbb{K}_2) + \nu^*(\mathbb{K}_1 \cap \mathbb{K}_2) = \nu^*(\mathbb{K}_1) + \nu^*(\mathbb{K}_2)$ for every $\mathbb{K}_1, \mathbb{K}_2$.*
5. *ν^* is inf-continuous: for every filtered family $(\mathbb{K}_i)_{i\in I}$, $\nu^*\left(\bigcap_{i\in I} \mathbb{K}_i\right) = \inf_{i\in I} \nu^*(\mathbb{K}_i)$.*

Proof. 1. Clearly $\nu^*(\mathbb{K}) \leq \nu(O)$ for every $\mathbb{K} \subseteq O$, so $\sup_{\mathbb{K}\subseteq O} \nu^*(\mathbb{K}) \leq \nu(O)$. Conversely, for every open O, since X is locally finitary, O is the directed union of all $\overset{\circ}{\mathbb{K}}$, for $\mathbb{K} \subseteq O$. Indeed, for each $x \in O$, let $\mathbb{K}_x$ be such that $x \in \overset{\circ}{\mathbb{K}_x} \subseteq \mathbb{K}_x \subseteq O$, then $O \subseteq \bigcup_{x\in O} \overset{\circ}{\mathbb{K}_x} \subseteq \bigcup_{\mathbb{K}\subseteq O} \overset{\circ}{\mathbb{K}}$; the converse inclusion is obvious. So $\nu(O) = \sup_{\mathbb{K}\subseteq O} \nu(\overset{\circ}{\mathbb{K}}) \leq \sup_{\mathbb{K}\subseteq O} \nu^*(\mathbb{K})$.

2. is obvious, since $\emptyset$ is finitary compact. 3. is also straightforward.

4. We first make a few observations:

(a) For every $\mathbb{K} \subseteq O$, $\mathbb{K} \subseteq \overset{\circ}{\mathbb{K}'} \subseteq \mathbb{K}' \subseteq O$ for some finitary compact $\mathbb{K}'$. Indeed, write $\mathbb{K} = \uparrow\{x_1, \ldots, x_n\}$. Since X is locally finitary, for every i with $1 \leq i \leq n$, there is a finitary compact $\mathbb{K}_i$ such that $x_i \in \overset{\circ}{\mathbb{K}_i} \subseteq \mathbb{K}_i \subseteq O$. So we may take $\mathbb{K}' = \bigcup_{i=1}^n \mathbb{K}_i$, noticing that $\bigcup_{i=1}^n \overset{\circ}{\mathbb{K}_i} \subseteq \overset{\circ}{\mathbb{K}'}$.

(b) Every finitary compact $\mathbb{K}$ is the filtered intersection of all $\mathbb{K}'$ such that $\mathbb{K} \subseteq \overset{\circ}{\mathbb{K}'}$. Since $\mathbb{K}$ is saturated, $\mathbb{K}$ is the intersection of all opens $O \supseteq \mathbb{K}$; on the one hand every $\mathbb{K}'$ such that $\overset{\circ}{\mathbb{K}'} \supseteq \mathbb{K}$ contains some open $O \supseteq \mathbb{K}$, trivially, on the other hand every open $O \supseteq \mathbb{K}$ contains some finitary compact $\mathbb{K}'$ such that $\overset{\circ}{\mathbb{K}'} \supseteq \mathbb{K}$ by (a); so $\mathbb{K}$ is the claimed intersection. This intersection is filtered because the intersection of two finitary compacts is finitary compact, and interior distributes over intersection.

(c) If $\mathbb{K}_1 \cap \mathbb{K}_2 \subseteq O$, then there are two opens O_1 and O_2 such that $\mathbb{K}_1 \subseteq O_1$, $\mathbb{K}_2 \subseteq O_2$, and $O_1 \cap O_2 \subseteq O$. Indeed, by (b) $\mathbb{K}_i$ is the filtered intersection of its finitary compact neighborhoods, $i = 1, 2$. So $\mathbb{K}_1 \cap \mathbb{K}_2$ is the intersection of all $\mathbb{K}'_1 \cap \mathbb{K}'_2$, where $\mathbb{K}'_i$ ranges over the finitary compact neighborhoods of $\mathbb{K}_i$, $i = 1, 2$; moreover, this intersection is filtered. Since the intersection of any pair of finitary compacts is finitary compact, each such $\mathbb{K}'_1 \cap \mathbb{K}'_2$ is finitary compact. Since X is finitarily sober, some such $\mathbb{K}'_1 \cap \mathbb{K}'_2$ is included in O, so we let O_i be the interior of $\mathbb{K}'_i$, $i = 1, 2$.

