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Reading Notes: Why is Cpo Cocomplete?

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Reading Notes: Why is Cpo Cocomplete?

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Abstract. We describe a construction of coequalizers in the category ***Cpo*** of complete partial orders and continuous maps, showing that this category is cocomplete. This is not new: Meseguer proved it in 1977, using fairly abstract category-theoretic tools, and Fiech reproved it in 1996, using fairly intricate by-hand constructions. We hope to give a clearer and better structured presentation of the construction.

1 Introduction

The category ***Cpo*** of (unpointed) complete partial orders (cpos) and continuous maps as morphisms is cocomplete. This is not a new statement, and it has been proved several times before (Meseguer [4], Fiech [2], Jung [unpublished, cited by Fiech]). However Meseguer's proof is non-constructive and applies strictly speaking only to ω-chain-complete partial orders, not to general directed-complete partial orders. On the other hand Fiech's proof is rather intricate and abounds in technical details. The aim of this note is to explain Meseguer and Fiech's construction of colimits in ***Cpo*** in a way that exhibits the essential ingredients of the construction. These are: the construction of coequalizers, and therefore of colimits, in the category ***Ord*** of partial orders and monotone maps (Section 3), the construction of free cpos over a given partial order (Section 4), and an epi-mono factorization in ***Cpo*** (Section 5). Mix the ingredients in an appropriate diagram (Section 6) and lo, here are our coequalizers in ***Cpo***.

Disclaimer. No theorem is specially new here. In fact Fiech's construction was essentially not new compared to Meseguer's either. We merely aim at giving a clearer and better structured presentation of the construction.

More concretely, the construction of free cpos in Section 4 is the same as that of [4], Proposition 1.2, only applied to complete, not just ω-complete partial orders. It is also extremely close to the classic construction of free *algebraic* cpos over a given cpo. The structure of coequalizers in ***Ord*** is also given in [4], proof of Theorem 1.1, and the epi-mono factorization of Section 5 in ***Cpo*** is, the difference between ω-complete and complete partial orders notwithstanding, described in [4] as its extremal epi-mono factorization (op.cit., proof of Theorem 1.1). The existence of coequalizers in the category of ω-complete partial orders is then deduced from a fairly abstract theorem of category theory.

While familiarity with category theory is a plus, it should not be necessary to know too much of it to understand the main ideas of this paper.

What use is this result? That ***Cpo*** is cocomplete is little known. Accordingly, it has been little used. We may therefore presume that it is of little use. However, the presence of (finite) colimits is required in the institution-based accounts of modularity, and gives an elegant formalization of modules and sharing between modules. Fiech [2] cites another application of this result to the semantics of subtyping. We have come to need this result while trying to prove that a given functor from ***Cpo*** to some category of presheaves had a left adjoint, using a left Kan extension argument. Such an argument relies on the existence of colimits in ***Cpo***, and is likely to be required more in the future. (Yes, this *does* use category theory, so we won't talk about this further.)

2 Definitions

A *cpo*, a.k.a. a *complete partial order* $(A, \leq)$ is a partial order such that every directed subset of A has a supremum $\bigsqcup D$. A subset D of A is *directed* if and only if D is not empty, and for every $x, y \in D$, there is a

$z \in D$ such that $x, y \leq z$. Cpos are variously called directed cpos, unpointed cpos, or even inductive partial orders (ipos) in the literature. When there is no risk of confusion, we shall plainly write $\leq$ the underlying ordering relation. Otherwise we shall write $\leq_A$ the ordering on A.

A function f from the cpo A to B is *continuous* if and only if, for every directed subset D of A, $f(\bigsqcup D) = \bigsqcup f(D)$.

There is a category $\mathbf{Cpo}$ whose objects are the cpos and whose morphisms are exactly the continuous functions. $\mathbf{Cpo}$ has many nice properties, of which we list some. We shall make extensive use of categorical language (please refer to standard sources, e.g. [3]), although we shall explain most intuitions with the help of analogies with similar constructions in the category $\mathbf{Set}$ of sets and functions.

First, $\mathbf{Cpo}$ has *products*. If $(A_i)_{i \in I}$ is a family of cpos, its product is the Cartesian product $\prod_{i \in I} A_i$, with ordering given by $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ if and only if $x_i \leq y_i$ for every $i \in I$.

$\mathbf{Cpo}$ also has *equalizers*. Consider the diagram on the right in $\mathbf{Cpo}$, meaning that we assume given two continuous functions f and g from the cpo A to B.

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad (1)$$

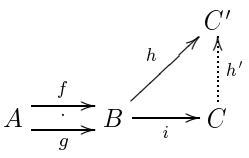
Consider the set C consisting of all those $x \in A$ such that $f(x) = g(x)$, equipped with the ordering $\leq$ inherited from A. Note that if D is a directed subset of C, then it has a sup $x = \bigsqcup D$ in A, and that by continuity $f(x) = \bigsqcup f(D) = \bigsqcup g(D)$ (because $f(y) = g(y)$ for every $y \in D$) $= g(x)$, so x is in C. It follows that C is a cpo. Call i the obvious embedding of C inside A. Then $f \circ i = g \circ i$. Moreover, i is universal in the sense that, if $h : C' \to A$ is any other continuous function such that $f \circ h = g \circ h$, by definition every element $h(x)$ is in C, so h factors as $i \circ h'$, where $h' : C' \to C$ takes the same values as h on C'. Therefore C (and i) is an *equalizer* of the pair f, g.

