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ON THE GEOMETRY OF INTUITIONISTIC S4 PROOFS

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(*communicated by Gunnar Carlsson*)

Abstract

The Curry-Howard correspondence between formulas and types, proofs and programs, proof simplification and program execution, also holds for intuitionistic modal logic S4. It turns out that the S4 modalities translate as a monoidal comonad on the space of proofs, giving rise to a canonical augmented simplicial structure. We study the geometry of these augmented simplicial sets, showing that each type gives rise to an augmented simplicial set which is a disjoint sum of nerves of finite lattices of points, plus isolated (-1)-dimensional subcomplexes. As an application, we give semantics of modal proofs (a.k.a., programs) in categories of augmented simplicial set and of topological spaces, and prove a completeness result in the style of Friedman: if any two proofs have the same denotations in each augmented simplicial model, then they are convertible. This result rests both on the fine geometric structure of the constructed spaces of proofs and on properties of subscone categories—the categorical generalization of the notion of logical relations used in lambda-calculus. Finally, we address a similar completeness theorem in models of so-called labeled topological spaces, and show that completeness holds provided a certain conjecture holds.

Contents

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1. Introduction

One of the most successful paradigms in modern theoretical computer science is the so-called *Curry-Howard* isomorphism [**28**], an easy but surprising connection between proofs in intuitionistic logics and functional programs, which has far-reaching consequences. One cardinal principle here is that logics help design well-crafted programming constructs, with good semantical properties. In intuitionistic logic, implications denote function spaces, conjunctions give rise to cartesian products, disjunctions are disjoint sums, false is the empty type, true is the unit type, universal quantifications are polymorphic types, existential quantifications are abstract data types. Classical logic in addition introduces the rich concept of *continuation* [**24, 42**], while the modal logic S4 introduces a form of staged computation [**44, 11**].

Our aim in this paper is to show that S4 proofs are also *geometric* objects. To be precise, S4 formulas correspond to augmented simplicial sets, and S4 proofs correspond to maps between these spaces. In particular, this extends the Curry-Howard picture to:

Logic		Programming		Geometry
Formulae	=	Types	=	Augmented Simplical Sets
Proofs	=	Programs	=	Augmented Simplicial Maps
Equality of Proofs	=	Convertibility	=	Equality of Maps

The = signs are exact, except possibly for the Programs=Augmented Simplicial Maps one (we only get *definable* augmented simplicial maps). In particular, it is well-known that equality of proofs, as defined by the symmetric closure of detour, or cut-elimination [**47**], is exactly convertibility of terms (programs). We shall in addition show that two (definable) augmented simplicial maps are equal if and only if their defining terms are convertible, i.e., equal as proofs (bottom right = sign). This will be Theorem 74 and Corollary 75, an S4 variant of Friedman's Theorem [**16**], which will constitute the main goal of this paper.

While Friedman's Theorem in the ordinary, non-modal, intuitionistic case can be proved in a relatively straightforward way using logical relations [**40**], the S4 case is more complex, and seems to require one to establish the existence of a certain strong retraction of one augmented simplicial set $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])$ onto another $\mathcal{S4}[F \supset G]$ (Corollary 48). By the way, we invite the reader to check that the existence of the corresponding strong retraction in the category of sets (as would be needed to map our techniques to the non-modal case) is trivial. The existence of the announced retraction in the category $\widehat{\Delta}$ of augmented simplicial sets is more involved, and prompts us to study the geometry of S4 proofs themselves.

The plan of the paper is as follows. After we review related work in Section 2, we deal with all logical preliminaries in Section 3. We start by recalling some basic concepts in logics in Section 3.1, and go on to the Curry-Howard correspondence between types and formulae, proofs and programs, equality of proofs and convertibility in Section 3.2. We also introduce the logic we shall use, namely minimal intuitionistic S4, giving its Kripke semantics (Section 3.4) as well as a natural deduction system and a proof term language λ_{S4}, essentially due to [**7**], for it. This is in Section 4.1, where we also prove basic properties about λ_{S4}—confluence, strong normalization of typed terms—and study the structure of normal and so-called η-long normal forms.

We come to the meat of this paper in Section 4, where we observe that each type F induces an augmented simplicial set whose q-simplices are terms of type $\Box^{q+1}F$ modulo $\approx$. We characterize exactly the computation of faces and degeneracies on terms written in η-long normal form in Section 4.1, where they take a particularly simple form. This allows us to study the geometry of these terms in a precise way in Section 4.2. The crucial notion here is *oriented contiguity*, which is an oriented form of path-connectedness. It turns out that this allows us to characterize the simplicial part of these augmented simplicial sets as the nerve of its points ordered by contiguity—this is an oriented simplicial complex. In dimension -1, we get all connected components of these simplicial complexes, as we show in Section 4.3. We also show that each non-empty connected component is a finite lattice of points (0-simplices). In Section 4.4 we turn to another important construction in these augmented simplicial sets, that of *planes*. Using the lattice structure, we are able to show that there are augmented simplicial maps projecting the whole space onto planes, under mild conditions. This is the essential ingredient in showing that $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])$ strongly retracts onto $\mathcal{S4}[F \supset G]$, as announced above.

Section 5 reverses the picture and shows that we may always interpret proofs as augmented simplicial maps. In general, we may always interpret proofs in any cartesian closed category (CCC) with a (strict) monoidal comonad—so-called *strict CS4 categories*—, as shown in Section 5.1 and Section 5.2. We give examples of strict CS4 categories in Section 5.1. In Section 5.2, we show additionally that the typed λ_{S4} calculus is a way of describing the free strict CS4 category on a given set of base types. In particular, strict CS4 categories offer a sound a complete way of describing λ_{S4} terms and equalities between them. However, these categories are general constructions that need to be made more concrete. We would like to be able to compare proofs in S4 by looking at them not in any strict CS4 category, but in more concrete ones, in particular in the category $\widehat{\Delta}$ of augmented simplicial sets. We show that λ_{S4} terms still get interpreted faithfully in $\widehat{\Delta}$ in Section 5.7—this is Friedman's Theorem for S4, which we prove using a variant of Kripke logical relations indexed over the category Δ, and using in an essential way the strong retraction of $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])$ onto $\mathcal{S4}[F \supset G]$ that we constructed in Section 4.4. We review logical relations in Section 5.3, explain how they work and why they should be generalized to some form of Kripke logical relation in our case. This is complex, and better viewed from an abstract, categorical viewpoint: this is why we use *subscones* (presented in Section 5.4), establish the Basic Lemma in Section 5.5 and the Bounding Lemma in Section 5.6, the main two ingredients in the equational completeness theorem of Section 5.7. Finally, we address the question of equational completeness in categories of topological spaces in Section 6. While this is still open, we shall show that equational completeness in so-called *labeled* compactly generated topological spaces, provided a certain conjecture (Conjecture 92) holds.

2. Related Work

First, let us dispel a possible misunderstanding. The part of this paper concerned with logic is about the proof theory of S4, that is, the study of proof terms as a programming language, not about validity or provability. The reader interested in categorical models of validity in the modal case is referred to [**51**] and the references therein. In this line, a well-known topological interpretation of the $\Box$ modality of S4, due to Kuratowski, is as follows: interpret each formula F as a subset of some topological space, and $\Box F$ as the interior of F. (In general, any coclosure operator works here.) Note that this interpretation collapses $\Box F$ with $\Box\Box F$, while our interpretations of $\Box$ will not. In fact no $\Box^p F$ can be compared with no $\Box^q F$ in our interpretations unless $p = q$.

It is easier to reason on proof terms than directly on proofs. In particular, it is more convenient to reason on Church's λ-calculs than on natural deduction proofs. This is why we use Bierman and de Paiva's λ_{S4} language [**7**] of proof terms for S4. There would have been many other suitable proposals, e.g., [**44, 11, 38**]. In particular, [**21**] dispenses with boxes around terms to represent $\Box$-introduction rules, by using operators with non-negative indices corresponding to dimensions. The augmented simplicial structure of the language is apparent in the syntax of this language; however λ_{S4} turned out to be more convenient technically.

S4 proof terms have been used for partial evaluation [**49**], run-time program generation [**11**], in higher-order abstract syntax [**34**], etc. The idea is that whereas F is a type of values, $\Box F$ is a type of delayed computations of values of type F, or of terms denoting values of type F; $\mathbf{d}$ evaluates these computations or these terms to return their values, and $\mathbf{s}$ lifts any delayed computation M to a doubly delayed computation whose value is M itself. This is similar to eval/quote in Lisp [**35**], or to processes evolving through time, say, starting at $t = 0$ and homing in on their values at $t = 1$, as argued in [**23**]. This is also similar to the viewpoint of Brookes and Geva [**9**], where comonads $(\Box, \mathbf{d}, \mathbf{s})$ are enriched into so-called *computational comonads*, by adding a natural transformation γ from the identity functor to $\Box$ allowing to lift any value, not just any computation, to a computation; γ must be such that $\mathbf{d} \circ \gamma_F = id_F$ and $\mathbf{s} \circ \gamma_F = \gamma_{\Box F} \circ \gamma_F$. In $\widehat{\Delta}$, such a γ induces a *contracting homotopy* $s_q^{-1} : K_q \to K_{q+1}$ for every $q \geqslant -1$, by $s_q^{-1} \hat{=} \Box^{q+1}(\gamma_{K_{-1}})$; these are usually used to build resolutions of chain complexes. While our comonads need not be computational in this sense, the role of contracting homotopies should become clearer by pondering over Proposition 68 and the construction of Lemma 46.

It is tempting to compare the computational comonads to E. Moggi's computational λ-calculus, i.e. CCCs with a strong monad. [**6**] is a nice introduction to the latter, and to their relation with Fairtlough and Mendler's propositional lax logic. According to Brookes and Geva, there is no special connection between computational comonads and strong monads. However, in a sense they do express similar concerns in programming language theory. Moreover, as shown in [**6**], strong monads are best understood as the existential dual $\Diamond$ ("in some future") to $\Box$ ("in all futures"). Kobayashi [**32**] deals with a calculus containing both $\Box$ and $\Diamond$. Pfenning and Davies [**43**] give an improved framework for combining both $\Box$ and $\Diamond$, and show how lax logic is naturally embedded in it. While classical negation provides a natural link between both modalities, in intuitionistic logic the link is more tenuous. Following R. Goré, there is a more cogent, intuitionistically valid connection between an existential and a universal modality, provided the existential modality is defined as a monad that is left-adjoint to $\Box$. In this sense, Moggi's strong monad should be written as the tense logic modality $\blacklozenge$ ("in some past"), so that $\blacklozenge F \supset G$ is provable if and only if $F \supset \Box G$ is. This duality is reflected in programming-language semantics, where $\Box F$ is the type of computations whose values are in F, while $\blacklozenge G$ is the type of values produced by computations in G. Geometrically, $\blacklozenge F$ builds a space of cones over the space F (at least as soon as F is connected), and this may be defined in categories of topological spaces or of augmented simplicial sets [**46**]; it turns out that the cone modality is indeed a strong monad. However existentials, and therefore also $\blacklozenge$, are hard to deal with in establishing equational completeness results, and this is why we won't consider them in this paper.

We hope that studies of the kind presented here will help understand connections between computation, logic and geometry. The relation to other geometric ways of viewing computation, such as [**26**] on distributed computation, is yet to be clarified. Another interesting piece of work at the intersection of logic (here, linear logic) and simplicial geometry is [**3, 4**], which provides sophisticated models for the multiplicative-exponential fragment of linear logic [**17**] based on affine simplicial spaces with an extra homological constraint. Note that there are strong links between S4 and linear logic, see e.g., [**37**].

3. Logics, the Curry-Howard Correspondence, and S4

3.1. Logics

Consider any logic, specified as a set of deduction rules. So we have got a notion of *formula*, plus a notion of *deduction*, or *proof*. Those formulas that we can deduce are called *theorems*. For example, in minimal propositional logic, one of the smallest non-trivial logics, the formulas are given by the grammar:

$$F, G ::= A | F \supset G$$

where A ranges over *propositional variables* in some fixed set Σ, and $\supset$ is implication. (This logic is called *minimal* because removing the only operator, $\supset$, would leave us with something too skinny to be called a logic at all.) The deductions in the standard Hilbert system for *intuitionistic* minimal logic are given by the following axioms:

$$F \supset G \supset F \tag{1}$$

$$(F \supset G \supset H) \supset (F \supset G) \supset F \supset H \tag{2}$$

where F, G, H range over all formulas, and $\supset$ associates to the right, that is, e.g., $F \supset G \supset H$ abbreviates $F \supset (G \supset H)$; and the *modus ponens* rule:

$$\frac{F \supset G \qquad F}{G}\,(MP)$$

which allows one to deduce G from two proofs, one of $F \supset G$, the other of F. Now there is a third level, apart from formulas and proofs, namely *proof simplifications*. Consider for example the following proof:

$$\dfrac{\dfrac{\dfrac{}{F \supset G \supset F}\,(1) \qquad \begin{array}{c}\vdots\,\pi_1\\ F\end{array}}{G \supset F}\,(MP) \qquad \begin{array}{c}\vdots\,\pi_2\\ G\end{array}}{F}\,(MP)$$

This may be simplified to just the proof π_1. The idea that proofs may be simplified until no simplification can be made any longer, obtaining equivalent *normal proofs*, was pioneered by Gerhard Gentzen [**48**] to give the first finitist proof (in the sense of Hilbert) of the consistency of first-order Peano arithmetic. If the logical system is presented in a proper way, as with Gentzen's sequent calculus, it is easy to see that false has no normal proof (no rule can lead to a proof of false). So false has no proof, otherwise any proof π of false could be simplified to a normal proof of false, which does not exist. Hilbert systems as the one above are not really suited to the task, and we shall instead use *natural deduction systems* [**47**] in Section 3.3.

3.2. The Curry-Howard Correspondence

Note that there is another reading of the logic. Consider any formula as being a set: $F \supset G$ will denote the set of all total functions from the set F to the set G. Then proofs are inhabitants of these sets: interpret the one-step proof (1) as the function taking $x \in F$ and returning the function that takes $y \in G$ and returns x, interpret (2) as the more complex functional that takes $x \in F \supset G \supset H$, $y \in F \supset G$, and $z \in F$, and returns $x(z)(y(z))$; finally, if π_1 is a proof of $F \supset G$—a *function* from F to G—and π_2 is in F, then (MP) builds $\pi_1(\pi_2)$, an element of G. Syntactically, we build a *programming language* by defining terms:

$$M, N, P ::= K|S|MN$$

where K and S are constants and MN denotes the *application* of M to N. (This language is called *combinatory logic.*) We may restrict to well-typed terms, where the typing rules are: K has any type $F \supset G \supset F$, S has any type $(F \supset G \supset H) \supset (F \supset G) \supset F \supset H$, and if M has type $F \supset G$ and N has type F, then MN has type G. This may be written using typing judgments $M : F$, which assign each term M a type F, using typing rules:

$$K : F \supset G \supset F \tag{3}$$

$$S : (F \supset G \supset H) \supset (F \supset G) \supset F \supset H \tag{4}$$

$$\frac{M : F \supset G \qquad N : F}{MN : G}\,(MP)$$

Note the formal similarity with the proof rules (1), (2), and (MP). Any typing rule can be converted to a proof, by forgetting terms. Conversely, any proof can be converted to a typing derivation by labeling judgments with suitable terms. Furthermore, given a typable term M, there is a unique so-called *principal typing* from which all other typings can be recovered (Hindley's Theorem). This constitutes half of the so-called *Curry-Howard correspondence* between programs (terms) and proofs. Subscripting K and S with the types they are meant to have at each occurrence in a term even makes this an isomorphism between typable terms and proofs.

Let us introduce the second half of the Curry-Howard correspondence: the proof simplification steps give rise to program *reduction rules*; here, the natural choice is $KMN \to M$, $SMNP \to MP(NP)$. It turns out that these reduction rules give rise to a notion of *computation*, where a term (a program) is reduced until a normal term is reached. This reduction process is then exactly the proof simplification process described above.

3.3. Natural Deduction and the Lambda-Calculus

The language of Hilbert systems and combinatory logic is not easy to work with, although this can be done [**27**]. A more comfortable choice is given by Church's λ-calculus [**2**], the programming language associated with minimal logic in *natural deduction* format [**47**]. Its terms are given by the grammar:

$$M, N, P ::= x | \lambda x \cdot M | MN$$

where x ranges over variables, $\lambda x \cdot M$ is λ-*abstraction* ("the function that maps x to M", where M typically depends on x). For convenience, we write $MN_1N_2 \dots N_k$ instead of $(\dots((MN)N_1)N_2 \dots)N_k$ (application associates to the left), and $\lambda x_1, x_2, \dots, x_k \cdot M$ instead of $\lambda x_1 \cdot \lambda x_2 \cdot \dots \lambda x_k \cdot M$.

Typing, i.e., proofs, are described using *sequents* instead of mere formulae. A sequent is an expression of the form $x_1 : F_1, \dots, x_n : F_n \vdash M : F$, meaning that M has type F *under the assumptions* that x_1 has type F_1, ..., x_n has type F_n. This is needed to type λ-abstractions. In this paper, the *context* $x_1 : F_1, \dots, x_n : F_n$ will be a *list* of *bindings* $x_i : F_i$, where the x_i's are distinct. We shall usually write Γ, Θ for contexts. The notation $\Gamma, x : F$ then denotes $x_1 : F_1, \dots, x_n : F_n, x : F$, provided x is not one of $x_1, \dots, x_n$. The typing rules are:

$$\frac{}{x_1 : F_1, \dots, x_n : F_n \vdash x_i : F_i}\,(Ax) \qquad (1 \leqslant i \leqslant n)$$

$$\frac{\Gamma, x : F \vdash M : G}{\Gamma \vdash \lambda x \cdot M : F \supset G}\,(\supset I) \qquad \frac{\Gamma \vdash M : F \supset G \quad \Gamma \vdash N : F}{\Gamma \vdash MN : G}\,(\supset E)$$

Finally, computation, i.e., proof simplification, is described by the *β-reduction rule* $(\lambda x \cdot M)N \to M[x := N]$, where $M[x := N]$ denotes the (capture-avoiding) substitution of N for x in M. We may also add the *η-reduction rule* $\lambda x \cdot Mx \to M$, provided x is not free in M. Although this is not necessary for proof normalization, η-reduction allows one to get an extensional notion of function, where two functions are equal provided they return equal results on equal arguments. (This also corresponds to reducing proofs of axiom sequents to proofs consisting of just the (Ax) rule, proof-theoretically.)

3.4. Minimal Intuitionistic S4

The topic of this paper is the intuitionistic modal logic S4. For simplicity, we consider *minimal* intuitionistic S4, which captures the core of the logic: formulae, a.k.a. types, are defined by: $F ::= A \mid F \supset F \mid \Box F$ where A ranges over a fixed set Σ of *base types*. (While adding conjunctions $\wedge$, and truth $\top$ do not pose any new problems, it should be noted that adding disjunctions $\vee$, falsehood $\bot$ or $\blacklozenge$ would not be as innocuous for some of the theorems of this paper.)

The usual semantics of (classical) S4 is its Kripke semantics. For any *Kripke frame* $(\mathcal{W}, \trianglerighteq)$ (a preorder), and a *valuation* ρ mapping base types $A \in \Sigma$ to sets of worlds (those intended to make A true), define when a formula F holds at a *world* $w \in \mathcal{W}$ in ρ, abbreviated $w, \rho \models F$: $w, \rho \models A$ if and only if $w \in \rho(A)$; $w, \rho \models F \supset G$ if and only if, if $w, \rho \models F$ then $w, \rho \models G$; and $w, \rho \models \Box F$ if and only for every $w' \trianglerighteq w$, $w', \rho \models F$. Think of $\Box F$ as meaning "from now on, in every future F holds".

The idea that the truth value of a formula F may depend on time is natural, e.g. in physics, where "the electron has gone through the left slit" may hold at time t but not at t'.

In *intuitionistic* S4 we may refine the semantics of formulae to include another preordering $\geqslant$ on worlds, accounting for intuitionistic forcing. Intuitively, $\geqslant$ may be some ordering on states of mind of a mathematician, typically the $\supseteq$ ordering on sets of basic facts that the mathematician knows (the analogy is due to Brouwer). Then if the mathematician knows $F \supset G$ when he is in some state of mind w (abbreviated $w \models F \supset G$), and if he knows F, he should also know G. Further, knowing $F \supset G$ in state of mind w also means that, once the mathematician has extended his state of mind to a larger w', if this w' allows him to deduce F, then he should be able to deduce G in the w' state of mind. The intuitionistic meaning of $F \supset G$ is therefore that $w \models F \supset G$ if and only if, *for every* $w' \geqslant w$, if $w' \models F$ then $w' \models G$. Knowing the negation of F in state of mind w not only means knowing that F does not hold in w, but also that it cannot hold in any state of mind $w' \geqslant w$, i.e., any w' extending w. One distinguishing feature of intuitionistic logic is that it may be the case that there are formulae F such that neither F nor its negation hold in some state of mind w—think of F as an unsolved conjecture—, so the classical tautology $F \vee \neg F$ does not hold in general.

The Kripke semantics of intuitionistic S4 is as follows.

Definition 1 (Kripke Semantics). *An* intuitionistic Kripke frame *is a triple* $(\mathcal{W}, \trianglerighteq, \geqslant)$, *where* $\trianglerighteq$ *and* $\geqslant$ *are preorderings on* $\mathcal{W}$ *such that* $\geqslant \subseteq \trianglerighteq$.

A valuation ρ *on* $\mathcal{W}$ *is a map from base types in* Σ *to upper sets of* worlds *in* $\mathcal{W}$*; an* upper set *is any subset of* $\mathcal{W}$ *such that whenever* $w \in W$ *and* $w' \geqslant w$, $w' \in W$.

The semantics of S4 formulas is given by:

$$w, \rho \models A \text{ iff } w \in \rho(A)$$
$$w, \rho \models F \supset G \text{ iff for every } w' \geqslant w, \text{ if } w', \rho \models F \text{ then } w', \rho \models G$$
$$w, \rho \models \Box F \text{ iff for every } w' \trianglerighteq w,\ w', \rho \models F$$

An S4 formula F *is* valid, *written* $\models F$, *if and only if* $w, \rho \models F$ *in every frame* $(\mathcal{W}, \trianglerighteq, \geqslant)$, *for every* $w \in \mathcal{W}$, *for every valuation* ρ.

The resulting logic is called **IntS4** in [**50**], and the condition relating $\geqslant$ and $\trianglerighteq$ there is $(\geqslant \circ \trianglerighteq \circ \geqslant) = \trianglerighteq$. In the S4 case where $\trianglerighteq$ is a preorder, this is equivalent to our $\geqslant \subseteq \trianglerighteq$.

For all our analogy with states of mind of a mathematician is worth, the condition $\geqslant \subseteq \trianglerighteq$ intuitively states that you can only learn new basic facts (increase w.r.t. $\geqslant$) while time passes ($\trianglerighteq$), but time may pass without you learning any new facts.

An example of such a Kripke frame is as follows. Let $(T, \trianglerighteq_0)$ (time) and $(S, \geqslant_0)$ be two orders, and let $\mathcal{W}$ be $T \times S$, with $(t, s) \trianglerighteq (t', s')$ if and only if $t \trianglerighteq_0 t'$ (whatever s and s'), and $(t, s) \geqslant_0 (t', s')$ if and only if $t = t'$ and $s \geqslant_0 s'$. Call *split* such a Kripke model. Think of pairs (t, s) as a time t, paired with a state of mind, i.e., a set s of basic facts that an observer (a physicist) knows to be true at time t. Order sets s by $\supseteq$ for $\geqslant_0$. Then $\Box F$ holds at (t, s) if and only if F holds at any later time $t' \trianglerighteq_0 t$, whatever the state of mind of the observer might be at that time.

And $F \supset G$ holds at (t, s) if and only if, for any expanded set of basic facts $s' \supseteq s$ the observer may discover, if F is discovered to hold in s' at the same time t, then G must hold in s' at time t, too.

There are other instances of Kripke structures of the form given in Definition 1. For instance, let $(A, \succeq)$ be any preorder. Define $\mathcal{W} \hat{=} \mathbb{P}(A)$. Let $w \trianglerighteq w'$ if and only if every element x' of w' is $\preceq$ some element x of w (this is a form of the *Hoare ordering*, or by reversing orderings, the *Smyth ordering*; both of them play a central role in the denotational semantics of programming languages), and let $\geqslant$ be $\supseteq$. Again $\geqslant \subseteq \trianglerighteq$, but $\mathcal{W}$ is not split.

We have mentioned the $\blacklozenge$ modality in Section 2. This would have the expected semantics: $w, \rho \models \blacklozenge F$ if and only if for some w' with $w \trianglerighteq w'$, $w', \rho \models F$. The other two modalities $\blacksquare$ ("in all pasts") and $\Diamond$ ("in some future") are naturally defined in intuitionistic modal logic by introducing a new binary relation $\trianglelefteq$ on $\mathcal{W}$, which needs not be the converse of $\trianglerighteq$, letting $w, \rho \models \blacksquare F$ if and only if for every $w' \trianglelefteq w$, $w', \rho \models F$, and $w, \rho \models \Diamond F$ if and only if for every w' with $w \trianglelefteq w'$, $w', \rho \models F$ [**50**]. The only constraints on $\geqslant$, $\trianglerighteq$ and $\trianglelefteq$ are that, in addition to $\geqslant \subseteq \trianglerighteq$, we should have $\geqslant \subseteq \trianglelefteq$, $\trianglelefteq \subseteq (\trianglelefteq \cap \trianglerighteq^o) \circ \geqslant$, and $\trianglerighteq \subseteq (\trianglerighteq \cap \trianglelefteq^o) \circ \geqslant$, where R^o denotes the converse of relation R.

3.5. Natural Deduction for Intuitionistic S4

In this paper, we shall not be so much interested in *validity* of S4 formulas as in actual *proofs* of S4 formulas. So let us talk about proofs. In Appendix A we show that validity coincides with existence of proofs; this essentially standard argument is for example not explicit in [**50**].

We use λ_{S4} as a language of proof terms for S4 [**7**]. The raw terms are:

$$M, N, P ::= x \mid MN \mid \lambda x \cdot M \mid dM \mid \boxed{M} \cdot \theta$$

where θ is an *explicit substitution*, that is, a substitution that appears as an explicit component of terms. A substitution θ is any finite mapping from variables x_i to terms t_i, $1 \leqslant i \leqslant n$, and is written $\{x_1 := t_1, \ldots, x_n := t_n\}$; its *domain* $\mathrm{dom}\,\theta$ is the set $\{x_1, \ldots, x_n\}$. (We omit the type subscript of variables whenever convenient.) The *yield* $\mathrm{yld}\,\theta$ is defined as $\bigcup_{x \in \mathrm{dom}\,\theta} \mathrm{fv}(\theta(x))$, mutually recursively with the set of *free variables* $\mathrm{fv}(M)$ of the term M: $\mathrm{fv}(x) \hat{=} \{x\}$, $\mathrm{fv}(MN) \hat{=} \mathrm{fv}(M) \cup \mathrm{fv}(N)$, $\mathrm{fv}(\lambda x \cdot M) \hat{=} \mathrm{fv}(M) \setminus \{x\}$, $\mathrm{fv}(dM) \hat{=} \mathrm{fv}(M)$, $\mathrm{fv}(\boxed{M} \cdot \theta) \hat{=} \mathrm{yld}\,\theta$. (We use $\hat{=}$ for *equality by definition.*) Moreover, we assume that, in any term $\boxed{M} \cdot \theta$, $\mathrm{fv}(M) \subseteq \mathrm{dom}\,\theta$; we also assume Barendregt's naming convention: no variable occurs both free and bound, or bound at two different places—bound variables are x in $\lambda x \cdot M$ and all variables in $\mathrm{dom}\,\theta$ in $\boxed{M} \cdot \theta$.

Our notations differ from [**7**]. There $\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\}$ is written **box** M **with** $N_1, \ldots, N_n$ **for** $x_1, \ldots, x_n$. The new notation allows one, first, to materialize the explicit substitution more naturally, and second the frame notation will be put to good use to explain what simplices look like. Also, dM is written **unbox** M in [**7**]; we use dM because it is more concise and hints that some face operator is at work.

Substitution application $M\theta$ is defined by: $x\theta \hat{=} \theta(x)$ if $x \in \mathrm{dom}\,\theta$, $x\theta \hat{=} x$ otherwise;

$(MN)\theta \hat{=} (M\theta)(N\theta)$; $(\lambda x \cdot M)\theta \hat{=} \lambda x \cdot (M\theta)$ provided $x \notin \mathrm{dom}\,\theta \cup \mathrm{yld}\,\theta$; $(dM)\theta \hat{=} d(M\theta)$; $(\boxed{M} \cdot \theta')\theta \hat{=} \boxed{M} \cdot (\theta' \cdot \theta)$, where *substitution concatenation* $\theta' \cdot \theta$ is defined as $\{x_1 := M_1, \ldots, x_n := M_n\} \cdot \theta \hat{=} \{x_1 := M_1\theta, \ldots, x_n := M_n\theta\}$.

Terms are equated modulo α-*conversion*, the smallest congruence $\equiv$ such that:

$$\lambda x \cdot M \equiv \lambda y \cdot (M\{x := y\})$$

$$\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} \equiv \boxed{M\{x_1 := y_1, \ldots, x_n := y_n\}} \cdot \{y_1 := N_1, \ldots, y_n := N_n\}$$

provided y is not free in M in the first case, and $y_1, \ldots, y_n$ are not free in M and are pairwise distinct in the second case, with identical type subscripts as $x_1, \ldots, x_n$ respectively.

The d operator is a kind of "eval", or also of "comma" operator in the language Lisp [**35**]. The $M, \theta \mapsto \boxed{M} \cdot \theta$ operator is more complex. Let's first look at a special case: for any term M such that $\mathrm{fv}(M) = \{x_1, \ldots, x_n\}$, let $\boxed{M}$ be $\boxed{M} \cdot \{x_1 := x_1, \ldots, x_n := x_n\}$—or, more formally, $\boxed{M\{x_1 := x'_1, \ldots, x_n := x'_n\}} \cdot \{x'_1 := x_1, \ldots, x'_n := x_n\}$. Then $\boxed{M}$ behaves like "quote" M in Lisp, or more exactly, "backquote" M; and provided $\mathrm{dom}\,\theta = \mathrm{fv}(M)$, $\boxed{M} \cdot \theta$ is exactly $(\boxed{M})\theta$: this is a *syntactic closure* in the sense of [**5**], namely a quoted term M together with an environment θ mapping free variables of M to their values.

$$\frac{}{\Gamma, x : F, \Theta \vdash x : F}\,(Ax)$$

$$\frac{\Gamma \vdash M : F \supset G \quad \Gamma \vdash N : F}{\Gamma \vdash MN : G}\,(\supset E) \qquad \frac{\Gamma, x : F \vdash M : G}{\Gamma \vdash \lambda x \cdot M : F \supset G}\,(\supset I)$$

$$\frac{\Gamma \vdash M : \Box F}{\Gamma \vdash dM : F}\,(\Box E) \qquad \frac{\overbrace{\Gamma \vdash N_i : \Box F_i}^{1 \leqslant i \leqslant n} \quad x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash M : G}{\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} : \Box G}\,(\Box I)$$

Figure 1: Typing λ_{S4} terms

The typing rules [**7**], defining a natural deduction system for minimal S4 are as in Figure 1, where Γ, Θ, ..., are *typing contexts*, i.e. lists of *bindings* $x : F$, where x is a variable, F is a type, and no two bindings contain the same variable in any given context. The *exchange rule*:

$$\frac{\Gamma, x : F, y : G, \Theta \vdash M : H}{\Gamma, y : G, x : F, \Theta \vdash M : H}$$

is easily seen to be admissible, so we can consider typing contexts as multisets instead of lists. In particular, there is no choice to be made as to the order of the variables $x_1, \ldots, x_n$ in the right premise of rule $(\Box I)$.

Define the *reduction* relation $\to$ on λ_{S4}-terms as the smallest relation compatible with term structure (i.e., if $M \to N$ then $C[M] \to C[N]$, where $C[P]$ denotes any

$$
\begin{array}{llll}
(\beta) & (\lambda x \cdot M)N \to M\{x := N\} & (d) & d(\boxed{M} \cdot \theta) \to M\theta \\
(\mathrm{gc}) & \boxed{M} \cdot (\theta, \{x := N\}) \to \boxed{M} \cdot \theta & & \text{provided } x \notin \mathrm{fv}(M) \\
(\mathrm{ctr}) & \boxed{M} \cdot (\theta, \{x := N, y := N\}) \to \boxed{M\{y := x\}} \cdot (\theta, \{x := N\}) & & \\
(\boxed{\Box}) & \boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\}) \to \boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta') & & \\
(\eta) & \lambda x \cdot Mx \to M \text{ provided } x \notin \mathrm{fv}(M) & (\eta\Box) & \boxed{dx} \cdot \{x := N\} \to N
\end{array}
$$

Figure 2: The reduction relation of λ_{S4}

term with a distinguished occurrence of P) defined in Figure 2 [**7, 20**]. Terms that match the left-hand side of rules are called *redexes* (for *red*uction *ex*pression). The *convertibility* relation $\approx$ is the smallest congruence extending $\to$; in other words, $\approx$ is the reflexive symmetric transitive closure of $\to$. In addition, we write $\to^+$ the transitive closure of $\to$, and $\to^*$ its reflexive transitive closure.

Rule (d) is like Lisp's rule for evaluating quoted expressions: evaluating $\boxed{M}$, by $d\boxed{M}$, reduces to M. Rule $(\boxed{\Box})$ can be seen either as an inlining rule, allowing one to inline the definition of x as $\boxed{N}$ inside the body M of the box $\boxed{M}$, or logically as a box-under-box commutation rule. (gc) is a garbage collection rule, while (ctr) is a contraction rule. We use a new notation in these rules: if θ and θ' are two substitutions with disjoint domains, then θ, θ' denotes the obvious union.

The last two rules are so-called *extensional* equalities. Together with (gc), $(\eta\Box)$ allows us to deduce $\boxed{dx} \cdot \theta \approx x\theta$, but *not* $\boxed{dM} \cdot \theta \approx M\theta$ for any term M: M has to be a variable. For a discussion of this, see [**21**].

3.6. Standard Properties: Subject Reduction, Confluence, Strong Normalization

We now prove standard properties of proof simplification calculi.

The following lemma is by a series of easy but tedious computations; it says that reduction preserves typings, alternatively that it rewrites proofs to proofs of the same sequents.

Lemma 2 (Subject Reduction). *If the typing judgment $\Gamma \vdash M : F$ is derivable and $M \to N$ then $\Gamma \vdash N : F$ is derivable.*

Proposition 3 (Strong Normalization). *If M is typable, then it is* strongly normalizing*, i.e., every reduction sequence starting from M is finite.*

Proof. By the reducibility method [**18**]. Let SN be the set of strongly normalizing terms, and define an interpretation of types as sets of terms as follows: for every base type A, $||A|| \hat{=} SN$;
$||F \supset G|| \hat{=} \{M \in SN | \text{whenever } M \to^* \lambda x \cdot M_1 \text{ then for every } N \in ||F||, M_1\{x := N\} \in ||G||\}$;
$||\Box F|| \hat{=} \{M \in SN | \text{whenever } M \to^* \boxed{M_1} \cdot \theta \text{ then } M_1\theta \in ||F||\}$. Observe that:

(CR1) $||F|| \subseteq SN$ for every type F;

(CR2) For every $M \in ||F||$, if $M \to M'$ then $M' \in ||F||$. This is by structural induction on F. This is clear when F is a base type. For implications, assume $M \in ||F \supset G||$ and $M \to M'$; then $M' \in SN$, and if $M' \to^* \lambda x \cdot M_1$, then since $M \in ||F \supset G||$, $M \to^* \lambda x \cdot M_1$, so by definition $M_1\{x := N\} \in ||G||$ for every $N \in ||F||$; therefore $M' \in ||F \supset G||$. The case of box types is proved similarly.

(CR3) For every neutral term M, if $M' \in ||F||$ for every M' with $M \to M'$, then $M \in ||F||$. (Call a term *neutral* if and only if it is not of the form $\lambda x \cdot M$ or $\boxed{M} \cdot \theta$.) This is again by structural induction on F. This is clear when F is a base type. For implications, assume that every M' such that $M \to M'$ is in $||F \supset G||$, and show that $M \in ||F \supset G||$. Clearly $M \in SN$, since every reduction starting from M must be empty or go through some $M' \in ||F \supset G|| \subseteq SN$ by (CR1). So assume that $M \to^* \lambda x \cdot M_1$. Since M is neutral, the given reduction is non-empty, so there is an M' such that $M \to M' \to^* \lambda x \cdot M_1$. By assumption $M' \in ||F \supset G||$, so for every $N \in ||F||$, $M_1\{x := N\} \in ||G||$. It follows that $M \in ||F \supset G||$. The case of box types is similar.

Next we show that:

1. If $M \in ||F \supset G||$ and $N \in ||F||$, then $MN \in ||G||$. By (CR1), M and N are in SN, so we prove this by induction on the pair (M, N) ordered by $\to$, lexicographically. Note that MN is neutral, and may only rewrite in one step to $M'N$ where $M \to M'$, or to MN' where $N \to N'$, or to $M_1\{x := N\}$ by (β) (if $M = \lambda x \cdot M_1$). In the first two cases, $M' \in ||F \supset G||$, resp. $N' \in ||F||$ by (CR2), so we may apply the induction hypothesis. In the third case, this is by definition of $||F \supset G||$. In each case we get a term in $||G||$, so by (CR3) $MN \in ||G||$.

2. If $M\{x := N\} \in ||G||$ for every $N \in ||F||$, then $\lambda x \cdot M \in ||F \supset G||$. To show this, we show the converse of 1: if for every $N \in ||F||$, $MN \in ||G||$, then $M \in ||F \supset G||$. Indeed, first $M \in SN$: take any variable x; x is in $||F||$ by (CR3), so $Mx \in ||G||$ by assumption, so $Mx \in SN$ by (CR1), hence $M \in SN$. Second, assume that $M \to^* \lambda x \cdot M_1$, then for every $N \in ||F||$, $MN \to^* M_1\{x := N\} \in ||G||$ by (CR2). So $M \in ||F \supset G||$.
   Using this, assume that $M\{x := N\} \in ||G||$ for every $N \in ||F||$, and show that $\lambda x \cdot M \in ||F \supset G||$. It is enough to show that $(\lambda x \cdot M)N \in ||G||$ for every $N \in ||F||$. We do this by induction on (M, N) ordered by $\to$, which is well-founded: indeed, $N \in ||F|| \subseteq SN$ by (CR1), and $M = M\{x := x\} \in ||G|| \subseteq SN$ by (CR1), since $x \in ||F||$ by (CR3). Since $(\lambda x \cdot M)N$ is neutral, apply (CR3): $(\lambda x \cdot M)N$ may rewrite to $(\lambda x \cdot M')N$ with $M \to M'$ (this is in $||G||$ by (CR2) and the induction hypothesis), or to $(\lambda x \cdot M)N'$ with $N \to N'$ (similar), or to $M\{x := N\}$ by (β) (in $||G||$ by assumption), or to $M'N$ by (η) where $M = M'x$, x not free in M' (then $M'N = M\{x := N\}$, which is in $||G||$ by assumption).

3. If $M \in ||\Box F||$, then $dM \in ||F||$. This is by induction on M ordered by $\to$,

which is well-founded since by (CR1) $M \in SN$. Now dM may rewrite either to dM' with $M \to M'$ (then apply the induction hypothesis, noting that $M' \in ||\Box F||$ by (CR2), so $dM' \in ||F||$), or to $M_1\theta$, provided $M = \boxed{M_1} \cdot \theta$ (then $M_1\theta \in ||F||$ by definition). Since dM is neutral, by (CR3) $dM \in ||F||$.

4. If $M\theta \in ||F||$ and θ maps each variable $x \in \operatorname{dom}\theta$ to some strongly normalizing term, then $\boxed{M} \cdot \theta \in ||\Box F||$. First we show the converse of 3: if $dM \in ||F||$ then $M \in ||\Box F||$. First since $dM \in ||F|| \subseteq SN$ by (CR1), $M \in SN$. It remains to show that whenever $M \to^* \boxed{M_1} \cdot \theta$ then $M_1\theta \in ||F||$. However then $dM \to^* M_1\theta$ must be in $||F||$ by (CR2).
   Knowing this, let $M\theta$ be in $||F||$ and θ map each variable $x \in \operatorname{dom}\theta$ to some strongly normalizing term. Let us show that $\boxed{M} \cdot \theta \in ||\Box F||$. It is enough to show that $d\boxed{M} \cdot \theta \in ||F||$, using (CR3) since this term is neutral. Letting θ be $\{x_1 := N_1, \ldots, x_n := N_n\}$, we show this by induction on, first, $N_1, \ldots, N_n$ ordered by the multiset extension [**12**] of $\to \cup \rhd$, where $\rhd$ is the immediate superterm relation (it is well-known that as soon as N_i is in the well-founded part of $\to$, it is also in the well-founded part of $\to \cup \rhd$; the multiset extension allows one to replace any element N_i of the multiset by any finite number of smaller elements, and is well-founded on all multisets of elements taken from the well-founded part of the underlying ordering); and second on $M\theta$, lexicographically. Now $d\boxed{M} \cdot \theta$ may rewrite in one step to:
   - $M\theta$ by (d); this is in $||F||$ by assumption.
   - dN_1 by $(\eta\Box)$, where $M = dx_1$ and $n = 1$. Then $dN_1 = M\theta$ is in $||F||$ by assumption.
   - $d\boxed{M'} \cdot \theta$ where $M \to M'$. By (CR2) $M'\theta \in ||F||$, so we may apply the induction hypothesis.
   - $d\boxed{M} \cdot \theta'$ where $\theta' = \{x_1 := N_1, \ldots, x_i := N_i', \ldots, x_n := N_n\}$ and $N_i \to N_i'$. Since $N_i' \in SN$, we may apply the induction hypothesis.
   - $d\boxed{M} \cdot \theta'$ where $\theta = \theta', \{x := N\}$ and x is not free in M by (gc). This is by the induction hypothesis. The same argument applies for (ctr).
   - $d\boxed{M\{x := \boxed{N}\}} \cdot (\theta_1, \theta')$ where $\theta = \theta_1, \{x := \boxed{N} \cdot \theta'\}$ by $(\boxed{\Box})$. We wish to apply the induction hypothesis. For this, we have to check that $M\{x := \boxed{N}\}(\theta_1, \theta')$ is in $||F||$. But $M\theta$ is in $||F||$ and equals $M(\theta_1, \{x := \boxed{N} \cdot \theta'\}$. The latter is equal or rewrites by (gc) to $M(\theta_1, \{x := (\boxed{N})\theta'\}) = M\{x := \boxed{N}\}(\theta_1, \theta')$, so the latter is in $||F||$ by (CR2).