(d) $\nu^*(\mathbb{K}_1 \cap \mathbb{K}_2) = \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} \nu(O_1 \cap O_2)$. Indeed by (c) the two filtered families $\{O | O \supseteq \mathbb{K}_1 \cap \mathbb{K}_2\}$ and $\{O_1 \cap O_2 | O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2\}$ are coinitial, i.e. for every element of one there is a smaller element in the other.

(e) $\nu^*(\mathbb{K}_1 \cup \mathbb{K}_2) = \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} \nu(O_1 \cup O_2)$. Indeed the two filtered families $\{O | O \supseteq \mathbb{K}_1 \cup \mathbb{K}_2\}$ and $\{O_1 \cup O_2 | O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2\}$ are clearly coinitial.

So

$$\begin{aligned}
\nu^*(\mathbb{K}_1 \cup \mathbb{K}_2) + \nu^*(\mathbb{K}_1 \cap \mathbb{K}_2) &= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} \nu(O_1 \cup O_2) + \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} \nu(O_1 \cap O_2) \quad \text{(by (d), (e))} \\
&= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} (\nu(O_1 \cup O_2) + \nu(O_1 \cap O_2)) \\
&= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} (\nu(O_1) + \nu(O_2)) \qquad \text{(by modularity)} \\
&= \inf_{O_1 \supseteq \mathbb{K}_1} \nu(O_1) + \inf_{O_2 \supseteq \mathbb{K}_2} \nu(O_2)) \\
&= \nu^*(\mathbb{K}_1) + \nu^*(\mathbb{K}_2)
\end{aligned}$$

5. We observe that $\nu^*\left(\bigcap_{i \in I} \mathbb{K}_i\right) = \inf_{O \supseteq \bigcap_{i \in I} \mathbb{K}_i} \nu(O) = \inf_{\exists i \in I \cdot O \supseteq \mathbb{K}_i} \nu(O)$ (since X is finitarily coherent, $O \supseteq \bigcap_{i \in I} \mathbb{K}_i$ is equivalent to $\exists i \in I \cdot O \supseteq \mathbb{K}_i$) $= \inf_{i \in I} \nu^*(\mathbb{K}_i)$. □

Remark 13. There is a subtle point in the inf-continuity property 5., insofar as $\bigcap_{i \in I} \mathbb{K}_i$ needs not be finitarily compact, although each $\mathbb{K}_i$ is. Therefore item 5. is not a property of inf-continuity of ν^* *on the finitary compacts*.

As an example of space where the filtered intersection of finitary compacts is not necessarily finitary compact, let $(X, \leq)$ be the set of finite and infinite words over a two-letter alphabet $\mathbb{B} = \{0, 1\}$, ordered by the prefix ordering $\leq$, seen as a dcpo. Let $\mathbb{K}_i$, $i \in \mathbb{N}$ be the set of finite words of length at least i: this is finitarily compact, since there are only finitely many words of length exactly i. However, the filtered intersection $\bigcap_{i \in \mathbb{N}} \mathbb{K}_i$ is the infinite set of all infinite words $\mathbb{B}^\omega$, which is not finitary compact. (It is compact, however.)