Given products and equalizers, we can build all *limits*, making $\mathbf{Cpo}$ a *complete* category [3].

We won't insist on the fact that $\mathbf{Cpo}$ is also cartesian-closed, rather we shall explore the dual notion of colimits.

It is easy to see that $\mathbf{Cpo}$ has *coproducts*, a.k.a., disjoint sums. If $(A_i)_{i \in I}$ is a family of cpos, its coproduct is the disjoint sum $\coprod_{i \in I} A_i$ (i.e., the set whose elements are pairs (i, x) with $x \in A_i$), with ordering given by $(i, x) \leq (j, y)$ if and only if $i = j$ and $x \leq y$ in A_i.

Dually to the equalizer case, a *coequalizer* of f and g as shown in (1) would be a cpo C together with a map $i : B \to C$ such that $i \circ f = i \circ g$, which are universal in the sense that whenever $h : B \to C'$ is such that $h \circ f = h \circ g$, then h factors uniquely through i, as shown on the right.



(2)

It is well-known that a category is *cocomplete*, i.e., it has all *colimits*, if and only if it has all coproducts and all coequalizers. To show that $\mathbf{Cpo}$ is cocomplete, we shall therefore concentrate on constructing coequalizers.

Note that in the category of sets, coequalizers are just *quotients*. That is, define $\sim$ as the smallest equivalence relation on B such that $f(x) \sim g(x)$ for every $x \in A$, then C would be $B/\sim$, and i would be the map sending each $y \in B$ to its equivalence class $[y]$. Colimits in sets are then quotients of disjoint sums, a.k.a. inductive limits. The structure of coequalizers in $\mathbf{Cpo}$ is more complicated.

We shall build coequalizers in $\mathbf{Cpo}$ in three steps: construct coequalizers in the very close category $\mathbf{Ord}$ of partial orders and monotone maps between them; build free cpos on given partial orders; then lift the coequalizers in $\mathbf{Ord}$ to $\mathbf{Cpo}$, using free cpos and an epi-mono factorization in $\mathbf{Cpo}$. This is exactly what Fiech does [2], although the construction, and in particular the required ingredients, are not so clear.

3 Coequalizers in $\mathbf{Ord}$

Let f, g be as in (1), in the category $\mathbf{Ord}$ of partial orders and monotone functions. To build its coequalizer in $\mathbf{Ord}$, do much as in sets. However, instead of building an equivalence relation $\sim$ on B, build the smallest preordering $\precsim$ on B extending the ordering $\leq_B$ on B and such that $f(x) \approx g(x)$ for every $x \in A$, writing $\approx$ for the equivalence relation $\precsim \cap \succsim$ associated with $\precsim$.

Now consider the quotient set $C = B/\approx$. Clearly $\precsim$ induces a partial ordering $\leq_C$ on C, defined by $[x] \leq [y]$ if and only if $x \precsim y$, where $[x]$ denotes the equivalence class of x by $\approx$. Since $\precsim$ extends $\leq_B$, the

map $i : x \in B \mapsto [x]$ is monotonic. By construction, $i(f(x)) = i(g(x))$ for every $x \in A$, so $i \circ f = i \circ g$. It remains to show that i is universal with this property.

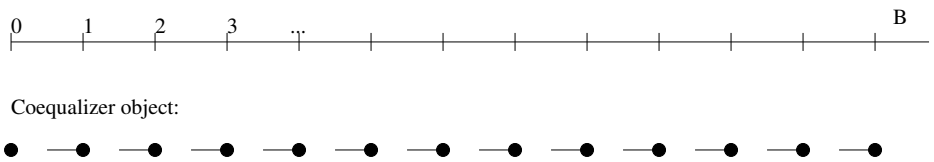
Consider any monotone map h from B to some partial order C', such that $h \circ f = h \circ g$ (compare with (2)). Let $\sqsubseteq$ be the relation on B defined by $x \sqsubseteq y$ if and only if $h(x) \leq h(y)$. This is clearly a preordering. Moreover, $\sqsubseteq$ extends $\leq_B$, since h is monotone: if $x \leq y$ in B, then $h(x) \leq h(y)$, i.e., $x \sqsubseteq y$. Also, $f(x) \sqsubseteq g(x)$ and $g(x) \sqsubseteq f(x)$ for every $x \in A$, since $h \circ f = h \circ g$. Since $\precsim$ is by definition the smallest preordering extending $\leq$ such that $f(x) \precsim g(x)$ and $g(x) \precsim f(x)$ for all $x \in A$, it follows that $\precsim$ is a smaller relation than $\sqsubseteq$. In other words, if $x \precsim y$ then $h(x) \leq h(y)$. In particular, if $x \approx y$ then $h(x) = h(y)$; this allows us to define $h' : C \to C'$ by $h'([x]) = h(x)$, and h' is automatically monotonic. Thus we are done: the h' arrow in Diagram (2) exists, and is clearly unique.