We now check that, given any typing derivation π of $x_1 : F_1, \ldots, x_n : F_n \vdash M : F$, for every $N_1 \in ||F_1||$, ..., $N_n \in ||F_n||$, $M\{x_1 := N_1, \ldots, x_n := N_n\} \in ||F||$. This is by structural induction on π. The (Ax) cas is obvious, while the other cases are dealt with by using items 1–4 above. Since $x_i \in ||F_i||$ by (CR3), it follows that $M \in ||F||$. By (CR1), $M \in SN$. $\Box$

Proposition 4 (Confluence). *If M is typable, and if $M \to^* N_1$ and $M \to^* N_2$, then there is P such that $N_1 \to^* P$ and $N_2 \to^* P$.*

Proof. A long and uninteresting series of computations shows that all critical pairs are joinable, hence λ_{S4} is locally confluent [**13**]. See Appendix B. This holds also on typable terms because of Lemma 2. By Newman's Lemma (every locally confluent and strongly normalizing rewrite system is confluent) and Proposition 3, λ_{S4} is confluent on typed λ_{S4}-terms. ☐

It is tempting to try and prove that λ_{S4} is confluent even in the untyped case. The standard attack using a form of finite developments theorem [**2**] does not work, because of the non-linearity of rule (ctr), i.e., the fact that N must occur twice on the left-hand side. (In particular, the confluence proof of [**20**] is faulty. However the related calculus $\lambda\,\mathrm{ev}\,\mathrm{Q}$ does not suffer from this non-linearity, and is confluent [**22**] even in the untyped case.) Note that a similar non-linearity problem plagues the λ-calculus with surjective pairing (rules $\pi_1\langle M, N\rangle \to M$, $\pi_2\langle M, N\rangle \to N$, $\langle \pi_1 M, \pi_2 M\rangle \to M$), which is *not* confluent [**31**]. However, it is confluent in the typed case, because it is locally confluent and strongly normalizing in this case.

Corollary 5. *Every typable λ_{S4}-term has a unique normal form.*

3.7. The Shape of Normal Forms, η-Long Normal Forms

One way of describing normal forms for typed terms is by the typing system BN of Figure 3.

$$\frac{}{\Gamma, x : F, \Theta \vdash_E x : F}\,(Ax_E) \qquad \frac{\Gamma \vdash_E M : F \supset G \quad \Gamma \vdash_I N : F}{\Gamma \vdash_E MN : G}\,(\supset E_E)$$

$$\frac{\Gamma \vdash_E M : F}{\Gamma \vdash_I M : F}\,(Flip) \qquad \frac{\Gamma \vdash_E M : \Box F}{\Gamma \vdash_E dM : F}\,(\Box E_E)$$

$$\frac{\Gamma, x : F \vdash_I M : G}{\Gamma \vdash_I \lambda x \cdot M : F \supset G}\,(\supset I_I) \qquad \frac{\overbrace{\Gamma \vdash_E N_i : \Box F_i}^{1 \leqslant i \leqslant n} \quad x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_I M : G}{\Gamma \vdash_I \boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} : \Box G}\,(\Box I_I)$$
$$(\mathrm{fv}(M) = \{x_1, \ldots, x_n\})$$

Figure 3: Typing beta-normal forms: System BN

Lemma 6. *Call a term* beta-normal *if and only if it contains no* (β), (d), (gc), $(\boxed{\Box})$ *redex (i.e., no redex except possibly* (ctr), (η) *or* $(\eta\Box)$ *redexes).*

If $\Gamma \vdash M : F$ *and* M *is beta-normal, then* $\Gamma \vdash_I M : F$. *Moreover, if* M *is* neutral, *i.e., not starting with a* λ *or a box, then* $\Gamma \vdash_E M$.

Conversely, if $\Gamma \vdash_I M : F$ *or* $\Gamma \vdash_E M : F$, *then* $\Gamma \vdash M : F$ *and* M *is beta-normal.*

Proof. By structural induction on the given derivation of $\Gamma \vdash M : F$. The cases M a variable, and M of the form $\lambda x \cdot M_1$ are trivial. If $M = M_1 M_2$, with $\Gamma \vdash M_1 : G \supset H$ and $\Gamma \vdash M_2 : G$, then M_1 must be neutral, otherwise by typing M_1

would start with a λ, and then M would be a (β)-redex. So by induction hypothesis $\Gamma \vdash_E M_1 : G \supset H$. Since by induction hypothesis $\Gamma \vdash_I M_2 : G$, it follows by rule $(\supset E_E)$ that $\Gamma \vdash_E M : H$. The case where $M = dM_1$ is similar. Finally, when M is of the form $\boxed{M_1} \cdot \theta$, with $\theta = \{x_1 : N_1, \ldots, x_n : N_n\}$, $\Gamma \vdash N_i : \Box F_i$ $(1 \leqslant i \leqslant n)$, and $x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash M_1 : F$, then by induction hypothesis $x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_I M_1 : F$. Moreover, since M is not a (gc) redex, $\mathrm{fv}(M) = \{x_1, \ldots, x_n\}$. Also, every N_i must be neutral, otherwise by typing they would start with a box, which is forbidden because M is not a $(\boxed{\Box})$ redex, so by induction hypothesis $\Gamma \vdash_E N_i : \Box F_i$. It follows that rule $(\Box I_I)$ applies, therefore $\Gamma \vdash_I M : \Box F$.

Conversely: if $\Gamma \vdash_I M : F$ or $\Gamma \vdash_E M : F$, then it is obvious that $\Gamma \vdash M : F$: erase all E and I subscripts, and remove all instances of $(Flip)$. It remains to show that M is beta-normal. Consider any subterm of M. If it is of the form $M_1 M_2$, then its type must have been derived using the $(\supset E_E)$ rule, which implies that M_1 is typed as in $\Gamma \vdash_E M_1 : F \supset G$; but no rule in BN (Figure 3) would allow one to derive such a judgment if M_1 began with λ; so $M_1 M_2$ is not a (β)-redex. Similarly, no subterm of M can be a (d) redex. The side-conditions on rule $(\Box I_I)$ entail that no subterm of M is a (gc) redex, while the fact that $N_i : \Box F_i$ must have been derived using a $\vdash_E$ judgment entails that no N_i starts with a box, hence that no subterm of M is a $(\boxed{\Box})$ redex. So M is beta-normal. $\Box$

A more convenient form than normal forms is the *η-long normal form*, imitating that of [**29**] in the non-modal case. In the S4 case, η-long normal forms are slightly more complex, but can be described as follows, including an additional linearity constraint on boxes.

Definition 7 (η-long normal form). *Call a term M* linear *if and only if every free variable of M occurs exactly once in M. Formally, define the notion of being* linear in W, *where W is a finite set of variables, as follows. Every variable is linear in W, $\lambda x \cdot M$ is linear in W provided M is linear in $W \setminus \{x\}$, MN is linear in W provided M and N are and $\mathrm{fv}(M) \cap \mathrm{fv}(N) \cap W = \emptyset$, ∂M is linear in W provided M is, $\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\}$ is linear in W provided each N_i, $1 \leqslant i \leqslant n$, is linear in W, and $\mathrm{fv}(N_i) \cap \mathrm{fv}(N_j) \cap W = \emptyset$ for every $1 \leqslant i \neq j \leqslant n$. A term M is* linear *provided it is linear in* $\mathrm{fv}(M)$.

Call $(Flip_0)$ the rule $(Flip)$ restricted to the case where F is in the set Σ of base types, and $(\Box I_0)$ the rule $(\Box I_I)$ restricted to the case where M is linear. Call BN_0 the typing system BN where all instances of $(Flip)$ are instances of $(Flip_0)$, and all instances of $(\Box I_I)$ are instances of $(\Box I_0)$.

*A term M is said to be η-*long normal *of type F in Γ if and only if we can derive $\Gamma \vdash_I M : F$ in system BN_0.*

Lemma 8 (Weakening). *For every BN derivation of $\Gamma \vdash_* M : F$ $(* \in \{I, E\})$, for every context Θ, there is a BN derivation of $\Gamma, \Theta \vdash_* M : F$.*

Proof. By structural induction on the given derivation. This is mostly obvious, provided we assume all bound variables have been renamed so as to be distinct from the ones in Θ. $\Box$

Lemma 9. *For every M such that $\Gamma \vdash M : F$, M has an η-long normal form $\eta[M]$. That is, there is a term $\eta[M]$ such that $M \approx \eta[M]$ and $\Gamma \vdash \eta[M] : F$.*

Proof. First by Proposition 3 and Lemma 6, we may assume that $\Gamma \vdash_I M : F$. The idea is then, first, to rewrite every instance of $(Flip)$ on non-base types F using only instances of $(Flip)$ on smaller types F, until all we get is instances of $(Flip_0)$. This is done using the following two rules:

$$\dfrac{\Gamma \vdash_E M : F \supset G}{\Gamma \vdash_I M : F \supset G}\,(Flip) \longrightarrow \dfrac{\dfrac{\dfrac{\Gamma, x : F \vdash_E M : F \supset G \quad \dfrac{\dfrac{}{\Gamma, x : F \vdash_E x : F}\,(Ax_E)}{\Gamma, x : F \vdash_I x : F}\,(Flip)}{\Gamma, x : F \vdash_E Mx : G}\,(\supset E_E)}{\Gamma, x : F \vdash_I Mx : G}\,(Flip)}{\Gamma \vdash_I \lambda x \cdot Mx : F \supset G}\,(\supset I_I) \tag{5}$$

$$\dfrac{\Gamma \vdash_E M : \Box F}{\Gamma \vdash_I M : \Box F}\,(Flip) \longrightarrow \dfrac{\Gamma \vdash_E M : \Box F \quad \dfrac{\dfrac{\dfrac{}{x : \Box F \vdash_E x : \Box F}\,(Ax_E)}{x : \Box F \vdash_E dx : F}\,(\Box E_E)}{x : \Box F \vdash_I dx : F}\,(Flip)}{\Gamma \vdash_I \boxed{dx} \cdot \{x := M\} : \Box F}\,(\Box I_I) \tag{6}$$

where in the right-hand side of the first rule, the derivation of $\Gamma, x : F \vdash_E M : F \supset G$ is obtained from the one of $\Gamma \vdash_E M : F \supset G$ by weakening (Lemma 8). This terminates, because the sum of the sizes of formulae on the right-hand sides of judgments in $(Flip)$ decreases (define the size $|F|$ of a formula F by $|A| \hat{=} 1$, $|F \supset G| \hat{=} |F| + |G| + 1$, $|\Box F| \hat{=} |F| + 1$).

On the other hand, we make every instance of $(\Box I_I)$ one of $(\Box I_0)$ by linearizing the term M. That is, for each free variable x_i in M, $1 \leqslant i \leqslant n$, with $k_i \geqslant 1$ distinct occurrences in M, create k_i fresh variables $x_{i1}, \ldots, x_{ik_i}$, let M' be M where the jth occurrence of x_i is replaced by x_{ij}, for every $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k_i$, and rewrite the derivation:

$$\dfrac{\overbrace{\Gamma \vdash_E N_i : \Box F_i}^{1 \leqslant i \leqslant n} \quad x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash M : F}{\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_n : N_n\} : \Box F}\,(\Box I_I) \tag{7}$$

into:

$$\dfrac{\overbrace{\Gamma \vdash_E N_i : \Box F_i}^{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k_i} \quad (x_{ij} : \Box F_i)_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k_i} \vdash M' : F}{\Gamma \vdash \boxed{M'} \cdot \{(x_{ij} := N_i)_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k_i}\} : \Box F}\,(\Box I_0) \tag{8}$$

□

Lemma 10. *Let $\Gamma \vdash M : F$. Then M has at most one η-long normal form of type F in Γ.*

Proof. Let M' be an η-long normal form of M. M' is beta-normal by construction. Let R_η be the rewrite system consisting of rules (η) and $(\eta\Box)$. It is clear that

R_η terminates and rewrites beta-normal terms to beta-normal terms. Similarly the rewrite system R_{ctr} consisting of the sole rule (ctr) terminates and rewrites R_η-normal beta-normal terms to R_η-normal beta-normal terms. Let M'' be any R_η-normal form of M', and M''' be any R_{ctr}-normal form of M''. Then M''' is R_{ctr}-normal, R_η-normal and beta-normal, hence normal.

Since M' is an η-long normal form of M, $M \approx M'$, so $M \approx M'''$. By Proposition 4 and since M''' is normal, $M \to^* M'''$. Summing up, $M \to^* M'''{}^*_{R_{\text{ctr}}} \leftarrow M''{}^*_{R_\eta} \leftarrow M'$, where $\to_R$ denotes rewriting by R.

Observe now that the rewrite system R_{ctr}^{-1} on derivations defined by the transformation $\boxed{M_1\{y := x\}} \cdot (\theta, \{x := N\}) \to \boxed{M_1} \cdot (\theta, \{x := N, y := N\})$ (where both x and y are free in M_1) is locally confluent. Moreover, whenever M_1 is well-typed and beta-normal, and rewrites to M_2 by R_{ctr}, then M_2 rewrites to M_1 by R_{ctr}^{-1}. Finally, R_{ctr}^{-1} terminates: for any term M_1, let $\mu(M_1)$ be $\sum_{x \in \text{fv}(M_1)} (n(x, M_1) - 1)$ where $n(x, M_1)$ is the number of occurrences of x in M_1; by induction on $\mu(M_1)$ followed lexicographically by the multiset of the terms $x\theta$, $x \in \text{dom}\,\theta$ ordered by $\to_{R_{\text{ctr}}^{-1}}$, $\boxed{M_1} \cdot \theta$ is R_{ctr}^{-1}-terminating as soon as each $x\theta$ is, $x \in \text{dom}\,\theta$; it follows by structural induction on terms that every term is R_{ctr}^{-1}-terminating.

Similarly, the rewrite system R_η^{-1} on derivations defined by (5) and (6) is terminating (as already noticed in Lemma 9), locally confluent, and whenever M_1 is well-typed and beta-normal, and rewrites to M_2 by R_η, then M_2 rewrites to M_1 by R_η^{-1}

So if M' is any η-long normal form of M, then $M \to^* M''' \to^*_{R_{\text{ctr}}^{-1}} M'' \to^*_{R_\eta^{-1}} M'$.

In general, if M'_1 and M'_2 are two η-long normal forms of M, we get $M \to^* M''' \to^*_{R_{\text{ctr}}^{-1}} M''_1 \to^*_{R_\eta^{-1}} M'_1$ and $M \to^* M''' \to^*_{R_{\text{ctr}}^{-1}} M''_2 \to^*_{R_\eta^{-1}} M'_2$. Since R_{ctr}^{-1} is confluent and M''_1 and M''_2 are R_{ctr}^{-1}-normal, $M''_1 = M''_2$. Since R_η^{-1} is confluent and M'_1 and M'_2 are η-long normal, hence R_η^{-1}-normal, $M'_1 = M'_2$. □

Lemmas 9 and 10 entail:

Proposition 11 (η-long normalization). *For every term M such that $\Gamma \vdash M : F$ is derivable, M has a unique η-long normal form of type F in Γ, which we write $\eta[M]$. In particular, whenever $\Gamma \vdash M : F$ and $\Gamma \vdash M' : F$, $M \approx M'$ if and only if $\eta[M] = \eta[M']$.*

The value of η-long normal forms is that substituting terms N_i of a certain form for variables in any η-long normal form yields an η-long normal form again:

Lemma 12. *If $x_1 : F_1, \ldots, x_n : F_n, \Theta \vdash_* M : F$ ($* \in \{I, E\}$) and $\Gamma \vdash_E N_i : F_i$ in system BN_0 for every i, $1 \leqslant i \leqslant n$, then $\Gamma, \Theta \vdash_* M\{x_1 := N_1, \ldots, x_n := N_n\} : F$ in system BN_0.*

Proof. By structural induction on the given derivation of $x_1 : F_1, \ldots, x_n : F_n, \Theta \vdash_* M : F$ in BN_0. If this was derived by (Ax_E), then $* = E$; if $M = x_i$ for some i, then $F = F_i$, $M\{x_1 := N_1, \ldots, x_n := N_n\} = N_i$ and we may indeed deduce $\Gamma, \Theta \vdash_E N_i : F$, by weakening from $\Gamma \vdash_E N_i : F$ (Lemma 8); otherwise let M be variable x, then $M\{x_1 := N_1, \ldots, x_n := N_n\} = x$, and we get $\Gamma, \Theta \vdash_E x : F$

by (Ax_E). If the last rule is $(\supset E_E)$, $(\Box E_E)$ or $(Flip_0)$, this is by the induction hypothesis, straightforwardly. If the last rule is $(\supset I_I)$, then $* = I$, M is of the form $\lambda x \cdot M_1$, F is of the form $G \supset H$, and by induction hypothesis we have been able to derive $\Gamma, \Theta, x : G \vdash_I M_1\{x_1 := N_1, \ldots, x_n := N_n\} : H$, from which we get $\Gamma, \Theta \vdash_I M\{x_1 := N_1, \ldots, x_n := N_n\} : G \supset H$ by $(\supset I_I)$. Finally, if the last rule is $(\Box I_0)$, then $* = I$, F is of the form $\Box G$, M is of the form $\boxed{M_1} \cdot \{y_1 := P_1, \ldots, y_k := P_k\}$, $\text{fv}(M_1) = \{y_1, \ldots, y_k\}$, M_1 is linear, and the typing derivation ends in:

$$\frac{\overbrace{x_1 : F_1, \ldots, x_n : F_n, \Theta \vdash_E P_j : \Box G_j}^{1 \leqslant j \leqslant k} \quad y_1 : \Box G_1, \ldots, y_k : \Box G_k \vdash_I M_1 : G}{x_1 : F_1, \ldots, x_n : F_n, \Theta \vdash_I \boxed{M_1} \cdot \{y_1 := P_1, \ldots, y_k := P_k\} : \Box G} (\Box I_0)$$

By induction hypothesis, we have got a derivation in BN_0 of $\Gamma, \Theta \vdash_E P_j\{x_1 := N_1, \ldots, x_n := N_n\} : \Box G_j$. Together with the derivation above of $y_1 : \Box G_1, \ldots, y_k : \Box G_k \vdash_I M_1 : G$, and since $\text{fv}(M_1) = \{y_1, \ldots, y_k\}$, M_1 is linear, we may apply $(\Box I_0)$ and derive

$$\begin{aligned} \Gamma, \Theta \vdash_I \boxed{M_1} \cdot \{ \quad & y_1 := P_1\{x_1 := N_1, \ldots, x_n := N_n\}, \\ & \ldots, \\ & y_n := P_n\{x_1 := N_1, \ldots, x_n := N_n\}\} : \Box G \end{aligned}$$

But this is precisely $\Gamma, \Theta \vdash_I M\{x_1 := N_1, \ldots, x_n := N_n\} : \Box G$. $\Box$

Lemma 13. *If M is η-long normal of type $\Box F$ in Γ, then M is of the form $\boxed{M_1} \cdot \theta$. Moreover, $\eta[dM] = M_1\theta$.*

Proof. The first part is obvious: $\Gamma \vdash_I M : \Box F$ in system BN_0, but only rule $(\Box I_0)$ can lead to this. Also, letting θ be $\{x_1 := N_1, \ldots, x_n := N_n\}$, we have $x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_I M_1 : F$ in BN_0, and $\Gamma \vdash_E N_i : \Box F_i$ in BN_0 for each i, $1 \leqslant i \leqslant n$. By Lemma 12, $\Gamma \vdash_I M_1\theta : F$ in BN_0. Since $dM \approx M_1\theta$, by Proposition 11 $\eta[dM] = M_1\theta$. $\Box$

The crucial thing in Lemma 13 is not so much that $dM \approx M_1\theta$, which is obvious. Rather, it is the fact that once we have reduced $d(\boxed{M_1} \cdot \theta)$ to $M_1\theta$ by (d), we have already reached its η-long normal form.

Similarly, we obtain:

Lemma 14. *Let $sM \hat{=} \boxed{x} \cdot \{x := M\}$. If $M \hat{=} \boxed{M_1} \cdot \theta$ is η-long normal of type $\Box F$ in Γ, then $\eta[sM] = \boxed{\boxed{M_1}} \cdot \theta$.*

Proof. First $sM = \boxed{x} \cdot \{x := \boxed{M_1} \cdot \theta\} \approx \boxed{\boxed{M_1}} \cdot \theta$. Then since M is η-long normal, letting θ be $\{x_1 := N_1, \ldots, x_n := N_n\}$, we have $x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_I M_1 : F$ in BN_0, and $\Gamma \vdash_E N_i : \Box F_i$ in BN_0 for each i, $1 \leqslant i \leqslant n$. So we can produce the following BN_0 derivation:

$$\dfrac{\overbrace{\begin{array}{c}\vdots\\ \Gamma \vdash_E N_i : \Box F_i\end{array}}^{1\leqslant i\leqslant n} \quad \dfrac{\overbrace{\dfrac{}{x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_E x_i : \Box F_i}(Ax_E)}^{1\leqslant i\leqslant n} \quad \begin{array}{c}\vdots\\ x_1 : \Box F_1,\ldots,x_n:\Box F_n \vdash_I M_1 : F\end{array}}{x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash_I \boxed{M_1} : \Box F}(\Box I_0)}{\Gamma \vdash_I \boxed{\boxed{M_1}}\cdot\theta : \Box\Box F}(\Box I_0)$$

so $\boxed{\boxed{M_1}}\cdot\theta$ is η-long normal of type $\Box\Box F$ in Γ. The claim then follows by Proposition 11. $\Box$

4. The Augmented Simplicial Structure of λ_{S4}

We define an augmented simplicial set consisting of typed λ_{S4}-terms. Recall that:

Definition 15 (A.s. set, a.s. map). *An* augmented simplicial set K *is a family of sets* K_q, $q \geqslant -1$, *of* q-simplices, *a.k.a. simplices of* dimension q, *with* face maps $\partial_q^i : K_q \to K_{q-1}$ *and* degeneracy maps $s_q^i : K_q \to K_{q+1}$, $0 \leqslant i \leqslant q$, *such that:*

(i) $\partial^i_{q-1} \circ \partial^j_q = \partial^{j-1}_{q-1} \circ \partial^i_q$ (ii) $s^i_{q+1} \circ s^{j-1}_q = s^j_{q+1} \circ s^i_q$ (iii) $\partial^i_{q+1} \circ s^j_q = s^{j-1}_{q-1} \circ \partial^i_q$
(iv) $\partial^i_{q+1} \circ s^i_q = \mathrm{id}$ (v) $\partial^{i+1}_{q+1} \circ s^i_q = \mathrm{id}$ (vi) $s^i_{q-1} \circ \partial^j_q = \partial^{j+1}_{q+1} \circ s^i_q$

where $0 \leqslant i \leqslant q$ *in* (iv), (v), *and* $0 \leqslant i < j \leqslant q$ *in the others.*

An augmented simplicial map $f : K \to L$ *is a family of maps* $f_q : K_q \to L_q$, $q \geqslant -1$, *such that* $\partial^i_q \circ f_q = f_{q-1} \circ \partial^i_q$ *and* $s^i_q \circ f_q = f_{q+1} \circ s^i_q$, $0 \leqslant i \leqslant q$.

Subscripts start at -1, which is standard and allows one to have q match the geometric dimension. We sometimes abbreviate "augmented simplicial" as "a.s." in the sequel. Also, when we run a risk of confusion, we write $\partial^i_{K_q}$ for ∂^i_q, and $s^i_{K_q}$ for s^i_q.

The category $\widehat{\Delta}$ of augmented simplicial sets as objects, and augmented simplicial maps as morphisms (see [**36**], VII.5), can also be presented as follows. Let Δ be the category whose objects are finite ordinals $[q] \hat{=} \{0, 1, \ldots, q\}$, $q \geqslant -1$, and whose morphisms are monotonic (i.e., non-decreasing) maps. This category is generated by morphisms $[q-1] \xrightarrow{\delta^i_q} [q]$ (mapping $j < i$ to j and $j \geqslant i$ to $j+1$) and $[q+1] \xrightarrow{\sigma^i_q} [q]$ (mapping $j \leqslant i$ to j and $j > i$ to $j-1$), and relations that are most succinctly described as (i)–(vi) where ∂ is replaced by δ, s by σ, and composition order is reversed. Then $\widehat{\Delta}$ is the category of functors from the opposite category Δ^o to the category **Set** of sets.

In general, $\widehat{\mathcal{C}}$ denotes the category of functors from $\mathcal{C}^o$ to **Set**, a.k.a. *presheaves* over $\mathcal{C}$. $\widehat{\mathcal{C}}$ is always an elementary topos [**33**], hence is a cartesian-closed category (CCC). The terminal object $\mathbf{1}$ of $\widehat{\Delta}$ is such that $\mathbf{1}_q$ is a singleton $\{*\}$ for every $q \geqslant -1$. The product $K \times L$ is such that $(K \times L)_q \hat{=} K_q \times L_q$, $\partial^i_{(K\times L)_q}(u, v) \hat{=} (\partial^i_{K_q} u, \partial^i_{L_q} v)$ and $s^i_{(K\times L)_q}(u, v) \hat{=} (s^i_{K_q} u, s^i_{L_q} v)$: i.e., product is component-wise.

The structure of exponentials, i.e., internal homs $\mathbf{Hom}_{\widehat{\Delta}}(K, L)$ is given by general constructions [**33**], which will be largely irrelevant here. For now, let us just say that we have got *a.s. application maps* $App : \mathbf{Hom}_{\widehat{\Delta}}(K, L) \times K \to L$, and an abstraction operator Λ on a.s. maps $f : K \times L \to M$, so that Λf is an a.s. map from K to $\mathbf{Hom}_{\widehat{\Delta}}(L, M)$, satisfying certain equations to be specified below. Furthermore, (-1)-simplices of $\mathbf{Hom}_{\widehat{\Delta}}(K, L)$ are just simplicial maps from K to L, while 0-simplices are homotopies between maps, and q-simplices for $q \geqslant 0$ are higher-dimensional homotopies.

In general, in any CCC $\mathcal{C}$—not just $\widehat{\Delta}$—, let ! denote the unique morphism $X \xrightarrow{!} \mathbf{1}$. For cartesian products, we have a *pair* $X \xrightarrow{\langle f,g\rangle} Y \times Z$ for every $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$, and *projections* $X_1 \times X_2 \xrightarrow{\pi_i} X_i$, $i \in \{1, 2\}$. We also have internal hom objects (exponentials) $\mathbf{Hom}_{\mathcal{C}}(X, Y)$, application $\mathbf{Hom}_{\mathcal{C}}(X, Y) \times X \xrightarrow{App} Y$, and abstraction $X \xrightarrow{\Lambda f} \mathbf{Hom}_{\mathcal{C}}(Y, Z)$ for every $X \times Y \xrightarrow{f} Z$. These obey the following *categorical combinator equations* [**10**], where we omit types (objects) for the sake of conciseness:

(a) $\mathrm{id} \circ f = f$	(b) $f \circ \mathrm{id} = f$	(c) $f \circ (g \circ h) = (f \circ g) \circ h$
(d) $\forall f : X \to \mathbf{1} \cdot f =!$	(e) $\pi_1 \circ \langle f, g\rangle = f$	(f) $\pi_2 \circ \langle f, g\rangle = f$
(g) $\langle \pi_1, \pi_2\rangle = \mathrm{id}$	(h) $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$	
(k) $\Lambda f \circ h = \Lambda(f \circ \langle h \circ \pi_1, \pi_2\rangle)$	(l) $App \circ \langle \Lambda f, g\rangle = f \circ \langle \mathrm{id}, g\rangle$	(m) $\Lambda(App \circ \langle f \circ \pi_1, \pi_2\rangle) = f$

For reasons of convenience, we shall abbreviate $\langle f \circ \pi_1, g \circ \pi_2\rangle$ as $f \times g$. Then, the following are derived equations:

(g') $\mathrm{id} \times \mathrm{id} = \mathrm{id}$	(h') $(f \times g) \circ (f' \times g') = (f \circ f') \times (g \circ g')$	
(k') $\Lambda f \circ h = \Lambda(f \circ (h \times \mathrm{id}))$		
(l') $App \circ \langle \Lambda f \circ h, g\rangle = f \circ \langle h, g\rangle$	(l'') $App \circ (\Lambda f \times \mathrm{id}) = f$	(m') $\Lambda(App \circ (f \times \mathrm{id})) = f$

4.1. The Augmented Simplicial Sets $\mathcal{S4}[\Gamma \vdash F]$

We observe that the S4 modality allows us to exhibit an augmented simplicial structure. We shall see later on (Section 5.2) that this arises from a comonad through the use of certain resolution functors. However, for now we prefer to remain syntactic and therefore relatively concrete.

Definition 16 ($\mathcal{S4}[\Gamma \vdash F]$). *For every context Γ, for every type F, let $[\Gamma \vdash F]$ be the set of all equivalence classes of λ_{S4}-terms M such that $\Gamma \vdash M : F$ is derivable, modulo $\approx$.*

For every $q \geqslant -1$, let $\mathcal{S4}[\Gamma \vdash F]_q$ be $[\Gamma \vdash \Box^{q+1} F]$, and let $\mathcal{S4}[\Gamma \vdash F]$ be the family $(\mathcal{S4}[\Gamma \vdash F]_q)_{q \geqslant -1}$.

For every function f from $[\Gamma \vdash F_1] \times \ldots \times [\Gamma \vdash F_n]$ to $[\Gamma \vdash G]$, define $\Box f$ as the function from $[\Gamma \vdash \Box F_1] \times \ldots \times [\Gamma \vdash \Box F_n]$ to $[\Gamma \vdash \Box G]$ that maps the tuple $(M_1, \ldots, M_k)$ to $\boxed{f(dx_1, \ldots, dx_k)} \cdot \{x_1 := M_1, \ldots, x_k := M_k\}$.

Say that f is substitutive *whenever $f(M_1, \ldots, M_k)\theta \approx f(M_1\theta, \ldots, M_k\theta)$.*

Finally, let ∂^i_q be the function from $\mathcal{S4}[\Gamma \vdash F]_q$ to $\mathcal{S4}[\Gamma \vdash F]_{q-1}$, $0 \leqslant i \leqslant q$, defined by $\partial^i_q M \hat{=} (\Box^i d)M$; and let s^i_q be the function from $\mathcal{S4}[\Gamma \vdash F]_q$ to $\mathcal{S4}[\Gamma \vdash F]_{q+1}$, $0 \leqslant i \leqslant q$, defined by $s^i_q M \hat{=} (\Box^i s)M$, where $sM \hat{=} \boxed{x} \cdot \{x := M\}$.

Lemma 17. *The following hold: 1.* $\Box\mathrm{id} = \mathrm{id}$. *2. if* f *is substitutive, then* $\Box f \circ \Box g = \Box(f \circ g)$. *3. for every* f, $\Box f$ *is substitutive. 4.* ∂^i_q *and* s^i_q *are substitutive.*

Proof. 1. $\Box\mathrm{id}(M) = \boxed{dx} \cdot \{x := M\} \approx M$ (by $(\eta\Box)$), so $\Box\mathrm{id} = \mathrm{id}$.

2. For every functions f and g, provided f is substitutive, then

$$\begin{aligned}
\Box f(\Box g(M)) &= \boxed{f(dx)} \cdot \{x := \boxed{g(dy)} \cdot \{y := M\}\} \\
&\approx \boxed{f(dx)\{x := \boxed{g(dy)}\}} \cdot \{y := M\} \quad \text{(by } (\boxed{\Box})\text{)} \\
&\approx \boxed{f(d\boxed{g(dy)})} \cdot \{y := M\} \quad \text{(since } f \text{ is substitutive)} \\
&\approx \boxed{f(g(dy))} \cdot \{y := M\} \quad \text{(by } (d)\text{)}
\end{aligned}$$

So $\Box f \circ \Box g = \Box(f \circ g)$. 3. is obvious, and 4. follows from 3. $\Box$

Proposition 18. *For every* Γ *and* F, *the triple* $(\mathcal{S}4[\Gamma \vdash F], (\partial^i_q)_{0 \leqslant i \leqslant q}, (s^i_q)_{0 \leqslant i \leqslant q})$ *is an augmented simplicial set.*

Proof. Because of Lemma 17, it is enough to check (i)–(vi) in the case $i = 0$, as the general case then follows immediately by induction on i:

(i) $\partial^0_{q-1}(\partial^j_q M) = d(\Box\partial^{j-1}_{q-1}M) = d\boxed{\partial^{j-1}_{q-1}(dx)} \cdot \{x := M\} \approx \partial^{j-1}_{q-1}(dM)$ (by (d)) $= \partial^{j-1}_{q-1}(\partial^0_q M)$.

(ii) $s^0_{q+1}(s^{j-1}_q M) = \boxed{y} \cdot \{y := s^{j-1}_q M\}$, while $s^j_{q+1}(s^0_q M) = \Box s^{j-1}_q(s^0_q M)$ $= \boxed{s^{j-1}_q(dy)} \cdot \{y := s^0_q M\} = \boxed{s^{j-1}_q(dy)} \cdot \{y := \boxed{x} \cdot \{x := M\}\}$ $\approx \boxed{s^{j-1}_q(d\boxed{x})} \cdot \{x := M\}$ (by $(\boxed{\Box})$) $\approx \boxed{s^{j-1}_q x} \cdot \{x := M\}$ (by (d)).

If $j = 1$, then $s^0_{q+1}(s^{j-1}_q M) = \boxed{y} \cdot \{y := \boxed{x} \cdot \{x := M\}\} \approx \boxed{\boxed{x}} \cdot \{x := M\}$ (by $(\boxed{\Box})$) $= \boxed{s^0_q x} \cdot \{x := M\}$, and this is precisely $s^j_{q+1}(s^0_q M)$.

If $j > 1$, then it obtains $s^0_{q+1}(s^{j-1}_q M) = \boxed{y} \cdot \{y := s^{j-1}_q M\}$ $= \boxed{y} \cdot \{y := \Box s^{j-2}_{q-1} M\} = \boxed{y} \cdot \{y := \boxed{s^{j-2}_{q-1}(dx)} \cdot \{x := M\}\}$ $\approx \boxed{\boxed{s^{j-2}_{q-1}(dx)}} \cdot \{x := M\}$ (by $(\boxed{\Box})$) $= \boxed{\Box s^{j-2}_{q-1} x} \cdot \{x := M\}$ $= \boxed{s^{j-1}_q x} \cdot \{x := M\}$, which is exactly $s^j_{q+1}(s^0_q M)$.

(iii) $\partial^0_{q+1}(s^j_q M) = d(\Box s^{j-1}_{q-1} M) = d\boxed{s^{j-1}_{q-1}(dx)} \cdot \{x := M\} \approx s^{j-1}_{q-1}(dM)$ (by (d)) $= s^{j-1}_{q-1}(\partial^0_q M)$.

(iv) $\partial^0_{q+1}(s^0_q M) = d(\boxed{x} \cdot \{x := M\}) \approx M$ by (d).

(v) $\partial^1_{q+1}(s^0_q M) = \Box\partial^0_q(s^0_q M) = \boxed{d(dx)} \cdot \{x := \boxed{y} \cdot \{y := M\}\} \approx \boxed{d(d\boxed{y})} \cdot \{y := M\}$ (by $(\boxed{\Box})$) $\approx \boxed{dy} \cdot \{y := M\}$ (by (d)) $\approx M$ (by $(\eta\Box)$).

(*vi*) $s^0_{q-1}(\partial^j_q M) = s^0_{q-1}(\Box\partial^{j-1}_{q-1} M) = \boxed{x} \cdot \{x := \boxed{\partial^{j-1}_{q-1}(dy)} \cdot \{y := M\}\}$ $\approx \boxed{\boxed{\partial^{j-1}_{q-1}(dy)}} \cdot \{y := M\}$ (by ($\boxed{\Box}$)) $= \boxed{\partial^j_q y} \cdot \{y := M\}$, while on the other hand $\partial^{j+1}_{q+1}(s^0_q M) = \Box\partial^j_q(\boxed{y} \cdot \{y := M\}) = \boxed{\partial^j_q(dx)} \cdot \{x := \boxed{y} \cdot \{y := M\}\}$ $\approx \boxed{\partial^j_q(d\boxed{y})} \cdot \{y := M\}$ (by ($\boxed{\Box}$)) $\approx \boxed{\partial^j_q y} \cdot \{y := M\}$ (by (d)).

$\Box$

By Lemma 13, the η-long normal form of any term of type $\Box^{q+1}F$ in Γ can be written in a unique way $\boxed{\ldots \boxed{\boxed{M_0} \cdot \theta_q} \cdot \theta_{q-1} \ldots} \cdot \theta_0$. Fix a variable x_0, and let θ_{q+1} be $\{x_0 \mapsto M_0\}$. Then this is also $\boxed{\ldots \boxed{\boxed{x_0\theta_{q+1}} \cdot \theta_q} \cdot \theta_{q-1} \ldots} \cdot \theta_0$. Therefore q-simplices in $\mathcal{S}4[\Gamma \vdash F]$ are basically sequences of $q+2$ substitutions, with additional typing and linearity conditions and conditions on the domains of substitutions. Let us compute faces and degeneracies as they act on η-long normal forms. For short, call the *η-long normal form* of a q-simplex M in $\mathcal{S}4[\Gamma \vdash F]$ the unique η-long normal form of type $\Box^{q+1}F$ in Γ of M.

First the following lemma will help us compute η-long normal forms of $\Box f$ applied to arguments in η-long normal form themselves.

Lemma 19. *Let f be any function from $[\Gamma \vdash F_1] \times \ldots \times [\Gamma \vdash F_n]$ to $[\Gamma \vdash G]$. We say that f is* linearity-preserving *if and only if for every η-long normal form M_1 of type F_1 in Γ, ..., for every η-long normal form M_n of type F_n in Γ, if M_1, ..., M_n are linear, then the η-long normal form of $f(M_1, \ldots, M_n)$ of type G in Γ is linear, too. By abuse, write $f(M_1, \ldots, M_n)$ this η-long normal form again.*

Say that f is non-collapsing *if and only if, for every η-long normal forms M_1 of type F_1 in Γ, ..., M_n of type F_n in Γ, then* $\mathrm{fv}(f(M_1, \ldots, M_n)) = \mathrm{fv}(M_1) \cup \ldots \cup \mathrm{fv}(M_n)$.

Let $\boxed{M_1} \cdot \theta_1$, ..., $\boxed{M_n} \cdot \theta_n$ be η-long normal forms of respective types $\Box F_1$, ..., $\Box F_n$ in Γ. Assume without loss of generality that θ_1, ..., θ_n have pairwise disjoint domains. If f is substitutive, linearity-preserving and non-collapsing, then the η-long normal form of type G in Γ of $\Box f(\boxed{M_1} \cdot \theta_1, \ldots, \boxed{M_n} \cdot \theta_n)$ is exactly $\boxed{f(M_1, \ldots, M_n)} \cdot (\theta_1, \ldots, \theta_n)$.

Proof. $\Box f(\boxed{M_1} \cdot \theta_1, \ldots, \boxed{M_n} \cdot \theta_n)$
$= \boxed{f(dx_1, \ldots, dx_n)} \cdot \{x_1 := \boxed{M_1} \cdot \theta_1, \ldots, x_n := \boxed{M_n} \cdot \theta_n\}$
$\approx \boxed{f(dx_1, \ldots, dx_n)\{x_1 := \boxed{M_1}, \ldots, x_n := \boxed{M_n}\}} \cdot (\theta_1, \ldots, \theta_n)$ (by ($\boxed{\Box}$))
$\approx \boxed{f(d\boxed{M_1}, \ldots, d\boxed{M_n})} \cdot (\theta_1, \ldots, \theta_n)$ (since f is substitutive) $\approx \boxed{f(M_1, \ldots, M_n)} \cdot (\theta_1, \ldots, \theta_n)$ (by (d)). It remains to show that the latter is η-long normal of type $\Box G$ in Γ, which will allow us to use Proposition 11. We only have to check that

$\mathrm{fv}(f(M_1,\ldots,M_n)) = \mathrm{dom}\,\theta_1 \cup \ldots \cup \mathrm{dom}\,\theta_n$ and that $f(M_1,\ldots,M_n)$ is linear. The former is because f is non-collapsing, and $\mathrm{fv}(M_i) = \mathrm{dom}\,\theta_i$, $1 \leqslant i \leqslant n$. The latter is because f is linearity-preserving, and each M_i is linear. Indeed, $\mathrm{fv}(M_i) = \mathrm{dom}\,\theta_i$ and M_i is linear because $\boxed{M_i} \cdot \theta_i$ is η-long normal. □

Proposition 20. *Let $M \in \mathcal{S}4\,[\Gamma \vdash F]_q$, of η-long form* $\boxed{\cdots \boxed{\boxed{x_0\theta_{q+1}} \cdot \theta_q} \cdot \theta_{q-1} \cdots} \cdot \theta_0$. *Then the η-long form of $\partial_q^i M$ is:*

$$\boxed{\cdots \boxed{\boxed{\boxed{\cdots \boxed{\boxed{x_0\theta_{q+1}} \cdot \theta_q} \cdot \theta_{q-1} \cdots} \cdot \theta_{i+2}} \cdot (\theta_{i+1} \cdot \theta_i)} \cdot \theta_{i-1} \cdots} \cdot \theta_0 \tag{9}$$

and the η-long form of $s_q^i M$ is:

$$\boxed{\cdots \boxed{\boxed{\boxed{\cdots \boxed{\boxed{x_0\theta_{q+1}} \cdot \theta_q} \cdot \theta_{q-1} \cdots} \cdot \theta_{i+1}} \cdot \mathrm{id}} \cdot \theta_i \cdots} \cdot \theta_0 \tag{10}$$

where id *is the identity substitution on* $\mathrm{dom}\,\theta_i$.