Despite of this, X is finitarily coherent. It is finitarily sober, since it is a dcpo (Remark 11). In fact, since X is algebraic, it is also continuous, hence sober. Note that every Scott open O of X is such that, if some infinite word w is in O, then some prefix of it is in O already. It follows that X is locally finitary: if $x \in O$, then there is a finite prefix y of x in O, so $x \in \overset{\circ}{\mathbb{K}} = \mathbb{K} \subseteq O$, where $\mathbb{K} = \uparrow y$. Finally, the intersection of any two finitary compacts is finitary compact, since any two words have no common upper bound, or they have a sup (when they are comparable). □

Theorem 2. *Let $(X, \mathcal{O})$ be a finitarily coherent space. Every bounded continuous valuation on $(X, \mathcal{O})$ is quasi-simple.*

Proof. Define $\overline{\nu}$ on the Alexandroff extension of $\mathcal{O}$ by $\overline{\nu}(U) = \sup_{\mathbb{K} \subseteq U} \nu^*(\mathbb{K})$, for each upward closed set U. We claim that $\overline{\nu}$ is a continuous valuation extending ν. It is clear that $\overline{\nu}$ is strict and monotonic. It is continuous, because for any directed family $(U_i)_{i \in I}$ of upward closed sets, for any finitary compact $\mathbb{K}$, $\mathbb{K} \subseteq \bigcup_{i \in I} U_i$ if and only if $\mathbb{K} \subseteq U_i$ for some $i \in I$. Indeed, the if direction is clear; for the only if direction, since $\mathbb{K}$ is Alexandroff compact, $\mathbb{K} \subseteq \bigcup_{i \in I} U_i$ implies that $\mathbb{K}$ is included in some finite union of U_is, hence in some U_i, since $(U_i)_{i \in I}$ is directed. It follows

$$\overline{\nu}\left(\bigcup_{i \in I} U_i\right) = \sup_{\mathbb{K} \subseteq \bigcup_{i \in I} U_i} \nu^*(\mathbb{K}) = \sup_{i \in I, \mathbb{K} \subseteq U_i} \nu^*(\mathbb{K}) = \sup_{i \in I} \sup_{\mathbb{K} \subseteq U_i} \nu^*(\mathbb{K}) = \sup_{i \in I} \overline{\nu}(U_i)$$

Next, Proposition 2, item 1, entails that $\overline{\nu}$ extends ν.

Let us show finally that $\overline{\nu}$ is modular. Fix two arbitrary upward closed subsets U and V. For any two finitary compacts $\mathbb{K}_1 \subseteq U$ and $\mathbb{K}_2 \subseteq V$, there is a finitary compact $\mathbb{K} \subseteq U \cap V$ such that $\mathbb{K}_1 \cap \mathbb{K}_2 \subseteq \mathbb{K}$, namely $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$: this is finitary compact by assumption. Conversely, for every finitary compact $\mathbb{K} \subseteq U \cap V$, there are two finitary compacts $\mathbb{K}_1 \subseteq U$ and $\mathbb{K}_2 \subseteq V$ such that $\mathbb{K} \subseteq \mathbb{K}_1 \cap \mathbb{K}_2$: take $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$. Therefore

$$\overline{\nu}(U \cap V) = \sup_{\mathbb{K} \subseteq U \cap V} \nu^*(\mathbb{K}) = \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} \nu^*(\mathbb{K}_1 \cap \mathbb{K}_2)$$

A similar argument, using the fact that the union of two finitary compacts is always finitary compact, implies

$$\overline{\nu}(U \cup V) = \sup_{\mathbb{K} \subseteq U \cup V} \nu^*(\mathbb{K}) = \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} \nu^*(\mathbb{K}_1 \cup \mathbb{K}_2)$$

It follows

$$\begin{aligned}
\overline{\nu}(U \cup V) + \overline{\nu}(U \cap V) &= \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} \nu^*(\mathbb{K}_1 \cup \mathbb{K}_2) + \nu^*(\mathbb{K}_1 \cap \mathbb{K}_2) \\
&= \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} \nu^*(\mathbb{K}_1) + \nu^*(\mathbb{K}_2) \qquad \text{(by Proposition 2, item 4)} \\
&= \sup_{\mathbb{K}_1 \subseteq U} \nu^*(\mathbb{K}_1) + \sup_{\mathbb{K}_2 \subseteq V} \nu^*(\mathbb{K}_2) \\
&= \overline{\nu}(U) + \overline{\nu}(V)
\end{aligned}$$