Example 1. Let A be the one-element partial order $\{*\}$, B be the five-element partial-order $\{\bot, a, b, a', b'\}$, with $\bot < a < b$ and $\bot < a' < b'$ and all other elements incomparable. Let f map $*$ to b, g map $*$ to b'. Then $\approx$ equates only b and b', and $B/\approx$ is the four-element order $[\bot] < [a], [a'] < [b] = [b']$.

Example 2. Let A be $\mathbb{N} \cup \{+\infty\}$, B be $[0, 1]$. Let $f : A \to B$ map $n \in \mathbb{N}$ to $(n + 1)/(n + 2)$, and $+\infty$ to 1. Let g be the constant function 1. The preordering $\precsim$ is the smallest such that for every $x, y \in [0, 1]$, $x \leq y$ implies $x \precsim y$, and such that $1 \precsim (n + 1)/(n + 2)$. In particular, $1 \precsim 1/2$. In fact, $\approx$ equates all real numbers between $1/2$ and 1. The quotient $C = B/\approx$ is then just $[0, 1/2]$, and $i : x \mapsto [x]$ maps x to x itself if $x \leq 1/2$, and to $1/2$ otherwise.

Example 3. Let A and $B = \{0, 1\}^*$ both be the set of finite words over the two-element alphabet $\{0, 1\}$. The ordering $\leq$ is the graded lexicographic ordering: $w \leq w'$ if and only if w and w' have the same length n, and either $w = w'$ or we can write $w = a_0 \dots a_{m-1} 0 a_{m+1} \dots a_n$ and $w' = a_0 \dots a_{m-1} 1 a'_{m+1} \dots a'_n$. Let $f : A \to B$ map w to w, g map w to $w0$. The preordering $\precsim$ is the smallest such that $w \precsim w'$ whenever $w \leq w'$, and such that $w0 \approx w$ for all w. The equivalence relation $\approx$ equates $w0 \dots 0$ with w, and a canonical representative of the equivalence class $[w]$ of w is w with all its trailing zeros removed. Another way of describing $[w]$, where $w = a_0 \dots a_n$, is as the number $\sum_{i=0}^{n} a_i 2^{-i-1}$, equivalently the number written $0.a_0 \dots a_n$ in base 2. The quotient ordering $\precsim$ is then just the usual ordering on numbers: the quotient object $C = B/\approx$ is the set of numbers in $[0, 1)$ with a finite base 2 expansion, and $i : B \to C$ maps every word $a_0 \dots a_n$ to the number $0.a_0 \dots a_n$ in base 2. Note also that the coequalizer here coincides with the set-theoretic quotient by the smallest equivalence class such that $f(w) = g(w)$ for all w.

Example 4. Let A be $\mathbb{N}$ with $\leq$ as ordering. Let B be $[0, +\infty)$, $f : A \to B$ map $n \in \mathbb{N}$ to n, $g : A \to B$ map $n \in \mathbb{N}$ to $n + 1/2$. The ordering $\precsim$ is such that $n + 1/2 \precsim n$, and $\approx$ equates all reals between n and $n + 1/2$. A canonical representative of $[x]$ when $x \in B$ is $\lfloor x \rfloor$ if $x \leq \lfloor x \rfloor + 1/2$, x otherwise. Note that, as in Example 2, the coequalizer is not just the set-theoretic quotient.



4 Free Cpos

There is a forgetful functor $|_|$ from ***Cpo*** to ***Ord***, which maps every cpo to the underlying partial order, and every continuous function to the underlying monotone function. Conversely, is there a way to complete a given partial order, adding all required missing sups, to get a cpo? Formally, we are looking for a *left adjoint* F to $|_|$, the *free cpo* functor, mapping each partial order A to the *free cpo* $F(A)$ *over* A.

Curiously, although this is one essential part of the construction in [2], Fiech claims that "it is not immediately clear what 'free' means as there are no 'free dcpos' over posets" (p.2). We prove the contrary.

For every partial order A, let $F(A)$ be the set of all down-closed directed subsets Q of A, ordered by set inclusion. A subset Q of A is *down-closed* if and only if for every $x \in Q$, for every $y \leq_A x$, then $y \in Q$.

For any subset X of A, we write $\downarrow X$ for the *down-closure* of X in A, namely the set of all $x \in A$ such that $x \leq y$ for some $y \in X$. Note that a subset X is down-closed if and only if $X = \downarrow X$.

The intuition behind this is that, given a directed subset Q, its sup will just be represented by itself. We take Q to be down-closed to get a canonical representation of the sup. Note that for every $x \in A$, $\downarrow x = \downarrow \{x\}$ already has a sup, namely x itself: $\downarrow x$ will be the way we represent x in the free cpo $F(A)$.

The action of F on morphisms is as follows. Given any monotone function $f : A \to B$ (a morphism in ***Ord***), $F(f)$ maps $Q \in F(A)$ to $\downarrow f(Q)$.