Proof. By induction on i, simultaneously with the fact that ∂_q^i and s_q^i are substitutive, linearity-preserving and non-collapsing. The inductive case is by Lemma 19. In the base case, if $i = 0$, then (9) is by Lemma 13, and (10) is by Lemma 14. □

The geometry of $\mathcal{S}4\,[\Gamma \vdash F]$ is therefore very close to that of the nerve of a category whose objects are contexts Γ, and whose morphisms $\Gamma \xrightarrow{\theta} \Theta$ are substitutions θ such that, letting Θ be $y_1 : G_1, \ldots, y_m : G_m$, θ is of the form $\{y_1 := M_1, \ldots, y_m := M_m\}$ where $\Gamma \vdash M_i : G_i$ is derivable for each i, $1 \leqslant i \leqslant m$. Identities are the identity substitutions, composition is substitution concatenation. (It is not quite a nerve, because of the added conditions on substitutions.) The connection with nerves will be made precise in Theorem 30 below.

4.2. The Geometry of $\mathcal{S}4\,[\Gamma \vdash F]$

In Proposition 20, note that substitutions are taken as is, in particular not modulo $\approx$. This hints at the fact that $\mathcal{S}4\,[\Gamma \vdash F]$ will in general not be a Kan complex: recall that a nerve of a small category $\mathcal{C}$ is Kan if and only if $\mathcal{C}$ is a groupoid, i.e., if and only if all morphisms in $\mathcal{C}$ are isomorphisms. In the category of substitutions above, the only isomorphisms are renamings $\{x_1 := y_1, \ldots, x_n := y_n\}$, where $y_1, \ldots, y_n$ are pairwise distinct variables.

Proposition 21. $\mathcal{S}4\,[\Gamma \vdash F]$ *is not Kan in general.*

Proof. Being Kan would imply in particular that given any two 1-simplices M_0 and M_1 with $\partial_1^0 M_0 = \partial_1^0 M_1$, there should be a 2-simplex M such that $\partial_2^0 M = M_0$

and $\partial_2^1 M = M_1$. Write the η-long normal forms of M_0 as $\boxed{\boxed{x_0\theta_2} \cdot \theta_1} \cdot \theta_0$, of M_1 as $\boxed{\boxed{x_0\theta_2'} \cdot \theta_1'} \cdot \theta_0'$. The condition $\partial_1^0 M_0 = \partial_1^0 M_1$ means that $\boxed{x_0\theta_2} \cdot (\theta_1 \cdot \theta_0) = \boxed{x_0\theta_2'} \cdot (\theta_1' \cdot \theta_0')$. In particular, up to a renaming of the variables free in $x_0\theta_2'$:

$$\theta_2 = \theta_2', \quad \theta_1 \cdot \theta_0 = \theta_1' \cdot \theta_0' \tag{11}$$

If M exists, then M is of the form $\boxed{\boxed{\boxed{x_0\vartheta_3} \cdot \vartheta_2} \cdot \vartheta_1} \cdot \vartheta_0$, and up to renamings of bound variables, $\partial_2^0 M = M_0$ entails:

$$\vartheta_3 = \theta_2, \quad \vartheta_2 = \theta_1, \quad \vartheta_1 \cdot \vartheta_0 = \theta_0 \tag{12}$$

and $\partial_2^1 M = M_1$ entails:

$$\vartheta_3 = \theta_2', \quad \vartheta_2 \cdot \vartheta_1 = \theta_1', \quad \vartheta_0 = \theta_0' \tag{13}$$

It follows that θ_1' must be an instance of θ_1, in particular. (An *instance* of a substitution θ is a substitution of the form $\theta \cdot \theta'$.) But (11) does not guarantee this. For example, take $\theta_2 \hat{=} \theta_2' \hat{=} \{x_0 := dx_0\}$, $\theta_1' \hat{=} \{x_0 := x_1\}$, $\theta_1 \hat{=} \{x_0 := dx_1\}$, $\theta_0 \hat{=} \{x_1 := x_1\}$, $\theta_0' \hat{=} \{x_1 := dx_1\}$. It is easily checked that M_0 and M_1 are in $\mathcal{S}4[\Gamma \vdash F]_1$, i.e., they are of type $\Box^2 F$ in Γ, for any formula F, where $\Gamma \hat{=} x_1 : \Box^2 F$. $\square$

This settles the case, at least when Γ contains at least one formula of the form $\Box^2 F$. When Γ is empty, it is easy to see that $\mathcal{S}4[\Gamma \vdash F]$ is empty except possibly in dimension -1, so this is trivially Kan—but the geometry of such simplicial sets is uninteresting.

The following notion will be useful in studying the geometry of $\mathcal{S}4[\Gamma \vdash F]$:

Definition 22 (Contiguity). *Let K be an augmented simplicial set. The q-simplex x is* one-step contiguous *to the q-simplex y, in short $x \rightarrowtriangle y$, if and only if there is a $(q+1)$-simplex z in K, and two indices i, j with $0 \leqslant i < j \leqslant q+1$ such that $\partial_{q+1}^j z = x$ and $\partial_{q+1}^i z = y$, and $x \neq y$.*

The q-simplex x is contiguous *to y if and only if $x \rightarrowtriangle^* y$, and* strictly contiguous *if and only if $x \rightarrowtriangle^+ y$. We say that x and y* are contiguous *if and only if $x \leftrightarrowtriangle y$, where $\leftrightarrowtriangle$ is the reflexive symmetric transitive closure of $\rightarrowtriangle$.*

Contiguity is usually presented a bit differently. In particular, it is usually not required that $j > i$ in one-step contiguity. Then $\rightarrowtriangle^*$ is an equivalence relation. However we shall need the finer notion of $\rightarrowtriangle$ in the sequel.

Geometrically, the idea of contiguity is as shown on Figure 4, where the two shaded triangles are contiguous 2-simplices.

The following lemma, for example, is unusual:

Lemma 23. *The relation $\rightarrowtriangle$ is well-founded.*

Proof. Define the following measure $\mu(M)$ for q-simplices M in $\mathcal{S}4[\Gamma \vdash F]_q$. Whenever M has η-long normal form $\boxed{\ldots \boxed{x_0\theta_{q+1}} \cdot \theta_q \ldots} \cdot \theta_0$, let $\mu(M)$ be the $(q+2)$-tuple

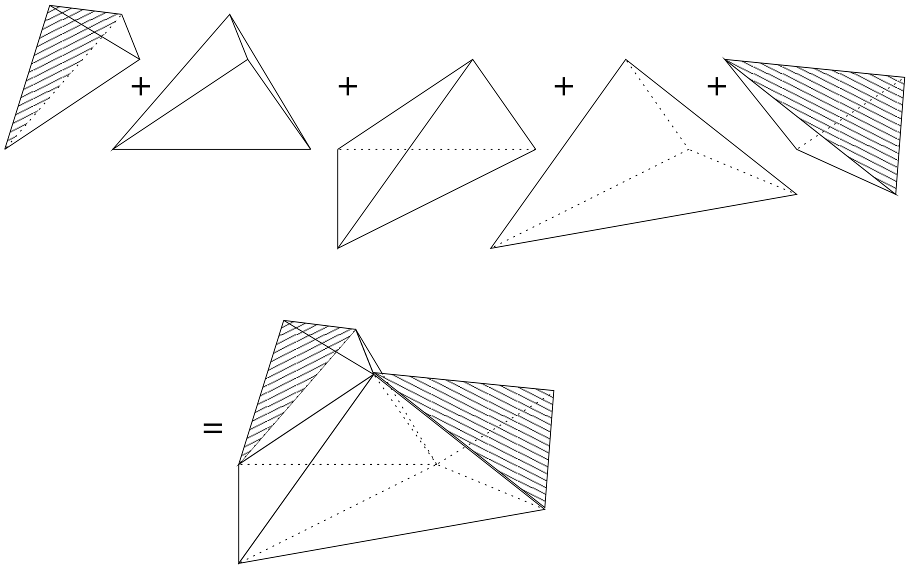

Figure 4: Contiguity

$(|\theta_0|, \ldots, |\theta_q|, |\theta_{q+1}|)$, ordered lexicographically from right to left, where $|\theta|$ is the *size* of θ, defined in any obvious way.

Let M and M' be two q-simplices in $\mathcal{S}\!\!/[\Gamma \vdash F]$, and assume that $M \twoheadrightarrow M'$. Then there is a $(q+1)$-simplex N, say:

$$\boxed{\boxed{\cdots \boxed{x_0\theta_{q+2}} \cdot \theta_{q+1} \cdots} \cdot \theta_1} \cdot \theta_0$$

and $i < j$ such that $\partial^j_{q+1} N = M$, $\partial^i_{q+1} N = M'$. That is:

$$M = \boxed{\cdots \boxed{\boxed{\cdots \boxed{\boxed{\cdots \boxed{x_0\theta_{q+2}} \cdot \theta_{q+1} \cdots} \cdot (\theta_{j+1} \cdot \theta_j)} \cdot \theta_{j-1} \cdots} \cdot \theta_{i+1}} \cdot \theta_i \cdots} \cdot \theta_0$$

$$M' = \boxed{\cdots \boxed{\boxed{\cdots \boxed{\boxed{\cdots \boxed{x_0\theta_{q+2}} \cdot \theta_{q+1} \cdots} \cdot \theta_{j+1}} \cdot \theta_j \cdots} \cdot \theta_{i+2}} \cdot (\theta_{i+1} \cdot \theta_i) \cdots} \cdot \theta_0$$

Clearly $\mu(M) \geqslant \mu(M')$. We claim that $\mu(M) > \mu(M')$. Since the lexicographic ordering on sizes is well-founded, this will establish the result.

Assume on the contrary that $\mu(M) = \mu(M')$. Then, $|\theta_{j+1} \cdot \theta_j| = |\theta_{j+1}|$, so up to a renaming of bound variables, $\theta_{j+1} \cdot \theta_j = \theta_{j+1}$, so $\theta_j = \mathrm{id}$. Then $|\theta_j| = |\theta_{j-1}|, \ldots,$ $|\theta_{i+2}| = |\theta_{i+1}|$, which imply that $\theta_{j-1}, \ldots, \theta_{i+1}$ must map variables to variables.

By the linearity constraints on η-long normal forms, they must be one-to-one. So up to renaming of bound variables, $\theta_{j-1} = \ldots = \theta_{i+1} = \mathrm{id}$. But then $M = M'$, contradicting $M \rightarrowtriangle M'$. $\square$

Corollary 24. *The relation $\rightarrowtriangle^+$ is a strict ordering.*

Proof. If it were reflexive, then we would have $M \rightarrowtriangle^+ M$ for some M, hence an infinite decreasing chain $M \rightarrowtriangle^+ M \rightarrowtriangle^+ M \rightarrowtriangle^+ \ldots$ $\square$

Definition 25 (Vertices, Component). *Let K be any augmented simplicial set. Given any q-simplex x of K, $q \geqslant 0$, the* vertices *of x are those 0-simplices that are iterated faces of x. The i*th vertex *of x, $0 \leqslant i \leqslant q$, is $\eth^i_q x \hat{=} \partial^0_1 \ldots \partial^{i-1}_i \partial^{i+1}_{i+1} \ldots \partial^q_q x$.*

The component *$\pi_0 x$ is $\partial^0_0 \ldots \partial^i_i \ldots \partial^q_q x$.*

It is well-known that each q-simplex has exactly $q+1$ vertices, and these are $\eth^0_q x$, $\ldots$, $\eth^q_q x$. Moreover, these vertices are contiguous:

Lemma 26. *Let K be any augmented simplicial set. Given any q-simplex x of K, $q \geqslant 0$,*

$$\eth^0_q x \rightarrowtriangle^* \eth^1_q x \rightarrowtriangle^* \ldots \rightarrowtriangle^* \eth^q_q x$$

Proof. To show that $\eth^i_q x \rightarrowtriangle^* \eth^{i+1}_q x$, let y be the 1-simplex $\partial^0_2 \ldots \partial^{i-1}_{i+1} \partial^{i+2}_{i+2} \ldots \partial^q_q x$. Then $\partial^0_1 y = \eth^{i+1}_q x$ and $\partial^1_1 y = \eth^i_q x$, so $\eth^i_q x \rightarrowtriangle^= \eth^{i+1}_q x$, where $\rightarrowtriangle^=$ is $\rightarrowtriangle \cup =$. $\square$

Observe that there is no need to take $\rightarrowtriangle^+$ instead of $\rightarrowtriangle$ in the case of 0-simplices in λ_{S4}:

Lemma 27. *The relation $\rightarrowtriangle$ is transitive on 0-simplices of $\mathcal{S}4[\Gamma \vdash F]$.*

Proof. Note that, if M and M' are two 0-simplices, then $M \rightarrowtriangle M'$ means that for some 1-simplex M_1, $\partial^1_1 M_1 = M$ and $\partial^0_1 M_1 = M'$.

So assume that $M \rightarrowtriangle M' \rightarrowtriangle M''$. There is a 1-simplex $\boxed{\boxed{N} \cdot \theta_1} \cdot \theta_2$ such that $M = \boxed{N\theta_1} \cdot \theta_2$ and $M' = \boxed{N} \cdot (\theta_1 \cdot \theta_2)$. There is also a 1-simplex $\boxed{\boxed{N'} \cdot \theta'_1} \cdot \theta'_2$ such that $M' = \boxed{N'\theta'_1} \cdot \theta'_2$ and $M'' = \boxed{N'} \cdot (\theta'_1 \cdot \theta'_2)$. Comparing both forms for M', we must have, up to renaming of bound variables, $N = N'\theta'_1$ and $\theta'_2 = \theta_1 \cdot \theta_2$. So $M_2 \hat{=} \boxed{\boxed{\boxed{N'} \cdot \theta'_1} \cdot \theta_1} \cdot \theta_2$ is a valid 2-simplex. Take $M_1 \hat{=} \partial^1_2 M_2 = \boxed{\boxed{N'} \cdot (\theta' \cdot \theta_1)} \cdot \theta_2$. Then $\partial^1_1 M_1 = M$ and $\partial^0_1 M_1 = M''$, so $M \rightarrowtriangle^= M''$. By Corollary 24, $M \rightarrowtriangle M''$. $\square$

The following lemma shows that, basically, if two simplices are contiguous, then they are so in a unique way:

Lemma 28 (Two-face Lemma). *Let M, M' be two q-simplices of $\mathcal{S}4[\Gamma \vdash F]$, $q \geqslant 0$. Then for any $0 \leqslant i < j \leqslant q+1$, there is at most one $(q+1)$-simplex N of $\mathcal{S}4[\Gamma \vdash F]$ such that $\partial^j_q N = M$ and $\partial^i_q N = M'$.*

Proof. Assume N exists, and write it as:

$$\boxed{\boxed{\ldots \boxed{x_0\vartheta_{q+2}} \cdot \vartheta_{q+1} \ldots} \cdot \vartheta_1} \cdot \vartheta_0$$

Also, write:

$$M = \boxed{\boxed{\ldots \boxed{x_0\theta_{q+1}} \cdot \theta_q \ldots} \cdot \theta_1} \cdot \theta_0$$

$$M' = \boxed{\boxed{\ldots \boxed{x_0\theta'_{q+1}} \cdot \theta'_q \ldots} \cdot \theta'_1} \cdot \theta'_0$$

Since $\partial^j_q N = M$, up to renaming of bound variables, $\theta_0 = \vartheta_0$, $\ldots$, $\theta_{j-1} = \vartheta_{j-1}$, $\theta_j = \vartheta_{j+1} \cdot \vartheta_j$, $\theta_{j+1} = \vartheta_{j+2}$, $\ldots$, $\theta_{q+1} = \vartheta_{q+2}$. And since $\partial^i_q N = M'$, up to renaming of bound variables, $\theta'_0 = \vartheta_0$, $\ldots$, $\theta'_{i-1} = \vartheta_{i-1}$, $\theta'_i = \vartheta_{i+1} \cdot \vartheta_i$, $\theta'_{i+1} = \vartheta_{i+2}$, $\ldots$, $\theta'_{q+1} = \vartheta_{q+2}$.

In particular, $\vartheta_0 = \theta_0$, $\ldots$, $\vartheta_{j-1} = \theta_{j-1}$, $\vartheta_{i+2} = \theta'_{i+1}$, $\ldots$, $\vartheta_{q+2} = \theta'_{q+1}$. So ϑ is determined uniquely as soon as $j > i+1$. If $j = i+1$, this determines ϑ uniquely, except possibly for ϑ_j. Now we use the additional equations $\theta_j = \vartheta_{j+1} \cdot \vartheta_j$ and $\theta'_{i+1} = \vartheta_{i+2}$. The latter means that $\theta'_j = \vartheta_{j+1}$. Since every variable of $\operatorname{dom} \vartheta_j$ is free in some term in the range of ϑ_{j+1}, this determines ϑ_j uniquely. $\square$

In the case of $\mathcal{S}4[\Gamma \vdash F]$, Lemma 26 is the only condition on vertices that needs to be satisfied for them to be vertices of a q-simplex:

Proposition 29. *Let M_0, M_1, $\ldots$, M_q be $q+1$ 0-simplices of $\mathcal{S}4[\Gamma \vdash F]$. If $M_0 \rightarrowtriangle^* M_1 \rightarrowtriangle^* \ldots \rightarrowtriangle^* M_q$, then there is a unique q-simplex M such that $\eth^i_q M = M_i$, $0 \leqslant i \leqslant q$.*

Proof. By Lemma 27, $M_0 \rightarrowtriangle^= M_1 \rightarrowtriangle^= \ldots \rightarrowtriangle^= M_q$.

We now show uniqueness by induction on $q \geqslant 0$. If $q = 0$, this is obvious. Otherwise, by induction there is at most one $(q-1)$-simplex M^0 with vertices $M_1 \rightarrowtriangle^= \ldots \rightarrowtriangle^= M_q$, and at most one $(q-1)$-simplex M^q with vertices $M_0 \rightarrowtriangle^= \ldots \rightarrowtriangle^= M_{q-1}$. If there is any q-simplex M with vertices M_0, M_1, $\ldots$, M_q, then $\partial^0_q M = M^0$ and $\partial^q_q M = M^q$, so there is at most one such M by Lemma 28.

Existence: write M_i as $\boxed{N_i} \cdot \theta_i$ for each i, $0 \leqslant i \leqslant q$. Since for each $i < q$, $M_i \rightarrowtriangle^= M_{i+1}$, there are (unique) 1-simplices $\boxed{\boxed{N'_i} \cdot \vartheta^i_1} \cdot \vartheta^i_0$ such that, up to renaming of bound variables, $N_i = N'_i \vartheta^i_1$, $\theta_i = \vartheta^i_0$, $N_{i+1} = N'_i$, $\theta_{i+1} = \vartheta^i_1 \cdot \vartheta^i_0$. Note that $N_i = N_{i+1}\vartheta^i_1$, and $\vartheta^{i+1}_0 = \vartheta^i_1 \cdot \vartheta^i_0$. So define:

$$M \mathrel{\hat{=}} \boxed{\boxed{\ldots \boxed{\ldots \boxed{N_q} \cdot \vartheta^{q-1}_1 \ldots} \cdot \vartheta^{i-1}_1 \ldots} \cdot \vartheta^0_1} \cdot \vartheta^0_0$$

In particular, $\eth^0_q M = \boxed{N_q \vartheta^{q-1}_1 \ldots \vartheta^{i-1}_1 \ldots \vartheta^0_1} \cdot \vartheta^0_0 = \boxed{N_{q-1}\vartheta^{q-2}_1 \ldots \vartheta^0_1} \cdot \vartheta^0_0 = \ldots = \boxed{N_0} \cdot \vartheta^0_0 = M_0$. And for every $i > 0$, $\eth^i_q M = \boxed{N_q \vartheta^{q-1}_1 \ldots \vartheta^i_1} \cdot (\vartheta^{i-1}_1 \cdot \ldots \cdot \vartheta^0_1 \cdot \vartheta^0_0)$

$= \boxed{N_i} \cdot (\vartheta_1^{i-1} \cdot \ldots \cdot \vartheta_1^1 \cdot \vartheta_1^0 \cdot \vartheta_0^0) = \boxed{N_i} \cdot (\vartheta_1^{i-1} \cdot \ldots \cdot \vartheta_1^1 \cdot \vartheta_0^1) = \ldots = \boxed{N_i} \cdot \vartheta_0^i = M_i$. □

To sum up, the non-augmented part of $\mathcal{S}4[\Gamma \vdash F]$ can be characterized as a particularly simple nerve:

Theorem 30 (Nerve Theorem). *Let $C[\Gamma \vdash F]$ be the partial order consisting of the 0-simplices of $\mathcal{S}4[\Gamma \vdash F]$, ordered by contiguity $\rightharpoonup^*$. Then the (non-augmented) simplicial set $(\mathcal{S}4[\Gamma \vdash F]_q)_{q \in \mathbb{N}}$ is (isomorphic to) the nerve $N(C[\Gamma \vdash F])$.*

Proof. Recall that the nerve of a category $\mathcal{C}$ has diagrams:

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{q-1}} A_{q-1} \xrightarrow{f_q} A_q$$

as q-simplices, where $f_1, f_2, \ldots, f_q$ are morphisms in $\mathcal{C}$, and $q \geqslant 0$. The ith face is obtained by removing A_i from the sequence, composing the neighboring arrows if $0 < i < q$, and dropping them if $i = 0$ or $i = q$. The ith degeneracy is obtained by duplicating A_i, adding an identity morphism.

By Lemma 26 and Proposition 29, q-simplices M are in bijection with ordered sequences of 0-simplices $M_0 \rightharpoonup^* M_1 \rightharpoonup^* \ldots \rightharpoonup^* M_q$. Moreover, for every j, $0 \leqslant j \leqslant q-1$, the jth vertex of $\partial_q^i M$ is:

$$\eth_q^j \partial_q^i M = \partial_1^0 \ldots \partial_j^{j-1} \partial_{j+1}^{j+1} \ldots \partial_{q-1}^{q-1} \partial_q^i M = \begin{cases} \eth_q^j M & \text{if } j < i \\ \eth_q^{j+1} M & \text{if } j \geqslant i \end{cases}$$

That is, the vertices of $\partial_q^i M$ are those of M except $\eth_q^i M$. Similarly, for every j, $0 \leqslant j \leqslant q+1$, the jth vertex of $s_q^i M$ is:

$$\eth_q^j s_q^i M = \partial_1^0 \ldots \partial_j^{j-1} \partial_{j+1}^{j+1} \ldots \partial_{q+1}^{q+1} s_q^i M = \begin{cases} \eth_q^j M & \text{if } j \leqslant i \\ \eth_q^i M & \text{if } j = i+1 \\ \eth_q^{j-1} M & \text{if } j > i+1 \end{cases}$$

That is, the vertices of $s_q^i M$ are those of M in sequence, with $\eth_q^i M$ occurring twice. □

In other words, $(\mathcal{S}4[\Gamma \vdash F]_q)_{q \in \mathbb{N}}$ is an *oriented simplicial complex.* Recall that an oriented simplicial complex is a family of linearly ordered sequences of so-called points, containing all one-element sequences, and such that any subsequence of an element of the family is still in the family. In fact, it is the *full* oriented simplicial complex, containing *all* linearly ordered sequences of points.

It is futile to study the classical notions of loop homotopy in such an oriented simplicial complex. Indeed, all loops are trivial: if the 1-simplex M is a loop, i.e., $\partial_1^0 M = \partial_1^1 M$ is some point N, then its sequence of vertices is $N \rightharpoonup^* N$, so M is a degenerate 1-simplex. Homotopies of loops, and in fact the natural extension of homotopies between 1-simplices, is trivial, too: let M, M' be two 1-simplices with $\partial_1^0 M = \partial_1^0 M' = M_0$ and $\partial_1^1 M = \partial_1^1 M' = M_1$; if there is a homotopy 2-simplex P connecting them, then its faces are M, M' plus some degenerate 1-simplex, so the sequence of vertices of P must be $M_1 \rightharpoonup^* M_0 \rightharpoonup^* M_0$ or $M_1 \rightharpoonup^* M_1 \rightharpoonup^* M_0$, from which it follows that the homotopy is one of the two degeneracies of $M = M'$.

In short, two 1-simplices are homotopic in the classical sense if and only if they are equal, and all homotopies are degenerate. However, studying homotopies of paths (not just loops) certainly remains interesting. In particular, the geometry of preorders and lattices viewed through their order complexes is a rich domain [**8**].

4.3. Components

This closes the case for non-negative dimensions. In dimension -1, recall that there are two extremal ways to build an augmentation of a simplicial set (see e.g., [**14**]). One, exemplified by the nerve construction for augmented simplicial sets, builds the augmentation of the simplicial set K, K_{-1}, as a one-element set (an empty set if K is empty), and ∂_0^0 is the unique function $K_0 \to K_{-1}$. The other builds K_{-1} as the set of connected components of K_0, that is, as the set of $\leftrightarrow$-equivalence classes of points. It turns out that the latter is how the augmentation is built in $\mathcal{S}4[\Gamma \vdash F]$. (Except that there might also be (-1)-simplices that are the component of no q-simplex for any $q \geqslant 0$.) In other words, components are exactly path connected components (plus isolated (-1)-dimensional simplices). This is shown in Proposition 32 below. First, we observe:

Proposition 31 (Lattice of points). *Let M_0 be a (-1)-simplex of $\mathcal{S}4[\Gamma \vdash F]$.*

The set $C(M_0)$ of 0-simplices M such that $\partial_0^0 M = M_0$, equipped with the ordering $\rightarrow^*$, is empty or is a finite lattice.

Proof. Every such M can be written in a unique way $\boxed{N} \cdot \theta$, with $N\theta = M_0$. But, up to the names of free variables in N, there are only finitely many such N's and θ's. So $C(M_0)$ is finite.

In the rest of the proof, fix a typing of M_0. This way, each subterm of M_0 gets a unique type. This will allow us to reason by induction on M_0—in general, on terms—instead on a BN_0 derivation of $\Gamma \vdash_I M_0 : F$.

Note that $\boxed{N} \cdot \theta \rightarrow^* \boxed{N'} \cdot \theta'$ if and only if $\boxed{N} \cdot \theta \rightarrow^= \boxed{N'} \cdot \theta'$ (by Lemma 27) if and only if for some $\boxed{\boxed{N_0} \cdot \vartheta_1} \cdot \vartheta_0$, $N = N_0\vartheta_1$, $\theta = \vartheta_0$, $N' = N_0$, $\theta' = \vartheta_1 \cdot \vartheta_0$ (up to renaming of bound variables), if and only if $N = N'\vartheta_1$ for some substitution ϑ_1. In other words, if and only if N is an *instance of* N'.

Then every pair of points M_1, M_2 of $C(M_0)$ has a supremum M. That is, $M \rightarrow^* M_1$, $M \rightarrow^* M_2$ and for every M' such that $M' \rightarrow^* M_1$ and $M' \rightarrow^* M_2$, $M \rightarrow^* M'$. Write M_1 as $\boxed{N_1} \cdot \theta_1$, M_2 as $\boxed{N_2} \cdot \theta_2$, with $N_1\theta_1 = N_2\theta_2$. Then, if M exists, M is a common instance of N_1 and N_2. It is easy to see that there is a *least* common instance $N_1 \wedge N_2$ of N_1 and N_2, i.e., one such that every other instance of N_1 and N_2 is an instance of $N_1 \wedge N_2$. More generally, given any finite set W of variables (used to collect λ-bound variables), call an instance of N *away from* W any term $N\theta$ such that $\operatorname{dom}\theta \cap W = \emptyset$. Then if there is a common instance of N_1 and N_2 away from W, then there is a least one $N_1 \wedge_W N_2$, where N_1 and N_2 are linear, and it is computed as in Figure 5. Then define $N_1 \wedge N_2$ as $N_1 \wedge_\emptyset N_2$. Since $N_1\theta = N_2\theta_2 = M_0$, there is a unique substitution θ with $\operatorname{dom}\theta = \operatorname{fv}(N_1 \wedge N_2)$, and the free variables of $N_1 \wedge N_2$ being free variables of N_1 or N_2, have boxed types. Therefore $M \hat{=} \boxed{N_1 \wedge N_2} \cdot \theta$ is a well-typed term, and $M \rightarrow^* M_1$ and $M \rightarrow^* M_2$, since $N_1 \wedge N_2$ is both an instance

$$
\begin{aligned}
x_1 \wedge_W N_2 &\hat{=} N_2 \qquad (x_1 \notin W)\\
N_1 \wedge_W x_2 &\hat{=} N_1 \qquad (x_2 \notin W)\\
(N_1 N_1') \wedge_W (N_2 N_2') &\hat{=} (N_1 \wedge_W N_2)(N_1' \wedge_W N_2')\\
(\lambda x \cdot N_1) \wedge_W (\lambda x \cdot N_2) &\hat{=} \lambda x \cdot (N_1 \wedge_{W \cup \{x\}} N_2)\\
dN_1 \wedge_W dN_2 &\hat{=} d(N_1 \wedge_W N_2)\\
\boxed{N} \cdot \{x_1 := N_{11}, \ldots, x_n := N_{n1}\} \wedge_W \boxed{N} &\cdot \{x_1 := N_{12}, \ldots, x_n := N_{n2}\}\\
&\hat{=} \boxed{N} \cdot \{x_1 := N_{11} \wedge_W N_{12}, \ldots,\\
&\qquad x_n := N_{n1} \wedge_W N_{n2}\}
\end{aligned}
$$

Figure 5: Least common instances

of N_1 and an instance of N_2. Moreover, by construction this is the least one, so M is the supremum of M_1 and M_2. (This is *unification* [**30**]. The key here is that we basically only need unification modulo an *empty* theory, instead of the theory of the relation $\approx$.) We write M as $M_1 \sqcup M_2$.

Symmetrically, every pair of points M_1, M_2 of $C(M_0)$ has an infimum M. That is, $M_1 \twoheadrightarrow^* M$, $M_2 \twoheadrightarrow^* M$ and for every M' such that $M_1 \twoheadrightarrow^* M'$ and $M_2 \twoheadrightarrow^* M'$, $M' \twoheadrightarrow^* M$. Write again M_1 as $\boxed{N_1} \cdot \theta_1$, M_2 as $\boxed{N_2} \cdot \theta_2$, with $N_1\theta_1 = N_2\theta_2$. Calling a *generalization* of a term N (away from W) any term having N as instance (away from W), we may compute a greatest common generalization $N_1 \vee_W N_2$ away from W of N_1 and N_2 as in Figure 6, where $N_1\theta_1 = N_2\theta_2$. As above, letting

$$
\begin{aligned}
x_1 \vee_W N_2 &\hat{=} x_1 \qquad (x_1 \notin W)\\
N_1 \vee_W x_2 &\hat{=} x_2 \qquad (x_2 \notin W)\\
(N_1 N_1') \vee_W (N_2 N_2') &\hat{=} (N_1 \vee_W N_2)(N_1' \vee_W N_2')\\
(\lambda x \cdot N_1) \vee_W (\lambda x \cdot N_2) &\hat{=} \lambda x \cdot (N_1 \vee_{W \cup \{x\}} N_2)\\
dN_1 \vee_W dN_2 &\hat{=} d(N_1 \vee_W N_2)\\
\boxed{N} \cdot \{x_1 := N_{11}, \ldots, x_n := N_{n1}\} \vee_W \boxed{N} &\cdot \{x_1 := N_{12}, \ldots, x_n := N_{n2}\}\\
&\hat{=} \boxed{N} \cdot \{x_1 := N_{11} \vee_W N_{12}, \ldots,\\
&\qquad x_n := N_{n1} \vee_W N_{n2}\}
\end{aligned}
$$

Figure 6: Greatest common generalizations

$N_1 \vee N_2 \hat{=} N_1 \vee_\emptyset N_2$, there is a unique substitution θ such that $\text{fv}(N_1 \vee N_2) = \text{dom}\,\theta$ and $(N_1 \vee N_2)\theta = M_0$, and $M \hat{=} \boxed{N_1 \vee N_2} \cdot \theta$ is a well-typed term, from which we conclude that M is indeed the infimum of M_1 and M_2. Write M as $M_1 \sqcap M_2$.

It remains to show that, if $C(M_0)$ is not empty, then it has a least element $\perp$ and a greatest element $\top$. This is obvious, as $\perp$ can be defined as the (finite) infimum of all elements of $C(M_0)$, and $\top$ as the (finite) supremum of all elements of $C(M_0)$. $\square$

Proposition 32. *Given any two* 0*-simplices* M_1 *and* M_2 *of* $\mathcal{S}4[\Gamma \vdash F]$, $M_1 \leftrightarrowtriangle M_2$ *if and only if* $\partial_0^0 M_1 = \partial_0^0 M_2$.

Proof. Clearly, if $M_1 \rightarrowtriangle^{=} M_2$, i.e. if $\partial_1^1 N = M_1$ and $\partial_1^0 N = M_2$ for some 1-simplex N, then $\partial_0^0 M_1 = \partial_0^0 \partial_1^1 N = \partial_0^0 \partial_1^0 N = \partial_0^0 M_2$. So if $M_1 \leftrightarrowtriangle M_2$, then $\partial_0^0 M_1 = \partial_0^0 M_2$.

Conversely, assume $\partial_0^0 M_1 = \partial_0^0 M_2$, and name M_0 this (-1)-simplex. By Proposition 31, there is an element $M_1 \sqcup M_2$ in $C(M_0)$, such that $M_1 \sqcup M_2 \rightarrowtriangle^* M_1$ and $M_1 \sqcup M_2 \rightarrowtriangle^* M_2$. In particular $M_1 \leftrightarrowtriangle M_2$. $\square$

In other words, non-empty components $C(M_0)$ coincide with path-connected components.

This generalizes to higher dimensions:

Proposition 33. *For any two* q*-simplices* M_1 *and* M_2 *of* $\mathcal{S}4[\Gamma \vdash F]$, $M_1 \leftrightarrowtriangle M_2$ *if and only if* $\pi_0 M_1 = \pi_0 M_2$.

Proof. Recall that $\pi_0 M$ denotes the component of M (Definition 25).

If $M_1 \rightarrowtriangle^{=} M_2$, then there is a $(q+1)$-simplex N and $j > i$ such that $M_1 = \partial_{q+1}^j N$, $M_2 = \partial_{q+1}^i N$. So $\pi_0 M_1 = \pi_0 N = \pi_0 M_2$.

Conversely, assume $\pi_0 M_1 = \pi_0 M_2 = M_0$. So every vertex of M_1 and M_2 is in $C(M_0)$. By the Nerve Theorem 30, we equate q-simplices with ordered sequences of $q+1$ vertices. Then notice that the sequence $N_0 \rightarrowtriangle^{=} N_1 \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_i \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_q$ is contiguous to $N_0 \rightarrowtriangle^{=} N_1 \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_i' \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_q$ as soon as $N_i \rightarrowtriangle^{=} N_i'$. Indeed the former is the $(i+1)$st face, and the latter is the ith face of the sequence $N_0 \rightarrowtriangle^{=} N_1 \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_i \rightarrowtriangle^{=} N_i' \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_q$. Iterating, we obtain that the sequence $N_0 \rightarrowtriangle^{=} N_1 \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_q$ is contiguous to $N_0' \rightarrowtriangle^{=} N_1' \rightarrowtriangle^{=} \ldots \rightarrowtriangle^{=} N_q'$ as soon as $N_i \rightarrowtriangle^* N_i'$ for every i, $0 \leqslant i \leqslant q$. Recall that every vertex of M_1 and M_2 is in $C(M_0)$. Using Proposition 31, M_1, viewed as the sequence $\eth_q^0 M_1 \rightarrowtriangle^* \eth_q^1 M_1 \rightarrowtriangle^* \ldots \rightarrowtriangle^* \eth_q^q M_1$, is contiguous to $(\eth_q^0 M_1 \sqcap \eth_q^0 M_2) \rightarrowtriangle^* (\eth_q^1 M_1 \sqcap \eth_q^1 M_2) \rightarrowtriangle^* \ldots \rightarrowtriangle^* (\eth_q^q M_1 \sqcap \eth_q^q M_2)$. Similarly for M_2. Since M_1 and M_2 are contiguous to the same q-simplex, $M_1 \leftrightarrowtriangle M_2$. $\square$

4.4. Planes and Retractions

Next, we show that certains subspaces of $\mathcal{S}4[\Gamma \vdash F]$ are retracts of the whole space, under some mild conditions.

Definition 34 (Planes). *Call a type* boxed *if and only if it is of the form* $\Box F$. *Call* Γ boxed *if and only if it maps variables to boxed types.*

Let Γ *be a context, and* $\Theta \hat{=} y_1 : \Box G_1, \ldots, y_p : \Box G_p$ *be a boxed subcontext of* Γ. *The* plane $\Theta^{\perp}$ *of* $\mathcal{S}4[\Gamma \vdash F]$ *is the set of* 0*-simplices of* $\mathcal{S}4[\Gamma \vdash F]$ *having and* η*-long normal form of the form* $\boxed{N} \cdot \theta$ *such that*

$$\textit{for every } y \in \operatorname{dom}\theta \cdot \textit{if } y_i \in \operatorname{fv}(y\theta) \textit{ then } y\theta = y_i \tag{14}$$

for every i, $1 \leqslant i \leqslant p$.

(Note that the types of variables y_j, $1 \leqslant j \leqslant p$, have to start with $\Box$ for this definition to make sense. To be fully formal, we should mention Γ and F in the notation for $\Theta^\perp$. However, Γ and F will be clear from context.) From the point of view of Gentzen-style sequents, a term in the given plane corresponds to a proof that ends in a $\Box$-introduction rule followed by series of cuts on formulae occurring on the left of the $\Box$-introduction rule, none of which being any of the $\Box G_i$s in Θ.

By extension, using Theorem 30 and Proposition 32, we define q-simplices of $\Theta^\perp$ as contiguous sequences of points $M_0 \multimap^* M_1 \multimap^* \ldots \multimap^* M_q$ of $\Theta^\perp$ for $q \geqslant 0$, and as components of points of $\Theta^\perp$ if $q = -1$.

It is not hard to see that $\Theta^\perp = \mathcal{S}4[\Gamma \vdash F]$ if Θ is empty. On the other hand, if $\Gamma = \Theta$, then the points of $\Theta^\perp$ are of the form $\boxed{N}$, with component N. In this case, any component N of the plane $\Theta^\perp$ contains exactly one point, namely $\boxed{N}$. It follows that in this case $\Theta^\perp$ is a discrete collection of points.

Lemma 35. *For any boxed subcontext Θ of Γ, $\Theta^\perp$ is a sub-a.s. set of Γ.*

Proof. Clearly every q-simplex of $\Theta^\perp$ is a q-simplex of $\mathcal{S}4[\Gamma \vdash F]$. That faces and degeneracies of q-simplices of $\Theta^\perp$ are still in $\Theta^\perp$ is by construction. $\Box$

Lemma 36. *Let Θ be any boxed subcontext of Γ, and $M_0 \in \mathcal{S}4[\Gamma \vdash F]_{-1}$. For any two 0-simplices M_1 and M_2 of $C(M_0)$:*

1. *if $M_1 \multimap^* M_2$ and $M_2 \in \Theta^\perp$ then $M_1 \in \Theta^\perp$.*
2. *if M_1 and M_2 are in $\Theta^\perp$ then so are $M_1 \sqcap M_2$ and $M_1 \sqcup M_2$.*

Proof. 1. Let $M_1 \hat{=} \boxed{N_1} \cdot \theta_1$ be in $C(M_0)$, and $M_2 \hat{=} \boxed{N_2} \cdot \theta_2$ be in $C(M_0)$ and in $\Theta^\perp$. So for every $z \in \operatorname{dom}\theta_2$ such that y_i, $1 \leqslant i \leqslant p$, is free in $z\theta_2$, $z\theta_2 = y_i$. Since $M_1 \multimap^* M_2$, not only is N_1 an instance of N_2, but there is also a substitution ϑ such that $\theta_2 = \vartheta \cdot \theta_1$, $\operatorname{dom}\vartheta = \operatorname{dom}\theta_2 = \operatorname{fv}(N_2)$ and $\operatorname{fv}(N_2\vartheta) = \operatorname{dom}\theta_1$. Assume that y_i is free in $y\theta_1$ for some $y \in \operatorname{dom}\theta_1 = \operatorname{fv}(N_2\vartheta)$. So y occurs free in some $z\vartheta$, $z \in \operatorname{fv}(N_2)$. In particular, y_i is free in $z\vartheta\theta_1 = z\theta_2$. Therefore $z\theta_2 = y_i$, recalling that $M_2 \in \Theta^\perp$. In other words, $z\vartheta\theta_1 = y_i$. It follows by standard size considerations that $z\vartheta$ is a variable. Since y occurs free in $z\vartheta$, it obtains $z\vartheta = y$. So $y\theta_1 = y_i$.

2. Let $M_1 \hat{=} \boxed{N_1} \cdot \theta_1$ and $M_2 \hat{=} \boxed{N_2} \cdot \theta_2$ in $\Theta^\perp$. For simplicity, assume that Θ contains exactly one variable y_1. This entails no loss of generality, as in general $\Theta^\perp$ is the intersection of all $(y_i : F_i)^\perp$, $y_i \in \operatorname{dom}\Theta$.
   That $M_1 \sqcup M_2$ is in $\Theta^\perp$ follows from 1, since $M_1 \sqcup M_2 \multimap^* M_1$.
   On the other hand $M_1 \sqcap M_2$ is of the form $\boxed{N_1 \vee N_2} \cdot \theta$ where θ is the unique substitution with domain $\operatorname{fv}(N_1 \vee N_2)$ such that $(N_1 \vee N_2)\theta = M_0$. In general, we may compute θ as $\theta_{\emptyset,M_0}(N_1, N_2)$, where $\theta_{W,M_0}(N_1, N_2)$ is the unique substitution with domain $\operatorname{fv}(N_1 \vee_W N_2) \setminus W$ such that $(N_1 \vee_W N_2)\theta_{W,M_0}(N_1, N_2) = M_0$, provided $N_1 \vee_W N_2$ exists and N_1 and N_2 are linear and have M_0 as

common instance. This parallels the computation of $N_1 \vee_W N_2$:

$$
\begin{aligned}
\theta_{W,M_0}(x_1, N_2) &\hat{=} \{x_1 := M_0\} \quad (x_1 \notin W)\\
\theta_{W,M_0}(N_1, x_2) &\hat{=} \{x_2 := M_0\} \quad (x_2 \notin W)\\
\theta_{W,M_0 M_0'}(N_1 N_1', N_2 N_2') &\hat{=} \theta_{W,M_0}(N_1, N_2) \cup \theta_{W,M_0'}(N_1', N_2')\\
\theta_{W,\lambda x \cdot M_0}(\lambda x \cdot N_1, \lambda x \cdot N_2) &\hat{=} \theta_{W \cup \{x\},M_0}(N_1, N_2)\\
\theta_{W,dM_0}(dN_1, dN_2) &\hat{=} \theta_{W,M_0}(N_1, N_2)
\end{aligned}
$$

$$
\begin{aligned}
\theta_{W, \boxed{N} \cdot \{x_1 := M_1, \ldots, x_n := M_n\}} & \Big(\boxed{N} \cdot \{x_1 := N_{11}, \ldots, x_n := N_{n1}\},\\
& \ldots\\
& \boxed{N} \cdot \{x_1 := N_{12}, \ldots, x_n := N_{n2}\} \Big)\\
& \hat{=} \bigcup_{j=1}^{n} \theta_{W,M_j}(N_{j1}, N_{j2})
\end{aligned}
$$

Notice that unions of substitutions are well-defined because we assume N_1 and N_2 are linear terms away from W (i.e., no two distinct subterms share any free variable except possibly for variables in W).