The conclusion follows by Theorem 1. □

Conversely to Fact 1, the construction in the proof of Theorem 2 entails:

Fact 2 *On a finitarily coherent space, every bounded continuous valuation has a* largest *A-extension.*

Proof. By Lemma 5, every A-extension $\overline{\nu}'$ of ν is such that $\overline{\nu}' \leq \nu^*$. In particular, $\overline{\nu}'(\mathbb{K}) \leq \nu^*(\mathbb{K})$ for every finitary compact $\mathbb{K}$. For every upward closed set U, then, $\overline{\nu}'(U) = \sup_{\mathbb{K} \subseteq U} \overline{\nu}'(\mathbb{K})$ (since $\overline{\nu}'$ is continuous) $\leq \sup_{\mathbb{K} \subseteq U} \nu^*(\mathbb{K}) = \overline{\nu}(U)$, where $\overline{\nu}$ was constructed in the proof of Theorem 2. □

Remark 14. Every bounded continuous valuation on the dcpo X of Remark 6 is quasi-simple, although X is not a continuous dcpo. This can be shown by hand (as we did in Remark 6), or using Theorem 2.

X is a dcpo, hence is finitarily sober (Remark 11). Closer examination shows that X is even sober. The Scott closed subsets are X and the complement of $A_{i,j}$, $i, j \in \mathbb{N}$. X is irreducible, as the closure of ω. The other irreducible closed subsets are of the form $F = X \setminus A_{i,j}$; then $F \subseteq (X \setminus A_{i,0}) \cup (X \setminus A_{0,j})$, so irreducibility implies that $i = 0$ or $j = 0$ (but not both, since F is irreducible, hence non-empty): but $X \setminus A_{i,0}$ is the closure of $i - 1$, and $X \setminus A_{0,j}$ is the closure of $(j-1)^*$. Therefore X is sober.

Observe that every upward closed subset is finitary compact: $\emptyset = \uparrow \emptyset$, $\{\omega\} = \uparrow \{\omega\}$, $[n, \omega] = \uparrow \{n\}$, $[n^*, \omega] = \uparrow \{n^*\}$, $A_{i,j} = \uparrow \{i, j^*\}$. In particular, X is locally finitary: for every open $x \in O$, $x \in \overset{\circ}{\mathbb{K}} \subseteq \mathbb{K} \subseteq O$ where we take the finitary compact $\mathbb{K}$ to be just O.

Since every upward closed subset is finitary compact, the compact saturated subsets are exactly the upward closed subsets, and are exactly the finitary compacts. In particular, the intersection of any two compact saturated subsets are compact saturated (hence X is a coherent space), and the intersection of any two finitary compacts is finitary compact (hence X is finitarily coherent, too). So Theorem 2 applies: every bounded continuous valuation on X is quasi-simple.

Note that Corollary 2 does not apply here, since X is not continuous. So Corollary 2 does not subsume Theorem 2. □

Remark 15. Conversely, every continuous dcpo $(X, \leq)$ is sober ([1], Proposition 7.2.27, p.77). X is also locally compact, in fact locally finitary: for any open O and $x \in O$, x is the sup of all elements $y \ll x$ since X is continuous; since O is open, some $y \ll x$ must then be in X, which entails that $x \in \Uparrow y \subseteq \uparrow y \subseteq O$; note that $\uparrow y$ is finitary compact and $\Uparrow y$ is its interior.

However, the interior of two finitary compacts may fail to be finitary compact. Let X be the space consisting of $\mathbb{N}$ union two fresh elements a and b, with the smallest ordering such that $a \leq n$ and $b \leq n$

for every $n \in \mathbb{N}$. This is a continuous dcpo, even an algebraic one: all elements are finite. As such, the Scott topology (which coincides with the Alexandroff topology here) is sober, locally compact and locally finitary, but the intersection of the two finitary compacts $\uparrow a$ and $\uparrow b$ is $\mathbb{N}$, which is not finitary, not even compact. So X is not finitarily coherent, and not coherent either.