We first have to verify that F is a functor. First, for every partial order A, we claim that $F(A)$ is indeed a cpo. Let D be a directed family of down-closet directed subsets Q_i of A, $i \in I$. Then build $Q = \bigcup_{i \in I} Q_i$: clearly, if Q is down-closed and directed, then it must be the sup of D. So let us show that Q is down-closed and directed. If $y \in Q$ then $y \in Q_i$ for some $i \in I$, so every $x \leq y$ is in $Q_i \subseteq Q$, since Q_i is down-closed; so Q is down-closed. Since D is a directed family, D and therefore also I are not empty; since no Q_i is empty, Q is not empty either. Let $x, y \in Q$, say $x \in Q_i$ and $y \in Q_j$; since D is directed, there is $k \in I$ such that $Q_i, Q_j \subseteq Q_k$, so $x, y \in Q_k$. Since Q_k is directed, there is $z \in Q_k \subseteq Q$ such that $x, y \leq z$. So Q is directed. We are done: every directed family D of elements Q_i of $F(A)$ has a sup, which is just the union of the Q_is.

We must then show that $F(f)$ is well-defined and continuous on morphisms f. First, $\downarrow f(Q)$ is clearly down-closed; it is also directed, since it is clearly non-empty, and for every $x, y \in \downarrow f(Q)$, by definition there are $x', y' \in Q$ such that $x \leq f(x')$, $y \leq f(y')$; since Q is directed, there is $z' \in Q$ such that $x', y' \leq z'$, so $x, y \leq z$, with $z = f(z') \in \downarrow f(Q)$. Since $\downarrow f(Q)$ is down-closed and directed, $F(f)$ is indeed a map from $F(A)$ to $F(B)$. It is continuous because $F(f)(\bigcup_{i \in I} Q_i) = \downarrow f(\bigcup_{i \in I} Q_i) = \{x \in B | \exists i \in I, x' \in Q_i \cdot x \leq f(x')\} = \bigcup_{i \in I} \{x \in B | \exists x' \in Q_i \cdot x \leq f(x')\} = \bigcup_{i \in I} \downarrow f(Q_i)$.

F preserves identities: $F(f)(Q) = \downarrow Q = Q$ whenever Q is down-closed. Before we show that F commutes with composition, observe that:

$$(*) \text{ for every monotone function } f : A \to B, \text{ for every } X \subseteq A, \downarrow f(X) = \downarrow f(\downarrow X)$$

The inclusion $\subseteq$ follows from $X \subseteq \downarrow X$, the other inclusion follows from $f(\downarrow X) \subseteq \downarrow f(X)$, which holds because f is monotone. It follows that F commutes with compositions: $F(g \circ f)(Q) = \downarrow g(f(Q)) = \downarrow g(\downarrow f(Q))$ by $(*)$.

To show that F is left adjoint to $|_|$, we show that there is a natural (in A) transformation $\dot{\eta}_A : A \to |F(A)|$ (the *unit* of the adjunction $F \dashv |_|$), and a natural transformation $\dot{\epsilon}_A : F(|A|) \to A$ (the *counit*), such that the following diagrams commute ([3], IV.1):

$$\begin{array}{ccc} F(E) & \xrightarrow{F(\dot{\eta}_E)} & F|F(E)| \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \dot{\epsilon}_{F(E)}} \\ & & F(E) \end{array} \qquad \begin{array}{ccc} |A| & & \\ {\scriptstyle \dot{\eta}_{|A|}}\downarrow & \searrow^{\mathrm{id}} & \\ |F|A|| & \xrightarrow[|\dot{\epsilon}_A|]{} & |A| \end{array} \tag{3}$$

The unit $\dot{\eta}_A : A \to |F(A)|$ maps $x \in A$ (where A is a partial order) to $\downarrow x$, where $\downarrow x = \downarrow \{x\} = \{y \in A | y \leq x\}$ is clearly down-closed and directed. This is also clearly natural in A, meaning that for every monotone $f : A \to B$, $\downarrow f(x) = \downarrow f(\downarrow x)$: this is a consequence of $(*)$ above.

The counit $\dot{\epsilon}_A : F(|A|) \to A$ (where A is a cpo) maps $Q \in F(|A|)$ to the sup of Q; this exists since A is a cpo and Q is directed. Naturality amounts to saying that, for every continuous $f : A \to B$, for every down-closed directed $Q \subseteq A$, $\bigsqcup f(Q) = f(\bigsqcup Q)$, which is obvious since f is continuous.

Now check diagrams (3). The first one is $\dot{\epsilon}_{F(A)} \circ F(\dot{\eta}_A) = \mathrm{id}_{F(A)}$, where A is a partial order. For every $Q \in F(A)$, $F(\dot{\eta}_A)(Q) = \{Q' | \exists x \in Q \cdot Q' \subseteq \dot{\eta}_A(x)\} = \{Q' | \exists x \in Q \cdot Q' \subseteq \downarrow x\}$. Now $\dot{\epsilon}_{F(A)}$ maps this to its sup (wrt. to $\subseteq$), i.e., to $\bigcup_{x \in Q, Q' \subseteq \downarrow x} Q' = \bigcup_{x \in Q} \downarrow x$ (since $\downarrow x$ is itself a possible Q') $= Q$ (since Q is directed and down-closed).

The second diagram is $|\dot{\epsilon}_A| \circ \dot{\eta}_{|A|} = \mathrm{id}_{|A|}$, for every cpo A. The left-hand side maps every $x \in |A|$ to $|\dot{\epsilon}_A|(\downarrow x) = \sup(\downarrow x) = x$.