Now, generalize the claim as follows. Assume that the common instance M_0 of N_1 and N_2 away from W is $M_0 = N_1\theta_1 = N_2\theta_2$ with $\operatorname{dom}\theta_1 = \operatorname{fv}(N_1)$, $\operatorname{dom}\theta_2 = \operatorname{fv}(N_2)$. Assume also that for every variable $z \in \operatorname{dom}\theta_i$ such that y_1 is free in $z\theta_i$ then $y_1 = z\theta_i$, for every $i \in \{1, 2\}$. Then an easy induction on terms shows that for every variable $y \in \operatorname{fv}(N_1 \vee_W N_2) \setminus W$ such that y_1 is free in $y\theta_{W,M_0}(N_1, N_2)$, then $y\theta_{W,M_0}(N_1, N_2) = y_1$. The crucial cases are the first two of the definition, which are symmetric. In particular in the first case, assume $N_1 = x_1 \notin W$. By assumption $x_1\theta_1 = N_2\theta_2 = M_0$; then $\theta_{W,M_0}(x_1, N_2) = \{x_1 := M_0\}$ is θ_1 restricted to x_1, therefore indeed $y\theta_{W,M_0}(N_1, N_2) = y_1$, whatever y may be.

The claim follows by taking $W = \emptyset$.

□

Proposition 37 (Projection). *If Γ is boxed, then for any $\Theta \subseteq \Gamma$, there is an augmented simplicial map $\pi_{\Theta^\perp}$,* projection onto $\Theta^\perp$*, from $\mathcal{S4}[\Gamma \vdash F]$ to its sub-a.s. set $\Theta^\perp$, which coincides with the identity on $\Theta^\perp$.*

Proof. Let us first define $\pi_{\Theta^\perp}$ on 0-simplices. Using Proposition 31, define $\pi_{\Theta^\perp}(\boxed{N} \cdot \theta)$ as the infimum $\bigsqcap S$ of the set S of elements $M \in \Theta^\perp$ such that $M \rightarrow^* \boxed{N} \cdot \theta$. Observe that, since Γ is boxed, $\boxed{N\theta}$ is well-typed, hence is a valid 0-simplex. Moreover, it is clear that $\boxed{N\theta} \in \Theta^\perp$, and $\boxed{N\theta} \rightarrow^* \boxed{N} \cdot \theta$. So S is not empty, therefore $\bigsqcap S$ exists and is a 0-simplex of $\mathcal{S4}[\Gamma \vdash F]$. By Lemma 36 any finite non-empty infimum of elements of $\Theta^\perp$ is in $\Theta^\perp$. By Proposition 31 $\mathcal{S4}[\Gamma \vdash F]$ is finite, so S is a finite non-empty infimum of elements of $\Theta^\perp$. So $\pi_{\Theta^\perp}(\boxed{N} \cdot \theta) = \bigsqcap S$ is in $\Theta^\perp$.

If $\boxed{N} \cdot \theta$ is already in $\Theta^\perp$, then it is in S. Since $M \rightarrow^* \boxed{N} \cdot \theta$ for every M in S by construction, $\boxed{N} \cdot \theta$ is the minimal element of S, hence $\boxed{N} \cdot \theta = \bigsqcap S = \pi_{\Theta^\perp}(\boxed{N} \cdot \theta)$. So $\pi_{\Theta^\perp}$ indeed coincides with the identity on $\Theta^\perp$.

To show that $\pi_{\Theta^\perp}$ extends to an a.s. map from $\mathcal{S4}[\Gamma \vdash F]$ to $\Theta^\perp$, it remains to show that $\pi_{\Theta^\perp}$ preserves components (obvious) and contiguity (for dimensions $\geqslant 1$). As far as the latter is concerned, let $M_1 \twoheadrightarrow^* M_2$ be 0-simplices in $\mathcal{S4}[\Gamma \vdash F]$. Since $M_1 \twoheadrightarrow^* M_2$, $\{M \in \Theta^\perp | M \twoheadrightarrow^* M_1\} \subseteq \{M \in \Theta^\perp | M \twoheadrightarrow^* M_2\}$, so $\bigsqcap\{M \in \Theta^\perp | M \twoheadrightarrow^* M_1\} \twoheadrightarrow^* \bigsqcap\{M \in \Theta^\perp | M \twoheadrightarrow^* M_2\}$. That is, $\pi_{\Theta^\perp}(M_1) \twoheadrightarrow^* \pi_{\Theta^\perp}(M_2)$. $\square$

The projection construction can be used to show a connection between the *syntactic function space* $\mathcal{S4}[\Gamma \vdash F \supset G]$ and $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[\Gamma \vdash F], \mathcal{S4}[\Gamma \vdash G])$. The first direction is easy:

Definition 38 (Syntactic application $\star$). *The* syntactic application map $\star$ *from* $\mathcal{S4}[\Gamma \vdash F \supset G] \times \mathcal{S4}[\Gamma \vdash F]$ *to* $\mathcal{S4}[\Gamma \vdash G]$ *(written infix) is defined by* $M \star_{-1} N \hat{=} MN$, $\star_q \hat{=} \Box^{q+1} \star_{-1}$.

Lemma 39. *For every substitutive function f from $[\Gamma \vdash F_1] \times \ldots \times [\Gamma \vdash F_n]$ to $[\Gamma \vdash F]$, the family of functions $(\Box^{q+1} f)_{q \geqslant -1}$ is an a.s. map from $\mathcal{S4}[\Gamma \vdash F_1] \times \ldots \times \mathcal{S4}[\Gamma \vdash F_n]$ to $\mathcal{S4}[\Gamma \vdash F]$.*

Proof. Write f_q for $\Box^{q+1} f$. Recall (Lemma 17) that, as soon as f is substitutive, then $\Box f \circ \Box g = \Box(f \circ g)$.

Let $M_1 \in \mathcal{S4}[\Gamma \vdash F_1]_q$, $\ldots$, $M_n \in \mathcal{S4}[\Gamma \vdash F_n]_q$. For any i with $0 \leqslant i \leqslant q$, $\partial_q^i(f_q(M_1, \ldots, M_n)) = \Box^i d(\Box^{q+1} f(M_1, \ldots, M_n)) = \Box^i (d \circ \Box^{q+1-i} f)(M_1, \ldots, M_n)$. So $\partial_q^i \circ f_q = \Box^i (d \circ \Box^{q+1-i} f)$. But

$$
\begin{aligned}
d(\Box^{q+1-i} f(N_1, \ldots, N_n)) &= d \boxed{\Box^{q-i} f(dx_1, \ldots, dx_n)} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} \\
&\approx \Box^{q-i} f(dx_1, \ldots, dx_n)\{x_1 := N_1, \ldots, x_n := N_n\} \\
&= \Box^{q-i} f(dN_1, \ldots, dN_n) \quad \text{(since f is substitutive)}
\end{aligned}
$$

so $d \circ \Box^{q+1-i} f = \Box^{q-i} f \circ d$. Therefore $\partial_q^i \circ f_q = \Box^i (d \circ \Box^{q+1-i} f) = \Box^i (\Box^{q-i} f \circ d) = \Box^q f \circ \Box^i d = f_{q-1} \circ \partial_q^i$ (using the fact that $\Box^{q-i} f$ is substitutive).

Similarly, we claim that $s_q^0(\Box^{k+1} f(N_1, \ldots, N_n)) \approx \Box^{k+2} f(s_q^0 N_1, \ldots, s_q^0 N_n)$. It will follow that $s_q^i \circ f_q = f_{q+1} \circ s_q^i$. The claim is proved by induction on $k \geqslant 0$. If $k = 0$, then

$$
\begin{aligned}
s_q^0(\Box f(N_1, \ldots, N_n)) &= \boxed{x} \cdot \{x := \boxed{f(dy_1, \ldots, dy_n)} \cdot \{y_1 := N_1, \ldots, y_n := N_n\}\} \\
&\approx \boxed{\boxed{f(dy_1, \ldots, dy_n)}} \cdot \{y_1 := N_1, \ldots, y_n := N_n\}
\end{aligned}
$$

while

$$
\begin{aligned}
&\Box^2 f(s^0_q N_1, \ldots, s^0_q N_n)\\
= &\boxed{\Box f(dx_1, \ldots, dx_n)} \cdot \{x_1 := s^0_q N_1, \ldots, x_n := s^0_q N_n\}\\
= &\boxed{\boxed{f(dz_1, \ldots, dz_n)} \cdot \{z_1 := dx_1, \ldots, z_n := dx_n\}}\\
&\qquad \{x_1 := \boxed{y_1} \cdot \{y_1 := N_1\}, \ldots, x_n := \boxed{y_n} \cdot \{y_n := N_n\}\}\\
\approx &\boxed{\boxed{f(dz_1, \ldots, dz_n)} \cdot \{z_1 := d\boxed{y_1}, \ldots, z_n := d\boxed{y_n}\}} \cdot \{y_1 := N_1, \ldots, y_n := N_n\}\\
\approx &\boxed{\boxed{f(dz_1, \ldots, dz_n)} \cdot \{z_1 := y_1, \ldots, z_n := y_n\}} \cdot \{y_1 := N_1, \ldots, y_n := N_n\}\\
= &\boxed{\boxed{f(dy_1, \ldots, dy_n)}} \cdot \{y_1 := N_1, \ldots, y_n := N_n\} \qquad \text{(by } \alpha\text{-renaming)}
\end{aligned}
$$

In the inductive case, $s^0_q(\Box^{k+1} f(N_1, \ldots, N_n)) = s^0_q(\Box(\Box^k f)(N_1, \ldots, N_n))$ $\approx \Box^2(\Box^k f)(s^0_q N_1, \ldots, s^0_q N_n)$ (by the above, replacing f by $\Box^k f$) $= \Box^{k+2} f(s^0_q N_1, \ldots, s^0_q N_n)$, as desired. $\Box$

Corollary 40. *The syntactic application map $\star$ is an a.s. map.*

The following shows how we may compute $\star$:

Lemma 41. *Let $M \in \mathcal{S4}[\Gamma \vdash F \supset G]_q$, $N \in \mathcal{S4}[\Gamma \vdash F]_q$ be η-long normal:*

$$M \hat{=} \boxed{\boxed{\ldots \boxed{M_1} \cdot \theta_q \ldots} \cdot \theta_1} \cdot \theta_0$$

$$N \hat{=} \boxed{\boxed{\ldots \boxed{N_1} \cdot \theta'_q \ldots} \cdot \theta'_1} \cdot \theta'_0$$

Then, provided $\operatorname{dom} \theta_i \cap \operatorname{dom} \theta'_i = \emptyset$ *for every i, $0 \leqslant i \leqslant q$,*

$$M \star_q N \approx \boxed{\boxed{\ldots \boxed{M_1 N_1} \cdot (\theta_q, \theta'_q) \ldots} \cdot (\theta_1, \theta'_1)} \cdot (\theta_0, \theta'_0)$$

Proof. This is clear if $q = -1$. If $q = 0$, $M \star_0 N = \boxed{M_1 N_1} \cdot (\theta_0, \theta'_0)$ by Lemma 19. Otherwise, this follows by the $q = 0$ case, using Theorem 30. $\Box$

From Corollary 40, it follows that application is uniquely determined by its values on components (simplices of dimension -1) and points (dimension 0). It also follows that $\Lambda(\star)$ is an a.s. map from $\mathcal{S4}[\Gamma \vdash F \supset G]$ to $\mathbf{Hom}_{\hat{\Delta}}(\mathcal{S4}[\Gamma \vdash F], \mathcal{S4}[\Gamma \vdash G])$.

There is a kind of converse to syntactic application. Intuitively, in the λ-calculus (the non-modal case), not only can you apply a term M to a term N, you can also build $\lambda x \cdot M$ from M: this term $\lambda x \cdot M$ is such that, once applied to N, you get $M\{x := N\}$. We can do almost the same thing here, except M has to be in some plane for this to work.

Proposition 42. *For any* 0*-simplex* P *in any a.s. set, define* $(P)_q$ *by*

$$(P)_{-1} \hat{=} \partial_0^0 P \qquad (P)_0 \hat{=} P \qquad (P)_{q+1} \hat{=} s(P)_q \quad (q \geqslant 0)$$

Say that a q*-simplex* M *of* $\mathcal{S}4[\Gamma, x : \Box F \vdash G]$ *is* abstractable *on* x *if and only if* $\pi_0 M$ *can be written as* $M_0\{y := dx\}$ *for some term* M_0 *such that* $\Gamma, y : F \vdash M_0 : G$ *is typable (in particular,* x *is not free in* M_0*), and* M *is in the plane* $(x : \Box F)^\perp$ *of* $\mathcal{S}4[\Gamma, x : \Box F \vdash G]$.

Then there is an a.s. map from the sub-a.s. set of terms M *in* $\mathcal{S}4[\Gamma, x : \Box F \vdash G]$ *that are abstractable on* x *to terms* $\lambda x^q \cdot M$*, such that* $(\lambda x^q \cdot M) \star^q (x)_q \approx M$.

Proof. Note that in syntactic a.s. sets as studied here, we may define $(P)_q$ more synthetically as $\boxed{\ldots \boxed{dP} \ldots}$, where dP is enclosed in $q+1$ boxes.

Case $q = 0$**.** Let us define $\lambda x^q \cdot M$ when $q = 0$. Write the η-long normal form of M as $\boxed{M_1} \cdot \theta$. Since M is in $(x : \Box F)^\perp$, for every free variable z of M_1 such that x is free in $z\theta$, $z\theta = x$. Let $x_1, \ldots, x_k$ be those free variables of M_1 such that $x_1\theta = \ldots = x_k\theta = x$. The restriction θ^* of θ to the remaining variables maps variables to terms where x is not free.

Moreover, by assumption $M_1\theta = M_0\{y := dx\}$, so x only occurs as direct argument of d in $M_1\theta$. By the definition of $x_1, \ldots, x_k$, there is a term M_2, obtained from M_1 by replacing each dx_i by y, such that $M_2\theta^* = M_0$. Moreover, by construction M_2 is η-long normal of type G under $\Gamma, y : F$, so:

$$\lambda x^0 \cdot M \hat{=} \boxed{\lambda y \cdot M_2} \cdot \theta^* \tag{15}$$

is a 0-simplex of the desired type. This is also η-long normal since $\mathrm{fv}(\lambda y \cdot M_2) = \mathrm{fv}(M_2) \setminus \{y\} = (\mathrm{fv}(M_1) \setminus \{x_1, \ldots, x_k\} \cup \{y\}) \setminus \{y\} = \mathrm{fv}(M_1) \setminus \{x_1, \ldots, x_k\} = \mathrm{dom}\,\theta \setminus \{x_1, \ldots, x_k\} = \mathrm{dom}\,\theta^*$, and $\lambda y \cdot M_2$ is linear since every free variable in M_2 except possibly y occurs exactly once.

We check that $(\lambda x^0 \cdot M) \star^0 (x)^0 \approx M$:

$$\begin{aligned}
(\lambda x^0 \cdot M) \star^0 (x)_0 &= \boxed{dz(dz')} \cdot \{z := \lambda x^0 \cdot M, z' := (x)_0\} \\
&\approx \boxed{d\boxed{\lambda y \cdot M_2}(d\boxed{dx})} \cdot (\theta^*, \{x := x\}) \qquad (\text{by } (\boxed{\Box})) \\
&\approx \boxed{(\lambda y \cdot M_2)(dx)} \cdot (\theta^*, \{x := x\}) \qquad (\text{by } (d)) \\
&\approx \boxed{M_2\{y := dx\}} \cdot (\theta^*, \{x := x\}) \qquad (\text{by } (\beta)) \\
&\approx \boxed{M_1} \cdot (\theta^*, \{x_1 := x, \ldots, x_k := x\}) \qquad (\text{by } (\text{ctr})) \\
&= \boxed{M_1} \cdot \theta = M
\end{aligned}$$

General Case. To extend $\lambda x^q \cdot M$ to $q = -1$, use Proposition 32. To extend this to $q \geqslant 1$, check that $M \mapsto \lambda x^0 \cdot M$ is $\multimap^*$-monotonic and use Theorem 30. Assume indeed $M \multimap^= M'$ are 0-simplices, where $M \hat{=} \boxed{M_1} \cdot \theta$ and $M' \hat{=} \boxed{M_1'} \cdot \theta'$. Then there is a 1-simplex $\boxed{\boxed{N} \cdot \vartheta_1} \cdot \vartheta_0$ such that $N\vartheta_1 = M_1$, $\vartheta_0 = \theta$, and $N = M_1'$,

$\vartheta_1 \cdot \vartheta_0 = \theta'$. In other words, there is a 1-simplex $\boxed{\boxed{M_1'} \cdot \vartheta_1} \cdot \theta$ such that $M_1'\vartheta_1 = M_1$ and $\vartheta_1 \cdot \theta = \theta'$.

Since M is in $(x : \Box F)^{\perp}$, let $x_1, \ldots, x_k$ be the free variables of M_1 such that $x_i\theta = x$, $1 \leqslant i \leqslant k$, as above. Similarly, let $x_1', \ldots, x_{k'}'$ be the free variables of M_1' such that $x_{i'}'\theta' = x$, $1 \leqslant i' \leqslant k'$.

Observe that for every i' with $1 \leqslant i' \leqslant k'$, $x_{i'}'\vartheta_1$ is such that $(x_{i'}'\vartheta_1)\theta = x_{i'}'\vartheta_1\theta = x_{i'}'\theta' = x$, so: (a) $x_{i'}'\vartheta_1$ is some x_i, $1 \leqslant i \leqslant k$. Conversely, if x_i is free in some $z'\vartheta_1$, $z' \in \operatorname{dom}\vartheta_1$, then x is free in $z'\vartheta_1\theta = z'\theta'$, so z' is some $x_{i'}'$. In brief: (b) if $x_i \in \operatorname{fv}(z'\vartheta_1)$, $z' \in \operatorname{dom}\vartheta_1$, then $z' = x_{i'}'$ for some i'. So we may write ϑ_1 as the disjoint union of the restriction ϑ_1^* of ϑ_1 to $\operatorname{dom}\vartheta_1 \setminus \{x_1', \ldots, x_{k'}'\}$ with a one-to-one (by (b)) substitution mapping each $x_{i'}'$ to some x_i (by (a)). In particular, $k' = k$ and without loss of generality, we may assume $x_i'\vartheta_1 = x_i$ for every i, $1 \leqslant i \leqslant k$. Moreover, by (b) no x_i is free in any $z'\vartheta_1^*$, $z' \in \operatorname{dom}\vartheta_1^*$.

Let θ^* be the restriction of θ to $\operatorname{fv}(M_1) \setminus \{x_1, \ldots, x_k\}$, θ'^* be that of θ' to $\operatorname{fv}(M_1') \setminus \{x_1', \ldots, x_k'\}$. Let M_2 be obtained from M_1 by replacing each dx_i by y, and M_2' be obtained from M_1' by replacing each dx_i' by y. Finally, let

$$P \hat{=} \boxed{\boxed{\lambda y \cdot M_2'} \cdot \vartheta_1^*} \cdot \theta^*$$

We first claim that P is a valid 1-simplex. Indeed, $\lambda y \cdot M_2'$ is linear; $\operatorname{fv}(\lambda y \cdot M_2') = \operatorname{fv}(M_2') \setminus \{y\} = \operatorname{fv}(M_1') \setminus \{x_1', \ldots, x_k'\} = \operatorname{dom}\vartheta_1 \setminus \{x_1', \ldots, x_k'\} = \operatorname{dom}\vartheta_1^*$; $\boxed{\lambda y \cdot M_2'} \cdot \vartheta_1^*$ is linear, since $\boxed{M_1'} \cdot \vartheta_1$ is; and $\operatorname{fv}(\boxed{\lambda y \cdot M_2'} \cdot \vartheta_1^*) = \bigcup_{z' \in \operatorname{fv}(\lambda y \cdot M_2')} \operatorname{fv}(z'\vartheta_1^*) = \bigcup_{z' \in \operatorname{fv}(M_1') \setminus \{x_1', \ldots, x_k'\}} \operatorname{fv}(z'\vartheta_1^*) = \bigcup_{z' \in \operatorname{fv}(M_1')} \operatorname{fv}(z'\vartheta_1) \setminus \{x_1, \ldots, x_k\}$ (since ϑ_1 is $\vartheta_1^* \uplus \{x_1' := x_1, \ldots, x_k' := x_k\}$ and no x_i is free in any $z'\vartheta_1^*$) $= \operatorname{fv}(\boxed{M_1'} \cdot \vartheta_1) \setminus \{x_1, \ldots, x_k\}$ $= \operatorname{dom}\theta \setminus \{x_1, \ldots, x_k\} = \operatorname{dom}\theta^*$.

We then claim that $\partial_1^1 P = \lambda x^0 \cdot M$ and $\partial_1^0 P = \lambda x^0 \cdot M'$.

For the first claim, notice that $M_2'\vartheta_1^*$ is obtained from $M_1'\vartheta_1^*$ by replacing each dx_i' by y, so $M_2'\vartheta_1$ is obtained from $M_1'\vartheta_1$ by replacing each dx_i by y. Since $M_1'\vartheta_1 = M_1$ and M_2 is obtained by replacing each dx_i in M_1 by y, it follows that $M_2'\vartheta_1 = M_2$. Therefore $\partial_1^1 P = \boxed{(\lambda y \cdot M_2')\vartheta_1^*} \cdot \theta^* = \boxed{\lambda y \cdot M_2'\vartheta_1} \cdot \theta^* = \boxed{\lambda y \cdot M_2} \cdot \theta^* = \lambda x^0 \cdot M$.

For the second claim, since $\vartheta_1 \cdot \theta = \theta'$ and $\vartheta_1 = \vartheta_1^* \uplus \{x_1' := x_1, \ldots, x_k' := x_k\}$, $\theta = \theta^* \uplus \{x_1 := x, \ldots, x_k := x\}$, $\theta' = \theta'^* \uplus \{x_1' := x, \ldots, x_k' := x\}$, and no x_i is free in any $z'\vartheta_1^*$, $z' \in \operatorname{dom}\vartheta_1^*$, it follows that $\vartheta_1^* \cdot \theta^* \uplus \{x_1 := x, \ldots, x_k := x\} = \theta'^* \uplus \{x_1 := x, \ldots, x_k := x\}$, whence $\vartheta_1^* \cdot \theta^* = \theta'^*$. So $\partial_1^0 P = \boxed{\lambda y \cdot M_2'} \cdot (\vartheta_1^* \cdot \theta^*) = \boxed{\lambda y \cdot M_2'} \cdot \theta'^* = \lambda x^0 \cdot M'$.

Therefore $M \twoheadrightarrow^= M'$. It follows that $M \mapsto \lambda x^0 \cdot M$ is indeed $\twoheadrightarrow^*$-monotonic, hence extends to a unique a.s. map in every dimension. $\Box$

It can be shown that $\Lambda(\star)_q$ is injective for every q, and we leave this to the reader. But we can say more, at the price of considering slightly looser a.s. sets:

Definition 43 ($\mathcal{S}4[F]$). *Let $(\mathcal{S}4[F])_q$, $q \geqslant -1$, be the set of all $\approx$-equivalence classes of λ_{S4}-terms M such that $\Gamma \vdash M : \Box^{q+1} F$ is derivable for some boxed context Γ.*

This gives rise to an a.s. set $\mathcal{S4}[F] \hat{=} (\mathcal{S4}[\Gamma]F, (\partial_q^i)_{0\leqslant i\leqslant q}, (s_q^i)_{0\leqslant i\leqslant q})$

Then, the $\star$ *map extends naturally to an a.s. map, written* $*$, *from* $\mathcal{S4}[F \supset G] \times \mathcal{S4}[F]$ *to* $\mathcal{S4}[G]$.

Note that we have defined simplices as typable λ_{S4}-terms, not typing derivations. The difference can be illustrated as follows: the variable x for instance is one λ_{S4}-term, while all typing derivations of $\Gamma, x : F \vdash x : F$ by (Ax) when Γ varies are all distinct. This will be made clearer, using categorical language, in Proposition 68.

The injectivity of $\Lambda(*)$ yields an embedding of $\mathcal{S4}[F \supset G]$ into $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])$. We shall show that this can be turned into the inclusion part of a strong retraction of $\mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])$ onto $\mathcal{S4}[F \supset G]$. First, we note some general results:

Definition 44 (Hull). *Let* K *be an a.s. set, and* $A \subseteq K_{-1}$. *The* hull $\overline{A}$ *is the a.s. subset of* K *whose* q*-simplices are all* q*-simplices* x *of* K *such that* $\pi_0 x \in A$.

This inherits face and degeneracy operators from K. Every a.s. set splits as a sum of hulls:

Proposition 45. *Every a.s. set* K *splits as a sum* $\coprod_{x\in K_{-1}} \overline{\{x\}}$. *In particular, for every* $A \subseteq K_{-1}$, $K = \overline{A} \amalg \overline{(K_{-1} \setminus A)}$.

The following lemma is the first one where the change from $\mathcal{S4}[\Gamma \vdash F]$ to $\mathcal{S4}[F]$ is required:

Lemma 46. *Let* A *be any subset of* $\mathcal{S4}[F]_{-1}$, *and assume that there is a* 0*-simplex* P *in* $\overline{A}$.

Then there is a strong retraction r_A *of* $\mathcal{S4}[F]$ *onto* $\overline{A}$. *In other words,* r_A *is an a.s. map such that, for every* $M \in \overline{A}$, $r_A(M) = M$.

Proof. For any q-simplex M of $\mathcal{S4}[F]$, then either M is in $\left(\overline{A}\right)_q$ and we let $r_A(M)$ be M, or M is in $\left(\overline{\mathcal{S4}[F]_{-1} \setminus A}\right)_q$ by Proposition 45, and we let $r_A(M)$ be $(P)_q \in \mathcal{S4}[F]_q$. Note that $r_A(M)$ is always in the hull of A.

Clearly, for every $M \in \overline{A}$, $r_A(M) = M$. It remains to show that r_A is a.s. If M is in $\left(\overline{A}\right)_q$, and $0 \leqslant i \leqslant q$, then $\partial_q^i M$ is in $\left(\overline{A}\right)_{q-1}$, so $\partial_q^i(r_A(M)) = \partial_q^i M = r_A(\partial_q^i M)$. Otherwise M is in $\left(\overline{\mathcal{S4}[F]_{-1} \setminus A}\right)_q$, so $\partial_q^i M$ is in $\left(\overline{\mathcal{S4}[F]_{-1} \setminus A}\right)_{q-1}$, therefore $\partial_q^i(r_A(M)) = \partial_q^i(P)_q \approx (P)_{q-1} = r_A(\partial_q^i M)$. Similarly for s_q^i. ☐

Proposition 47. $\mathcal{S4}[F \supset G]$ *is a strong retract of* $\overline{\operatorname{Im}\Lambda(*)_{-1}}$.

More precisely, there is an a.s. map $\mathfrak{R}^1_{F\supset G}$ *from* $\overline{\operatorname{Im}\Lambda(*)_{-1}}$ *to* $\mathcal{S4}[F \supset G]$ *such that* $\mathfrak{R}^1_{F\supset G} \circ \Lambda(*) = \mathrm{id}_{\mathcal{S4}[F\supset G]}$.

Proof. For every boxed context Γ, fix some variable ξ_Γ outside the domain of Γ.

Let f be any q-simplex of $\overline{\operatorname{Im}\Lambda(*)_{-1}}$. That is, first, $f \in \mathbf{Hom}_{\widehat{\Delta}}(\mathcal{S4}[F], \mathcal{S4}[G])_q$, and $\pi_0 f = \Lambda(*)_{-1}(M)$ for some term M and some boxed context Γ such that

$\Gamma \vdash_I M : F \supset G$ is derivable in BN_0. Since M is η-long normal of type $F \supset G$, M must be of the form $\lambda y \cdot M_1$, with $\Gamma, y : F \vdash_I M_1 : G$ derivable in BN_0. To sum up:

$$\Lambda(*)_{-1}(\lambda y \cdot M_1) = \pi_0 f \tag{16}$$

Now $App_q(f, (\xi_\Gamma)_q)$ is a q-simplex of $\mathcal{S}4[G]$. Its component is $\pi_0(App_q(f, (\xi_\Gamma)_q))$, which equals $App_{-1}(\pi_0 f, d\xi_\Gamma) = App_{-1}(\Lambda(*)_{-1}(\lambda y \cdot M_1), d\xi_\Gamma)$ (by (16)) $= (\lambda y \cdot M_1) \star^{\mathrm{II}}_{-1} d\xi_\Gamma$ (by the combinator equations, in particular (l)) $= (\lambda y \cdot M_1)(d\xi_\Gamma) \approx M_1\{y := d\xi_\Gamma\}$.

As far as typing is concerned, since $\Gamma, y : F \vdash_I M_1 : G$ is derivable in BN_0, $\Gamma, \xi_\Gamma : \Box F \vdash M_1\{y := d\xi_\Gamma\}$ is, too. So $\pi_0(App_q(f, (\xi_\Gamma)_q))$ is in $\mathcal{S}4[\Gamma, \xi_\Gamma : \Box F \vdash G]_{-1}$, from which it follows that $App_q(f, (\xi_\Gamma)_q)$ is in $\mathcal{S}4[\Gamma, \xi_\Gamma : \Box F \vdash G]_q$.

Using Proposition 37, let $M' \hat{=} \pi_{(\xi_\Gamma:\Box F)^\perp}(App_q(f, (\xi_\Gamma)_q))$. This is an element of $\mathcal{S}4[\Gamma, \xi_\Gamma : \Box F \vdash G]_q$. Since projection is a.s., it preserves components, so $\pi_0(M') = \pi_0(App_q(f, (\xi_\Gamma)_q)) = M_1\{y := d\xi_\Gamma\}$. By construction M' is in $(\xi_\Gamma : \Box F)^\perp$, so $M' = \pi_{(\xi_\Gamma:\Box F)^\perp}(App_q(f, (\xi_\Gamma)_q))$ is abstractable on ξ_Γ. We may therefore use Proposition 42, and let:

$$\mathfrak{R}^1_{F\supset G}(f) \hat{=} \lambda^q \xi_\Gamma \cdot \pi_{(\xi_\Gamma:\Box F)^\perp}(App_q(f, (\xi_\Gamma)_q)) \tag{17}$$

This is clearly a.s., as a composition of a.s. maps.

Check that $\mathfrak{R}^1_{F\supset G}$ is a left inverse to $\Lambda(*)$. It is enough to check this in dimension 0, by Theorem 30 and Proposition 32, since $\mathfrak{R}^1_{F\supset G}$ is a.s. So let f be any 0-simplex in $\mathrm{Im}(\Lambda(*)_0)$, i.e., $f = \Lambda(*)_0(P)$ with $P \in \mathcal{S}4[F \supset G]_0$. Write P in a unique way as the η-long normal form $\boxed{\lambda y \cdot P_1} \cdot \theta$. Then:

$$\begin{aligned}
\mathfrak{R}^1_{F\supset G}(f) &= \lambda^0 \xi_\Gamma \cdot \pi_{(\xi_\Gamma:\Box F)^\perp}(App_0(f, (\xi_\Gamma)_0)) \\
&= \lambda^0 \xi_\Gamma \cdot \pi_{(\xi_\Gamma:\Box F)^\perp}(P *_0 (\xi_\Gamma)_0) \\
&= \lambda^0 \xi_\Gamma \cdot \pi_{(\xi_\Gamma:\Box F)^\perp}(\boxed{P_1\{y := d\xi_\Gamma\}} \cdot \theta) \\
&= \lambda^0 \xi_\Gamma \cdot \boxed{P_1\{y := d\xi_\Gamma\}} \cdot \theta \quad \text{(since } \boxed{P_1\{y := d\xi_\Gamma\}} \cdot \theta \text{ is in } (\xi_\Gamma : \Box F)^\perp) \\
&= \boxed{\lambda y \cdot P_1} \cdot \theta \quad \text{(by (15))} \\
&= P
\end{aligned}$$

where for readability we have not converted $\boxed{P_1\{y := d\xi_\Gamma\}} \cdot \theta$ to its η-long normal form ($P_1\{y := d\xi_\Gamma\}$ is in general not linear in ξ_Γ). $\Box$

Combining Lemma 46 with $A \hat{=} \mathrm{Im}(\Lambda(*)_{-1})$ and Proposition 47, we get:

Corollary 48 (Strong Functional Retraction). *$\mathcal{S}4[F \supset G]$ is a strong retract of the a.s. set $\mathbf{Hom}_{\hat{\Delta}}(\mathcal{S}4[F], \mathcal{S}4[G])$: there is an augmented simplicial map $\mathfrak{R}_{F\supset G}$ from $\mathbf{Hom}_{\hat{\Delta}}(\mathcal{S}4[F], \mathcal{S}4[G])$ to $\mathcal{S}4[F \supset G]$ such that $\mathfrak{R}_{F\supset G} \circ \Lambda(*) = \mathrm{id}_{\mathcal{S}4[F\supset G]}$.*

Proof. Take $\mathfrak{R}_{F\supset G}$ as $\mathfrak{R}^1_{F\supset G} \circ r_{\overline{\mathrm{Im}(\Lambda(*)_{-1})}}$. Lemma 46 applies because there is indeed a 0-simplex in $\overline{\mathrm{Im}(\Lambda(*)_{-1})}$, e.g., $\Lambda(*)_0(x)$, where $x : \Box(F \supset G) \vdash x : \Box(F \supset G)$, so $x \in \mathcal{S}4[x : \Box(F \supset G) \vdash F \supset G]_0 \subseteq \mathcal{S}4[F \supset G]_0$. $\Box$

5. Augmented Simplicial and Other Models

There is a natural interpretation of (non-modal) types and typed λ-terms in the category **Set** of sets and total functions. Interpret base types as sets, interpret $F \supset G$ as the set of all total functions from F to G. Then λ-terms, or more precisely derivations of $x_1 : F_1, \ldots, x_n : F_n \vdash M : F$, are interpreted as total functions from $F_1 \times \ldots \times F_n$ to F. The variable x_i gets interpreted as the ith projection, application of M to N is interpreted as the function mapping $g \in F_1 \times \ldots \times F_n$ to $M(g)(N(g))$, and abstraction $\lambda x \cdot M : F \supset G$ is interpreted as the function mapping $g \in F_1 \times \ldots \times F_n$ to the function mapping $x \in F$ to $M(g, x)$ (*currying*). This is arguably *the* intended semantics of λ-terms.

In particular, if M and N are convertible λ-terms by the (β) and (η) rules (they are *$\beta\eta$-equivalent*), then they have the same interpretation. However, this interpretation is far from being onto: note that there are only countably many λ-terms, while as soon as some base type A gets interpreted as an infinite set, $A \supset A$ will not be countable, and $(A \supset A) \supset A$ will neither be countable nor even of the cardinality of the powerset of $\mathbb{N}$.

Nonetheless, it can be proved that this interpretation is *equationally complete*:

Theorem 49 ([16]). *If the two typed λ-terms M and N, of the same type F, have the same set-theoretic interpretation for every choice of the interpretation of base types, then M and N are $\beta\eta$-equivalent.*

In fact, there is even a fixed set-theoretic interpretation such that, if M and N have the same value in this interpretation, then they are $\beta\eta$-equivalent. Extending this result will be the topic of Section 5.3 and subsequent ones.

5.1. The $(\Box, \mathbf{d}, \mathbf{s})$ Comonad on $\widehat{\Delta}$, and Strict CS4 Categories

In the S4 case, given the fact that $\mathcal{S4}[\Gamma \vdash F]$ is an augmented simplicial set, it is natural to investigate the extension of the above constructions to intuitionistic S4 on the one hand and the category of augmented simplicial sets on the other hand.

In general, intuitionistic S4 proofs can be interpreted in any CCC with a monoidal comonad. While the CCC structure of $\widehat{\Delta}$, accounting for the non-modal part of S4 proofs, was recalled in Section 4, the monoidal comonad we use is:

Definition 50 ($\Box$ Comonad in $\widehat{\Delta}$). *For every a.s. set K, let $\Box K$ denote the a.s. set such that $(\Box K)_q \hat{=} K_{q+1}$, $\partial^i_{(\Box K)_q} \hat{=} \partial^{i+1}_{K_{q+1}}$, $s^i_{(\Box K)_q} \hat{=} s^{i+1}_{K_{q+1}}$. For any a.s. map $f : K \to L$, let $\Box f : \Box K \to \Box L$ be such that $(\Box f)_q \hat{=} f_{q+1}$. Let $\mathbf{d} : \Box K \to K$ and $\mathbf{s} : \Box K \to \Box^2 K$ be the a.s. maps such that $(\mathbf{d})_q \hat{=} \partial^0_{K_{q+1}}$ and $(\mathbf{s})_q \hat{=} s^0_{K_{q+1}}$ respectively, $q \geqslant -1$.*

CCCs with a monoidal comonad have already been argued to be the proper categorical models of intuitionistic S4 [**7**]. While Bierman and de Paiva only show that (β), (η), and (d) are sound, it is easy to check that the other equalities (gc), (ctr), $(\boxed{\Box})$ and $(\eta\Box)$ are also sound.

It is easy to check that the monoidal comonad of Definition 50 satisfies $\Box\mathbf{1} = \mathbf{1}$, $\Box(K \times L) = \Box K \times \Box L$ (up to natural isomorphisms that we will not make explicit, for readability purposes), and the following so-called *strict monoidal comonad* equations hold:

(n) $\Box\mathrm{id} = \mathrm{id}$	(o) $\Box(f \circ g) = \Box f \circ \Box g$	(p) $\mathbf{d} \circ \Box f = f \circ \mathbf{d}$	(q) $\mathbf{s} \circ \Box f = \Box^2 f \circ \mathbf{s}$
(r) $\Box\pi_1 = \pi_1$	(s) $\mathbf{d} \circ \mathbf{s} = \mathrm{id}$	(t) $\Box\mathbf{d} \circ \mathbf{s} = \mathrm{id}$	(u) $\Box\mathbf{s} \circ \mathbf{s} = \mathbf{s} \circ \mathbf{s}$
(v) $\Box\pi_2 = \pi_2$	(w) $\Box\langle f, g\rangle = \langle \Box f, \Box g\rangle$	(x) $\mathbf{d} \circ \langle f, g\rangle = \langle \mathbf{d} \circ f, \mathbf{d} \circ g\rangle$	
		(y) $\mathbf{s} \circ \langle f, g\rangle = \langle \mathbf{s} \circ f, \mathbf{s} \circ g\rangle$	

Definition 51 (Strict CS4 Category). *A* strict CS4 category *is any cartesian-closed category $\mathcal{C}$ together with a strict monoidal comonad* $(\Box, \mathbf{d}, \mathbf{s})$.

Strict CS4 categories are the categories in which we can interpret typed λ_{S4}-terms. Bierman and de Paiva considered non-strict CS4 categories [**7**]. We shall only need the strict variant; this will make our exposition simpler. In particular $\widehat{\Delta}$ with the comonad of Definition 50 is a strict CS4 category.

The $\Box$ functor on $\widehat{\Delta}$ is related to Duskin and Illusie's *décalage* functor $\blacksquare$ [**46**]. Standardly, décalage is dual to $\Box$. For every a.s. set K, the *converse* $\check{K}$ of K is obtained by letting $(\check{K})_q \hat{=} K_q$, $\partial^i_{\check{K}_q} \hat{=} \partial^{q-i}_{K_q}$, $s^i_{\check{K}_q} \hat{=} s^{q-i}_{K_q}$. That is, $\check{K}$ is obtained from K by reversing the order of faces. Then $\blacksquare K$ is the converse of $\Box\check{K}$. If $\Box K$ in a sense means "in every future, K", then it is natural to think of $\blacksquare K$ as "in every past, K". As announced in Section 2, we shall leave the task of investigating such other modalities to a future paper.

5.1.1. Topological Models

There are many other interesting strict CS4 Categories. Of interest in topology in the category **CGHaus** of *compactly generated topological spaces*, a.k.a., *Kelley spaces* ([**36**] VII.8). (It is tempting to use the category **Top** of topological spaces, however **Top** is not a CCC. It has sometimes been argued that **CGHaus** was the right category to do topology in.) Recall that a Kelley space is a Hausdorff topological space X whose closed subsets are exactly those subsets A whose intersection with every compact subspace of X is closed in X. **CGHaus** has Kelley spaces as objects and continuous functions as morphisms. Moreover, for every Hausdorff space X, there is a smallest topology containing that of X that makes it Kelley. The resulting Kelley space $K(X)$ is the *kelleyfication* of X, and is obtained by adding as closed sets every $A \subseteq X$ whose intersection with every compact subspace of X is closed in X. The terminal object $\mathbf{1}$ in **CGHaus** is the one-point topological space, while the product of X and Y is the kelleyfication of the product of X and Y as topological spaces, and the internal hom $\mathbf{Hom}_{\mathbf{CGHaus}}(X, Y)$ is the kelleyfication of the space of all continuous functions from X to Y with the compact-open topology. We may equip **CGHaus** with a structure of strict CS4 category as follows:

Definition 52 ($\Box$ Comonad in CGHaus). *For every topological space X, the* cocone $\Box X$ *over X is the disjoint sum $\coprod_{x_0 \in X} \Box_{x_0} X$, where the $\Box_{x_0} X$ is the space of all continuous functions α from $[0,1]$ to X such that $\alpha(0) = x_0$, with the compact-open topology.*

For every continuous function $f : X \to Y$, let $\Box f : \Box X \to \Box Y$ be the function mapping each $\alpha \in \Box X$ to $f \circ \alpha \in \Box Y$.