Therefore Theorem 2 does not apply, but Corollary 2 does: every bounded continuous valuation on X is quasi-simple. So Theorem 2 does not subsume Corollary 2 either. □

Remark 16. The space X of Remark 8 is often taken as the prime example of a non-sober dcpo. However the reason why the continuous valuation ν of Remark 8 is not quasi-simple does not lie in non-sobriety. Let m be a non-negative integer, and choose any sequence $k_m, k_{m+1}, \ldots$ of integers greater than or equal to m: the set O of pairs (i, j) with $i \geq m$, and $j \geq k_i$ or $j = \omega$, is Scott open. In fact, it is easy to see that each non-empty Scott open is of this form. It follows that no finitary compact can contain any non-empty Scott open, that is, the interior of any finitary compact is empty. So X is not locally finitary. Also, in X the intersection of two finitary compacts needs not be finitary compact: if $i \neq i'$, and $j, j' \in \mathbb{N}$, then $\uparrow (i, j) \cap \uparrow (i', j') = \{(n, \omega) | n \geq \max(j, j')\}$, which is compact but not finitary compact. These are the two assumptions of Theorem 2 that fail. Observe that the third is satisfied: X is finitarily sober, as a dcpo, although it is not sober. □

4 Other Extension Results

Theorem 1 is one characterization of a class of continuous valuations by existence of extensions. We would like to present two other similar theorems, which are much easier to prove.

4.1 Discrete Valuations

Let us call a *discrete valuation* any valuation of the form $\sum_{i=0}^{+\infty} a_i \delta_{x_i}$, where $(a_i)_{i \in \mathbb{N}}$ is a summable family of non-negative reals, i.e., $\sum_{i=0}^{+\infty} a_i < +\infty$. Recall that such a family is also *commutatively summable*, that is, for any permutation σ, of $\mathbb{N}$, $\sum_{i=0}^{+\infty} a_{\sigma(i)} = \sum_{i=0}^{+\infty} a_i < +\infty$.

Similarly to Theorem 1, we may show:

Proposition 3. *Let $(X, \mathcal{O})$ be a topological space. The bounded function $\nu : \mathcal{O} \to \mathbb{R}_+$ is a discrete valuation if and only if ν extends to a continuous valuation on the discrete topology.*

Proof. Only if: define $\overline{\nu}$ as mapping every subset Y of X to $\sum_{i \in \mathbb{N}, x_i \in Y} a_i$. Then $\overline{\nu}$ extends ν to the discrete topology, is strict and monotonic. It is continuous and modular because $(a_i)_{i \in \mathbb{N}}$ is commutatively summable.

If: assume ν extends to a continuous valuation $\overline{\nu}$ on the discrete valuation. For every $x \in X$, let a_x be $\overline{\nu}(\{x\})$. Then, for every $Y \subseteq X$, $\overline{\nu}(Y) = \sum_{x \in Y} a_x$. We claim that $a_x \neq 0$ for only countably many $x \in X$. Let B be $\nu(X)$. Since ν is bounded, $B \in \mathbb{R}_+$. For every $N \in \mathbb{N} \setminus \{0\}$, there can be at most B/N points x such that $\nu(\{x\}) \geq 1/N$. So $\{x | a_x \geq 1/N\}$ is finite. Therefore $\{x | a_x \neq 0\} = \bigcup_{N \geq 1} \{x | a_x \geq 1/N\}$ is countable. Let $i \mapsto x_i$ be a bijection from $\mathbb{N}$ to $\{x | a_x \neq 0\}$, then ν coincides with $\sum_{i=0}^{+\infty} a_{x_i} \delta_{x_i}$ on $\mathcal{O}$. □