Example 5. Consider the partial order $(\mathbb{N}, \leq)$. All non-empty down-closed subsets of $\mathbb{N}$ are of the form $[0, n]$ or $[0, +\infty)$. They are all directed. Writing n for $[0, n]$ and $+\infty$ for $[0, \infty)$, it is easy to see that $F(\mathbb{N})$ is $\mathbb{N} \cup \{+\infty\}$, with a new element $+\infty$, greater than all natural numbers.

Example 6. Let $T = \{0,1\}^*$ be the infinite binary tree, whose elements are finite words over the two-element alphabet $\{0,1\}$. The ordering $\leq$ is the prefix ordering: $w \leq w'$ if and only if $w' = ww''$ for some word w''. Let D be any directed set of words: for any two words w, w' in D, w and w' are prefixes of the same word (in D), so $w \leq w'$ or $w' \leq w$. Therefore D is a *chain*, i.e., a linearly ordered set of words. When D is additionally down-closed, D must be additionally a set of words $\{\epsilon, a_0, a_0a_1, \ldots, a_0a_1 \ldots a_n, \ldots\}$. This can be equated with the infinite word $a_0a_1 \ldots a_n \ldots$. It follows that $F(T)$ is the set of finite *and infinite* words over $\{0,1\}$.

Example 7. The previous examples basically created new sups "at infinity". This example shows that F may even create new sups of families that already had a sup. Consider the partial ordering $\mathbb{N} \cup \{+\infty\}$ of Example 5. Its directed down-closed subsets are $[0,n]$ ($n \in \mathbb{N}$), $[0,+\infty)$ as before, plus the whole set $\mathbb{N} \cup \{+\infty\}$. Therefore its free cpo can be written $\mathbb{N} \cup \{+\infty', +\infty\}$ where $n < +\infty' < +\infty$, and $+\infty'$ is a newly added infinity just below the old $+\infty$.

Example 8. The difference between T in Example 6 and A and B in Example 3 is that we use the prefix ordering in the former, and the graded lexicographic ordering in the latter. Let therefore A be $\{0,1\}^*$ with the graded lexicographic ordering. Any directed subset D of A is comprised of words of the same length. However any set of words over $\{0,1\}$ of a given length is finite. So every directed subset D of A has a sup x, which is in D. If D is down-closed, then $D = {\downarrow} x$. It follows that $F(A)$ is actually isomorphic to A itself in this case.

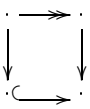
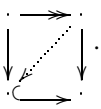
Example 9. Consider the partial order $[0,1]$, and let us compute the free cpo $F[0,1]$ on it. Any directed down-closed subset Q of $[0,1]$ already has a sup x in $[0,1]$. Unless $Q = \{0\}$ and $x = 0$, x may or may not be in Q; i.e., Q may be either ${\downarrow} x = [0,x]$ or the half-open interval $[0,x)$. It is not hard to see that $F[0,1]$ is isomorphic (as a cpo) to the disjoint union $\{x^+ | x \in [0,1]\} \cup \{x^- | x \in (0,1]\}$, with the ordering $x^- \leq y^-$ iff $x \leq y$, $x^+ \leq y^+$ iff $x \leq y$, $x^- \leq y^+$ iff $x \leq y$, $x^+ \leq y^-$ iff $x < y$ (equate x^+ with $[0,x]$, x^- with $[0,x)$). In other words, we retrieve every $x \in [0,1]$ as x^+ in $F[0,1]$, and new elements x^- "infinitesimally below x" have been added for all $x \neq 0$.

An interesting fact, although out of scope of this note, is that $F(A)$ is always an *algebraic* cpo, where the finite elements are just $\dot{\eta}_A(x) = {\downarrow} x$, $x \in A$. Example 7 shows that $F(F(A)) \neq A$ in general.

5 Epi-Mono Factorizations

In the category of sets, every function factors as the composition of a surjection and an injection: if f is a function from A to B, then let C be the range of f, e be the function from A to C mapping x to $f(x)$, and m be the inclusion of C into B, so that $f = m \circ e$.

In other categories, where surjectivity and injectivity might have less relevance or even be undefinable, we replace surjective functions by *epis* and injective functions by *monos*: a mono is a morphism $f : A \to B$ such that if $g, h : C \to A$ are such that $f \circ g = f \circ h$ then $g = h$, symmetrically f is epi if and only if whenever $g, h : B \to C$ are such that $g \circ f = h \circ f$ then $g = h$. A *factorization system* on a category is then a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms, the class $\mathcal{E}$ of *relevant epis*, written $\twoheadrightarrow$, and the class $\mathcal{M}$ of *relevant monos*, written $\hookrightarrow$, such that [1]:

- first, as in sets, every morphism $f : A \to B$ factors as the composition of a relevant epi e and a relevant mono m, $A \overset{e}{\twoheadrightarrow} C \overset{m}{\hookrightarrow} B$;
- second, all isos should be both in $\mathcal{E}$ and in $\mathcal{M}$, and both $\mathcal{E}$ and $\mathcal{M}$ should be stable by composition (i.e., if f and g are in one of these classes, and $f \circ g$ exists, then $f \circ g$ is in the same class). Recall that an *iso* is a morphism f such that there is a morphism f^{-1} with $f \circ f^{-1} = \mathrm{id}$, $f^{-1} \circ f = \mathrm{id}$.
- third, each commuting square  has a unique diagonal making both triangles commute:  .