The counit $\mathbf{d}$ *maps every* $\alpha \in \Box X$ *to* $\alpha(1) \in X$.

The comultiplication $\mathbf{s}$ *maps every* $\alpha \in \Box X$ *to the map* $t \mapsto (t' \mapsto \alpha(tt'))$ *in* $\Box^2 X$.

This comonad is in fact related to the décalage functor, through singular simplex and geometric realization functorsas we shall see in Section 6.

In terms of processes, we may think of $\alpha \in \Box X$ as some process that starts at time 0 and will produce a value at time 1. The counit $\mathbf{d}$ is the operator that extracts the final value of the process α as argument.

We have called this a *cocone* comonad because it can be shown that the $\Box$ functor admits a cone functor (strongly related to the familiar construction of cones) as a left adjoint (Proposition 54).

Proposition 53. *The construction* $(\Box, \mathbf{d}, \mathbf{s})$ *of Definition 52 is a strict monoidal comonad on* **CGHaus**, *making it a strict CS4 category.*

Proof. First show that $\Box_{x_0} X$ is Kelley. Since $[0,1]$ is compact, it is locally compact Hausdorff; it is then well-known that the space of continuous functions from $[0,1]$ to X is Kelley: this is $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X)$. (In general $\mathbf{Hom}_{\mathbf{CGHaus}}(Y, X)$ is the kelleyfication of the space of continuous functions from Y to X, not the space itself.) Since $\{x_0\}$ is closed in X, and the projection $\alpha \mapsto \alpha(0)$ is continuous, $\Box_{x_0} X$ is closed in $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X)$. As a closed subset of a Kelley space, $\Box_{x_0} X$ is then Kelley, too. Since every coproduct of Kelley spaces is also Kelley, it follows that $\Box X$ is Kelley.

Next we must show that $\Box f$ is continuous whenever $f : X \to Y$ is. We first show the auxiliary:

Claim A. For every function $f : X \to \Box Y$, f is continuous if and only if, for every connected component C of X:

(i) for every $x, y \in C$, $f(x)(0) = f(y)(0)$, and

(ii) the restriction $f_{|C}$ of f to C is continuous from C to $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], Y)$.

Only if: since C is connected, $f(C)$ is connected. But each $\Box_{x_0} X$ is both open and closed in $\Box X$ by construction, so $f(C) \subseteq \Box_{x_0} X$ for some $x_0 \in X$. By definition of $\Box_{x_0} X$, this means that $f(x)(0) = x_0$ for every $x \in C$, whence *(i)*. On the other hand, since f is continuous, $f_{|C}$ is also continuous from C to $\Box_{x_0} X$ for the x_0 above. Since every subset of $\Box_{x_0} X$ that is closed in $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], Y)$ is also closed in $\Box_{x_0} X$ by definition, *(ii)* holds. (We use the fact that f is continuous if and only if the inverse image of every closed set is closed.)

If: let x_0 be $f(x)(0)$ for some (and therefore all, by *(i)*) $x \in C$. Then $f(C) \subseteq \Box_{x_0} X$. By *(ii)*, and since every closed subset of $\Box_{x_0} X$ is closed in $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], Y)$, $f_{|C}$ is continuous from C to $\Box_{x_0} X$, hence to $\Box X$. For every open O of $\Box X$, $f^{-1}(O)$ is the union of $f_{|C}^{-1}(O)$ when C ranges over the connected components of X, and is therefore open. So f is indeed continuous from X to $\Box Y$. Claim A is proved.

Now let $f : X \to Y$ be continuous; we claim that $\Box f$ is continuous, too. Let C be a connected component of $\Box X$. Since every $\Box_{x_0} X$ is both open and closed in $\Box X$, C is included in some $\Box_{x_0} X$. So for every $\alpha, \beta \in C$, $\Box f(\alpha)(0) = f(\alpha(0)) = f(x_0) = f(\beta(0)) = \Box f(\beta)(0)$, therefore *(i)* holds. Moreover $f_{|C}$ is trivially continuous from C

to $\mathbf{Hom_{CGHaus}}([0,1], Y)$, since $f \circ \_$ is a continuous operation (this is the morphism $\Lambda(f \circ App)$ in **CGHaus**, which is a CCC). So Claim A applies, and $\Box f$ is continuous.

Let us now show that $\mathbf{d} : \Box X \to X$ is continuous. Let F be any closed subset of X, then $\mathbf{d}^{-1}(F) = \{\alpha \in \Box X | \alpha(1) \in F\} = \bigcup_{x_0 \in X} \{\alpha \in \Box_{x_0} X | \alpha(1) \in F\} = \bigcup_{x_0 \in X} \{\alpha \in \mathbf{Hom_{CGHaus}}([0,1], X) | \alpha(0) = x_0 \wedge \alpha(1) \in F\}$. Since the functions from $\mathbf{Hom_{CGHaus}}([0,1], X)$ to X mapping α to $\alpha(0)$ and $\alpha(1)$ respectively are continuous, each $\{\alpha \in \mathbf{Hom_{CGHaus}}([0,1], X) | \alpha(0) = x_0 \wedge \alpha(1) \in F\}$ is closed in $\mathbf{Hom_{CGHaus}}([0,1], X)$; hence in $\Box_{x_0} X$. Since a set is closed in a sum space if and only if its intersection with every summand is closed in the summand, $\mathbf{d}^{-1}(F)$ is closed.

Let us show that $\mathbf{s} : \Box X \to \Box^2 X$ is continuous. Let C be any connected component of $\Box X$. In particular $C \subseteq \Box_{x_0} X$ for some x_0. So for every $\alpha \in C$, $\alpha(0) = x_0$, therefore $\mathbf{s}(\alpha)(0)$ is the map $t' \mapsto \alpha(0t')$, i.e., the constant map $t' \mapsto x_0$. As this is independent of α, *(i)* holds. On the other hand, let F be any closed subset of $\mathbf{Hom_{CGHaus}}([0,1], \Box X)$. Then, letting f_0 be the constant map $t' \mapsto x_0$, $\mathbf{s}_{|C}^{-1}(F) = \mathbf{s}_{|C}^{-1}(F \cap \Box_{f_0} \Box X) = \mathbf{s}_{|C}^{-1}(F \cap \Box_{f_0} \Box_{x_0} X)$ is closed in $\mathbf{Hom_{CGHaus}}([0,1], X)$. Indeed $\Box_{f_0} \Box_{x_0} X$ is closed in $\mathbf{Hom_{CGHaus}}([0,1], \mathbf{Hom_{CGHaus}}([0,1], X))$, and $\mathbf{s}_{|C}$ is continuous from $\mathbf{Hom_{CGHaus}}([0,1], X)$ to $\mathbf{Hom_{CGHaus}}([0,1], \mathbf{Hom_{CGHaus}}([0,1], X))$, as a composition of continuous maps. So $\mathbf{s}_{|C}^{-1}(F)$ is also closed in C, hence *(ii)* holds. By Claim A $\mathbf{s}$ is continuous.

We now claim that $\Box$ is strict monoidal. The terminal object ! in **CGHaus** is any singleton $\{\bullet\}$; $\Box\{\bullet\}$ is the space of all paths from $\bullet$ to $\bullet$ in $\{\bullet\}$, and is therefore also a singleton set. On the other hand, products are slightly harder to deal with. Let $X * Y$ denote the product of X and Y as topological spaces; then $X \times Y$ is the kelleyfication of $X * Y$. We claim that the pair of functions:

$$\begin{aligned} F : \Box(X \times Y) &\to \Box X \times \Box Y \\ \gamma &\mapsto (\pi_1 \circ \gamma, \pi_2 \circ \gamma) \\ G : \Box X \times \Box Y &\to \Box(X \times Y) \\ (\alpha, \beta) &\mapsto (t \mapsto (\alpha(t), \beta(t))) \end{aligned}$$

defines a natural isomorphism between $\Box(X \times Y)$ and $\Box X$ and $\Box Y$. That they are inverse of each other is clear, it remains to show that they are continuous. For F, since $\times$ is a product in **CGHaus**, it is enough to show that $\gamma \mapsto \pi_1 \circ \gamma$ is continuous from $\Box(X \times Y)$ to $\Box X$, and similarly for $\gamma \mapsto \pi_2 \circ \gamma$. Apply Claim A: let C be any connected component of $\Box(X \times Y)$. Since each $\Box_{(x_0,y_0)}(X \times Y)$ is both open and closed in $\Box(X \times Y)$, C is included in some $\Box_{(x_0,y_0)}(X \times Y)$ for some $x_0 \in X$, $y_0 \in Y$. So $\gamma \mapsto \pi_1 \circ \gamma$ maps any $\gamma \in C$ to some path whose value at 0 is x_0, and is therefore independent of γ: *(i)* holds. And *(ii)* is obvious, so F is continuous. For G, this is subtler, and we require to prove the following first:

Claim B. Every connected component C of $X \times Y$ is a subset of some product $A * B$, where A is a connected component of X and B a connected component of Y.

Indeed, every connected component of X, resp. Y, is both open and closed in X, resp. Y. So every product $A * B$ is both open and closed in $X * Y$, when A and B

are connected components. Since the topology of $X \times Y$ is finer than that of $X * Y$, $A * B$ is also both open and closed in $X \times Y$. Let S be the set of pairs (A, B) of connected components such that $C \cap (A * B) \neq \emptyset$. Note that the union of all $A * B$ for $(A, B) \in S$ covers C. Since $C \cap (A * B)$ is both open and closed in S and C is connected, there can be at most one pair A, B of connected components such that $C \cap (A * B) \neq \emptyset$. It follows that $C \subseteq A * B$. Claim B is proved.

To show that G is continuous, apply Claim A. For every connected component C of $\Box X \times \Box Y$, using Claim B, C is included in some product of connected components of $\Box X$ and $\Box Y$ respectively. In particular $C \subseteq \Box_{x_0} X * \Box_{y_0} Y$ for some $x_0 \in X$, $y_0 \in Y$. It follows that for every $(\alpha, \beta) \in C$, $G(\alpha, \beta)(0) = (\alpha(0), \beta(0)) = (x_0, y_0)$ is independent of α and β. So *(i)* holds. Also, *(ii)* holds trivially. Therefore G is continuous.

It remains to check equations (n)–(y), which are easy and left to the reader. $\Box$

We have said that $\Box$ was dual to a cone functor. We make this precise now.

Proposition 54. *There is a strong monad* $(\blacklozenge, \mathfrak{e}, \mathfrak{m})$ *(the* cone *monad) on the category* **CGHaus** *of Kelley spaces such that* $\blacklozenge \dashv \Box$.

Proof. Define $\blacklozenge X$ for every topological space X as $([0,1] \times X)/ \leftrightsquigarrow$, where $\leftrightsquigarrow$ is the smallest equivalence relation such that $(0, x) \leftrightsquigarrow (0, y)$ provided x and y lie in the same connected component of X (we say x and y are *connected* in X). Since $[0, 1]$ is compact, hence locally compact and Hausdorff, the product $[0, 1] \times X$ in **CGHaus** is identical to the ordinary product in **Top**. Also, the quotient $([0, 1] \times X)/ \leftrightsquigarrow$, equipped with the quotient topology, is clearly Hausdorff. Since every Hausdorff quotient of a Kelley space is Kelley ([**36**], VII.8), $\blacklozenge X$ is Kelley. Let us write $|t, x|$ the equivalence class of (t, x).

This defines a functor $\blacklozenge$ from **CGHaus** to **CGHaus** by defining $\blacklozenge f$, for every continuous function $f : X \to Y$, as the function mapping $|t, x|$ to $|t, f(x)|$. This is well-defined because, whenever $(t, x) \leftrightsquigarrow (t', y)$, then either $(t, x) = (t', y)$ and we are done, or $t = t' = 0$, and x and y are connected in X, so $f(x)$ and $f(y)$ are connected in Y (because f is continuous), therefore $(t, f(x)) \leftrightsquigarrow (t', f(y))$. This is continuous: for every open O in $\blacklozenge X$, $(\blacklozenge f)^{-1}(O)$ open in $\blacklozenge X$ if and only if the set A of pairs (t, x) that are $\leftrightsquigarrow$-equivalent to some (t', x') such that $(t', f(x)) \in O'$, where O' is the open of all points $\leftrightsquigarrow$-equivalent to some point of O, is open. However by definition A is exactly the inverse image by $\mathrm{id} \times f$ of O', hence is open.

Define $\mathfrak{e} : X \to \blacklozenge X$ by $\mathfrak{e}(x) \hat{=} |1, x|$. This is clearly well-defined and continuous.

Before we continue, observe that $|t, x|$ and $|t', x'|$ are connected in $\blacklozenge X$ only if x and x' are connected in X. Assume indeed that x and x' are not connected in X: let O be the connected component of x, O' that of x'. Then $([0, 1] \times O)/ \leftrightsquigarrow$ consists of equivalence classes of pairs whose second component is in O (similarly with O' in place of O). Since $O \cap O' = \emptyset$, it follows that $([0, 1] \times O)/ \leftrightsquigarrow \cap ([0, 1] \times O)/ \leftrightsquigarrow = \emptyset$. Since $([0, 1] \times O)/ \leftrightsquigarrow$ and $([0, 1] \times O')/ \leftrightsquigarrow$ are both open in $\blacklozenge X$ and contain respectively $|t, x|$ and $|t', x'|$, it follows that these two points are not connected in $\blacklozenge X$.

Define $\mathfrak{m} : \blacklozenge^2 X \to \blacklozenge X$ by $\mathfrak{m}|t_1, |t_2, x|| \hat{=} |t_1 t_2, x|$. This is well-defined because whenever $(t_1, |t_2, x|) \leftrightsquigarrow (t'_1, |t'_2, x'|)$, then either $t_1 = t'_1$, $t_2 = t'_2$ and $x = x'$, and we are done; or $t_1 = t'_1$, $t_2 = t'_2 = 0$, and x and x' are connected, then $t_1t_2 = t'_1t'_2 = 0$ so $(t_1t_2, x) \leftrightsquigarrow (t'_1t'_2, x')$; or $t_1 = t'_1 = 0$ and $|t_2, x|$ and $|t'_2, x'|$ are connected in $\blacklozenge X$. In the latter case, in particular, x and x' are connected in X. Then $(t_1t_2, x) = (0, x)$, while $(t'_1t'_2, x') = (0, x')$. Since x and x' are connected, $|0, x| = |0, x'|$. To show that $\mathfrak{m}$ is continuous, observe that $(t_1, (t_2, x)) \mapsto (t_1t_2, x)$ is continuous, and use a similar argument as for the continuity of $\blacklozenge f$.

This defines a monad: that $\mathfrak{e}$ and $\mathfrak{m}$ define natural transformations is clear. Then, $\mathfrak{m} \circ \mathfrak{m} = \mathfrak{m} \circ \blacklozenge\mathfrak{m}$ since the left-hand side maps $|t_1, |t_2, |t_3, x|||$ to $|(t_1t_2)t_3, x|$, and the right-hand side maps it to $|t_1(t_2t_3), x|$. Also, $\mathfrak{m} \circ \mathfrak{e}$ maps $|t, x|$ to $|1.t, x| = |t, x|$, therefore equals id, while $\mathfrak{m} \circ \blacklozenge\mathfrak{e}$ maps $|t, x|$ to $|t.1, x| = |t, x|$, therefore also equals id.

This is a strong monad, that is, there is a continuous map $\tau_{X,Y} : X \times \blacklozenge Y \to \blacklozenge(X \times Y)$, natural in X and Y, such that: (a) $\blacklozenge\pi_2 \circ \tau_{\mathbf{1},Y} = \pi_2$; (b) $\tau_{X,Y} \circ (\mathrm{id} \times \mathfrak{e}) = \mathfrak{e}$; (c) letting α map $((x, y), z)$ to $(x, (y, z))$, $\tau_{X,Y\times Z} \circ (\mathrm{id} \circ \tau_{Y,Z}) \circ \alpha = \blacklozenge\alpha \circ \tau_{X\times Y,Z}$; and (d) $\mathfrak{m} \circ \blacklozenge\tau_{X,Y} \circ \tau_{X,\blacklozenge Y} = \tau_{X,Y} \circ (\mathrm{id} \times \mathfrak{m})$.

Indeed, let $\tau_{X,Y}$ map $(x, |t, y|)$ to $|t, (x, y)|$. This is well-defined, since if $|t, y| = |t', y'|$, then either $t = t'$ and $y = y'$ and we are done, or $t = t' = 0$ and y and y' are connected in Y; then (x, y) and (x, y') are connected in $X \times Y$, since the map $y \mapsto (x, y)$ is continuous, so $(t, (x, y)) \leftrightsquigarrow (t', (x, y))$. It is then easily seen to be continuous. It is also clearly natural in X and Y.

Condition (a) means that for the unique $* \in \mathbf{1}$, for every $|t, y| \in \blacklozenge Y$, $|t, \pi_2(*, y)| = |t, y|$; condition (b) means that for every $x \in X$, $y \in Y$, $\tau_{X,Y}(x, |1, y|) = |1, (x, y)|$; condition (c) means that for every $x \in X$, $y \in Y$, $|t, z| \in \blacklozenge Z$, $\tau_{X,Y\times Z}(x, |t, (y, z)|) = |t, (x, (y, z))|$; condition (d) means that for every $x \in X$, $|t_1, |t_2, y|| \in \blacklozenge^2 Y$, $|t_1t_2, (x, y)| = \tau_{X,Y}(x, |t_1t_2, y|)$. They are all obvious.

It now remains to check that $\blacklozenge \dashv \Box$. The counit $\eta : X \to \Box\blacklozenge X$ of the adjunction maps x to $t \mapsto |t, x|$. The unit $\epsilon : \blacklozenge\Box X \to X$ maps $|t, \alpha|$ to $\alpha(t)$.

Check that η is continuous: this requires us to show (Claim A) that for every connected component C of X, *(i)* for every $x, x' \in C$, $|0, x| = |0, x'|$ (by definition), and *(ii)* the map sending every $x \in C$ to $t \mapsto |t, x|$ in $\mathbf{Hom}_{\mathbf{CGHaus}}([0, 1], \blacklozenge X)$ is continuous. Since **CGHaus** is cartesian-closed, the latter is equivalent to the condition that the map sending $(x, t) \in C \times [0, 1]$ to $|t, x| \in \blacklozenge X$ is continuous. But $\blacklozenge X$ is defined using the quotient topology, the largest topology such that the quotient map $(t, x) \mapsto |t, x|$ is continuous.

Now we prove the:

Claim C: $\alpha, \alpha' \in \Box Z$ are connected if and only if $\alpha(0) = \alpha'(0)$.

Indeed, if they are connected, then they are in the same $\Box_{z_0} Z$, since the latter are both open and closed in $\Box Z$; therefore $\alpha(0) = \alpha'(0) = z_0$. Conversely, we show that $\alpha(0) = \alpha'(0)$ implies that α and α' are path-connected: let $h : [0, 1] \to \Box Z$ map $t \in [0, 1/2]$ to $t' \mapsto \alpha((1 - 2t)t')$, and $t \in [1/2, 1]$ to $t' \mapsto \alpha'((2t - 1)t')$. Using Claim A, it is easy to see that h is continuous. Since α and α' are path-connected, they are connected;

Check that ϵ is well-defined. If $(t, \alpha) \leftrightsquigarrow (t', \alpha')$, then either $t = t'$ and $\alpha = \alpha'$ and we are done, or $t = t' = 0$ and α and α' are connected in $\Box X$; then by Claim C $\alpha(0) = \alpha'(0)$, so $\alpha(t) = \alpha'(t')$. Then ϵ is clearly continuous, since application $(t, \alpha) \mapsto \alpha(t)$ is continuous in **CGHaus**.

It remains to check that $\epsilon \circ \blacklozenge\eta = \mathrm{id}$ (for every $|t, x| \in \blacklozenge X$, the left-hand side maps it to $\epsilon(|t, (t' \mapsto |t', x|)|) = |t, x|$) and $\Box\epsilon \circ \eta = \mathrm{id}$ (for every $\alpha \in \Box X$, the left-hand side maps it to $\Box\epsilon(t \mapsto |t, \alpha|) = t \mapsto \alpha(t) = \alpha$). $\Box$

Recalling the discussion of Section 2, this gives a way of interpreting Moggi's computational lambda-calculus in labeled Kelley spaces, together with λ_{S4}, and preserving a duality between the $\Box$ and $\blacklozenge$ forms of talking about computations and values (the adjunction $\blacklozenge \dashv \Box$). Such a cone monad also exists in $\widehat{\Delta}$, but is more complex (See Appendix C.)

It is instructing to see that if X is a space of points, $\Box X$ is a space of paths, then $\Box^2 X$ is a space of singular 2-simplices, and in general $\Box^q X$ will be a space of singular q-simplices.

Let's examine $\Box^2 X$ first. This is a space of paths β, such that each $\beta(t)$, $t \in [0, 1]$ is itself a path, so β is a kind of square, up to deformation. However, β is continuous and $[0, 1]$ is connected, so the range of β is connected as well. But the range of β is a subset of $\Box X$, which is the direct sum of spaces $\Box_{x_0} X$, $x_0 \in X$. In any direct sum of topological spaces, every summand is both open and closed, hence every connected subspace is in fact a subspace of some summand. In our case, this means that the range of β is a subset of some $\Box_{x_0} X$. In other words, $\beta(t)(0) = x_0$ for every t, so the range of β assumes the shape of a *triangle*, up to deformation: see Figure 7.

Note that this phenomenon is entirely due to the strange topology we take on $\Box X$, which separates completely paths α that do not have the same $\alpha(0)$. Had we just taken $\Box X$ to be the set of paths in X with the compact-open topology, $\Box^q X$ would have been a set of cubes, not simplices.

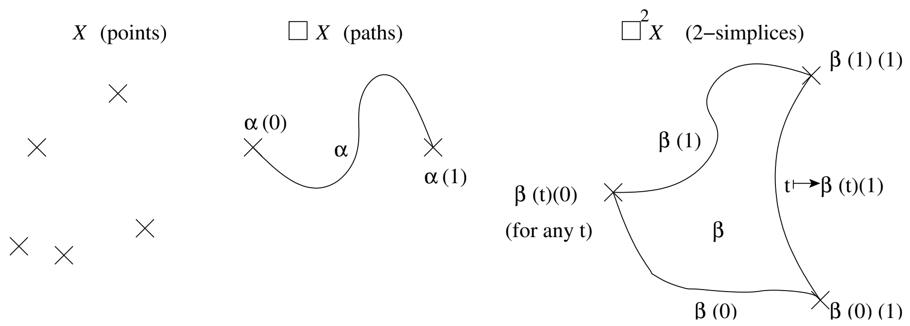

Figure 7: Extended singular simplices

In general, define $\mathbf{XSing}_q(X)$, for $q \geqslant -1$, as the set of all *extended singular q-simplices* in X:

Definition 55 (Extended Singular Simplices, XSing). *For every $q \geqslant -1$, the* extended singular q-simplices *are the continuous maps from $\Delta^+{}_q$ to $\mathcal{F}$, where $\Delta^+{}_q \hat{=} \{(t_0, \ldots, t_q) \mid t_0 \geqslant 0, \ldots, t_q \geqslant 0, t_0 + \ldots + t_q \leqslant 1\}$ is the* standard extended q-simplex.

$\Delta^+{}_{-1}$ is the singleton containing only the empty tuple (). Otherwise, $\Delta^+{}_q$ is a polyhedron whose vertices are $(0, \ldots, 0)$ first, and second the points $e_0, \ldots, e_q$, where $e_i \hat{=} (t_0, \ldots, t_q)$ with $t_i = 1$ and $t_j = 0$ for all $j \neq i$. This is analogous to the more usual notion of *standard q-simplices* Δ_q, for $q \geqslant 0$, which are the sub-polyhedra with vertices $e_0, \ldots, e_q$, namely $\Delta_q \hat{=} \{(t_0, \ldots, t_q) \mid t_0 \geqslant 0, \ldots, t_q \geqslant 0, t_0 + \ldots + t_q = 1\}$. The *singular q-simplices* of X are the continuous maps from Δ_q to X. See Figure 8 for an illustration of what the standard simplices, and standard extended simplices, look like.

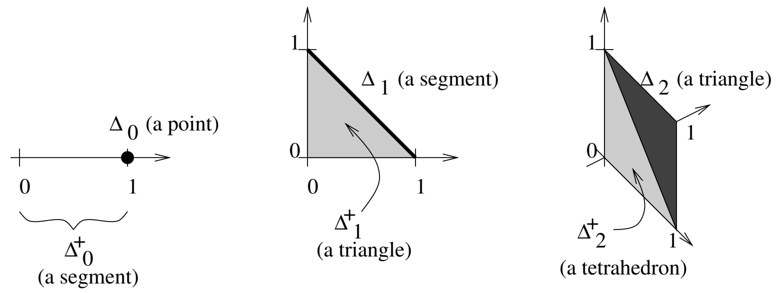

Figure 8: Standard and extended simplices

The topology on $\mathbf{XSing}_q(X)$ is given as follows. When $q = -1$, $\mathbf{XSing}_q(X)$ is isomorphic to X. Otherwise, $\mathbf{XSing}_q(X)$ is viewed as the topological sum of all spaces $\mathbf{XSing}_q^\gamma(X) \hat{=} \{f \in \mathbf{Hom}_{\mathbf{CGHaus}}(\Delta^+{}_q, X) \mid f_{|\Delta_q} = \gamma\}$, when γ ranges over all singular q-simplices of X.

Lemma 56. *$\Box^q X$ is homeomorphic to $\mathbf{XSing}_q(X)$.*

Proof. By induction on $q \geqslant -1$. The base case is by definition, so it remains to show that $\Box\,\mathbf{XSing}_q(X)$ is homeomorphic to $\mathbf{XSing}_{q+1}(X)$.

On the one hand, we map every $\beta \in \mathbf{XSing}_{n+1}(X)$ to $\varphi(\beta) \in \Box\,\mathbf{XSing}_n(X)$, by defining $\varphi(\beta)$ as the function mapping $t \in [0,1]$ to $(t_0, \ldots, t_{n-1}) \in \Delta^+{}_{n-1} \mapsto \beta(t_0, \ldots, t_{n-1}, (1-t).(1 - t_0 - \ldots - t_{n-1}))$. We must show that indeed $\varphi(\beta) \in \Box\,\mathbf{XSing}_n(X)$, and that φ is continuous.

First, β is in $\mathbf{XSing}_{n+1}^v X$, where $v \hat{=} \beta_{|\Delta_n}$. So $\varphi(\beta)(t)$ is $(t_0, \ldots, t_{n-1}) \in \Delta^+{}_{n-1} \mapsto \beta(t_0, \ldots, t_{n-1}, (1-t).(1 - t_0 - \ldots - t_{n-1}))$ is in $\mathbf{XSing}_n^\gamma(X)$ for every $t \in [0,1]$, where γ is the singular $(n-1)$-simplex $(t_0, \ldots, t_{n-1}) \in \Delta_{n-1} \mapsto \beta(t_0, \ldots, t_{n-1}, 0)$. Note that γ is independent of t, and that $\varphi(\beta)(t)$ is continuous in t (recall that abstraction and application are continuous for the compact-open topology, and that the topology of $\mathbf{XSing}_n^\gamma(X)$ is exactly the compact-open topology). So $\varphi(\beta)$ is in $\Box_w(\mathbf{XSing}_n^\gamma(X))$, where $w \hat{=} \varphi(\beta)(0)$, hence it is in $\Box\,\mathbf{XSing}_n(X)$.

Then, observe that φ maps every $\mathbf{XSing}_{n+1}^v(X)$ to $\Box_w\,\mathbf{XSing}_n^\gamma(X)$, where $v \in \mathbf{Hom}_{\mathbf{CGHaus}}(\Delta_n, X)$, γ is $(t_0, \ldots, t_{n-1}) \in \Delta_{n-1} \mapsto v(t_0, \ldots, t_{n-1}, 0)$, and w is $(t_0, \ldots, t_{n-1}) \in \Delta^+{}_{n-1} \mapsto v(t_0, \ldots, t_{n-1}, 1 - t_0 - \ldots - t_{n-1})$, and that these spaces have exactly the compact-open topology. Since abstraction and application are continuous, φ is continuous on each summand $\mathbf{XSing}_{n+1}^v(X)$; so φ is continuous from

$\mathbf{XSing}_{n+1}(X)$ to $\Box\,\mathbf{XSing}_n(X)$.

On the other hand, we define a map ψ from $\Box\,\mathbf{XSing}_n(X)$ to $\mathbf{XSing}_{n+1}(X)$ as follows. For every $\alpha \in \Box\,\mathbf{XSing}_n(X)$, α is in $\Box_w\,\mathbf{XSing}_n(X)$, where $w \hat{=} \alpha(0)$. Since α is continuous from $[0,1]$ to $\mathbf{XSing}_n(X)$, and since $[0,1]$ is connected, the range of α is, too, so it must be included in some $\mathbf{XSing}_n^\gamma(X)$. In particular, $\alpha(t)_{|\Delta_{n-1}} = \gamma$ for every $t \in [0,1]$. Then let $\psi(\alpha)$ map $(t_0,\ldots,t_n) \in {\Delta^+}_n$ to:

$$\begin{cases} \alpha\left(\frac{1-t_0-\ldots-t_{n-1}-t_n}{1-t_0-\ldots-t_{n-1}}\right)(t_0,\ldots,t_{n-1}) & \text{if } t_0+\ldots+t_{n-1} \neq 1 \\ \gamma(t_0,\ldots,t_{n-1}) & \text{if } t_0+\ldots+t_{n-1} = 1 \end{cases}$$

$\psi(\alpha)$ is in $\mathbf{XSing}_{n+1}^v(X)$, where $v \hat{=} \psi(\alpha)_{|\Delta_n}$. Indeed, we just have to check that $\psi(\alpha)$ is a continuous function. Clearly, $\psi(\alpha)$ is continuous over the set of all $(t_0,\ldots,t_n) \in {\Delta^+}_n$ such that $t_0+\ldots+t_{n-1} \neq 1$ (first case of the definition), as well as over the set of those such that $t_0+\ldots+t_{n-1} = 1$ (second case). Then, when $(t_0,\ldots,t_n) \in {\Delta^+}_n$ tends to $(t_0^0,\ldots,t_{n-1}^0,t_n^0)$ in ${\Delta^+}_n$ such that $t_0^0+\ldots+t_{n-1}^0 = 1$, then $\alpha((1-t_0-\ldots-t_{n-1}-t_n)/(1-t_0-\ldots-t_{n-1}))(t_0,\ldots,t_{n-1})$ tends to $\gamma(t_0^0,\ldots,t_{n-1}^0)$, although $(1-t_0-\ldots-t_{n-1}-t_n)/(1-t_0-\ldots-t_{n-1})$ may not tend to any limit. This is because α is continuous, application is continuous and $\alpha(t)(t_0^0,\ldots,t_{n-1}^0) = \gamma(t_0^0,\ldots,t_{n-1}^0)$ whatever the value of t is.

Then, ψ is continuous from $\Box_w\,\mathbf{XSing}_n^\gamma(X)$ to $\mathbf{XSing}_{n+1}^v(X)$, for every γ in $\mathbf{Hom}_{\mathbf{CGHaus}}(\Delta_{n-1},X)$, every w in $\mathbf{XSing}_n^\gamma(X)$, and where v is $(t_0,\ldots,t_n) \in \Delta_n \mapsto w(t_0,\ldots,t_{n-1})$, for similar reasons as φ. So ψ is a continuous map from $\Box\,\mathbf{XSing}_n(X)$ to $\mathbf{XSing}_{n+1}(X)$.

It remains to show that φ and ψ are mutually inverse:

$$\begin{aligned}
&\varphi(\psi(\alpha)) \\
&= \varphi\left((t_0,\ldots,t_n) \in {\Delta^+}_n \mapsto \begin{cases} \alpha\left(\frac{1-t_0-\ldots-t_{n-1}-t_n}{1-t_0-\ldots-t_{n-1}}\right)(t_0,\ldots,t_{n-1}) \\ \qquad\text{if } t_0+\ldots+t_{n-1} \neq 1 \\ \gamma(t_0,\ldots,t_{n-1}) \quad \text{if } t_0+\ldots+t_{n-1} = 1 \end{cases}\right) \\
&= t \in [0,1] \mapsto (t_0,\ldots,t_{n-1}) \in {\Delta^+}_{n-1} \mapsto \\
&\quad \begin{cases} \alpha\left(\frac{1-t_0-\ldots-t_{n-1}-(1-t).(1-t_0-\ldots-t_{n-1})}{1-t_0-\ldots-t_{n-1}}\right)(t_0,\ldots,t_{n-1}) & \text{if } t_0+\ldots+t_{n-1} \neq 1 \\ \gamma(t_0,\ldots,t_{n-1}) & \text{if } t_0+\ldots+t_{n-1} = 1 \end{cases} \\
&= t \in [0,1] \mapsto \big((t_0,\ldots,t_{n-1}) \in {\Delta^+}_{n-1} \mapsto \alpha(t)(t_0,\ldots,t_{n-1})\big) = \alpha
\end{aligned}$$

$$\begin{aligned}
\psi(\varphi(\beta)) &= \psi(t \in [0,1] \mapsto (t_0,\ldots,t_{n-1}) \in {\Delta^+}_{n-1} \mapsto \\
&\qquad\qquad \beta(t_0,\ldots,t_{n-1},(1-t).(1-t_0-\ldots-t_{n-1})))) \\
&= (t_0,\ldots,t_n) \in {\Delta^+}_n \mapsto \\
&\qquad \begin{cases} \beta\left(t_0,\ldots,t_{n-1},\left(1-\frac{1-t_0-\ldots-t_{n-1}-t_n}{1-t_0-\ldots-t_{n-1}}\right).(1-t_0-\ldots-t_{n-1}),\right) \\ \qquad \text{if } t_0+\ldots+t_{n-1} \neq 1 \\ \beta(t_0,\ldots,t_{n-1},0) \quad \text{if } t_0+\ldots+t_{n-1} = 1 \text{ (hence } t_n = 0) \end{cases} \\
&= (t_0,\ldots,t_n) \in {\Delta^+}_n \mapsto \beta(t_0,\ldots,t_{n-1},t_n) = \beta
\end{aligned}$$

□

Note that (extended) simplices over a space of functions $X \to Y$ also have an elegant geometric interpretation. While $X \to Y$ is a set of continuous functions, $\Box(X \to Y)$ is a set of continuous paths from functions f to functions g in $X \to Y$, so $\Box(X \to Y)$ is a set of *homotopies* between continuous functions from X to Y. The elements of $\Box^q(X \to Y)$, $q \geqslant 1$, are then known as *higher-order homotopies*: $\Box^2(X \to Y)$ is the set of homotopies between homotopies, etc. This is a classical construction in algebraic topology [**39**].

In terms of proof theory, there is a translation of intuitionistic proofs to S4 proofs which replaces every base type A by $\Box A$ and every implication by a corresponding boxed implication. At the level of proof terms, this yields the SKInT calculus of [**19**], which interprets (slightly more than) Plotkin's call-by-value λ-calculus [**45**]. The present constructions give rise to a model in terms of paths (elements of base types) and homotopies (implications) for SKInT. This is left to the reader.

5.1.2. Models in Categories of Orders, Cpos, and Categories

More cogent to computer science are models of the λ-calculus based on complete partial orders. Here, too, we may define strict monoidal comonads as follows. First recall that **Ord**, the category whose objects are partial orders and whose morphisms are monotonic functions, is a CCC. Similarly, **Cat**, the category of small categories, is a CCC. The category **Cpo** of *complete partial orders* (cpos) has cpos as objects and continuous functions as morphisms. Recall that a cpo is any partial order in which every infinite increasing chain $x_0 \leqslant x_1 \leqslant \ldots \leqslant x_i \leqslant \ldots$ has a least upper bound. (We don't require our cpos to be *pointed*, i.e., to have a least element.) A function is *continuous* provided it preserves all least upper bounds; in particular, a continuous function is monotonic. Again, **Cpo** is a CCC. A variant is the category **DCpo** of *directed cpos*, where it is instead required that all non-empty directed subsets have a least upper bound; a *directed* subset E is one where any two elements in E have a least upper bound in E. Continuous functions are then required to preserve least upper bounds of all directed sets. Again, **DCpo** is a CCC.

Definition 57 (□ Comonad in Ord, Cpo, DCpo). *For every partial order* $(X, \leqslant)$*, let* $\Box X$ *be the set of all pairs* (x_0, x_1) *of elements of* X *such that* $x_0 \leqslant x_1$*, ordered by* $(x_0, x_1) \leqslant (y_0, y_1)$ *if and only if* $x_0 = y_0$ *(not* $x_0 \leqslant y_0$*) and* $x_1 \leqslant y_1$*. For every monotonic function* $f : X \to Y$ *(resp. continuous), let* $\Box f$ *map* (x_0, x_1) *to* $(f(x_0), f(x_1))$*.*

The counit $\mathbf{d} : \Box X \to X$ *maps* (x_0, x_1) *to* x_1*.*

The comultiplication $\mathbf{s} : \Box X \to \Box^2 X$ *maps* (x_0, x_1) *to* $((x_0, x_0), (x_0, x_1))$*.*

It is easily checked that this defines a strict monoidal comonad on **Ord**, **Cpo**, **DCpo**. As for the topological case, we may give a synthetic description of $\Box^q X$: this is isomorphic to the partial order (resp. cpo, resp. dcpo) of all chains $x_{-1} \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_q$ of elements of X, ordered by:

$$(x_{-1}, x_0, \ldots, x_q) \leqslant (x'_{-1}, x'_0, \ldots, x'_q)$$

if and only if $x_{-1} = x'_{-1}$, $x_0 = x'_0, \ldots, x_{q-1} = x'_{q-1}$, and $x_q \leqslant x'_q$. Just like iterating $\Box$ in the topological case allowed us to retrieve a form of of singular simplex functor, we retrieve a form of nerve functor.

In passing, we invite the reader to check that there is also a cone monad $\blacklozenge$ in **Ord**, **Cpo** and **DCpo**: $\blacklozenge X$ is X with a new bottom element added below every connected component of X. (Connected components are the equivalence classes of the symmetric closure of $\leqslant$.) The unit $X \to \blacklozenge X$ is the natural inclusion of orders. The multiplication $\blacklozenge^2 X \to \blacklozenge X$ squashes the two additional bottoms of $\blacklozenge^2 X$ to the one coming from $\blacklozenge X$. Again, this is a strong monad left adjoint to $\Box$.

We leave it to the reader to check that similar constructions work in **Cat**: for every small category $\mathcal{C}$, let $\Box\mathcal{C}$ be the category of all morphisms of $\mathcal{C}$; morphisms from $X \to X_0$ to $X \to X_1$ are all commuting triangles:

$$\begin{tikzcd}
X \arrow[d] \arrow[dr] & \\
X_0 \arrow[r] & X_1
\end{tikzcd}$$

In short, $\Box\mathcal{C}$ is the coproduct of all coslices over $\mathcal{C}$. The counit is given by: $\mathbf{d}$ is the functor mapping $X \to X_0$ to X_0, and the diagram above to the morphism $X_0 \to X_1$ in $\mathcal{C}$. Comultiplication maps every object $X \to X_0$ in $\mathcal{C}$ to the obvious commuting triangle $X \genfrac{}{}{0pt}{}{\to}{\to} X_0$, and morphisms as given by the triangle above to commuting tetrahedra:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] & X_1 \\
X_0 \arrow[r, "\mathrm{id}"'] \arrow[ur] & X_0 \arrow[u]
\end{tikzcd}$$

5.2. Interpreting S4 Proofs into CCCs with Monoidal Comonads

Fix an arbitrary strict CS4 category $\mathcal{C}$, calling its strict monoidal comonad $(\Box, \mathbf{d}, \mathbf{s})$. Our prime example is $\widehat{\Delta}$, but we do not restrict to it here. We reuse the CCC notations of Section 4 and the strict monoidal comonad notations of Section 5.1, together with equations (a)–(m) and (n)–(y).

Extend the set-theoretic interpretation of λ-terms to an interpretation of formulas as objects in $\mathcal{C}$, and of terms as morphisms in $\mathcal{C}$; this interpretation is shown in Figure 9. This is parameterized by an *environment* ρ mapping each base type $A \in \Sigma$ to some object $\rho(A)$. Our notations match standard meaning functions in denotational semantics.

We let $X_1 \times \ldots \times X_n \hat{=} (\ldots(\mathbf{1} \times X_1) \times \ldots \times X_{n-1}) \times X_n$, and $\langle f_1, \ldots, f_n\rangle \hat{=} \langle\langle \ldots \langle !, f_1\rangle \ldots, f_{n-1}\rangle, f_n\rangle$. We actually make an abuse of language by considering that this is an interpretation of typed λ_{S4}-terms instead of of typing derivations.