Although every discrete valuation is quasi-simple, the converse does not hold: on $[0, 1]$ with the Scott topology on $\leq$, the Scott opens are $[0, 1]$, $\emptyset$, and $(a, 1]$ for each $a \in [0, 1)$. The *Lebesgue valuation* ν maps $[0, 1]$ to 1, $\emptyset$ to 0, and $(a, 1]$ to $1 - a$. It extends to all upward closed sets by $\nu[a, 1] = 1 - a$. So by Jones' Theorem (Corollary 2), this is a quasi-simple valuation; but it is not discrete: otherwise by Proposition 3 ν extends to a continuous valuation $\overline{\nu}$ on the discrete topology. Then $\overline{\nu}\{0\} = \nu[0, 1] - \nu(0, 1] = 0$. And for every $a \in (0, 1]$, $\overline{\nu}\{a\} \leq \nu(a - \epsilon, 1] - \nu(a, 1]$ for every sufficiently small $\epsilon > 0$, so $\overline{\nu}\{a\} \leq \epsilon$. Therefore $\overline{\nu}$ must be the everywhere zero valuation, hence also ν, which is impossible. (The purpose of this discussion is not to show that the Lebesgue measure is not discrete, which is well-known, but to illustrate the use of Proposition 3.)

Although every simple valuation is discrete, the converse fails. E.g., on the dcpo $[0, 1]$ again, $\sum_{i=0}^{+\infty} 1/(n+1)^2 \delta_{1/n}$ is clearly discrete, but takes infinitely many values, so it cannot be simple.

4.2 Point-Continuous Valuations

A valuation ν is *point-continuous* [7] if and only if, for every open O, for every $r \in \mathbb{R}_+$ such that $\nu(O) > r$, there is a finitary compact $\mathbb{K} \subseteq O$ such that $\nu(O') > r$ for every open $O' \supseteq \mathbb{K}$.

Lemma 11. *Let $(X, \mathcal{O})$ be a topological space. Given a function $\nu : \mathcal{O} \to \mathbb{R}_+$, the following propositions are equivalent:*

1. *ν is a point-continuous valuation;*
2. *ν is a valuation, and there is a function f mapping each finitary compact $\mathbb{K}$ to a non-negative real such that, for every open O, $\nu(O) = \sup_{\mathbb{K} \subseteq O} f(\mathbb{K})$;*
3. *ν is a valuation, and for every open O, $\nu(O) = \sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K})$;*
4. *ν is modular on $\mathcal{O}$, and extends to a strict, monotonic and continuous function $\overline{\nu}$ on the Alexandroff extension of $\mathcal{O}$.*

Proof. The implication $3 \Rightarrow 2$ is clear. Let us show $2 \Rightarrow 1$, and assume $\nu(O) = \sup_{\mathbb{K} \subseteq O} f(\mathbb{K})$. If $\nu(O) > r$, then there is a finitary compact $\mathbb{K} \subseteq O$ such that $f(\mathbb{K}) > r$. But then, for every open $O' \supseteq \mathbb{K}$, $\nu(O') \geq f(\mathbb{K}) > r$. Let us show $1 \Rightarrow 3$. Assume that ν is point-continuous. Recall that $\nu^*(\mathbb{K}) = \inf_{O' \text{ open} \supseteq \mathbb{K}} \nu(O')$. In particular (take $O' = O$), $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K}) \leq \nu(O)$. Conversely, $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K}) \leq \nu(O)$ is equivalent to the property that for every $r < \nu(O)$, there is a finitary compact $\mathbb{K} \subseteq O$ such that $\nu^*(\mathbb{K}) \geq r$: this is a direct consequence of 1.

To show the equivalence between 4. and 1.–3., we prove $4 \Rightarrow 2$ and $3 \Rightarrow 4$. $4 \Rightarrow 2$: let f be $\overline{\nu}$, then $\sup_{\mathbb{K} \subseteq O} f(\mathbb{K}) = \sup_{\mathbb{K} \subseteq O} \overline{\nu}(\mathbb{K}) = \overline{\nu}(O)$ (since $\overline{\nu}$ is continuous) $= \nu(O)$ (since $\overline{\nu}$ extends ν). $3 \Rightarrow 4$: let $\overline{\nu}$ be defined by $\overline{\nu}(U) = \sup_{\mathbb{K} \subseteq U} \nu^*(\mathbb{K})$. This clearly extends ν to the Alexandroff extension of $\mathcal{O}$, and is strict and monotonic. It is continuous because $\mathbb{K}$ is compact in the Alexandroff topology: see the proof of Theorem 2 for a similar argument. □

Item 4. of the previous lemma shows in particular that point-continuous valuations can be characterized by the fact that they are exactly the valuations that can be extended to some specific kind of function (strict, monotonic, continuous) on some topology (Alexandroff). Compared to Theorem 1, we allow the extension to fail to be modular here.