  This is called the *diagonal fill-in* condition.

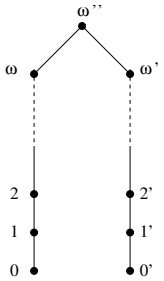
In the standard definition of factorization systems, neither are morphisms in $\mathcal{E}$ required to be epis, nor are morphisms in $\mathcal{M}$ required to be monos. However they will be epis and monos respectively here (in which case $(\mathcal{E}, \mathcal{M})$ is an *epi-mono factorization*), justifying the names "relevant epis" and "relevant monos".

There are several factorization systems on $\boldsymbol{Cpo}$. As in any category, we have two trivial ones: the one where $\mathcal{E}$ contains all morphisms and $\mathcal{M}$ is the class of all isos, and the one where $\mathcal{E}$ is the class of all isos and $\mathcal{E}$ contains all morphisms.

We shall be interested in one that looks like the one that mentioned in sets above. Take any morphism $f : A \to B$ in $\boldsymbol{Cpo}$. We cannot take C to be the range of f in B, because this may not be a cpo: some sups may be lacking. Instead, take C to be the smallest sub-cpo of B. In other words, let C be the *sup-closure* of the range $\mathrm{rng}(f)$ of f in B, where the sup-closure of $C' \subseteq B$ in B is the smallest subset $\overline{C'}$ of B containing C' and such that the sup of every directed family of elements of $\overline{C'}$ is again in $\overline{C'}$.

A slight trap here is that $\overline{C'}$ is not just C' plus all sups of directed families of elements of C', and sups of directed families of sups of directed families of ... of sups of directed families have to be added, transfinitely. Another trap is that, although any cpo, in particular B, can be seen as a topological space equipped with the Scott topology (where the closed sets are sup-closed subsets with up-closed complements), the sup-closure of C' in B is *not* the topological closure of C' in B: indeed there is no reason why $\overline{C'}$ should have an up-closed complement.

Example 10. Consider the cpo on the right, and let C' be $\mathbb{N} \cup \mathbb{N}'$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}' = \{0', 1', 2', \ldots\}$. Any directed subset of C' must be contained either in $\mathbb{N}$ or in $\mathbb{N}'$, so the set of all sups of directed subsets of C' is $\mathbb{N} \cup \{\omega\}$ union $\mathbb{N}' \cup \{\omega'\}$. So the sup-closure of C' is *strictly smaller* than the set of all sups of elements of C', which would additionally contain ω''.



Example 11. Consider the cpo $(\mathbb{N} \cup \{+\infty\}) \times \mathbb{N}$ union $\{+\infty\}$, with the ordering $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j = j'$, or $i = i' = +\infty$ and $j \leq j'$; and $(i, j) \leq +\infty$. Let C' be the subset $\mathbb{N} \times \mathbb{N}$. Every directed subset of C' must be included in some $\mathbb{N} \times \{j\}$, so the set C'_1 of sups of directed subsets of C' is $(\mathbb{N} \cup \{+\infty\}) \times \mathbb{N}$. Now the set $\{(+\infty, j) | j \in \mathbb{N}\}$ is directed in C'_1, and has $+\infty$ as sup. So the sup-closure of C' is the whole space. Note that $+\infty$ is in the sup-closure of C' but is not a sup of a directed subset of C', since it is not in C'_1. In other words, the sup-closure of C' is in general *strictly larger* than the set of sups of directed subsets of C'.

Note that:

$$(**) \text{ for every morphism } f : B \to D, \text{ and every } C' \subseteq B,\ f(\overline{C'}) \subseteq \overline{f(C')}.$$

Indeed, let X be $f^{-1}(\overline{f(C')})$. For every $x \in C'$, $f(x) \in f(C') \subseteq \overline{f(C')}$, so $x \in X$: as x is arbitrary, $C' \subseteq X$. Then, let D be any directed subset of X, and $x = \bigsqcup D$. By continuity $f(x) = \bigsqcup f(D)$, and by definition of X, $f(D) \subseteq \overline{f(C')}$; so $f(x) = \bigsqcup f(D) \in \overline{f(C')}$, that is, $x \in X$: so X is sup-closed. As $\overline{C'}$ is the smallest sup-closed set containing C', $\overline{C'} \subseteq X$. So by definition of X, $f(\overline{C'}) \subseteq \overline{f(C')}$.

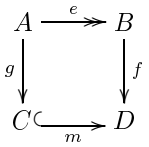
Let $\mathcal{E}$ be the class of all morphisms $e : A \to B$ such that $B = \overline{\mathrm{rng}(e)}$. In an sense, these are morphisms whose range is *dense* in their codomain; it is easy to see that e is then epi in $\boldsymbol{Cpo}$. Let $\mathcal{M}$ be the class of all cpo *embeddings*, i.e., continuous functions $m : A \to B$ such that $x \leq y$ *if and only if* $m(x) \leq m(y)$. Notice that m is in particular injective, hence mono. Notice also that not only is m continuous, but $\mathrm{rng}(m)$ is a sub-cpo of B and $m^{-1} : \mathrm{rng}(m) \to A$ is continuous. Indeed, let $D = (y_i)_{i \in I}$ be a directed family in $\mathrm{rng}(m)$, and let $x_i = m^{-1}(y_i)$; the family $(x_i)_{i \in I}$ is directed, too: for any i, j, there is k such that $y_i, y_j \leq_B y_k$, i.e., $m(x_i), m(x_j) \leq m(x_k)$, therefore $x_i, x_j \leq_A x_k$. Then since m is continuous, $m(\bigsqcup_{i \in I} x_i) = \bigsqcup_{i \in I} m(x_i)$, that is, $m(\bigsqcup_{i \in I} m^{-1}(y_i)) = \bigsqcup_{i \in I} y_i$, which implies that $\bigsqcup_{i \in I} y_i$ is in $\mathrm{rng}(m)$ and that $\bigsqcup_{i \in I} m^{-1}(y_i) = m^{-1}(\bigsqcup_{i \in I} y_i)$ by applying m^{-1} to both sides.