If Γ is $x_1 : F_1, \ldots, x_n : F_n$, we also let $\mathcal{C}\,[\![\Gamma]\!]\,\rho$ be the product $\mathcal{C}\,[\![F_1]\!]\,\rho \times \ldots \times \mathcal{C}\,[\![F_n]\!]\,\rho$.

$$\mathcal{C}\llbracket A\rrbracket\rho\hat{=}\rho(A) \quad \mathcal{C}\llbracket F\supset G\rrbracket\rho\hat{=}\mathbf{Hom}_{\mathcal{C}}(\mathcal{C}\llbracket F\rrbracket\rho,\mathcal{C}\llbracket G\rrbracket\rho) \quad \mathcal{C}\llbracket\Box F\rrbracket\rho\hat{=}\Box\mathcal{C}\llbracket F\rrbracket\rho$$

$$\begin{aligned}
\mathcal{C}\llbracket\Gamma\vdash x_i:F_i\rrbracket\rho &\hat{=} \pi_2\overbrace{\circ\pi_1\circ\ldots\circ\pi_1}^{n-i} \quad \text{where } \Gamma\hat{=}x_1:F_1,\ldots,x_n:F_n\\
\mathcal{C}\llbracket\Gamma\vdash MN:G\rrbracket\rho &\hat{=} App\circ\langle\mathcal{C}\llbracket\Gamma\vdash M:F\supset G\rrbracket\rho,\mathcal{C}\llbracket\Gamma\vdash N:F\rrbracket\rho\rangle\\
\mathcal{C}\llbracket\Gamma\vdash\lambda x\cdot M:F\supset G\rrbracket\rho &\hat{=} \Lambda(\mathcal{C}\llbracket\Gamma,x:F\vdash M:G\rrbracket\rho)\\
\mathcal{C}\llbracket\Gamma\vdash dM:F\rrbracket\rho &\hat{=} \mathbf{d}\circ\mathcal{C}\llbracket\Gamma\vdash M:\Box F\rrbracket\rho\\
\mathcal{C}\left\llbracket\Gamma\vdash\boxed{M}\cdot\theta:\Box G\right\rrbracket\rho &\hat{=} \Box\mathcal{C}\llbracket\Theta\vdash M:G\rrbracket\rho\circ\mathbf{s}\\
&\qquad\circ\langle\mathcal{C}\llbracket\Gamma\vdash N_1:\Box F_1\rrbracket\rho,\ldots,\mathcal{C}\llbracket\Gamma\vdash N_n:\Box F_n\rrbracket\rho\rangle\\
&\quad\text{where } \Theta\hat{=}x_1:\Box F_1,\ldots,x_n:\Box F_n,\\
&\qquad\theta\hat{=}\{x_1:=N_1,\ldots,x_n:=N_n\}
\end{aligned}$$

Figure 9: Interpreting S4 proof terms

Lemma 58 (Soundness). *The interpretation of Figure 9 is sound in every strict CS4 category $\mathcal{C}$: if $\Gamma\vdash M:F$ is derivable, then $\mathcal{C}\llbracket M\rrbracket\rho$ is a morphism from $\mathcal{C}\llbracket\Gamma\rrbracket\rho$ to $\mathcal{C}\llbracket F\rrbracket\rho$; and if $M\approx N$ then $\mathcal{C}\llbracket M\rrbracket\rho=\mathcal{C}\llbracket N\rrbracket\rho$.*

Proof. The typing part is immediate. For the equality part, standard arguments [**10**] show that:

$$\begin{aligned}
&\mathcal{C}\llbracket\Gamma\vdash M\{x_1:=N_1,\ldots,x_n:=N_n\}:F\rrbracket\rho\\
=\ &\mathcal{C}\llbracket x_1:F_1,\ldots,x_n:F_n\vdash M:F\rrbracket\rho\\
&\circ\langle\mathcal{C}\llbracket\Gamma\vdash N_1:F_1\rrbracket\rho,\ldots,\mathcal{C}\llbracket\Gamma\vdash N_n:F_n\rrbracket\rho\rangle
\end{aligned}\tag{18}$$

where the indicated sequents are derivable; and that:

$$\mathcal{C}\llbracket\Gamma,x:F\vdash M:G\rrbracket\rho=\mathcal{C}\llbracket\Gamma\vdash M:G\rrbracket\rho\circ\pi_1$$

if x is not free in M. We check that if $M\to N$ then $\mathcal{C}\llbracket M\rrbracket\rho=\mathcal{C}\llbracket N\rrbracket\rho$, which entails the second claim. See Appendix D. ☐

If we are allowed to vary the strict CS4 category $\mathcal{C}$, then there are converses to Lemma 58. The idea is that we can always define a syntactic category $\mathcal{C}$ as follows:

Definition 59 ($\mathcal{S4}_\Sigma$ Category). *Let $\mathcal{S4}_\Sigma$ be the category whose objects are contexts mapping variables to types built on the set Σ of base types, and whose morphisms are:*

$$\Gamma\hat{=}x_1:F_1,\ldots,x_n:F_n \xrightarrow{\theta\hat{=}\{y_1:=M_1,\ldots,y_m:=M_m\}} \Theta\hat{=}y_1:G_1,\ldots,y_m:G_m$$

where θ is a substitution such that $\Gamma\vdash M_j:G_j$ for every j, $1\leqslant j\leqslant m$, modulo $\approx$.

The identity on Γ is the identity substitution $\mathrm{id}_\Gamma\hat{=}\{x_1:=x_1,\ldots,x_n:=x_n\}$*, and composition $\theta\circ\theta'$ is substitution concatenation $\theta\cdot\theta'$.*

This is a CCC with a strict monoidal comonad. The terminal object $\mathbf{1}$ is the empty context, and the unique morphism $\Gamma\xrightarrow{!}\mathbf{1}$ is the empty substitution. To define products, notice that contexts are isomorphic up to renaming of variables. In other words, $x_1:F_1,\ldots,x_n:F_n$ is isomorphic to $x_1':F_1,\ldots,x_n':F_n$. This allows us to only define $\Gamma \times \Gamma'$ when Γ and Γ' have disjoint domains. Then $\Gamma \times \Gamma'$ is the concatenation Γ, Γ' of contexts, and for any $\Theta \xrightarrow{\theta} \Gamma$ and $\Theta \xrightarrow{\theta} \Gamma'$, $\langle \theta, \theta' \rangle$ is the morphism (θ, θ'). Projections are restrictions:

$$\Gamma \times \Gamma' \xrightarrow{\pi_1 \hat{=} \{x_1 := x_1, \ldots, x_n := x_n\}} \Gamma$$

$$\Gamma \times \Gamma' \xrightarrow{\pi_2 \hat{=} \{x'_1 := x'_1, \ldots, x'_{n'} := x'_{n'}\}} \Gamma'$$

Given that $\Gamma = x_1 : F_1, \ldots, x_n : F_n$ and $\Gamma' = x'_1 : F'_1, \ldots, x'_{n'} : F'_{n'}$, the internal hom object $\mathbf{Hom}_{\mathcal{S4}_\Sigma}(\Gamma, \Gamma')$ is the context $z_1 : F_1 \supset \ldots \supset F_n \supset F'_1, \ldots, z_{n'} : F_1 \supset \ldots \supset F_n \supset F'_{n'}$. Application $\mathbf{Hom}_{\mathcal{S4}_\Sigma}(\Gamma, \Gamma') \times \Gamma \xrightarrow{App} \Gamma'$ is built from syntactic application, as $\{x'_1 := z_1 x_1 \ldots x_n, \ldots, x'_{n'} := z_{n'} x_1 \ldots x_n\}$, while abstraction is built from λ-abstraction as follows. For every $\Theta \times \Gamma \xrightarrow{\theta} \Gamma'$, where Γ and Γ' are as above, and $\Theta \hat{=} y_1 : G_1, \ldots, y_m : G_m$,

$$\Theta \xrightarrow{\begin{array}{l} \{\quad z_1 := \lambda x_1, \ldots, x_n \cdot x'_1 \theta, \\ \quad\ \ldots, \\ \quad\ z_{n'} := \lambda x_1, \ldots, x_n \cdot x'_{n'} \theta \quad \} \end{array}} \mathbf{Hom}_{\mathcal{S4}_\Sigma}(\Gamma, \Gamma')$$

This only uses the non-modal part of S4, and in particular only the computation rules (β) and (η).

The strict monoidal comonad $(\Box, \mathbf{d}, \mathbf{s})$ on $\mathcal{S4}_\Sigma$ is defined using the S4 $\Box$ modality: on objects, $\Box(x_1 : F_1, \ldots, x_n : F_n)$ is defined as $x_1 : \Box F_1, \ldots, x_n : \Box F_n$; on morphisms, for any θ as given in Definition 59, $\Box\theta$ is:

$$\Box\Gamma \xrightarrow{\Box\theta \hat{=} \{y_1 := \Box M_1, \ldots, y_m := \Box M_m\}} \Box\Theta$$

where $\Box M$ is $\boxed{M\{x_1 := dx_1, \ldots, x_n := dx_n\}}$ for any M such that $\Gamma \vdash M : G$ is derivable. The counit $\mathbf{d}$ is:

$$\Box\Gamma \xrightarrow{\mathbf{d} \hat{=} \{x_1 := dx_1, \ldots, x_n := dx_n\}} \Gamma$$

while comultiplication is:

$$\Box\Gamma \xrightarrow{\mathbf{s} \hat{=} \{x_1 := sx_1, \ldots, x_n := sx_n\}} \Box^2\Gamma$$

Recall that sM is $\boxed{x} \cdot \{x := M\}$.

It trivially follows:

Proposition 60 (Existential Completeness). *Let ρ map every base type $A \in \Sigma$ to the context $z : A$. If there is a morphism from $\mathcal{S4}_\Sigma [\![\Gamma]\!] \rho$ to $\mathcal{S4}_\Sigma [\![F]\!] \rho$ in $\mathcal{S4}_\Sigma$ then F is provable from Γ, i.e., there is a λ_{S4}-term M such that $\Gamma \vdash M : F$ is derivable.*

Proposition 61 (Evaluation Functor). *For every strict CS4 category $\mathcal{C}$, and every $\rho : \Sigma \to \mathcal{C}$, $\mathcal{C} [\![\_]\!] \rho$ extends ρ to a representation of strict CS4 categories from $\mathcal{S4}_\Sigma$ to $\mathcal{C}$.*

Proof. For every morphism

$$\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n \xrightarrow{\theta \hat{=} \{y_1 := M_1, \ldots, y_m := M_m\}} y_1 : G_1, \ldots, y_m : G_m$$

define $\mathcal{C}\llbracket\theta\rrbracket\rho$ as $\langle\mathcal{C}\llbracket\Gamma\vdash M_1 : G_1\rrbracket\rho,\ldots,\mathcal{C}\llbracket\Gamma\vdash M_m : G_m\rrbracket\rho\rangle$. This is functorial: indeed $\mathcal{C}\llbracket\mathrm{id}_\Gamma\rrbracket\rho = \langle\pi_2\circ\pi_1^{m-1},\ldots,\pi_2\circ\pi_1,\pi_2\rangle = \mathrm{id}$, and $\mathcal{C}\llbracket\_\rrbracket\rho$ preserves composition by (18). This preserves cartesian products by construction.

This preserves $\Box$. Indeed,

$$\begin{aligned}
&\mathcal{C}\llbracket\Box\Gamma\vdash\Box M : \Box F\rrbracket\rho\\
&= \mathcal{C}\left\llbracket\Box\Gamma\vdash\boxed{M\{x_1 := dx_1,\ldots,x_n := dx_n\}} : \Box F\right\rrbracket\rho\\
&= \Box\mathcal{C}\llbracket\Gamma\vdash M\{x_1 := dx_1,\ldots,x_n := dx_n\} : F\rrbracket\rho\\
&\qquad\circ\mathbf{s}\circ\langle\pi_2\circ\pi_1^{m-1},\ldots,\pi_2\circ\pi_1,\mathbf{s}\circ\pi_2\rangle\\
&= \Box\mathcal{C}\llbracket\Gamma\vdash M\{x_1 := dx_1,\ldots,x_n := dx_n\} : F\rrbracket\rho\circ\mathbf{s}\\
&= \Box\left(\mathcal{C}\llbracket\Gamma\vdash M : F\rrbracket\rho\circ\langle\mathbf{d}\circ\pi_2\circ\pi_1^{m-1},\ldots,\mathbf{d}\circ\pi_2\circ\pi_1,\mathbf{d}\circ\pi_2\rangle\right)\circ\mathbf{s}\\
&= \Box(\mathcal{C}\llbracket\Gamma\vdash M : F\rrbracket\rho\circ\mathbf{d})\circ\mathbf{s}\\
&= \Box\mathcal{C}\llbracket\Gamma\vdash M : F\rrbracket\rho\circ\Box\mathbf{d}\circ\mathbf{s} = \Box\mathcal{C}\llbracket\Gamma\vdash M : F\rrbracket\rho\quad\text{(by (t))}
\end{aligned}$$

So:

$$\begin{aligned}
\mathcal{C}\llbracket\Box\theta\rrbracket\rho &= \langle\mathcal{C}\llbracket\Box\Gamma\vdash\Box M_1 : \Box G_1\rrbracket\rho,\ldots,\mathcal{C}\llbracket\Box\Gamma\vdash\Box M_m\rrbracket\rho\rangle\\
&= \langle\Box\mathcal{C}\llbracket\Gamma\vdash M_1 : G_1\rrbracket\rho,\ldots,\Box\mathcal{C}\llbracket\Gamma\vdash M_m : G_m\rrbracket\rho\rangle\\
&= \Box\mathcal{C}\llbracket\theta\rrbracket\rho\quad\text{(by (w))}
\end{aligned}$$

$\mathcal{C}\llbracket\_\rrbracket\rho$ preserves $\mathbf{d}$:

$$\begin{aligned}
\mathcal{C}\llbracket\mathbf{d}\rrbracket\rho &= \langle\mathcal{C}\llbracket\Box\Gamma\vdash dx_1 : F_1\rrbracket\rho,\ldots,\mathcal{C}\llbracket\Box\Gamma\vdash dx_n : F_n\rrbracket\rho\rangle\\
&= \langle\mathbf{d}\circ\pi_2\circ\pi_1^{n-1},\ldots,\mathbf{d}\circ\pi_2\circ\pi_1,\mathbf{d}\circ\pi_2\rangle\\
&= \mathbf{d}\circ\langle\pi_2\circ\pi_1^{n-1},\ldots,\pi_2\circ\pi_1,\pi_2\rangle = \mathbf{d}
\end{aligned}$$

$\mathcal{C}\llbracket\_\rrbracket\rho$ preserves $\mathbf{s}$. Indeed,

$$\begin{aligned}
\mathcal{C}\left\llbracket\Gamma\vdash sM : \Box^2F\right\rrbracket\rho &= \mathcal{C}\left\llbracket\Gamma\vdash\boxed{x}\cdot\{x := M\} : \Box^2F\right\rrbracket\rho\\
&= \Box\mathcal{C}\llbracket x : \Box F\vdash x : \Box F\rrbracket\rho\circ\mathbf{s}\circ\mathcal{C}\llbracket\Gamma\vdash M : \Box F\rrbracket\rho\\
&= \Box\mathrm{id}\circ\mathbf{s}\circ\mathcal{C}\llbracket\Gamma\vdash M : \Box F\rrbracket\rho\\
&= \mathbf{s}\circ\mathcal{C}\llbracket\Gamma\vdash M : \Box F\rrbracket\rho
\end{aligned}$$

So:

$$\begin{aligned}
\mathcal{C}\llbracket\mathbf{s}\rrbracket\rho &= \langle\mathbf{s}\circ\pi_2\circ\pi_1^{n-1},\ldots,\mathbf{s}\circ\pi_2\circ\pi_1,\mathbf{s}\circ\pi_2\rangle\\
&= \mathbf{s}\circ\langle\pi_2\circ\pi_1^{n-1},\ldots,\pi_2\circ\pi_1,\pi_2\rangle = \mathbf{s}
\end{aligned}$$

The functor $\mathcal{C}\llbracket\_\rrbracket\rho$ also preserves internal homs, application *App* and abstraction Λ. This is standard, tedious and uninstructive, hence omitted. $\Box$

Proposition 62 (Free Strict CS4 Category). *$\mathcal{S}4_\Sigma$ is the free strict CS4 category on Σ.*

More precisely, for every set Σ of base types, seen as a discrete category, let $\subseteq$ denote the natural inclusion functor of Σ into $\mathcal{S}4_\Sigma$. Then for every strict CS4 category $\mathcal{C}$, for every functor $\rho : \Sigma\to\mathcal{C}$, there is a unique functor Φ that makes the *following diagram commute:*

$$
\begin{tikzcd}
\Sigma \arrow[r, "\subseteq"] \arrow[d, "\rho"'] & \mathcal{S4}_\Sigma \arrow[dl, "\Phi"] \\
\mathcal{C} &
\end{tikzcd}
$$

Furthermore, Φ is exactly the $\mathcal{C}\,[\![\_]\!]\,\rho$ functor as defined in Figure 9.

Proof. Uniqueness: assume Φ exists, we shall show that it is uniquely determined. On objects, Φ must map every formula F to $\mathcal{C}\,[\![F]\!]\,\rho$, and in general every context Γ to $\mathcal{C}\,[\![\Gamma]\!]\,\rho$. On morphisms, since Φ must preserve products, Φ is uniquely determined by the images of morphisms in $\mathcal{S4}_\Sigma$ of the form

$$\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n \xrightarrow{\{y_1 := M_1, \ldots, y_m := M_m\}} \Theta \hat{=} y_1 : G_1, \ldots, y_m : G_m$$

with $m = 1$. In this case, equate the morphism with the judgment $\Gamma \vdash M_1 : G_1$. Then Φ is uniquely determined by its values on typed λ_{S4}-terms. (For readability, we make an abuse of language by equating terms with their typing derivations.)

Since Φ must preserve $\Box$ and $\mathbf{s}$, we must have:

$$\Phi\left(\boxed{M\{x_1 := dx_1, \ldots, x_n := dx_n\}}\right) = \Box\Phi(M)$$
$$\Phi\left(\boxed{x} \cdot \{x := M\}\right) = \mathbf{s} \circ \Phi(M)$$

Since, using ($\boxed{\Box}$), (d), and possibly (gc):

$$
\begin{aligned}
&\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} \\
\approx &\left(\boxed{M\{x_1 := dx_1, \ldots, x_n := dx_n\}}\right) \{x_1 := \boxed{y_1} \cdot \{y_1 := N_1\}, \ldots, \\
&\qquad x_n := \boxed{y_n} \cdot \{y_n := N_n\}\}
\end{aligned}
$$

it follows that:

$$\Phi\left(\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\}\right) = \Box\Phi(M) \circ \mathbf{s} \circ \langle \Phi(N_1), \ldots, \Phi(N_n) \rangle$$

Similarly, since Φ must preserve $\mathbf{d}$, we must have $\Phi(dM) = \mathbf{d} \circ \Phi(M)$. We recognize the clauses for $\mathcal{C}\,[\![\_]\!]\,\rho$ for boxes and d terms given in Figure 9. The case of internal homs, application and abstraction are equally easy and standard, whence Φ must be $\mathcal{C}\,[\![\_]\!]\,\rho$.

Existence: taking $\Phi \hat{=} \mathcal{C}\,[\![\_]\!]\,\rho$, this is by Lemma 58 and Proposition 61. $\Box$

Corollary 63. *Let $\subseteq$ denote the canonical inclusion $\Sigma \subseteq \mathcal{S4}_\Sigma$, mapping each base type A to A, seen as a formula. Then $\mathcal{S4}_\Sigma\,[\![\Gamma \vdash M : F]\!]\,(\subseteq) \approx M$.*

Proof. Apply Proposition 62 with $\mathcal{C} \hat{=} \mathcal{S4}_\Sigma$, $\rho \hat{=} (\subseteq)$. $\Box$

Corollary 63 immediately implies:

Proposition 64 (Equational Completeness). *Let M, N be two λ_{S4}-terms such that $\Gamma \vdash M : F$ and $\Gamma \vdash N : F$ are derivable.*

If $\mathcal{S4}_\Sigma\,[\![\Gamma \vdash M : F]\!]\,(\subseteq) = \mathcal{S4}_\Sigma\,[\![\Gamma \vdash N : F]\!]\,(\subseteq)$, then $M \approx N$.

While $\mathcal{S4}_\Sigma$ is characterized as the free strict CS4 category, we end this section by elucidating the construction of the augmented simplicial set $\mathcal{S4}[F]$ of Definition 43 from a categorical point of view. First, we note:

Lemma 65. *There is a functor $\mathcal{S4}[\_]$ mapping every formula F to $\mathcal{S4}[F]$, and more generally every context $\Gamma\hat{=}x_1 : F_1, \ldots, x_n : F_n$ to $\mathcal{S4}[\Gamma]\hat{=}\mathcal{S4}[F_1] \times \ldots \times \mathcal{S4}[F_n]$, and every morphism*

$$\Gamma\hat{=}x_1 : F_1, \ldots, x_n : F_n \xrightarrow{\theta\hat{=}\{y_1 := M_1, \ldots, y_m := M_m\}} \Theta\hat{=}y_1 : G_1, \ldots, y_m : G_m$$

to the morphism $\mathcal{S4}[\theta]$ in $\widehat{\Delta}$ which, as an augmented simplicial map, sends $(N_1, \ldots, N_n) \in \mathcal{S4}[\Gamma]_q$ to $(\Box^{q+1} M_1\varphi, \ldots, \Box^{q+1} M_m\varphi)$, where $\varphi\hat{=}\{x_1 := N_1, \ldots, x_n := N_n\}$.

Moreover, $\mathcal{S4}[\_]$ is faithful, preserves all finite products and the given comonads in the source and target categories.

Proof. That it is a functor follows from equations (n) and (o). It clearly preserves all finite products and maps the syntactic comonad $(\Box, \mathbf{d}, \mathbf{s})$ to the (dual) décalage comonad $(\Box, \mathbf{d}, \mathbf{s})$ in $\widehat{\Delta}$, as an easy check shows. Finally, it is faithful: in the definition of $\mathcal{S4}[\theta]$ above, we retrieve θ uniquely from $\mathcal{S4}[\theta]$ by looking at the image of the tuple $(x_1, \ldots, x_n)$ by $\mathcal{S4}[\theta]_{-1}$. $\Box$

We can give an even more abstract description of $\mathcal{S4}[\_]$ as follows, which is essentially a way of generalizing the familiar hom-set functor $\mathrm{Hom}_{\mathcal{C}}(\_, \_)$ to the augmented simplicial case. In this way, we shall see that it is related to the standard resolution of any comonad ([**36**], VII.6):

Definition 66 (Resolution Functor Res**).** *Let $(\mathcal{C}, \Box, \mathbf{d}, \mathbf{s})$ be any strict CS4 category. There is a* resolution functor $\mathrm{Res}_{\mathcal{C}} : \mathcal{C}^o \times \mathcal{C} \to \widehat{\Delta}$ *which maps every pair A, B of objects in $\mathcal{C}$ to the augmented simplicial set $((\mathrm{Hom}_{\mathcal{C}}(A, \Box^{q+1} B))_{q \geqslant -1}, \partial^i_q\hat{=}(\Box^i \mathbf{d} \circ \_), s^i_q\hat{=}(\Box^i \mathbf{s} \circ \_))$, and every pair of morphisms $A' \xrightarrow{f} A$, $B \xrightarrow{g} B'$ to the a.s. map $\mathrm{Res}_{\mathcal{C}}(f, g)$ given in dimension $q \geqslant -1$ by $\mathrm{Res}_{\mathcal{C}}(f, g)_q(a)\hat{=}\Box^{q+1} g \circ a \circ f$ for every $a \in \mathrm{Hom}_{\mathcal{C}}(A, \Box^{q+1} B)$.*

For instance, $\mathrm{Res}_{\mathcal{S4}_\Sigma}(\mathbf{1}, F)$ is the augmented simplicial set of ground λ_{S4}-terms of type $\Box^{q+1} F$, $q \geqslant -1$. (A term is *ground* provided it has no free variable.) However we have seen in Lemma 46 that this would not be enough for our purposes. The minimal augmented simplicial set that seems to work is as follows:

Definition 67 (Contracting Resolution Functor CRes**).** *Let $(\mathcal{C}, \Box, \mathbf{d}, \mathbf{s})$ be a small strict CS4 category. Then the* contracting resolution functor $\mathrm{CRes}_{\mathcal{C}} : \mathcal{C} \to \widehat{\Delta}$ *is the colimit $\varinjlim(\Lambda(\mathrm{Res}_{\mathcal{C}}) \circ \Box)$ in the category $\mathbf{Hom_{Cat}}(\mathcal{C}, \widehat{\Delta})$ of functors from $\mathcal{C}$ to $\widehat{\Delta}$.*

This definition makes sense, provided we take Λ as meaning abstraction in **Cat**: while $\mathrm{CRes}_{\mathcal{C}}$ is a functor from $\mathcal{C}^o \times \mathcal{C}$ to $\widehat{\Delta}$, $\Lambda(\mathrm{CRes}_{\mathcal{C}})$ is a functor from $\mathcal{C}^o$ to $\mathbf{Hom_{Cat}}(\mathcal{C}, \widehat{\Delta})$; since $\Box$ is an endofunctor in $\mathcal{C}$, it also defines an endofunctor in $\mathcal{C}^o$. Finally, the indicated colimit exists because $\mathbf{Hom_{Cat}}(\mathcal{C}, \widehat{\Delta}) = \mathbf{Hom_{Cat}}(\mathcal{C}, \mathbf{Hom_{Cat}}(\Delta^o, \mathbf{Set})) \cong \mathbf{Hom_{Cat}}(\mathcal{C} \times \Delta^o, \mathbf{Set})$ is a category of presheaves, hence a topos, hence is small cocomplete; and $\mathcal{C}$, therefore also $\mathcal{C}^o$ is small.

Geometrically, the idea is that instead of taking resolutions from a one-point space $\mathbf{1}$ (as in $\mathrm{Res}_{\mathcal{C}}(\mathbf{1}, \_)$), we take all resolutions from enough spaces with a contracting homotopy, properly amalgamated. Recall that a *contracting homotopy* on an augmented simplicial set $K \hat{=} (K_q)_{q \geqslant -1}$ is an a.s. map from K to $\Box K$ that is a right inverse to $\mathbf{d}$ in $\widehat{\Delta}$. More concretely, this is a family of maps $s_q^{-1} : K_q \to K_{q+1}$, $q \geqslant -1$, such that $s_{q+1}^{-1} \circ s_q^j = s_{q+1}^{j+1} \circ s_q^{-1}$ and $s_{q-1}^{-1} \circ \partial_q^j = \partial_{q+1}^{j+1} \circ s_q^{-1}$, for all $0 \leqslant j \leqslant q$, and $\partial_{q+1}^0 \circ s_q^{-1} = \mathrm{id}$. (This is exactly what is needed to build the more standard notion of contracting homotopy in simplicial homology.) Then a trivial way of ensuring that $\mathrm{Res}_{\mathcal{C}}(A, B)$ has a contracting homotopy is to take A of the form $\Box A'$: indeed, for any $f \in \mathrm{Res}_{\mathcal{C}}(\Box A', B)_q = \mathrm{Hom}_{\mathcal{C}}(\Box A', \Box^{q+1} B)$, we may then define $s_q^{-1}(f)$ as $\Box f \circ \mathbf{s}$.

Proof-theoretically, when $\mathcal{C}$ is $\mathcal{S}4_\Sigma$, $s_q^{-1}(M)$ is the term $\boxed{M}$. This is the manifestation of the $(\Box I)$ rule. At the level of programs, this is Lisp's `quote` operator.

Proposition 68. *For every context Θ, the a.s. set $\mathcal{S}4[\Theta]$ is exactly* $\mathrm{CRes}_{\mathcal{S}4_\Sigma}(\Theta)$.

Proof. We deal with the case where Θ is of the form $z : F$ for a single formula F, for readability purposes. The general case is similar.

Colimits in functor categories are taken pointwise, so $\mathrm{CRes}_{\mathcal{S}4_\Sigma}(F)$ is the colimit of the functor that maps every context Γ to the a.s. set $\mathrm{Res}_{\mathcal{S}4_\Sigma}(\Box\Gamma, F)$ of all λ_{S4}-terms M such that $\Box\Gamma \vdash M : \Box^{q+1} F$, modulo $\approx$. On the one hand, $\mathcal{S}4[F]$ is the apex of a cocone consisting of morphisms $\mathrm{Res}_{\mathcal{S}4_\Sigma}(\Box\Gamma, F) \longrightarrow \mathcal{S}4[F]$ that map each typing derivation of $\Box\Gamma \vdash M : \Box^{q+1} F$ to the term M itself. On the other hand, we claim that $\mathcal{S}4[F]$ is universal among all such apexes. Let indeed K be any a.s. set such that there are morphisms $\mathrm{Res}_{\mathcal{S}4_\Sigma}(\Box\Gamma, F) \xrightarrow{f_\Gamma} K$, where Γ ranges over contexts; and such that these morphisms define a cocone: whenever $\Gamma \xrightarrow{\theta} \Gamma'$ is a morphism in $\mathcal{S}4_\Sigma$, for every $q \geqslant -1$, $(\Box\Gamma' \xrightarrow{M} F) \in \mathrm{Res}_{\mathcal{S}4_\Sigma}(\Box\Gamma', F)_q$, $f_{\Gamma'}(\Box\Gamma' \xrightarrow{M} F) = f_\Gamma(\Box\Gamma \xrightarrow{M \cdot \Box\theta} F)$. Taking for θ all substitutions mapping variables to variables, and noticing that for any variable x, $\Box x = \boxed{dx} \approx x$, it follows that f_Γ depends only on M, not on $\Box\Gamma' \xrightarrow{M} F$: this defines the unique morphism from $\mathcal{S}4[F]$ to K. Therefore $\mathcal{S}4[F]$ is a colimit of the desired functor. By the uniqueness of colimits (up to isomorphism), the result obtains. $\Box$

5.3. A Review of Logical Relations

While the $\mathcal{C}[\![\_]\!]$ interpretation is complete when we are allowed to take $\mathcal{S}4_\Sigma$ for $\mathcal{C}$, we are interested in taking more geometrical categories for $\mathcal{C}$, in particular $\widehat{\Delta}$ or **CGHaus**.

Let us first review the standard way of proving Friedman's Theorem 49 ([**40**], Chapter 8) using logical relations. We shall then discuss why this proof cannot be replayed directly in our case, and do appropriate modifications.

Friedman's result is for the non-modal part of λ_{S4}, the λ-calculus with $\beta\eta$-equality, interpreted in **Set**. Let us spell out the relevant part of the interpretation of Figure 9 in detail. Given a map from base types $A \in \Sigma$ to sets $\rho(A)$, let $\mathbf{Set}[\![F \supset G]\!]\rho$ be the set of all functions from $\mathbf{Set}[\![F]\!]\rho$ to $\mathbf{Set}[\![G]\!]\rho$. Then $\mathbf{Set}[\![\Gamma]\!]\rho$, where $\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n$, is a mapping from each variable x_i to

$\mathbf{Set}\,[\![F_i]\!]\,\rho$: this is a Γ-*environment* ε. The interpretation in **Set** then maps every typing derivation of a λ-term M of type F in Γ, and every Γ-environment ε to an element $\mathbf{Set}\,[\![\Gamma \vdash M : F]\!]\,\rho\varepsilon$ (for short, $\mathbf{Set}\,[\![M]\!]\,\rho\varepsilon$) of $\mathbf{Set}\,[\![F]\!]\,\rho$: $\mathbf{Set}\,[\![x]\!]\,\rho\varepsilon$ is $\varepsilon(x)$, $\mathbf{Set}\,[\![MN]\!]\,\rho\varepsilon$ is $\mathbf{Set}\,[\![M]\!]\,\rho\varepsilon$ applied to $\mathbf{Set}\,[\![N]\!]\,\rho\varepsilon$, and $\mathbf{Set}\,[\![\Gamma \vdash \lambda x \cdot M : F \supset G]\!]\,\rho\varepsilon$ is the function that maps each $v \in \mathbf{Set}\,[\![F]\!]\,\rho$ to $\mathbf{Set}\,[\![M]\!]\,\rho(\varepsilon[x \mapsto v])$.

Let $\mathbf{Set}[F]$ be defined as the set of all λ-terms of type F, modulo $\beta\eta$-conversion. We get an interpretation of λ-terms in the free CCC over Σ by mapping every term M to $\mathbf{Set}[M]\theta \hat{=} M\theta$, where the Γ-environment θ is just a substitution.

A *logical relation* is a family of binary relations R^F indexed by formulae F, between $\mathbf{Set}[F]$ and $\mathbf{Set}\,[\![F]\!]\,\rho$, such that $M\ R^{F \supset G}\ f$ if and only if $MN\ R^G\ f(a)$ for every M and a such that $N\ R^F\ a$. (In general, logical relations are relations indexed by types between Henkin models, or between CCCs. We specialize the notion to our problem at hand.) The fundamental lemma of logical relations (the Basic Lemma of [**40**]) states that, when ε is a Γ-environment ($\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n$) and θ a substitution mapping each x_i to a term of type F_i, whenever $x_i\theta\ R^{F_i}\ \varepsilon(x_i)$ for each i, then $\mathbf{Set}[M]\theta\ R^F\ \mathbf{Set}\,[\![M]\!]\,\rho\varepsilon$ for any term M of type F in Γ. In other words, as soon as environments are related through the logical relation, then so are the values of any term in both models.

To show that $\mathbf{Set}\,[\![\_]\!]\,\rho$ is equationally complete, it is enough to show that we can build a *functional* logical relation, i.e., one such that for every $a \in \mathbf{Set}\,[\![F]\!]\,\rho$, there is at most one $M \in \mathbf{Set}[F]$ (up to $\approx$) such that $M\ R^F\ a$. Note that any logical relation is uniquely determined by the relations R^A with $A \in \Sigma$. The trick is to choose R^A so that not only R^A but every R^F is functional. It turns out that asking that R^F be functional only does not carry through, and we must require R^F to be functional and *onto*: for every $M \in \mathbf{Set}[F]$, there must be at least one $a \in \mathbf{Set}\,[\![F]\!]\,\rho$ such that $M\ R^F\ a$. Under these assumptions, $R^{F \supset G}$ is then functional and onto as soon as R^F and R^G are. First, it is functional: choose $f \in \mathbf{Set}\,[\![F \supset G]\!]\,\rho$, a function from $\mathbf{Set}\,[\![F]\!]\,\rho$ to $\mathbf{Set}\,[\![G]\!]\,\rho$, then every term M such that $M\ R^{F \supset G}\ f$ must be such that for every $N\ R^F\ a$, $MN\ R^G\ f(a)$. Using the Axiom of Choice and the fact that R^F is onto, we may define a function $i_F : \mathbf{Set}[F] \to \mathbf{Set}\,[\![F]\!]\,\rho$ such that $N\ R^F\ i_F(N)$. Then M must be such that for every N, $MN\ R^G\ f(i_F(N))$. Since R^G is functional, we may define a projection $p_G : \mathbf{Set}\,[\![F]\!]\,\rho \to \mathbf{Set}[F]$ such that $P\ R^G\ a$ implies $P = p_G(a)$ (when there is no P such that $P\ R^G\ a$, $p_G(a)$ is arbitrary). So M must be such that for every N, $MN = p_G(f(i_F(N)))$. This determines MN uniquely, hence M too, provided it exists. So $R^{F \supset G}$ is functional. To show that it is onto, map $M \in \mathbf{Set}[F \supset G]$ to the function $f \in \mathbf{Set}\,[\![F \supset G]\!]\,\rho$ mapping a to $i_G(M\ p_F(a))$.

This is essentially the line of proof that we shall follow. However, in our case **Set** is replaced by $\widehat{\Delta}$, resp. **CGHaus**, and in neither case is the Axiom of Choice valid: if $p : K \to L$ is an epi in $\widehat{\Delta}$, there is no a.s. map $i : L \to K$ in general such that $p \circ i = \mathrm{id}$. The same happens in **CGHaus** (although the Axiom of Choice allows us to pick a function i that is left-inverse to p, there may be no such *continuous* left-inverse). Therefore we have to build i_F and p_F explicitly by induction on formulae. The important property that needs to be preserved for each formula F is what we shall call the Bounding Lemma: if $a = i_F(M)$ then $M\ R^F\ a$, and if $M\ R^F\ a$ then

$M = p_F(a)$. Retracing the argument above, we find that this requires us to define $i_{F\supset G}(M)$ as the function mapping a to $i_G(M\ p_F(a))$, however the obvious definition for $p_{F\supset G}$: $p_{F\supset G}(f)\hat{=}\lambda x\cdot p_G(f(i_F(x)))$ is *wrong*. This is because this is incompatible with α-renaming in general, and therefore does not map functions to $\approx$-classes of λ-terms. Indeed, compatibility with α-renaming imposes that $\lambda x\cdot p_G(f(i_F(x))) = \lambda y\cdot p_G(f(i_F(x)))\{x := y\}$, but there is no reason why $p_G\circ f\circ i_F$ should be substitutive. The solution is to define $p_{F\supset G}$ by $p_{F\supset G}(f)\hat{=}\mathfrak{R}_{F\supset G}(N\mapsto p_G(f(i_F(N))))$, where $\mathfrak{R}_{F\supset G}$ is a retraction of the set of functions from $\mathbf{Set}[F]$ to $\mathbf{Set}[G]$ onto the syntactic function space $\mathbf{Set}[F\supset G]$—retraction meaning that $\mathfrak{R}_{F\supset G}(N\mapsto MN)$ should be the term M exactly. This is exactly what we have taken the pain of constructing in the augmented simplicial case in Corollary 48.

One final note before we embark on actually proving the theorem. The right notion of logical relation here is one of *Kripke logical relation*, a more complex notion than ordinary logical relations. Moreover, contrarily to more usual cases, the set of worlds we use for this Kripke logical relation cannot just be a preorder: it has to be a category, in fact the augmented simplicial category Δ. Concretely, we have to use families of relations R_q^F indexed by both formulae F and dimensions $q\geqslant -1$, such that:

(a.s.) for every a, a', if $a\ R_q^F\ a'$ then, for every i, $0\leqslant i\leqslant q$, $\partial_q^i a\ R_{q-1}^F\ \partial_q^i a'$ and $s_q^i a\ R_{q+1}^F\ s_q^i a'$;

($\Box$ logical) for every a, a', $a\ R_q^{\Box F}\ a'$ if and only if $a\ R_{q+1}^F\ a'$.

($\supset$ logical) for every f, f', $f\ R_q^{F\supset G}\ f'$ if and only if for every monotonic function $\mu : [p]\to[q]$, for every a, a' such that $a\ R_p^F\ a'$,

$$\Box^{p+1}App(\hat{\mu}(f), a)\ R_p^G\ \Box^{p+1}App(\hat{\mu}(f'), a')$$

where $\hat{\mu}$ is defined in the unique way so that $\widehat{\delta_q^i} = \partial_q^i$, $\widehat{\sigma_q^i} = s_q^i$, $\widehat{\mathrm{id}} = \mathrm{id}$, and $\widehat{\mu\circ\mu'} = \widehat{\mu'}\circ\widehat{\mu}$.

The latter condition is particularly unwieldy. We prefer to use a more categorical notion, which will factor out all irrelevant details. It turns out that logical relations and Kripke logical relations are special cases of *subscones* [**41, 1**]: these are the right notion here.

5.4. Scones and Subscones

Given any two categories $\mathcal{C}$ and $\mathcal{D}$ having all finite cartesian products, and such that $\mathcal{D}$ has all pullbacks, given any functor $\mathfrak{F} : \mathcal{C}\to\mathcal{D}$ that preserves all finite cartesian products, the *subscone* $\mathcal{D}\downarrow\mathfrak{F}$ [**41**] has as objects all triples (d, c, m) where d is an object of $\mathcal{D}$, c is an object of $\mathcal{C}$, and

$$d \overset{m}{\hookrightarrow} \mathfrak{F}(c)$$

is monic in $\mathcal{D}$. The *scone* $\mathcal{D}\downarrow\mathfrak{F}$ is defined similarly, only without the requirement that m be monic.

Let $\mathcal{C}$ be any strict CS4 category. Given any set Σ of base types, and a mapping that assigns each base type an object in $\mathcal{C}$ (this can be seen as a functor from Σ, seen as the trivial category with elements of Σ as objects and only identity morphisms):

$$\Sigma \xrightarrow{\rho} \mathcal{C}$$

there is a unique representation of strict CS4 categories $\mathcal{C}\,[\![\_]\!]\,\rho$ from the free strict CS4 category $\mathcal{S}4_\Sigma$ on Σ to $\mathcal{C}$:

$$\begin{tikzcd} \Sigma \arrow[r, "\subseteq"] \arrow[d, "\rho"'] & \mathcal{S}4_\Sigma \arrow[dl, "\mathcal{C}\,[\![\_]\!]\,\rho"] \\ \mathcal{C} & \end{tikzcd}$$

where $\subseteq$ denotes the canonical inclusion functor from Σ to $\mathcal{S}4_\Sigma$.

If, in the diagram above, we replace $\mathcal{C}$ by a subscone category $\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F}$, we get a diagram:

$$\begin{tikzcd} \Sigma \arrow[r, "\subseteq"] \arrow[d, "\tilde{\rho}"'] & \mathcal{S}4_\Sigma \arrow[dl, "(\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F})\,[\![\_]\!]\,\tilde{\rho}"] \\ \mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F} & \end{tikzcd} \tag{19}$$

for each given $\tilde{\rho}$, and where $(\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F})\,[\![\_]\!]\,\tilde{\rho}$ is uniquely determined as a representation of strict CS4 categories: this will be the right notion of *Kripke logical relation*.

It is well-known ([**41**], Proposition 4.2) that, provided that $\mathcal{C}$ and $\mathcal{D}$ are cartesian-closed, and $\mathcal{D}$ has equalizers (i.e., $\mathcal{D}$ is finitely complete), and provided $\mathfrak{F}$ preserves finite products, then $\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F}$ is a CCC, and the forgetful functor $U : \mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F} \longrightarrow \mathcal{C}$, which maps every object (d, c, m) to c, is a representation of CCCs. (Similarly for the scone $\mathcal{D} \downarrow \mathfrak{F}$.) We make explicit the construction of terminal objects, products and internal homs in $\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F}$:

5.4.0.1. TERMINAL OBJECT. This is $(\mathbf{1}_\mathcal{D}, \mathbf{1}_\mathcal{C}, \mathrm{id})$.

5.4.0.2. BINARY PRODUCTS. The product of (d, c, m) with (d', c', m') is $(d \times d', c \times c', m \times m')$.

5.4.0.3. INTERNAL HOMS. Build $(d'', c'', m'') = \mathbf{Hom}_{\mathcal{D} \mathbin{\downharpoonleft} \mathfrak{F}}((d, c, m), (d', c', m'))$ as follows. (The careful reader will note that the construction is the same in the scone $\mathcal{D} \downarrow \mathfrak{F}$.) First, $c'' \hat{=} \mathbf{Hom}_\mathcal{C}(c, c')$.