Remark 17. Every point-continuous valuation is a continuous valuation, but the converse fails: the example of Remark 3 yields a continuous valuation ν such that $\nu(O)$ is in general different from $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K})$, showing that item 3. of the previous lemma fails. □

Remark 18. Every quasi-simple valuation is point-continuous, but the converse fails: the example of Remark 8 is not quasi-simple but is point-continuous. Indeed, for every open O, if $\nu(O) > r$ then O is not empty and $r < 1$; take $x \in O$, then the finitary compact $\uparrow x$ is included in O, and for every open O' containing it, O' is not empty, hence $\nu(O') = 1 > r$. □

Remark 19. Item 4. of Lemma 11 shows that quasi-simple and point-continuous valuations are rather similar, although the two notions differ. There are more puzzling similarities. In particular, Heckmann [7] shows that every bounded point-continuous valuation is the sup of a family of simple valuations (op.cit., Lemma 5.3). This may seem a claim that point-continuous valuations are quasi-simple. However, every bounded point-continuous valuation is the sup of a family, but not necessarily a *directed* family of simple valuations. This actually makes a difference: as shown in Remark 9, the valuation ν of Remark 1 is a sup of simple valuations but is not quasi-simple, although it is point-continuous (Remark 18). □

Remark 20. The set of point-continuous valuations is always a dcpo. Indeed, the point of [7] is precisely that this set is the soberification of the set of simple valuations, and its specialization preordering is just the ordinary ordering of valuations. That this is a dcpo can also be proved directly using Lemma 11. We have to show that any sup ν of a directed family $(\nu_i)_{i \in I}$ of point-continuous valuations is again point-continuous. By Lemma 11, item 3, $\nu_i(O) = \sup_{\mathbb{K} \subseteq O} \nu_i^*(\mathbb{K})$, so $\nu(O) = \sup_{i \in I} \sup_{\mathbb{K} \subseteq O} \nu_i^*(\mathbb{K}) = \sup_{\mathbb{K} \subseteq O} \sup_{i \in I} \nu_i^*(\mathbb{K})$. So ν is point-continuous by Lemma 11, item 2, taking $f(\mathbb{K}) = \sup_{i \in I} \nu_i^*(\mathbb{K})$.

On the contrary, it is unknown whether the set of quasi-simple valuations is a dcpo, except in special cases (when all continuous valuations are quasi-simple for example, see Corollary 2, Theorem 2, or when the given quasi-simple valuations are bicontinuous, see Proposition 10). □

5 Related Work and Conclusion

Claire Jones' PhD thesis [8] is a milestone, and contains a wealth of fundamental results on continuous valuations. She also cites previous works by Graham, Saheb-Djahromi, Plotkin, and Frutos Escrig, who were the first to study continuous valuations as models of probabilistic non-determinism. Among other results, Jones shows that every bounded continuous valuation on a continuous dcpo is quasi-simple. This is one of the steps necessary to show that the dcpo of all bounded continuous valuations on a continuous dcpo is again a continuous dcpo, one of Jones' major achievements.

A classical theme in continuous valuation theory is to look for conditions under which continuous valuations extend to a (unique) measure on the Borel σ-algebra generated by the topology. This was initiated by Saheb-Djahromi, pursued by Jones, and corrected and extended by Alvarez-Manilla, Edalat and Saheb-Djahromi: the PhD thesis [2] contains an impressive collection of results and pointers to relevant literature. Among the results that one can found there, any locally finite continuous valuation on a T_3 space extends uniquely to a regular τ-smooth Borel measure (Theorem 3.23, p.84); every locally finite valuation on a locally compact sober space extends uniquely to a τ-smooth Borel measure (Theorem 3.27, p.86); every locally finite valuation extends uniquely to a Radon measure on the Borel σ-algebra generated by the union of the topology and its cocompact topology (Proposition 3.34, p.89, attributed to Norberg and Vervaat), and also to a Radon measure on the Borel σ-algebra generated by the patch topology of the space (Corollary 3.40, p.92, an extension of a result by Lawson). As we noticed in Remark 4, a difference between these results and the ones presented here is that when extensions to measures exist, they seem to be unique: this is not the case with extensions to Alexandroff extensions of topologies. However there are intriguing connections to measures: as noticed in Remark 5, on monotone convergence spaces every quasi-simple valuation extends to a unique Borel measure ([2], Theorem 2.16).