There is a strengthening of $(**)$ when f is a relevant mono:

$$(***) \text{ for every } m : C \hookrightarrow D \text{ , and every } C' \subseteq C,\ m(\overline{C'}) = \overline{m(C')}.$$

Indeed, by $(**)$ with $f = m$ the $\subseteq$ inclusion holds. Conversely, let C'' be $m(C')$. This is a subset of the cpo $\mathrm{rng}(m)$, and by $(**)$ applied to $f = m^{-1}$, $m^{-1}(\overline{C''}) \subseteq \overline{m^{-1}(C'')}$, therefore $\overline{C''} \subseteq m(\overline{m^{-1}(C'')})$. Since $C'' = m(C')$, $m^{-1}(C'') = C'$, so $\overline{m(C')} \subseteq m(\overline{C'})$.

Now $f : A \to B$ factors as $A \xrightarrow{e} C \xhookrightarrow{m} B$, where $C = \overline{\mathrm{rng}(f)}$, $e(x) = f(x)$ for all $x \in A$, and m is the standard inclusion of C in B. It is clear that $e \in \mathcal{E}$ and $m \in \mathcal{M}$.
Let us check the diagonal fill-in condition: consider any diagram as on the right in ***Cpo***.



By $(**)$, $f(\overline{\mathrm{rng}(e)}) \subseteq \overline{f(\mathrm{rng}(e))}$. By the diagram above $f(\mathrm{rng}(e)) = f(e(A)) = m(g(A))$, so $f(\overline{\mathrm{rng}(e)}) \subseteq \overline{m(g(A))} = \overline{m(\mathrm{rng}(g))} = m(\overline{\mathrm{rng}(g)})$ (by $(***)$). Since e is a relevant epi, $\mathrm{rng}(e) = B$, so $\mathrm{rng}(f) = f(B) = f(\overline{\mathrm{rng}(e)}) \subseteq m(\overline{\mathrm{rng}(g)})$. In particular, $\mathrm{rng}(f) \subseteq \mathrm{rng}(m)$. So function $h : B \to C$ defined by $h(x) = m^{-1}(f(x))$ is well-defined and continuous. This is our diagonal fill-in: first $m \circ h = f$ by construction; then $m \circ h \circ e = f \circ e = m \circ g$, so since m is mono, $h \circ e = g$.

Notice finally that this diagonal fill-in is unique, as m is mono.

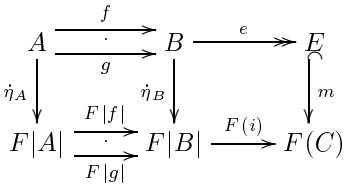
We still have to check that every iso is both a relevant epi and a relevant mono, and that compositions of relevant epis (resp. monos) are relevant epis (resp. monos):

- Let $f : A \to B$ be an iso, with inverse f^{-1}: then $\mathrm{rng}(f) = B$, so $\overline{\mathrm{rng}(f)} = B$, therefore f is a regular epi; also f is easily seen to be a relevant mono.
- Let $A \xrightarrow{e} B$ and $B \xrightarrow{e'} C$ be relevant epis. Then $\overline{\mathrm{rng}(e' \circ e)} = \overline{e'(e(A))} \supseteq e'(\overline{e(A)})$ (by $(**)) = e'(B)$ (since e is a relevant epi) $= \mathrm{rng}(e')$. In particular, taking the sup-closure of both sides, $\overline{\overline{\mathrm{rng}(e' \circ e)}} \supseteq \overline{\mathrm{rng}(e')}$. The left-hand side is clearly just $\overline{\mathrm{rng}(e' \circ e)}$, while the right-hand side is C since e' is a relevant epi, so $\overline{\mathrm{rng}(e' \circ e)} = C$, showing that $e' \circ e$ is a relevant epi.
- That the composition of two relevant monos is again a relevant mono is obvious.

6 Constructing Coequalizers in *Cpo*

The rest of the construction is now easy. Let f and g be two morphisms from A to B.

As shown in Section 3, $|f|$ and $|g|$ have a coequalizer $i : |B| \to C$. (C was some suitable quotient $B/\approx$.) Using the unit $\dot{\eta}$ of the adjunction $F \dashv |_|$ as seen in Section 4, we build the diagram on the right.