Then, we build two morphisms. We build the first one from:

$$\mathfrak{F}(\mathbf{Hom}_\mathcal{C}(c, c')) \times d \xrightarrow{\mathrm{id} \times m} \mathfrak{F}(\mathbf{Hom}_\mathcal{C}(c, c')) \times \mathfrak{F}(c) \xrightarrow{\mathfrak{F}(App)} \mathfrak{F}(c')$$

by currying, getting:

$$\mathfrak{F}(\mathbf{Hom}_\mathcal{C}(c, c')) \xrightarrow{\Lambda(\mathfrak{F}(App) \circ (\mathrm{id} \times m))} \mathbf{Hom}_\mathcal{D}(d, \mathfrak{F}(c')) \tag{20}$$

The second one is built from:

$$\mathbf{Hom}_\mathcal{D}(d, d') \times d \xrightarrow{App} d' \xhookrightarrow{m'} \mathfrak{F}(c')$$

again by currying:

$$\mathbf{Hom}_{\mathcal{D}}(d, d') \xrightarrow{\Lambda(m' \circ App)} \mathbf{Hom}_{\mathcal{D}}(d, \mathfrak{F}(c')) \tag{21}$$

We claim that this morphism is monic. Indeed, consider two morphisms f, g such that $\Lambda(m' \circ App) \circ f = \Lambda(m' \circ App) \circ g$. Applying $App \circ (\_ \times \mathrm{id})$ on the left-hand side, we get $App \circ ((\Lambda(m' \circ App) \circ f) \times \mathrm{id}) = App \circ (\Lambda(m' \circ App \circ (f \times \mathrm{id})) \times \mathrm{id})$ (by (k')) $= m' \circ App \circ (f \times \mathrm{id})$ (by (l'')). Applying to both sides of the equation, we therefore get $m' \circ App \circ (f \times \mathrm{id}) = m' \circ App \circ (g \times \mathrm{id})$, therefore $App \circ (f \times \mathrm{id}) = App \circ (g \times \mathrm{id})$, because m' is monic. Applying Λ on both sides, the left-hand side simplifies to f and the right-hand side to g by (m'), therefore $f = g$. So $\Lambda(m' \circ App)$ is indeed monic.

We now build (d'', c'', m'') by the following pullback diagram:

$$\begin{tikzcd}
d'' \arrow[r, hook, "m''"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}(\mathbf{Hom}_{\mathcal{C}}(c, c')) \arrow[d, "\Lambda(\mathfrak{F}(App) \circ (\mathrm{id} \times m))"', "(20)"] \\
\mathbf{Hom}_{\mathcal{D}}(d, d') \arrow[r, hook, "\Lambda(m' \circ App)", "(21)"'] & \mathbf{Hom}_{\mathcal{D}}(d, \mathfrak{F}(c'))
\end{tikzcd}$$

where the upper morphism m'' is monic because pullbacks preserve monics.

Application in the subscone is given by the pair of morphisms $App \circ (s \times \mathrm{id})$ from $d'' \times d$ to d' and App from $c'' \times c$ to c'.

Conversely, given any morphism (u, v) in the subscone from $(d_0, c_0, m_0) \times (d, c, m)$ to (d', c', m'), we build its curried morphism from (d_0, c_0, m_0) to (d'', c'', m'') as follows. Recall that since (u, v) is a morphism, the following square commutes:

$$\begin{tikzcd}
d_0 \times d \arrow[r, hook, "m_0 \times m"] \arrow[d, "u"'] & \mathfrak{F}(c_0) \times \mathfrak{F}(c) \arrow[d, "\mathfrak{F}(v)"] \\
d' \arrow[r, hook, "m'"] & \mathfrak{F}(c')
\end{tikzcd}$$

The curried version of the morphism (u, v) is then $(\hat{u}, \Lambda(v))$, where $\hat{u}$ is given as the unique morphism that makes the following diagram commute:

$$\begin{tikzcd}
d_0 \arrow[rr, "m_0"] \arrow[dr, dashed, "\hat{u}"] \arrow[ddr, "\Lambda(u)"'] & & \mathfrak{F}(c_0) \arrow[d, "\mathfrak{F}(\Lambda(v))"] \\
& d'' \arrow[r, "m''"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}(c'') \arrow[d, "\Lambda(\mathfrak{F}(App) \circ (\mathrm{id} \times m))"'] \\
& \mathbf{Hom}_{\mathcal{D}}(d, d') \arrow[r, "\Lambda(m' \circ App)"'] & \mathbf{Hom}_{\mathcal{D}}(d, \mathfrak{F}(c'))
\end{tikzcd}$$

where the bottom pullback diagram is given by the definition of internal homs in the subscone. (The outer diagram commutes: $\Lambda(m' \circ App) \circ \Lambda(u) = \Lambda(m' \circ App \circ (\Lambda(u) \times \mathrm{id}))$ [by (k')] $= \Lambda(m' \circ u)$ [by (l'')] $= \Lambda(\mathfrak{F}(v) \circ (m_0 \times m))$ since (u, v) is a morphism, while $\Lambda(\mathfrak{F}(App) \circ (\mathrm{id} \times m)) \circ \mathfrak{F}(\Lambda(v)) \circ m_0 = \Lambda(\mathfrak{F}(App) \circ (\mathrm{id} \times m) \circ ((\mathfrak{F}(\Lambda(v)) \circ m_0) \times \mathrm{id}))$ [by (k')] $= \Lambda(\mathfrak{F}(App) \circ (\mathfrak{F}(\Lambda(v)) \times \mathrm{id}) \circ (m_0 \times m)) = \Lambda(\mathfrak{F}(App \circ (\Lambda(v) \times \mathrm{id})) \circ (m_0 \times m))$ [because $\mathfrak{F}$ is a functor that preserves products] $= \Lambda(\mathfrak{F}(v) \circ (m_0 \times m))$ [by (l'')].)

5.4.0.4. Comonad $(\Box, \mathbf{d}, \mathbf{s})$. Whereas there is at most one CCC structure on any given category, there are in general many choices for a strict monoidal comonad. A standard choice for defining a comonad on $\mathcal{D} \downarrow \mathfrak{F}$ based on a given comonad $(\Box, \mathbf{d}, \mathbf{s})$ on $\mathcal{C}$ works by defining $\Box(d, c, m)$ as $(d', \Box c, m')$, where d' and m' are given by the pullback diagram:

$$\begin{tikzcd} d' \arrow[r, "m'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{F}(\mathbf{d})"] \\ d \arrow[r, hook, "m"'] & \mathfrak{F}(c) \end{tikzcd}$$

This would not work for our purposes: intuitively, if (d, c, m) represents a relation R^F, defining $R^{\Box F}$ this way as $(d', \Box c, m')$ would mean replacing the **($\Box$ logical)** condition by: for every a, a', $a\ R_q^{\Box F}\ a'$ if and only if $\partial_q^0 a\ R_q^F\ \partial_q^0 a'$. However with this definition $R^{\Box F}$ would be too large, and the second implication $M\ R^F\ a \Longrightarrow M = p_F(a)$ of the Bounding Lemma would not hold in general. Note that this would also be unsatisfying as this definition does not use the comonad we may have on $\mathcal{D}$.

Instead, we notice that there is a simpler solution as soon as $\mathcal{D}$ also comes equipped with a comonad, provided $\mathfrak{F}$ preserves $\Box$, $\mathbf{d}$ and $\mathbf{s}$ (we shall say that $\mathfrak{F}$ *preserves the comonad* $(\Box, \mathbf{d}, \mathbf{s})$), and also that $\Box$ preserves monics. Then letting $\Box(d, c, m)$ be $(\Box d, \Box c, \Box m)$ defines an object in the subscone. Indeed, $\Box m$ is a morphism from $\Box d$ to $\Box\mathfrak{F}(c) = \mathfrak{F}(\Box c)$, and is monic since m is and $\Box$ preserves monics. This is what will work here. (In the case of scones, the requirement that $\Box$ preserves monics is not necessary.)

This simple case will work for completeness in the augmented simplicial case. In general, $\mathfrak{F}(\Box c)$ and $\Box\mathfrak{F}(c)$ won't be isomorphic, and we shall only have a monic $\mathfrak{I}_c$ from $\mathfrak{F}(\Box c)$ to $\Box\mathfrak{F}(c)$ for each object c. We then insist on the following condition (whose role will be apparent from the proof of Proposition 70): the image by $\Box$ of a pullback with one side equal to $\mathfrak{I}_c$ should again be a pullback square. This subsumes the above, simple case where $\mathfrak{I}_c$ is the identity, since pullbacks in this case are flat (top and bottom rows equal, linked by identities). We also require a series of equations relating $\mathfrak{F}$, $\mathfrak{I}$ and the comonads:

Definition 69 (($\mathfrak{F}, \Box$)-Commutator). *Let $\mathcal{C}$ and $\mathcal{D}$ are strict CS4 categories, and $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ a functor.*

An ($\mathfrak{F}, \Box$)-commutator *is a natural transformation $\mathfrak{I}$ from $\mathfrak{F}\Box$ to $\Box\mathfrak{F}$ such that* *the following equations hold:*

$$\mathbf{d} \circ \mathfrak{I}_c = \mathfrak{F}(\mathbf{d}) \tag{22}$$

$$\mathbf{s} \circ \mathfrak{I}_c = \Box\mathfrak{I}_c \circ \mathfrak{I}_{\Box c} \circ \mathfrak{F}(\mathbf{s}) \tag{23}$$

We say, additionally, that $\Box$ preserves pullbacks along $\mathfrak{I}$, *if and only if for each object c of $\mathcal{C}$, for every pullback:*

$$\begin{tikzcd} d' \arrow[r, "m'"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{I}_c"] \\ \Box d \arrow[r, "\Box m"'] & \Box(\mathfrak{F}(c)) \end{tikzcd}$$

the following commuting square is again a pullback:

$$\begin{tikzcd} \Box d' \arrow[r, "\Box m'"] \arrow[d, "\Box s"'] & \Box\mathfrak{F}(\Box c) \arrow[d, "\Box\mathfrak{I}_c"] \\ \Box^2 d \arrow[r, "\Box^2 m"'] & \Box^2(\mathfrak{F}(c)) \end{tikzcd}$$

Proposition 70. *Assume that $\mathcal{C}$ and $\mathcal{D}$ are strict CS4 categories, that $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ preserves finite products, that $\mathcal{D}$ is finitely complete, that $\Box$ preserves monics and pullbacks along $\mathfrak{I}$, where $\mathfrak{I}$ is an $(\mathfrak{F}, \Box)$-commutator.*

Then the subscone $\mathcal{D}\wr\mathfrak{F}$ is a strict CS4 category when equipped with the comonad $(\Box, \mathbf{d}, \mathbf{s})$ defined by:

- *On objects, $\Box(d, c, m)$ is given as the unique morphism $(d', \Box c, m')$ that makes the following a pullback diagram:*

$$\begin{tikzcd} d' \arrow[r, "m'"] \arrow[d, "\hat{m}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{I}_c"] \\ \Box d \arrow[r, "\Box m"'] & \Box\mathfrak{F}(c) \end{tikzcd} \tag{24}$$

On morphisms, if (u, v) is a morphism from (d_1, c_1, m_1) to (d_2, c_2, m_2), then $\Box(u, v)$ is the unique morphism $(\hat{u}, \Box v)$ that makes the following diagram com-

mute:

$$\begin{tikzcd}
d'_1 \arrow[rr, "m'_1"] \arrow[dr, dotted, "\hat{u}"'] \arrow[dd, "\hat{m}_1"'] & & \mathfrak{F}(\Box c_1) \arrow[dr, "\mathfrak{F}(\Box v)"] \arrow[dd, "\mathfrak{I}_{c_1}"' near start] & \\
& d'_2 \arrow[rr, "m'_2" near start] \arrow[dd, "\hat{m}_2"' near start] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathfrak{F}(\Box c_2) \arrow[dd, "\mathfrak{I}_{c_2}"] \\
\Box d_1 \arrow[rr, "\Box m_1" near end] \arrow[dr, "\Box u"'] & & \Box\mathfrak{F}(c_1) \arrow[dr, "\Box\mathfrak{F}(v)"] & \\
& \Box d_2 \arrow[rr, "\Box m_2"] & & \Box\mathfrak{F}(c_2)
\end{tikzcd} \tag{25}$$

where the front face defines $\Box(d_2, c_2, m_2)$, the back face defines $\Box(d_1, c_1, m_1)$, the bottom face is by the definition of (u, v) as a morphism, the right face is by naturality of $\mathfrak{I}$, and $\hat{u}$ is then determined uniquely by the fact that the front face is a pullback.

- *The counit* $\mathbf{d}$ *from* $\Box(d, c, m)$ *to* (d, c, m) *is* $(\mathbf{d} \circ \hat{m}, \mathbf{d})$*, where* $\hat{m}$ *is given as in Diagram (24).*
- *Comultiplication* $\mathbf{s}$ *is* $(\bar{\mathbf{s}}, \mathbf{s})$ *where* $\mathbf{s}'$ *is uniquely determined so that the following diagram commutes:*

$$\begin{tikzcd}
d' \arrow[rr, "m'"] \arrow[dr, dashed, "\bar{\mathbf{s}}"] \arrow[ddd, "\hat{m}"'] & & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{F}(\mathbf{s})"] \\
& d'' \arrow[r, "m''"] \arrow[d, "\hat{m}'"'] \arrow[dr, phantom, "(26.1)"] & \mathfrak{F}(\Box^2 c) \arrow[d, "\mathfrak{I}_{\Box c}"] \\
& \Box d' \arrow[r, "\Box m'"] \arrow[d, "\Box\hat{m}"'] \arrow[dr, phantom, "(26.2)"] & \Box\mathfrak{F}(\Box c) \arrow[d, "\Box\mathfrak{I}_c"] \\
\Box d \arrow[r, "\mathbf{s}"'] & \Box^2 d \arrow[r, "\Box^2 m"'] & \Box^2\mathfrak{F}(c)
\end{tikzcd} \tag{26}$$

where (26.1) is the defining square for $\Box(d', \Box c, m') = \Box^2(d, c, m)$ and (26.2) is the image under $\Box$ of the defining square for $\Box(d, c, m)$.

Moreover, the forgetful functor $U : \mathcal{D} \wr \mathfrak{F} \longrightarrow \mathcal{C}$ mapping every object (d, c, m) in *the scone (resp. subscone) to c and every morphism (u, v) to v is a representation of strict CS4 categories.*

Proof. The CCC structure on $\mathcal{D} \downarrow \mathfrak{F}$ (resp. $\mathcal{D} \downharpoonleft \mathfrak{F}$) as well as the fact that U is a representation of CCCs is well-known. Let us check the comonad.

First, in Diagram (24) since m is monic, $\Box m$ is, too, because $\Box$ preserves monics; so m' must be monic, since pullbacks preserve monics. It follows that $(d', \Box c, m)$ is indeed an object in $\mathcal{D} \downharpoonleft \mathfrak{F}$.

We claim that $\Box$ is a functor: this is because $\Box$ on morphisms is defined as the solution to a universal problem. Moreover, $U(\Box(d, c, m)) = \Box c$, so U preserves $\Box$.

Then, $\mathbf{d} = (\mathbf{d} \circ \hat{m}, \mathbf{d})$ is well-defined. Indeed, pile the definition (24) of $\Box(d, c, m)$ atop the square saying that $\mathbf{d} \circ \Box m = m \circ \mathbf{d}$:

$$\begin{tikzcd}
d' \arrow[r, "m'"] \arrow[d, "\hat{m}"'] & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{I}_c"] \\
\Box d \arrow[r, "\Box m"'] \arrow[d, "\mathbf{d}"'] & \Box \mathfrak{F}(c) \arrow[d, "\mathbf{d}"] \\
d \arrow[r, "m"'] & \mathfrak{F}(c)
\end{tikzcd}$$

Since $\mathbf{d} \circ \mathfrak{I}_c = \mathfrak{F}(\mathbf{d})$ by (22), this implies that $m \circ (\mathbf{d} \circ \hat{m}) = \mathfrak{F}(\mathbf{d}) \circ m'$, which means exactly that $(\mathbf{d} \circ \hat{m}, \mathbf{d})$ is a morphism from $(d', \Box c, m')$ to (d, c, m). Also, U maps $\mathbf{d} = (\mathbf{d} \circ \hat{m}, \mathbf{d})$ to $\mathbf{d}$ in $\mathcal{C}$: U preserves $\mathbf{d}$.

To show that $\mathbf{s}$ is well-defined is a bit more intricate. Since $\Box$ preserves pullbacks along $\mathfrak{I}$, (26.2) is a pullback diagram, so the square obtained by piling (26.1) above (26.2) is again a pullback square. Then $\bar{\mathbf{s}}$ will be uniquely defined as soon as we show that the outer square commutes. Indeed, the leftmost path from d' to $\Box^2 \mathfrak{F}(c)$ is $\Box^2 m \circ \mathbf{s} \circ \hat{m} = \mathbf{s} \circ \Box m \circ \hat{m}$ by (q), while the rightmost path is $\Box \mathfrak{I}_c \circ \mathfrak{I}_{\Box c} \circ \mathfrak{F}(\mathbf{s}) \circ m' = \mathbf{s} \circ \mathfrak{I}_c \circ m'$ by (23). These two are equal since $\Box m \circ \hat{m} = \mathfrak{I}_c \circ m'$ by the definition of $\Box(d, c, m)$ (Diagram (24)). Moreover, $U(\mathbf{s}) = \mathbf{s}$, as Diagram (26) indicates.

It remains to check the comonad equations (n)–(y). We have already checked (n), (o); (r), (v), (w), (x), (y) follow from the fact that $\Box$ and $\mathfrak{F}$ preserve finite products. The others are all checked in the same fashion.

Let us check (p): $\mathbf{d} \circ \Box f = f \circ \mathbf{d}$. Write the morphism f as (u, v), from (d, c, m) to (d', c', m'). Then $\Box f$ is given by $(\hat{u}, \Box v)$ as shown in Diagram (25). Writing the top face of the latter diagram above the square saying that $\mathbf{d}$ is a morphism in the subscone yields the diagram below (left), where we have taken the notations of Diagram (25). On the other hand, putting the definition of $\mathbf{d}$ in the subscone atop the defining square for (u, v) yields the diagram on the right:

$$
\begin{tikzcd}
d'_1 \arrow[r, tail, "m'_1"] \arrow[d, "\hat{u}"'] & \mathfrak{F}(\Box c_1) \arrow[d, "\mathfrak{F}(\Box v)"] \\
d'_2 \arrow[r, tail, "m'_2"] \arrow[d, "\mathbf{d} \circ \hat{m}_2"'] & \mathfrak{F}(\Box c_2) \arrow[d, "\mathfrak{F}(\mathbf{d})"] \\
d_2 \arrow[r, tail, "m_2"'] & \mathfrak{F}(c_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
d'_1 \arrow[r, tail, "m'_1"] \arrow[d, "\mathbf{d} \circ \hat{m}_1"'] & \mathfrak{F}(\Box c_1) \arrow[d, "\mathfrak{F}(\mathbf{d})"] \\
d_1 \arrow[r, tail, "m_1"] \arrow[d, "u"'] & \mathfrak{F}(c_1) \arrow[d, "\mathfrak{F}(v)"] \\
d_2 \arrow[r, tail, "m_2"'] & \mathfrak{F}(c_2)
\end{tikzcd}
$$

Since $\mathbf{d} \circ \Box v = v \circ \mathbf{d}$, the righmost paths from d'_1 to $\mathfrak{F}(c_2)$ in each diagram are equal, so the leftmost ones are equal, too. Since m_2 is monic, the left vertical paths (from d'_1 to d_2) are equal. This means exactly that $(\mathbf{d} \circ \hat{m}_2) \circ \hat{u} = u \circ (\mathbf{d} \circ \hat{m}_1)$, therefore $\mathbf{d} \circ \Box(u, v) = (u, v) \circ \mathbf{d}$ in the subscone.

Let us check (q), and go a bit faster. Taking $f \hat{=} (u, v)$, and the notations of Diagram (24) and Diagram (26), we have got the following two diagrams:

$$
\begin{tikzcd}
d'_1 \arrow[r, tail, "m'_1"] \arrow[d, "\hat{u}"'] & \mathfrak{F}(\Box c_1) \arrow[d, "\mathfrak{F}(\Box v)"] \\
d'_2 \arrow[r, tail, "m'_2"] \arrow[d, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c_2) \arrow[d, "\mathfrak{F}(\mathbf{s})"] \\
d''_2 \arrow[r, tail, "m''_2"] & \mathfrak{F}(\Box^2 c_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
d'_1 \arrow[r, tail, "m'_1"] \arrow[d, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c_1) \arrow[d, "\mathfrak{F}(\mathbf{s})"] \\
d''_1 \arrow[r, tail, "m''_1"] \arrow[d, "\hat{\hat{u}}"'] & \mathfrak{F}(\Box^2 c_1) \arrow[d, "\mathfrak{F}(\Box^2 v)"] \\
d''_2 \arrow[r, tail, "m''_2"'] & \mathfrak{F}(\Box^2 c_2)
\end{tikzcd}
$$

Again the rightmost paths from d'_1 to $\mathfrak{F}(\Box^2 c_2)$ are equal, since $\mathbf{s} \circ \Box v = \Box^2 v \circ \mathbf{s}$. So the leftmost ones are equal, too. Since m''_2 is monic, $\bar{\mathbf{s}} \circ \hat{u} = \hat{\hat{u}} \circ \bar{\mathbf{s}}$. Therefore (q) holds in the subscone.

Equation (s) follows from the diagram:

$$
\begin{tikzcd}
d' \arrow[r, tail, "m'"] \arrow[d, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c) \arrow[d, "\mathfrak{F}(\mathbf{s})"] \\
d'' \arrow[r, tail, "m''"] \arrow[d, "\mathbf{d} \circ \hat{m}'"'] & \mathfrak{F}(\Box^2 c) \arrow[d, "\mathfrak{F}(\mathbf{d})"] \\
d' \arrow[r, tail, "m'"'] & \mathfrak{F}(\Box c)
\end{tikzcd}
$$

where the lower square is the definition of $\mathbf{d}$ from $\Box^2(d,c,m) = \Box(d',\Box c, m') = (d'',\Box^2 c, m'')$ to $\Box(d,c,m) = (d',\Box c, m')$. The rightmost path from $\mathfrak{F}(\Box c)$ to $\mathfrak{F}(\Box c)$ is $\mathfrak{F}(\mathbf{d}) \circ \mathfrak{F}(\mathbf{s}) = \mathfrak{F}(\mathbf{d} \circ \mathbf{s}) = \mathfrak{F}(\mathrm{id}) = \mathrm{id}$. So $m' = m' \circ (\mathbf{d} \circ \hat{m}') \circ \bar{\mathbf{s}}$. Since m' is monic, $(\mathbf{d} \circ \hat{m}') \circ \bar{\mathbf{s}} = \mathrm{id}$, so $\mathbf{d} \circ \mathbf{s} = \mathrm{id}$ in the subscone.

For Equation (t), similarly, consider the diagram:

$$\begin{tikzcd}
d' \arrow[r, tail, "m'"] \arrow[d, tail, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c) \arrow[d, tail, "\mathfrak{F}(\mathbf{s})"] \\
d'' \arrow[r, tail, "m''"] \arrow[d, tail, "\widehat{\mathbf{d} \circ \hat{m}}"'] & \mathfrak{F}(\Box^2 c) \arrow[d, tail, "\mathfrak{F}(\Box \mathbf{d})"] \\
d' \arrow[r, tail, "m'"'] & \mathfrak{F}(\Box c)
\end{tikzcd}$$

Again the right-hand vertical path is the identity, so the left-hand one is the identity, too, since m' is monic. So $\Box\mathbf{d} \circ \mathbf{s} = \mathrm{id}$ in the subscone.

For (u), as for (p) and (q), this follows from the two diagrams:

$$\begin{tikzcd}
d' \arrow[r, tail, "m'"] \arrow[d, tail, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c) \arrow[d, tail, "\mathfrak{F}(\mathbf{s})"] \\
d'' \arrow[r, tail, "m''"] \arrow[d, tail, "\hat{\bar{\mathbf{s}}}"'] & \mathfrak{F}(\Box^2 c) \arrow[d, tail, "\mathfrak{F}(\Box \mathbf{s})"] \\
d''' \arrow[r, tail, "m'''"'] & \mathfrak{F}(\Box^3 c)
\end{tikzcd}
\qquad
\begin{tikzcd}
d' \arrow[r, tail, "m'"] \arrow[d, tail, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box c) \arrow[d, tail, "\mathfrak{F}(\mathbf{s})"] \\
d'' \arrow[r, tail, "m''"] \arrow[d, tail, "\bar{\mathbf{s}}"'] & \mathfrak{F}(\Box^2 c) \arrow[d, tail, "\mathfrak{F}(\mathbf{s})"] \\
d''' \arrow[r, tail, "m'''"'] & \mathfrak{F}(\Box^3 c)
\end{tikzcd}$$

$\Box$

Here is a standard way of building $(\mathfrak{F},\Box)$-commutators from $(\mathfrak{U},\Box)$-cocommutators (the dual notion) through an adjunction $\mathfrak{F} \dashv \mathfrak{U}$.

Proposition 71. *Let $\mathfrak{U} : \mathcal{D} \to \mathcal{C}$ be a functor. An $(\mathfrak{U},\Box)$-*cocommutator *is a natural transformation $\mathfrak{S}$ from $\Box\mathfrak{U}$ to $\mathfrak{U}\Box$ such that the following equations hold:*

$$\mathfrak{U}(\mathbf{d}) \circ \mathfrak{S}_d = \mathbf{d} \tag{27}$$

$$\mathfrak{U}(\mathbf{s}) \circ \mathfrak{S}_d = \mathfrak{S}_{\Box d} \circ \Box\mathfrak{S}_d \circ \mathbf{s} \tag{28}$$

If $\mathfrak{U}$ has a left-adjoint $\mathfrak{F}$, then the natural transformation $\mathfrak{I}$ from $\mathfrak{F}\Box$ to $\Box\mathfrak{F}$ defined by:

$$\mathfrak{I}_c \hat{=} \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c)$$

where η *is the unit of the adjunction, and* lad *is the natural isomorphism from* $\mathrm{Hom}_{\mathcal{C}}(\_, \mathfrak{U}\_)$ *to* $\mathrm{Hom}_{\mathcal{D}}(\mathfrak{F}\_, \_)$*, is an* $(\mathfrak{F}, \Box)$*-commutator.*

Proof. Recall ([**36**], IV.1) the following equations, which express the naturality of lad:

$$\mathrm{lad}(g \circ h) = \mathrm{lad}\, g \circ \mathfrak{F}(h) \tag{29}$$

$$\mathrm{lad}(\mathfrak{U}(k) \circ g) = k \circ \mathrm{lad}\, g \tag{30}$$

Also, recall that $\mathrm{lad}(\eta_c) = \mathrm{id}_{\mathfrak{F}(c)}$.

We first check that $\mathfrak{I}$ is a natural transformation. For any morphisms f from c to c' in $\mathcal{C}$, observe that: $(*)$ $\eta_{c'} \circ f = \mathfrak{U}\mathfrak{F}(f) \circ \eta_c$. This is because η is a natural transformation from id to $\mathfrak{U}\mathfrak{F}$. Then $\mathfrak{I}_{c'} \circ \mathfrak{F}(\Box f) = \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c')} \circ \Box\eta_{c'}) \circ \mathfrak{F}(\Box f) = \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c')} \circ \Box\eta_{c'} \circ \Box f)$ (by (29)) $= \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c')} \circ \Box(\eta_{c'} \circ f)) = \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c')} \circ \Box(\mathfrak{U}\mathfrak{F}(f) \circ \eta_c))$ (by $(*)$) $= \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c')} \circ \Box\mathfrak{U}\mathfrak{F}(f) \circ \Box\eta_c) = \mathrm{lad}(\mathfrak{U}(\Box\mathfrak{F}(f)) \circ \mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c)$ (since $\mathfrak{S}$ is natural) $= \Box\mathfrak{F}(f) \circ \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c)$ (by (30)) $= \Box\mathfrak{F}(f) \circ \mathfrak{I}_c$. So $\mathfrak{I}$ is natural.

Let us check Equation (22):

$$\begin{aligned}
\mathbf{d} \circ \mathfrak{I}_c &= \mathbf{d} \circ \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c) \\
&= \mathrm{lad}(\mathfrak{U}(\mathbf{d}) \circ \mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c) \qquad \text{(by (30))} \\
&= \mathrm{lad}(\mathbf{d} \circ \Box\eta_c) \qquad \text{(by (27))} \\
&= \mathrm{lad}(\eta_c \circ \mathbf{d}) \qquad \text{(by (p))} \\
&= \mathrm{lad}(\eta_c) \circ \mathfrak{F}(\mathbf{d}) \qquad \text{(by (29))} \\
&= \mathfrak{F}(\mathbf{d})
\end{aligned}$$

since $\mathrm{lad}(\eta_c) = \mathrm{id}_{\mathfrak{F}(c)}$.

Let us check Equation (23):

$$\begin{aligned}
\mathbf{s} \circ \mathfrak{I}_c &= \mathbf{s} \circ \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c) \\
&= \mathrm{lad}(\mathfrak{U}(\mathbf{s}) \circ \mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c) \qquad \text{(by (30))} \\
&= \mathrm{lad}(\mathfrak{S}_{\Box\mathfrak{F}(c)} \circ \Box\mathfrak{S}_{\mathfrak{F}(c)} \circ \mathbf{s} \circ \Box\eta_c) \qquad \text{(by (28))} \\
&= \mathrm{lad}(\mathfrak{S}_{\Box\mathfrak{F}(c)} \circ \Box\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box^2\eta_c \circ \mathbf{s}) \qquad \text{(by (q))} \\
&= \mathrm{lad}(\mathfrak{S}_{\Box\mathfrak{F}(c)} \circ \Box(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c) \circ \mathbf{s})
\end{aligned}$$

while

$$\begin{aligned}
\Box\mathfrak{I}_c \circ \mathfrak{I}_{\Box c} \circ \mathfrak{F}(\mathbf{s}) &= \Box\mathfrak{I}_c \circ \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(\Box c)} \circ \Box\eta_{\Box c}) \circ \mathfrak{F}(\mathbf{s}) \\
&= \Box\mathfrak{I}_c \circ \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(\Box c)} \circ \Box\eta_{\Box c} \circ \mathbf{s}) \qquad \text{(by (29))} \\
&= \mathrm{lad}(\mathfrak{U}(\Box\mathfrak{I}_c) \circ \mathfrak{S}_{\mathfrak{F}(\Box c)} \circ \Box\eta_{\Box c} \circ \mathbf{s}) \qquad \text{(by (30))} \\
&= \mathrm{lad}(\mathfrak{S}_{\Box\mathfrak{F}(c)} \circ \Box\mathfrak{U}(\mathfrak{I}_c) \circ \Box\eta_{\Box c} \circ \mathbf{s}) \qquad \text{(by naturality of } \mathfrak{S}\text{)} \\
&= \mathrm{lad}(\mathfrak{S}_{\Box\mathfrak{F}(c)} \circ \Box(\mathfrak{U}(\mathfrak{I}_c) \circ \eta_{\Box c}) \circ \mathbf{s})
\end{aligned}$$

Equation (23) will therefore follow from $\mathfrak{U}(\mathfrak{I}_c) \circ \eta_{\Box c} = \mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c$, which we now show. Since lad is one-to-one, it suffices to show $\mathrm{lad}(\mathfrak{U}(\mathfrak{I}_c) \circ \eta_{\Box c}) = \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c)$, and indeed $\mathrm{lad}(\mathfrak{U}(\mathfrak{I}_c) \circ \eta_{\Box c}) = \mathfrak{I}_c \circ \mathrm{lad}(\eta_{\Box c})$ (by (30)) $= \mathfrak{I}_c = \mathrm{lad}(\mathfrak{S}_{\mathfrak{F}(c)} \circ \Box\eta_c)$. $\Box$

5.5. The Basic Lemma

The argument here is the same as [**41**]. Given any functor $\Sigma \xrightarrow{\tilde{\rho}} \mathcal{D}\upharpoonleft\mathfrak{F}$, we get a functor from Σ to $\mathcal{C}$ by composition with U. By the freeness of $\mathcal{S}4_\Sigma$, there are unique functors $\mathcal{C}\,[\![\_]\!]\,\rho$ (where $\rho \hat{=} U \circ \tilde{\rho}$) and $(\mathcal{D}\upharpoonleft\mathfrak{F})\,[\![\_]\!]\,\tilde{\rho}$ which make the upper left and upper right triangles in the following diagram (which is a diagram in the category **Cat** of categories) commute:

$$\begin{tikzcd}
\Sigma \arrow[rr, "\subseteq"] \arrow[dd, "\tilde{\rho}"'] \arrow[ddrr, "\rho" near start] & & \mathcal{S}4_\Sigma \arrow[dd, "\mathcal{C}\,[\![_]\!]\,\rho"] \arrow[ddll, "(\mathcal{D}\upharpoonleft\mathfrak{F})\,[\![_]\!]\,\tilde{\rho}"' near end] \\
& & \\
\mathcal{D}\upharpoonleft\mathfrak{F} \arrow[rr, "U"'] & & \mathcal{C}
\end{tikzcd}$$

Since U is a representation of strict CS4 categories, $U \circ (\mathcal{D}\upharpoonleft\mathfrak{F})\,[\![\_]\!]\,\tilde{\rho}$, too, therefore by the uniqueness of the $\mathcal{C}\,[\![\_]\!]\,\rho$ arrow on the right as a representation of strict CS4 categories, we get:

Lemma 72 (Basic Lemma). $U \circ (\mathcal{D}\upharpoonleft\mathfrak{F})\,[\![\_]\!]\,\tilde{\rho} = \mathcal{C}\,[\![\_]\!]\,(U \circ \tilde{\rho})$

5.6. The Bounding Lemma

Now we consider the case where $\mathcal{C}$ is of the form $\mathcal{C}_1 \times \mathcal{D}$, and $\mathfrak{F} \hat{=} \mathfrak{F}_1 \otimes \mathrm{id}$, where $\mathfrak{F}_1 : \mathcal{C}_1 \to \mathcal{D}$ is a functor that preserves all finite products, and $\mathfrak{I}^1$ is an $(\mathfrak{F}_1, \Box)$-commutator. (Whenever $\mathfrak{F}_1 : \mathcal{C}_1 \to \mathcal{D}$, $\mathfrak{F}_2 : \mathcal{C}_2 \to \mathcal{D}$, we let $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ be the functor mapping C_1, C_2 to $\mathfrak{F}_1(C_1) \times \mathfrak{F}_2(C_2)$.)

Typically, $\mathcal{C}_1$ will be $\mathcal{S}4_\Sigma$, $\mathfrak{F}_1 \hat{=} \mathrm{CRes}_{\mathcal{S}4_\Sigma}$, $\mathcal{D} \hat{=} \widehat{\Delta}$, and $\mathfrak{I}_{(c_1,d)} \hat{=} (\mathfrak{I}^1_{c_1}, \mathrm{id}_d)$—in this case $\mathfrak{I}^1_{c_1}$ will be an identity morphism.

We shall prove:

Lemma 73 (Bounding Lemma). *Let $\mathcal{C}_1$ and $\mathcal{D}$ be strict CS4 categories, $\mathfrak{F}_1 : \mathcal{C}_1 \to \mathcal{D}$ preserve finite products . Assume also that $\mathcal{D}$ is finitely complete and that $\Box$ preserves monics in $\mathcal{D}$ as well as pullbacks along some $(\mathfrak{F}_1, \Box)$-commutator $\mathfrak{I}^1$. Fix $\rho_1 : \Sigma \to \mathcal{C}_1$. Assume finally that for every formulae F and G, $\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\,[\![F]\!]\,\rho_1)$, $\mathfrak{F}_1(\mathcal{C}_1\,[\![G]\!]\,\rho_1))$ retracts strongly onto $\mathfrak{F}_1(\mathcal{C}_1\,[\![F \supset G]\!]\,\rho_1)$, meaning that there is a family of morphisms $\mathfrak{R}_{F \supset G}$ in $\mathcal{D}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\,[\![F]\!]\,\rho_1), \mathfrak{F}_1(\mathcal{C}_1\,[\![G]\!]\,\rho_1)) \arrow[d, "\mathrm{id}"'] & \mathfrak{F}_1(\mathcal{C}_1\,[\![F \supset G]\!]\,\rho_1) \arrow[l, "\Lambda(\mathfrak{F}_1(App))"'] \arrow[d, "\mathrm{id}"] \\
\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\,[\![F]\!]\,\rho_1), \mathfrak{F}_1(\mathcal{C}_1\,[\![G]\!]\,\rho_1)) \arrow[r, "\mathfrak{R}_{F \supset G}"'] & \mathfrak{F}_1(\mathcal{C}_1\,[\![F \supset G]\!]\,\rho_1)
\end{tikzcd} \tag{31}$$

and similarly, that for every formula F, $\Box\mathfrak{F}_1(\mathcal{C}_1\,[\![F]\!]\,\rho_1)$ retracts strongly onto $\mathfrak{F}_1(\mathcal{C}_1\,[\![\Box F]\!]\,\rho_1)$, meaning that there is a family of morphisms $\mathfrak{R}_{\Box F}$ such that the *following diagram commutes:*

$$
\begin{tikzcd}
\Box\mathfrak{F}_1(\mathcal{C}_1 \llbracket F \rrbracket \rho_1) \arrow[d, "\mathrm{id}"'] & \mathfrak{F}_1(\mathcal{C}_1 \llbracket \Box F \rrbracket \rho_1) \arrow[l, tail, "\mathfrak{I}^1_{\mathcal{C}_1 \llbracket F \rrbracket \rho_1}"'] \arrow[d, "\mathrm{id}"] \\
\Box\mathfrak{F}_1(\mathcal{C}_1 \llbracket F \rrbracket \rho_1) \arrow[r, two heads, "\mathfrak{R}_{\Box F}"] & \mathfrak{F}_1(\mathcal{C}_1 \llbracket \Box F \rrbracket \rho_1)
\end{tikzcd}
\tag{32}
$$

Let $\rho_2 : \Sigma \to \mathcal{D}$ *be* $\mathfrak{F}_1 \circ \rho_1$, *and* $\tilde{\rho} : \Sigma \to \mathcal{D}\wr(\mathfrak{F}_1 \otimes \mathrm{id})$ *map every* $A \in \Sigma$ *to* $(\rho_2(A), (\rho_1(A), \rho_2(A)), \langle \mathrm{id}, \mathrm{id}\rangle)$.

For every formula F, *write* $(\mathcal{D}\wr(\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket F \rrbracket \tilde{\rho}$ *as* $(D_F, (\mathcal{C}_1 \llbracket F \rrbracket \rho_1, \mathcal{D} \llbracket F \rrbracket \rho_2), \langle m'_F, m''_F\rangle)$.

Then there are families of morphisms i_F *and* p_F *in* $\mathcal{D}$, *and monics* h^1_F *and* h^2_F *that make the following diagrams commute for each formula* F*:*

$$
\begin{tikzcd}
 & \mathfrak{F}_1(\mathcal{C}_1 \llbracket F \rrbracket \rho_1) & \\
\mathfrak{F}_1(\mathcal{C}_1 \llbracket F \rrbracket \rho_1) \arrow[ur, "\mathrm{id}"] \arrow[r, hook, "h^1_F"] \arrow[dr, "i_F"'] & D_F \arrow[u, "m'_F"'] \arrow[r, hook, "h^2_F"] \arrow[d, "m''_F"'] & \mathcal{D} \llbracket F \rrbracket \rho_2 \arrow[ul, "p_F"'] \arrow[dl, "\mathrm{id}"] \\
 & \mathcal{D} \llbracket F \rrbracket \rho_2 &
\end{tikzcd}
\tag{33}
$$

Proof. We first build i_F and p_F for each formula F so that $p_F \circ i_F = \mathrm{id}$. This is indeed required for the result to hold, since Diagram 33 implies $p_F \circ h^2_F \circ h^1_F = \mathrm{id}$ and $\mathrm{id} \circ h^2_F \circ h^1_F = i_F$.

When F is a base type $A \in \Sigma$, define i_A and p_A in $\mathcal{D}$ so that the following diagrams commute:

$$
\begin{tikzcd}
\rho_2(A) = \mathfrak{F}_1(\rho_1(A)) \arrow[d, "\mathrm{id}"'] & \mathfrak{F}_1(\rho_1(A)) \arrow[l, tail, "i_A"'] \arrow[d, "\mathrm{id}"] \\
\rho_2(A) \arrow[r, two heads, "p_A"'] & \rho_2(A) = \mathfrak{F}_1(\rho_1(A))
\end{tikzcd}
$$

by just taking i_A and p_A to be id.

When $F = \Box G$, let $i_{\Box G} \hat{=} \Box i_G \circ \mathfrak{I}^1_{\mathcal{C}_1 \llbracket G \rrbracket \rho_1}$, $p_{\Box G} \hat{=} \mathfrak{R}_{\Box G} \circ \Box p_G$, so $p_{\Box G} \circ i_{\Box G} = \mathfrak{R}_{\Box G} \circ \Box p_G \circ \Box i_G \circ \mathfrak{I}^1_{\mathcal{C}_1 \llbracket G \rrbracket \rho_1} = \mathfrak{R}_{\Box G} \circ \Box(p_G \circ i_G) \circ \mathfrak{I}^1_{\mathcal{C}_1 \llbracket G \rrbracket \rho_1} = \mathfrak{R}_{\Box G} \circ \Box \mathrm{id} \circ \mathfrak{I}^1_{\mathcal{C}_1 \llbracket G \rrbracket \rho_1} = \mathfrak{R}_{\Box G} \circ \mathfrak{I}^1_{\mathcal{C}_1 \llbracket G \rrbracket \rho_1} = \mathrm{id}$.