Sünderhauf [10] studies spaces of valuations from the quasi-metric point of view. Our Corollary 3, stating that on algebraic dcpos, every bounded continuous valuations is the sup of a directed family of simple valuations based on finite elements, is part of his Theorem 3.4. We have noted in Remark 10 that there remained a gap in his proof, which we have filled.

Section 3.4 owes a lot to Tix's Diplomarbeit [11]. Alvarez-Manilla [2] notes that Tix's results were independently discovered by Norberg and Vervaat.

Finally, we have discussed the subtle relationship between quasi-simple and point-continuous valuations in Section 4.2. The latter were introduced in [7], where it is proved that the space of point-continuous valuations is exactly the soberification of the set of simple valuations. Heckmann also shows (Theorem 4.1 in op.cit.) that every continuous valuation on a locally finitary space is point-continuous. We have shown (Theorem 2) that every continuous valuation is even quasi-simple (a stronger property) if X is finitarily coherent (a stronger requirement).

To finish, let us just sum up the main results of this paper in the following table, where classes of valuations above the double line are equivalent to extension conditions under the double line:

valuation:	simple $\subsetneq$	discrete $\subsetneq$	quasi-simple $\subsetneq$	point-continuous $\subsetneq$	continuous
extends to:		continuous valuation	continuous valuation	strict, monotonic, continuous function	
on:		discrete topology	Alexandroff extension	Alexandroff extension	

It is in particular remarkable that several classes of valuations can be characterized by the existence of extensions.

References

1. S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Oxford University Press, 1994.
2. M. Alvarez-Manilla. *Measure Theoretic Results for Continuous Valuations on Partially Ordered Spaces*. PhD thesis, University of London and Imperial College, Sept. 2000.

3. M. A. Bukatin and S. Y. Shorina. Partial metrics and co-continuous valuations. In M. Nivat, editor, *Foundations of Software Sciences and Computation Structures*, volume 1378 of *Lecture Notes in Computer Science*, pages 125–139. Springer-Verlag, 1998.
4. A. Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.
5. A. Edalat, A. Jung, K. Keimel, and M. Kwiatkowska, editors. *Proceedings of the 3rd Workshop on Computation and Approximation (Comprox III)*, volume 13 of *Electronic Notes in Theoretical Computer Science*, Birmingham, England, Sept. 1997. Elsevier. http://www.elsevier.com/gej-ng/31/29/23/37/23/show/Products/notes/index.htt.
6. R. Heckmann. *Power Domain Constructions (Potenzbereich-Konstruktionen)*. PhD thesis, Univerität des Saarlandes, Dec. 1990. http://rw4.cs.uni-sb.de/ heckmann/diss/diss.html.
7. R. Heckmann. Spaces of valuations. In S. Andima, R. C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and Applications: 11th Summer Conference at the University of Southern Maine*, volume 806 of *Annals of the New York Academy of Sciences*, pages 174–200, New York, USA, 1996.
8. C. Jones. *Probabilistic Non-Determinism*. PhD thesis, University of Edinburgh, 1990. Technical Report ECS-LFCS-90-105.
9. K. Keimel. Bi-continuous valuations. In Edalat et al. [5]. http://www.elsevier.com/gej-ng/31/29/23/37/23/show/Products/notes/index.htt.
10. P. Sünderhauf. Spaces of valuations as quasimetric domain. In Edalat et al. [5]. http://www.elsevier.com/gej-ng/31/29/23/37/23/show/Products/notes/index.htt.
11. R. Tix. *Stetige Bewertungen auf topologischen Räumen*. PhD thesis, TH Darmstadt, June 1995.