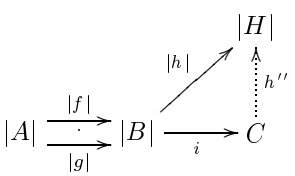


(4)

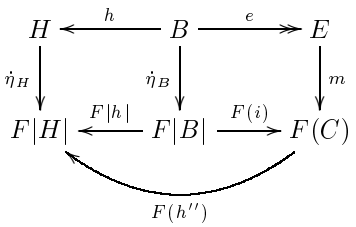
Beware that not everything commutes here! However, there are two commuting squares on the left: $\dot{\eta}_B \circ f = F|f| \circ \dot{\eta}_A$ and $\dot{\eta}_B \circ g = F|g| \circ \dot{\eta}_A$, coming from the naturality of $\dot{\eta}$. We define e and m so that the right square commutes: this is the epi-mono factorization of $F(i) \circ \dot{\eta}_B$. Moreover, we define i as the coequalizer of $|f|$ and $|g|$ in ***Ord***, so $F(i) \circ F|f| = F(i) \circ F|g|$. Therefore the two paths from A to $F(C)$ through $F|A|$ in the diagram, namely $F(i) \circ F|f| \circ \dot{\eta}_A$ and $F(i) \circ F|g| \circ \dot{\eta}_A$, are the same. Using the two commuting squares on the left, it follows that $F(i) \circ \dot{\eta}_B \circ f = F(i) \circ \dot{\eta}_B \circ g$. Using the right commuting square, $m \circ e \circ f = m \circ e \circ g$. Since m is mono, $e \circ f = e \circ g$.

We claim that E, together with e, is the desired coequalizer of f and g. We just have to show that any morphism $h : B \to H$ such that $h \circ f = h \circ g$ factors through e, as $h' \circ e$ for some unique h'.

Since i is a coequalizer in ***Ord*** (compare (2)), and $|h| \circ |f| = |h| \circ |g|$, we can construct a morphism h'' in ***Ord*** such that the diagram on the right commutes.
The diagram (6) below then commutes, too.



(5)

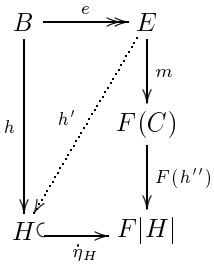


(6) Indeed the top left square is a naturality square for $\dot\eta$, the top right square was already the top right square of (4) and was the defining square for the epi-mono factorization of $F(i) \circ \dot\eta_B$, and the bottom cup is F applied to the right-hand triangle of (5).

Now notice that:

($\dagger$) the unit $\dot\eta_H$ is a relevant mono.

Indeed, $\dot\eta_H(x) \leq_{F|H|} \dot\eta_H(y)$ if and only if $\downarrow x \subseteq \downarrow y$, if and only if $x \leq_H y$.
We can then reorganize the outer contour of (6) to form the diagram on the right, where additionally we have marked that $\dot\eta_H$ was a relevant mono explicitly.
Here the morphism h' is given by the diagonal fill-in property of such squares, and the upper triangle gives the desired factorization of h through e. Note that any other morphism $h'_1 : E \to H$ making the upper triangle commute is such that $F(h'') \circ m \circ e = \dot\eta_H \circ h = \dot\eta_H \circ h'_1 \circ e$, so $F(h'') \circ m = \dot\eta_H \circ h'_1$ since e is epi: this means that h'_1 also makes the lower triangle commute, therefore $h' = h'_1$ since the fill-in is unique.



This terminates our proof that E, e indeed defines the coequalizer of f and g in ***Cpo***.

More generally, we have proved in this section:

Theorem 1. *Let $\mathcal{C}$ be a category with coequalizers, $\mathcal{D}$ a category with an epi-mono factorization system $(\mathcal{E}, \mathcal{M})$. Assume there is adjunction $F \dashv |_| : \mathcal{D} \to \mathcal{C}$ whose unit is in $\mathcal{M}$. Then $\mathcal{D}$ has coequalizers.*

Proof. The argument of this section applies, replacing ***Ord*** by $\mathcal{C}$ and ***Cpo*** by $\mathcal{D}$. □

Example 12. Let A be $\mathbb{N} \cup \{+\infty\}$, B be $[0,1]$. Let $f : A \to B$ map $n \in \mathbb{N}$ to $(n+1)/(n+2)$, and $+\infty$ to 1, and let g be the constant function 1 (see Example 2). Recall that $C = [0, 1/2]$ and that i maps x to $\min(x, 1/2)$. On the other hand, $F|B|$ (see Example 9) is the set of all elements x^+ ($x \in [0,1]$) and x^- ($x \in (0,1]$), with x^- "just below" x^+; the unit $\dot\eta_B$ maps $x \in [0,1]$ to x^+. So $F(i) \circ \dot\eta_B$ maps x to $\min(x, 1/2)^+ \in F[0, 1/2]$. Note that the elements of $F[0, 1/2]$ are the x^+ ($x \in [0, 1/2]$) and the x^- ($x \in (0, 1/2]$). The coequalizer E is obtained as the sup-closure in $F[0, 1/2]$ of the range $\{\min(x, 1/2)^+ | x \in [0,1]\}$. This is the sup-closure in $F[0, 1/2]$ of $\{x^+ | x \in [0, 1/2]\}$, namely $F[0, 1/2]$ itself; the coequalizer map e sends $x \in [0,1]$ to $\min(x, 1/2)^+$.

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