When F is of the form $G \supset H$, we build i_F and p_F in the unique type-consistent way. I.e., we have the following diagram:

$$\begin{tikzcd}
\mathcal{D}\llbracket H\rrbracket\rho_2 & & \mathfrak{F}_1(\mathcal{C}_1\llbracket H\rrbracket\rho_1) \arrow[ll, "i_H"'] \\
\begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1),\mathfrak{F}_1(\mathcal{C}_1\llbracket H\rrbracket\rho_1))\\ \times\mathcal{D}\llbracket G\rrbracket\rho_2\end{array} \arrow[rr, "\mathrm{id}\times p_G"'] & & \begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1),\mathfrak{F}_1(\mathcal{C}_1\llbracket H\rrbracket\rho_1))\\ \times\mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1)\end{array} \arrow[u, "App"']
\end{tikzcd}$$

using i_H and p_G from the induction hypothesis. Apply Λ to the resulting composite morphism, and compose with $\Lambda(\mathfrak{F}_1(App))$; this yields $i_{G\supset H}$, defined as:

$$\begin{tikzcd}
\mathcal{D}\llbracket G\supset H\rrbracket\rho_2 & \begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1),\\ \mathfrak{F}_1(\mathcal{C}_1\llbracket H\rrbracket\rho_1))\end{array} \arrow[l, "\Lambda(i_H\circ App"', "\circ(\mathrm{id}\times p_G))"] & \mathfrak{F}_1(\mathcal{C}_1\llbracket G\supset H\rrbracket\rho_1) \arrow[l, "\Lambda(\mathfrak{F}_1(App))"']
\end{tikzcd}$$

Similarly, we define a morphism $p_{G\supset H}$ from $\mathcal{D}\llbracket G\supset H\rrbracket\rho_2$ to $\mathfrak{F}_1(\mathcal{C}_1\llbracket G\supset H\rrbracket\rho_1)$ as the composite:

$$\begin{tikzcd}
\mathcal{D}\llbracket G\supset H\rrbracket\rho_2 \arrow[r, "\Lambda(p_H\circ App", "\circ(\mathrm{id}\times i_G))"'] & \begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1),\\ \mathfrak{F}_1(\mathcal{C}_1\llbracket H\rrbracket\rho_1))\end{array} \arrow[r, "\mathfrak{R}_{G\supset H}"'] & \mathfrak{F}_1(\mathcal{C}_1\llbracket G\supset H\rrbracket\rho_1)
\end{tikzcd}$$

Superposing both diagrams, together with (31), we get:

$$\begin{tikzcd}
\mathcal{D}\llbracket G\supset H\rrbracket\rho_2 \arrow[d, "\mathrm{id}"'] & \begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket F\rrbracket\rho_1),\\ \mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1))\end{array} \arrow[l, "\Lambda(i_H\circ App"', "\circ(\mathrm{id}\times p_G))"] \arrow[d, "\mathrm{id}"'] & \mathfrak{F}_1(\mathcal{C}_1\llbracket F\supset G\rrbracket\rho_1) \arrow[l, "\Lambda(\mathfrak{F}_1(App))"'] \arrow[d, "\mathrm{id}"] \\
\mathcal{D}\llbracket G\supset H\rrbracket\rho_2 \arrow[r, "\Lambda(p_H\circ App", "\circ(\mathrm{id}\times i_G))"'] & \begin{array}{c}\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{C}_1\llbracket F\rrbracket\rho_1),\\ \mathfrak{F}_1(\mathcal{C}_1\llbracket G\rrbracket\rho_1))\end{array} \arrow[r, "\mathfrak{R}_{F\supset G}"'] & \mathfrak{F}_1(\mathcal{C}_1\llbracket F\supset G\rrbracket\rho_1)
\end{tikzcd}$$

where $i_{G\supset H}$ is the top line, $p_{G\supset H}$ is the bottom line, the right square commutes by (31), and the left square commutes, as calculation shows:

$$\begin{aligned}
&\Lambda(p_H\circ App\circ(\mathrm{id}\times i_G))\circ\Lambda(i_H\circ App\circ(\mathrm{id}\times p_G))\\
&=\Lambda(p_H\circ App\circ(\mathrm{id}\times i_G)\circ(\Lambda(i_H\circ App\circ(\mathrm{id}\times p_G))\times\mathrm{id}))\qquad(\text{by }(k'))\\
&=\Lambda(p_H\circ App\circ(\Lambda(i_H\circ App\circ(\mathrm{id}\times p_G))\times i_G))\\
&=\Lambda(p_H\circ App\circ(\Lambda(i_H\circ App\circ(\mathrm{id}\times p_G))\times\mathrm{id})\circ(\mathrm{id}\times i_G))\\
&=\Lambda(p_H\circ i_H\circ App\circ(\mathrm{id}\times p_G)\circ(\mathrm{id}\times i_G))\qquad(\text{by }(l''))\\
&=\Lambda(p_H\circ i_H\circ App\circ(\mathrm{id}\times(p_G\circ i_G)))\\
&=\Lambda(App\circ(\mathrm{id}\times\mathrm{id}))\qquad(\text{since }p_H\circ i_G=\mathrm{id},\ p_G\circ i_G=\mathrm{id})\\
&=\mathrm{id}\qquad(\text{by }(l''))
\end{aligned}$$

So $p_{G\supset H}\circ i_{G\supset H}=\mathrm{id}$.

We now build h^1_F and h^2_F. Note that as soon as Diagram 33 commutes, h^1_F and h^2_F will automatically be monic. Indeed, since $p_F\circ i_F=\mathrm{id}$, i_F is monic; as $i_F=m''_F\circ h^1_F$, h^1_F will be monic, too. Similarly, since $\langle m'_F, m''_F\rangle$ is a monic (because it is part of the definition of an object in the subscone), and $\langle m'_F, m''_F\rangle=\langle p_F,\mathrm{id}\rangle\circ h^2_F$, h^2_F will be a monic, too.

Also, that h^1_F and h^2_F are monic will imply that $\tilde{h}^1_F\hat{=}(h^1_F,\mathrm{id})$ and $\tilde{h}^2_F\hat{=}(h^2_F,\mathrm{id})$ will be monic in $(\mathcal{D}\wr(\mathfrak{F}_1\otimes\mathrm{id}))\llbracket F\rrbracket\tilde{\rho}$. Therefore, that Diagram 33 is commutative is equivalent to showing the existence of the following diagram in $\mathcal{D}{\downarrow}(\mathfrak{F}_1 \otimes \mathrm{id})$:

$$\tilde{I}_F \xhookrightarrow{(h_F^1, \mathrm{id})} (\mathcal{D}{\downarrow}(\mathfrak{F}_1 \otimes \mathrm{id}))\,[\![F]\!]\,\tilde{\rho} \xhookrightarrow{(h_F^2, \mathrm{id})} \tilde{P}_F$$

where

$$\begin{aligned}
\tilde{I}_F &\hat{=} (\mathfrak{F}_1(\mathcal{C}_1\,[\![F]\!]\,\rho_1), (\mathcal{C}_1\,[\![F]\!]\,\rho_1, \mathcal{D}\,[\![F]\!]\,\rho_2), \langle \mathrm{id}, i_F \rangle) \\
\tilde{P}_F &\hat{=} (\mathcal{D}\,[\![F]\!]\,\rho_1, (\mathcal{C}_1\,[\![F]\!]\,\rho_1, \mathcal{D}\,[\![F]\!]\,\rho_2), \langle p_F, \mathrm{id} \rangle)
\end{aligned}$$

We build h_F^1 and h_F^2 by structural induction on F.

If F is a base type A, notice that $i_A = p_A = m'_A = m''_A = \mathrm{id}$, so take $h_A^1 \hat{=} h_A^2 \hat{=} \mathrm{id}$.

If F is a box formula $\Box G$. Recall first that $p_{\Box G} = \mathfrak{R}_{\Box G} \circ \Box p_G$, and $i_{\Box G} = \Box i_G \circ \mathfrak{I}^1$ (we drop indices to $\mathfrak{I}^1$ to avoid clutter). On the other hand, the definition of $\Box$ in the subscone is by the following pullback diagram:

$$\begin{tikzcd}
D_{\Box G} \arrow[r, "\langle m'_{\Box G}, m''_{\Box G} \rangle"] \arrow[d, "\widehat{\langle m'_G, m''_G \rangle}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}_1(\mathcal{C}_1\,[\![\Box G]\!]\,\rho_1) \times \mathcal{D}\,[\![\Box G]\!]\,\rho_2 \arrow[d, "\mathfrak{I}^1 \times \mathrm{id}"] \\
\Box D_G \arrow[r, "\langle \Box m'_G, \Box m''_G \rangle"'] & \Box \mathfrak{F}_1(\mathcal{C}_1\,[\![G]\!]\,\rho_1) \times \Box \mathcal{D}\,[\![G]\!]\,\rho_2
\end{tikzcd} \tag{34}$$

This allows us to define $h_{\Box G}^1$ by the universal property of the pullback:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1\,[\![\Box G]\!]\,\rho_1 \arrow[dr, dashed, "h_{\Box G}^1"] \arrow[drr, "{\langle \mathrm{id}, \Box i_G \circ \mathfrak{I}^1 \rangle}"] \arrow[ddr, "\Box h_G^1 \circ \mathfrak{I}^1"'] & & \\
& D_{\Box G} \arrow[r, "\langle m'_{\Box G}, m''_{\Box G} \rangle"] \arrow[d, "\widehat{\langle m'_G, m''_G \rangle}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}_1(\mathcal{C}_1\,[\![\Box G]\!]\,\rho_1) \times \mathcal{D}\,[\![\Box G]\!]\,\rho_2 \arrow[d, "\mathfrak{I}^1 \times \mathrm{id}"] \\
& \Box D_G \arrow[r, "\langle \Box m'_G, \Box m''_G \rangle"'] & \Box \mathfrak{F}_1(\mathcal{C}_1\,[\![G]\!]\,\rho_1) \times \Box \mathcal{D}\,[\![G]\!]\,\rho_2
\end{tikzcd}$$

The upper triangle is exactly the left part of the diagram we are looking for:

$$\begin{tikzcd}
& \mathfrak{F}_1(\mathcal{C}_1\,[\![\Box G]\!]\,\rho_1) \\
\mathfrak{F}_1(\mathcal{C}_1\,[\![\Box G]\!]\,\rho_1) \arrow[ur, "\mathrm{id}"] \arrow[r, "h_{\Box G}^1"] \arrow[dr, "i_{\Box G}"'] & D_{\Box G} \arrow[u, "m'_{\Box G}"'] \arrow[d, "m''_{\Box G}"] \\
& \mathcal{D}\,[\![\Box G]\!]\,\rho_2
\end{tikzcd}$$

On the other hand, the right part is given by $h_{\Box G}^2 \hat{=} m''_{\Box G}$, provided we can show that $p_{\Box G} \circ m''_{\Box G} = m'_{\Box G}$. To show this, note that we can rewrite Diagram (34) as:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket \Box G \rrbracket \rho_1) \arrow[d, "\mathfrak{I}^1"'] & D_{\Box G} \arrow[l, "m'_{\Box G}"'] \arrow[r, "m''_{\Box G}"] \arrow[d, "\widehat{\langle m'_G, m''_G \rangle}"'] & \mathcal{D} \llbracket \Box G \rrbracket \rho_2 \arrow[d, "\mathrm{id}"] \\
\Box\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \rrbracket \rho_1) & \Box D_G \arrow[l, "\Box m'_G"] \arrow[r, "\Box m''_G"'] & \mathcal{D} \llbracket \Box G \rrbracket \rho_2
\end{tikzcd}$$

Since by induction hypothesis $p_G \circ m''_G = m'_G$, adding the morphism $\Box p_G$ from the lower right $\mathcal{D} \llbracket \Box G \rrbracket \rho_2$ to the lower left $\Box\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \rrbracket \rho_1)$ makes another commutative diagram. It follows that (reading from the topmost $D_{\Box G}$ to the lower left $\Box\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \rrbracket \rho_1)$ in two different ways) $\Box p_G \circ m''_{\Box G} = \mathfrak{I}^1 \circ m'_{\Box G}$. Composing with $\mathfrak{R}_{\Box G}$ on the left we get $\mathfrak{R}_{\Box G} \circ \Box p_G \circ m''_{\Box G} = \mathfrak{R}_{\Box G} \circ \mathfrak{I}^1 \circ m'_{\Box G}$, that is, $p_{\Box G} \circ m''_{\Box G} = m'_{\Box G}$, as desired. This terminates the box case.

If F is an arrow type $G \supset H$. We first build $h^1_{G \supset H}$. Construct the morphism:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \times \mathfrak{F}_1(\mathcal{C}_1 \llbracket G \rrbracket \rho_1) \arrow[d, "\mathfrak{F}_1(App)"'] & \mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \times D_G \arrow[l, "\mathrm{id} \times m'_G"'] \\
\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1) \arrow[r, "h^1_H"] & D_H
\end{tikzcd}$$

in $\mathcal{D}$. For short, let us call this morphism u temporarily.

Also construct the morphism $v \hat{=} (App, App)$ from $(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1, \mathcal{D} \llbracket G \supset H \rrbracket \rho_2) \times (\mathcal{C}_1 \llbracket G \rrbracket \rho_1, \mathcal{D} \llbracket G \rrbracket \rho_2)$ to $(\mathcal{C}_1 \llbracket H \rrbracket \rho_1, \mathcal{D} \llbracket H \rrbracket \rho_2)$ in $\mathcal{C}_1 \times \mathcal{D}$.

We claim that (u, v) is a morphism in $\mathcal{D} \downarrow (\mathfrak{F}_1 \otimes \mathrm{id})$ (if so, this is from $\tilde{I}_{G \supset H} \times (\mathcal{D} \downarrow (\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket G \rrbracket \tilde{\rho}$ to $(\mathcal{D} \downarrow (\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket H \rrbracket \tilde{\rho}$). This requires us to show that the following diagram commutes (where we have split products so as to increase readability):

$$\begin{tikzcd}
\begin{array}{c}\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \\ \times \mathfrak{F}_1(\mathcal{C}_1 \llbracket G \rrbracket \rho_1)\end{array} \arrow[d, "\mathfrak{F}_1(App)"'] & \begin{array}{c}\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \\ \times D_G\end{array} \arrow[l, "\mathrm{id} \times m'_G"'] \arrow[r, "i_{G \supset H}", "\times m''_G"'] \arrow[d, "u"'] & \begin{array}{c}\mathcal{D} \llbracket G \supset H \rrbracket \rho_2 \\ \times \mathcal{D} \llbracket G \rrbracket \rho_2\end{array} \arrow[d, "App"] \\
\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1) & D_H \arrow[l, "m'_H"] \arrow[r, "m''_H"'] & \mathcal{D} \llbracket H \rrbracket \rho_2
\end{tikzcd} \tag{35}$$

The left square of (35) commutes because $m'_H \circ h^1_H$ is the identity on D_H, by induction hypothesis, so

$$m'_H \circ u = m'_H \circ h^1_H \circ \mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G) = \mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)$$

For the right square of (35), note that $App \circ (i_{G \supset H} \times \mathrm{id}) = i_H \circ \mathfrak{F}_1(App) \circ (\mathrm{id} \times p_G)$, by the definition of $i_{G \supset H}$. Composing with $\mathrm{id} \times m''_G$ on the right, it follows $App \circ (i_{G \supset H} \times m''_G) = i_H \circ \mathfrak{F}_1(App) \circ (\mathrm{id} \times (p_G \circ m''_G))$. However $p_G \circ m''_G = m'_G$ by induction hypothesis. So we get:

$$App \circ (i_{G \supset H} \times m''_G) = i_H \circ \mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)$$

Since $m''_H \circ h^1_H = i_H$ by induction hypothesis, the right-hand side is exactly $m''_H \circ u$, so the right square of (35) commutes.

As (35) commutes, (u, v) is indeed a morphism in $\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id})$. We may then curry it in $\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id})$, getting a morphism from $\tilde{I}_{G\supset H}$ to the internal hom object $\mathbf{Hom}_{\mathcal{D}\downarrow(\mathfrak{F}_1\otimes\mathrm{id})}((\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id}))\,[\![G]\!]\,\tilde{\rho}, (\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id}))\,[\![H]\!]\,\tilde{\rho})$, that is, from $\tilde{I}_{G\supset H}$ to $(\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id}))\,[\![G \supset H]\!]\,\tilde{\rho}$.

Let us call this latter morphism $\tilde{h}^1_{G\supset H}$. As recalled in Paragraph 5.4.0.3, this morphism is of the form $(\hat{u}, \Lambda(v))$, where $\Lambda(v)$ is taken in the product category $\mathcal{C}_1 \times \mathcal{D}$. Since v was application is this category, $\Lambda(v) = \mathrm{id}$. So $\tilde{h}^1_{G\supset H}$ is of the required form $(h^1_{G\supset H}, \mathrm{id})$; i.e., we let $h^1_{G\supset H}$ be $\hat{u}$. Because this is a morphism in $\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id})$, the following diagram commutes:

$$
\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \arrow[r, "\langle \mathrm{id}, i_{G\supset H}\rangle"] \arrow[d, "h^1_{G\supset H}"'] & \mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \times \mathcal{D}\,[\![G \supset H]\!]\,\rho_2 \arrow[d, "\mathrm{id}"] \\
D_{G\supset H} \arrow[r, "\langle m'_{G\supset H}, m''_{G\supset H}\rangle"'] & \mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \times \mathcal{D}\,[\![G \supset H]\!]\,\rho_2
\end{tikzcd}
$$

This is exactly the left part of the desired diagram:

$$
\begin{tikzcd}
& \mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \\
\mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \arrow[ur, "\mathrm{id}"] \arrow[r, "h^1_{G\supset H}"] \arrow[dr, "i_{G\supset H}"'] & D_{G\supset H} \arrow[u, "m'_{G\supset H}"'] \arrow[d, "m''_{G\supset H}"] \\
& \mathcal{D}\,[\![G \supset H]\!]\,\rho_2
\end{tikzcd}
$$

Let us now build $h^2_{G\supset H}$. Define $h^2_{G\supset H} \hat{=} m''_{G\supset H}$. Adapting the definition of internal homs to the subscone category $\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id})$, and using a few trivial isomorphisms, $(\mathcal{D}\downarrow(\mathfrak{F}_1 \otimes \mathrm{id}))\,[\![G \supset H]\!]\,\tilde{\rho}$ is given by the following pullback diagram:

$$
\begin{tikzcd}
D_{G\supset H} \arrow[r, "\langle m'_{G\supset H}, m''_{G\supset H}\rangle"] \arrow[d, "s_{G\supset H}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{F}_1(\mathcal{C}_1\,[\![G \supset H]\!]\,\rho_1) \times \mathcal{D}\,[\![G \supset H]\!]\,\rho_2 \arrow[d, "\Lambda(\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)) \times \Lambda(App \circ (\mathrm{id} \times m''_G))"] \\
\mathbf{Hom}_{\mathcal{D}}(D_G, D_H) \arrow[r, "\langle \Lambda(m'_H \circ App), \Lambda(m''_H \circ App)\rangle"'] & \mathbf{Hom}_{\mathcal{D}}(D_G, \mathfrak{F}_1(\mathcal{C}_1\,[\![H]\!]\,\rho_1)) \times \mathbf{Hom}_{\mathcal{D}}(D_G, \mathcal{D}\,[\![H]\!]\,\rho_2)
\end{tikzcd}
$$

Splitting products, we may rewrite this as:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \arrow[d, "\Lambda(\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G))"] & D_{G \supset H} \arrow[l, "m'_{G \supset H}"'] \arrow[r, "m''_{G \supset H}"] \arrow[d, "s_{G \supset H}"] & \mathcal{D} \llbracket G \supset H \rrbracket \rho_2 \arrow[d, "\Lambda(App \circ (\mathrm{id} \times m''_G))"'] \\
\mathbf{Hom}_{\mathcal{D}}(D_G, \mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1)) & \mathbf{Hom}_{\mathcal{D}}(D_G, D_H) \arrow[l, "\Lambda(m'_H \circ App)"'] \arrow[r, "\Lambda(m''_H \circ App)"] & \mathbf{Hom}_{\mathcal{D}}(D_G, \mathcal{D} \llbracket H \rrbracket \rho_2)
\end{tikzcd}$$

Take the product with D_G:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \times D_G \arrow[d, "\Lambda(\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)) \times \mathrm{id}"] & D_{G \supset H} \times D_G \arrow[l, "m'_{G \supset H} \times \mathrm{id}"'] \arrow[r, "m''_{G \supset H} \times \mathrm{id}"] \arrow[d, "s_{G \supset H} \times \mathrm{id}"] & \mathcal{D} \llbracket G \supset H \rrbracket \rho_2 \times D_G \arrow[d, "\Lambda(App \circ (\mathrm{id} \times m''_G)) \times \mathrm{id}"'] \\
\mathbf{Hom}_{\mathcal{D}}(D_G, \mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1)) \times D_G & \mathbf{Hom}_{\mathcal{D}}(D_G, D_H) \times D_G \arrow[l, "\Lambda(m'_H \circ App) \times \mathrm{id}"'] \arrow[r, "\Lambda(m''_H \circ App) \times \mathrm{id}"] & \mathbf{Hom}_{\mathcal{D}}(D_G, \mathcal{D} \llbracket H \rrbracket \rho_2) \times D_G
\end{tikzcd}$$

Putting this above the following diagram:

$$\begin{tikzcd}
\mathbf{Hom}_{\mathcal{D}}(D_G, \mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1)) \times D_G \arrow[d, "App"'] & \mathbf{Hom}_{\mathcal{D}}(D_G, D_H) \times D_G \arrow[l, "\Lambda(m'_H \circ App) \times \mathrm{id}"'] \arrow[r, "\Lambda(m''_H \circ App) \times \mathrm{id}"] \arrow[d, "App"'] & \mathbf{Hom}_{\mathcal{D}}(D_G, \mathcal{D} \llbracket H \rrbracket \rho_2) \times D_G \arrow[d, "App"] \\
\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1) & D_H \arrow[l, "m'_H"] \arrow[r, "m''_H"'] & \mathcal{D} \llbracket H \rrbracket \rho_2
\end{tikzcd}$$

which is easily seen to commute, we obtain:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) \times D_G \arrow[d, "\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)"] & D_{G \supset H} \times D_G \arrow[l, "m'_{G \supset H} \times \mathrm{id}"'] \arrow[r, "m''_{G \supset H} \times \mathrm{id}"] \arrow[d, "s_{G \supset H}"'] & \mathcal{D} \llbracket G \supset H \rrbracket \rho_2 \times D_G \arrow[d, "App \circ (\mathrm{id} \times m''_G)"'] \\
\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1) & D_H \arrow[l, "m'_H"] \arrow[r, "m''_H"'] & \mathcal{D} \llbracket H \rrbracket \rho_2
\end{tikzcd} \qquad (36)$$

Indeed, the leftmost vertical morphism is $App \circ (\Lambda(\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)) \times \mathrm{id}) = \mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G)$, while the rightmost vertical morphism is obtained similarly.

By induction hypothesis $m'_H = p_H \circ m''_H$. So we may complete Diagram (36) by adding a p_H arrow from the lower right $\mathcal{D} \llbracket H \rrbracket \rho_2$ to the lower left $\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1)$, and get a commutative diagram again. Looking at the leftmost and the rightmost paths from the upper $D_{G \supset H} \times D_G$ to the lower left $\mathfrak{F}_1(\mathcal{C}_1 \llbracket H \rrbracket \rho_1)$, it follows:

$$\mathfrak{F}_1(App) \circ (\mathrm{id} \times m'_G) \circ (m'_{G \supset H} \times \mathrm{id}) = p_H \circ App \circ (\mathrm{id} \times m''_G) \circ (m''_{G \supset H} \times \mathrm{id})$$

Composing with $\mathrm{id} \times h^1_G$ on the right and simplifying, we obtain $\mathfrak{F}_1(App) \circ (m'_{G \supset H} \times (m'_G \circ h^1_G)) = p_H \circ App \circ (m''_{G \supset H} \times (m''_G \circ h^1_G))$. By induction hypothesis $m'_G \circ h^1_G = \mathrm{id}$, and $m''_G \circ h^1_G = i_G$, so:

$$\mathfrak{F}_1(App) \circ (m'_{G \supset H} \times \mathrm{id}) = p_H \circ App \circ (m''_{G \supset H} \times i_G) \qquad (37)$$

This entails that $\Lambda(p_H \circ App \circ (\mathrm{id} \times i_G)) \circ m''_{G \supset H} = \Lambda(p_H \circ App \circ (\mathrm{id} \times i_G) \circ (m''_{G \supset H} \times \mathrm{id})) = \Lambda(p_H \circ App \circ (m''_{G \supset H} \times i_G)) = \Lambda(\mathfrak{F}_1(App) \circ (m'_{G \supset H} \times \mathrm{id}))$ (using (37)) $= \Lambda(\mathfrak{F}_1(App)) \circ m'_{G \supset H}$. Composing with $\mathfrak{R}_{G \supset H}$ on the left, remembering that $\mathfrak{R}_{G \supset H} \circ \Lambda(\mathfrak{F}_1(App)) = \mathrm{id}$ (Diagram (31)), it obtains:

$$\mathfrak{R}_{G \supset H} \circ \Lambda(p_H \circ App \circ (\mathrm{id} \times i_G)) \circ m''_{G \supset H} = m'_{G \supset H}$$

That is, $p_{G \supset H} \circ m''_{G \supset H} = m'_{G \supset H}$.

On the other hand, recall that $h^2_{G \supset H} = m''_{G \supset H}$. So we have got the right part of the desired diagram:

$$\begin{tikzcd}
\mathfrak{F}_1(\mathcal{C}_1 \llbracket G \supset H \rrbracket \rho_1) & \\
D_{G \supset H} \arrow[u, "m'_{G \supset H}"] \arrow[r, "h^2_{G \supset H}"] \arrow[d, "m''_{G \supset H}"] & \mathcal{D} \llbracket G \supset H \rrbracket \rho_2 \arrow[ul, "p_{G \supset H}"'] \arrow[dl, "\mathrm{id}"] \\
\mathcal{D} \llbracket G \supset H \rrbracket \rho_2 &
\end{tikzcd}$$

This terminates the implication case. □

5.7. Equational Completeness

We can now prove:

Theorem 74 (Equational Completeness). *Let $\mathcal{D}$ be a strict CS4 category, $\mathfrak{F}_1 : \mathcal{S4}_\Sigma \to \mathcal{D}$ preserve finite products. Assume also that $\mathcal{D}$ is finitely complete, that $\Box$ preserves monics in $\mathcal{D}$ as well as pullbacks along some $(\mathfrak{F}_1, \Box)$-commutator $\mathfrak{I}^1$, and that for every formulae F and G, $\mathbf{Hom}_{\mathcal{D}}(\mathfrak{F}_1(\mathcal{S4}_\Sigma \llbracket F \rrbracket (\subseteq)), \mathfrak{F}_1(\mathcal{S4}_\Sigma \llbracket G \rrbracket (\subseteq)))$ retracts strongly onto $\mathfrak{F}_1(\mathcal{S4}_\Sigma \llbracket F \supset G \rrbracket (\subseteq))$, and $\Box\mathfrak{F}_1(\mathcal{S4}_\Sigma \llbracket F \rrbracket (\subseteq))$ retracts strongly onto $\mathfrak{F}_1(\mathcal{S4}_\Sigma \llbracket \Box F \rrbracket (\subseteq))$, where $\subseteq$ is the canonical inclusion of Σ into $\mathcal{S4}_\Sigma$.*

Assume finally that $\mathfrak{F}_1$ is faithful on morphisms with domain the empty context.

Then there is a valuation $\rho_2 : \Sigma \to \mathcal{D}$ such that, for every λ_{S4}-terms M and N of type F under Γ, $M \approx N$ if and only if $\mathcal{D} \llbracket M \rrbracket \rho_2 = \mathcal{D} \llbracket N \rrbracket \rho_2$.

Proof. The only if direction is soundness (Lemma 58). Let us deal with the if direction. Without loss of generality, assume M and N ground, and Γ the empty context: if Γ is not empty, say $\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n$, we reduce to the empty case by reasoning on $\lambda x_1, \ldots, x_n \cdot M$ and $\lambda x_1, \ldots, x_n \cdot N$ instead of M and N. Take $\mathcal{C}_1 \hat{=} \mathcal{S4}_\Sigma$, ρ_1 be $\subseteq$. As in Lemma 73, let ρ_2 be $\mathfrak{F}_1 \circ \rho_1$, and $\tilde{\rho}$ map $A \in \Sigma$ to $(\rho_2(A), (\rho_1(A), \rho_2(A)), \langle \mathrm{id}, \mathrm{id} \rangle)$.

By the Basic Lemma (Lemma 72),

$$U((\mathcal{D} \wr (\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket F \rrbracket \tilde{\rho}) = (\mathcal{S4}_\Sigma \llbracket F \rrbracket \rho_1, \mathcal{D} \llbracket F \rrbracket \rho_2)$$

where the forgetful functor U maps each morphism (u, v) in the subscone to v. Expanding the definition of U in this case, for every type derivation of $\vdash M : F$, the morphism $(\mathcal{D} \wr (\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket M \rrbracket \tilde{\rho}$ from $\mathbf{1}$ to $(\mathcal{D} \wr (\mathfrak{F}_1 \otimes \mathrm{id})) \llbracket F \rrbracket \tilde{\rho}$ can be written (u, v), where the Basic Lemma demands that $v = \mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1 \times \mathcal{D} \llbracket M \rrbracket \rho_2$. Since (u, v) is a morphism in $\mathcal{D}\mathcal{L}(\mathfrak{F}_1 \otimes \mathrm{id})$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, hook, "\mathrm{id}"] \arrow[d, "u"'] & \mathbf{1} \arrow[d, "\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1)"', "\times \mathcal{D} \llbracket M \rrbracket \rho_2"] \\
D_F \arrow[r, hook, "\langle m'_F, m''_F \rangle"'] & \mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket F \rrbracket \rho_1) \times \mathcal{D} \llbracket F \rrbracket \rho_2
\end{tikzcd}$$

That is,

$$m'_F \circ u = \mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1) \qquad m''_F \circ u = \mathcal{D} \llbracket M \rrbracket \rho_2$$

By the Bounding Lemma (Lemma 73, Diagram (33)), $p_F \circ m''_F = m'_F$, so:

$$\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1) = p_F \circ \mathcal{D} \llbracket M \rrbracket \rho_2$$

As this holds for every M such that $\vdash M : F$ is derivable, it follows immediately that if we take any two such terms M and N, such that $\mathcal{D} \llbracket M \rrbracket \rho_2 = \mathcal{D} \llbracket N \rrbracket \rho_2$, then $\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1) = \mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket N \rrbracket \rho_1)$. Since $\mathfrak{F}_1$ is faithful on morphisms with domain the empty context, $\mathcal{S}4_\Sigma \llbracket M \rrbracket \rho_1 = \mathcal{S}4_\Sigma \llbracket N \rrbracket \rho_1$. Since $\rho_1 = (\subseteq)$, by Proposition 64, $M \approx N$. $\Box$

Corollary 75 (Equational Completeness in $\widehat{\Delta}$). *There is a valuation $\rho_2 : \Sigma \to \widehat{\Delta}$ such that, for every λ_{S4}-terms M and N of type F under Γ, $M \approx N$ if and only if $\widehat{\Delta} \llbracket M \rrbracket \rho_2 = \widehat{\Delta} \llbracket N \rrbracket \rho_2$.*

Proof. Let us check all hypotheses. First, $\widehat{\Delta}$ is a strict CS4 category using Definition 50. Take $\mathfrak{F}_1 \hat{=} \mathcal{S}4\,[\_]$ (alternatively, the contracting resolution functor $\mathrm{CRes}_{\mathcal{S}4_\Sigma}$). By Lemma 65, $\mathfrak{F}_1$ preserves all finite products and the $(\Box, \mathbf{d}, \mathbf{s})$ comonad (so the canonical isomorphism from $\mathfrak{F}_1(\Box c)$ to $\Box\mathfrak{F}_1(c)$ is an $(\mathfrak{F}_1, \Box)$-commutator along which $\Box$ preserves all pullbacks, too), and is faithful. $\widehat{\Delta}$ is finitely complete (in fact a topos). And $\Box$ preserves monics: recall that a monic in $\widehat{\Delta}$ is an a.s. map $(f_q)_{q \geqslant -1}$ such that every f_q is one-to-one ([**15**] 1.462); it follows that $\Box$, which maps $(f_q)_{q \geqslant -1}$ to $(f_{q+1})_{q \geqslant -1}$, preserves monics in $\widehat{\Delta}$. $\Box\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket F \rrbracket (\subseteq))$ retracts strongly onto $\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket \Box F \rrbracket (\subseteq))$, because they are canonically isomorphic. Finally, by Corollary 48, $\mathbf{Hom}_{\widehat{\Delta}}(\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket F \rrbracket (\subseteq)), \mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket G \rrbracket (\subseteq)))$ retracts strongly onto $\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket F \supset G \rrbracket (\subseteq))$ (observe that $\mathfrak{F}_1(\mathcal{S}4_\Sigma \llbracket F \rrbracket (\subseteq)) = \mathcal{S}4\,[F]$, and similarly for G). $\Box$

6. Towards Equational Completeness in Topological Spaces

The equational completeness problem in **CGHaus** (with the comonad of Section 5.1.1) is currently open. We may attack the problem by using Theorem 74, and this requires us to find a suitable functor $\mathfrak{F}_1 : \mathcal{S}4_\Sigma \to \mathbf{CGHaus}$. A natural choice, considering that $\mathcal{S}4_\Sigma$ is already equipped with a functor $\mathcal{S}4\,[\_] : \mathcal{S}4_\Sigma \to \widehat{\Delta}$, is to compose it with some geometric realization functor. The standard geometric realization functor [**39, 36**] however confuses all (-1)-dimensional objects to one element, and is therefore a bad choice. A better choice is as follows. Notice that the extended singular simplex functor **XSing** of Definition 55 has a left adjoint, since **CGHaus** is small cocomplete ([**36**] VII.8), which may be expressed as a coend: for each a.s. set K,

$$\mathbf{XGeom}(K) \hat{=} \int^q K_q \cdot \Delta^+{}_q$$

where the copower $K_q \cdot \Delta^+{}_q$ is the topological sum of $|K_q|$ disjoint copies of $\Delta^+{}_q$. More concretely, the points of $\mathbf{XGeom}(K)$ are pairs (y, v) where $y \in K_q$ and $v \in \Delta^+{}_q$, $q \geqslant -1$, modulo the smallest equivalence relation $\sim$ such that $(\partial_q^i y, v) \sim (y, \delta_q^i v)$ and $(s_q^i y, v) \sim (y, \sigma_q^i v)$ for all $0 \leqslant i \leqslant q$, where

$$\delta_q^i(t_0, \ldots, t_{q-1}) \hat{=} (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1}) \tag{38}$$

$$\sigma_q^i(t_0, \ldots, t_{q+1}) \hat{=} (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{q+1}) \tag{39}$$

This is then topologized as a quotient space. By standard arguments (adapted from, e.g., [**39**], Lemma 14.2), every point can be written in a unique way as a *non-degenerate point* (y, v), that is one such that y is not the degeneracy of any simplex, and v is a tuple of non-zero elements.

The main problem in using **XGeom** for equational completeness purposes is that **XGeom** does *not* preserve terminal objects. Indeed, consider $\mathbf{XGeom}(\mathbf{1})$. For clarity, write $*_q$ the unique q-simplex of $\mathbf{1}$; it is non-degenerate if and only if $q \leqslant 0$. The non-degenerate points of $\mathbf{XGeom}(\mathbf{1})$ are $(*_{-1}, ())$ and $(*_0, (t))$ for all $0 < t \leqslant 1$. In other words, $\mathbf{XGeom}(\mathbf{1})$ is homeomorphic to $[0, 1]$, hardly the terminal object in **CGHaus**.

6.1. Labeled Topological Spaces

Nonetheless, a slight modification of the spaces we use seems to provide better models. We shall show that, under a plausible conjecture, equational completeness also holds when we interpret S4 proofs in this setting.

A *labeled topological space* is any tuple (X, E, ℓ) where X is a topological space, E is a set of so-called *labels*, and ℓ maps each connected component of X to some label in E. Note that some labels may not label any connected component in X, and that some labels may label more than one connected component. A more categorical definition of labeled topological spaces is as follows. Recall that the forgetful functor $|\_| : \mathbf{Top} \to \mathbf{Set}$ has a left-adjoint $\mathbf{D}$ mapping every set E to E with the discrete topology. This also works when replacing **Top** by **CGHaus**, as every discrete topological space is Kelley.

Definition 76 (Labeled Spaces). *The category of* labeled topological spaces *is the scone* $\mathbf{Top} \downarrow \mathbf{D}$.

The category of labeled Kelley spaces *is the scone* $\mathbf{CGHaus} \downarrow \mathbf{D}$.

This matches the description above. A labeled topological space is a triple (X, E, p) where p is a continuous function from X to E equipped with the discrete topology. So, for each label $e \in E$, $p^{-1}(e)$ is both an open and a closed subset of X, and is therefore a union of connected components of X. In this way, each connected component receives a unique label. Conversely, given any labeled space (X, E, ℓ) in the first sense, we obtain an object (X, E, p) in the scone where p maps each point of X, say in connected component c, to $\ell(c)$.

Morphisms of labeled spaces are pairs $(f, r) : (X, E, p) \to (X', E', p)$ such that the continuous function f is consistent with the *relabeling* function r, namely f sends any connected component of X of label e to a connected component with label $r(e)$.

We now define a structure of strict CS4 category on $\mathbf{CGHaus} \downarrow \mathbf{D}$. Note that $\mathbf{Set}$ and $\mathbf{CGHaus}$ are CCCs, $\mathbf{CGHaus}$ is finitely complete (in fact small complete), and $\mathbf{D}$ preserves all finites products, so $\mathbf{CGHaus} \downarrow \mathbf{D}$ is a CCC.

Definition 77 ($\Box$ Comonad in $\mathbf{CGHaus} \downarrow \mathbf{D}$). *Recall the* $(\Box, \mathbf{d}, \mathbf{s})$ *comonad in* $\mathbf{CGHaus}$ *(Definition 52).*

This defines a strict monoidal comonad on $\mathbf{CGHaus} \downarrow \mathbf{D}$ *by:*

- $\Box(X, E, p) \hat{=} (\Box X, |X|, e_0)$*, where* e_0 *maps* α *to* $\alpha(0)$*;*
- *for each morphism* $(f, r) : (X, E, p) \to (X', E', p')$*,* $\Box(f, r) \hat{=} (\Box f, |f|)$*;*
- $\mathbf{d} : \Box(X, E, p) \to (X, E, p)$ *is the morphism* $(\mathbf{d}, |p|)$*;*
- $\mathbf{s} : \Box(X, E, p) \to \Box^2(X, E, p)$ *is the morphism* $(\mathbf{s}, \mathbf{k})$*, where* $\mathbf{k}$ *maps each* $x \in X$ *to the constant path* $t \in [0, 1] \mapsto x$*.*

Note that this construction is distinct from the one of Proposition 70. In particular, we don't use any comonad on $\mathbf{Set}$.

Proposition 78. *The construction* $(\Box, \mathbf{d}, \mathbf{s})$ *of Definition 77 is a strict monoidal comonad on* $\mathbf{CGHaus}$*, making it a strict CS4 category.*

Proof. By Proposition 53, $\Box(X, E, p) \hat{=} (\Box X, |X|, e_0)$ is a tuple such that $\Box X$ is Kelley. Moreover, $e_0 : \Box X \to \mathbf{D}(|X|)$ is continuous: for every closed subset F of $|X|$, $e_0^{-1}(F)$ is the disjoint union of all cocones $\Box_{x_0} X$ with $x_0 \in F$, and is therefore closed in $\Box X$, which is equipped with the sum topology. So $\Box(X, E, p)$ is a labeled Kelley space.

If (f, r) is a morphism from (X, E) to (X', E') in $\mathbf{CGHaus} \downarrow \mathbf{D}$, that $\Box(f, r) \hat{=} (\Box f, |f|)$ is a morphism means that the following diagram commutes:

$$\begin{tikzcd}
\Box X \arrow[r, "e_0"] \arrow[d, "\Box f"'] & \mathbf{D}(|X|) \arrow[d, "\mathbf{D}(|f|)"] \\
\Box X' \arrow[r, "e_0"'] & \mathbf{D}(|X'|)
\end{tikzcd}$$

In other words, that for every α in $\Box X$, $f(\alpha(0)) = \Box f(\alpha)(0)$. This holds, since $\Box f(\alpha) = f \circ \alpha$ by definition. So $\Box$ indeed defines a functor from $\mathbf{CGHaus} \downarrow \mathbf{D}$ to $\mathbf{CGHaus} \downarrow \mathbf{D}$.

Then $\mathbf{d}$ is a morphism in the scone if and only if the following diagram commutes in **CGHaus**:

$$\begin{tikzcd} \Box X \arrow[r, "e_0"] \arrow[d, "\mathbf{d}"'] & \mathbf{D}(|X|) \arrow[d, "\mathbf{D}(|p|)"] \\ X \arrow[r, "p"'] & \mathbf{D}(E) \end{tikzcd}$$

This means that, for every α in $\Box X$, $p(\alpha(0))$ should equal $p(\alpha(1))$. Since α is a continuous path from $[0,1]$ to X, $\alpha(0)$ and $\alpha(1)$ are in the same connected component in X. So p labels them identically.

On the other hand, $\mathbf{s}$ is a morphism in the scone if and only if the following diagram commutes in **CGHaus**:

$$\begin{tikzcd} \Box X \arrow[r, "e_0"] \arrow[d, "\mathbf{s}"'] & \mathbf{D}(|X|) \arrow[d, "\mathbf{D}(\mathbf{k})"'] \\ \Box^2 X \arrow[r, "e_0"'] & \mathbf{D}(|\Box X|) \end{tikzcd}$$

In other words, for every α in $\Box X$, $\mathbf{k}(\alpha(0))$ should equal $\mathbf{s}(\alpha)(0)$. But the former is the function mapping each $t \in [0,1]$ to $\alpha(0)$, while the latter is $(t, t' \mapsto \alpha(tt'))(0) = (t' \mapsto \alpha(0))$. So $(\Box, \mathbf{d}, \mathbf{s})$ indeed defines a comonad on $\mathbf{CGHaus} \downarrow \mathbf{D}$.

We now claim that the defined comonad is strict monoidal. Recall that $\mathbf{1}$ in the scone is $(\mathbf{1}_{\mathbf{CGHaus}}, \mathbf{1}_{\mathbf{Set}}, \mathrm{id})$, that $\mathbf{1}_{\mathbf{CGHaus}}$ is the one-point Kelley space, that $\mathbf{1}_{\mathbf{Set}}$ is the one-point set. Then $\Box(\mathbf{1})$ is $(\Box \mathbf{1}_{\mathbf{CGHaus}}, |\mathbf{1}_{\mathbf{CGHaus}}|, e_0)$, where $\Box \mathbf{1}_{\mathbf{CGHaus}}$ is naturally isomorphic to $\mathbf{1}_{\mathbf{CGHaus}}$, $|\mathbf{1}_{\mathbf{CGHaus}}| = \mathbf{1}_{\mathbf{Set}}$, and e_0 is the unique functor to $\mathbf{1}_{\mathbf{Set}}$. So $\Box(\mathbf{1})$ is naturally isomorphic to $\mathbf{1}$. As far as binary products are concerned, note that $\Box((X, E, p) \times (X', E', p')) = \Box(X \times X', E \times E', p \times p')$ (by Paragraph 5.4.0.2) $= (\Box(X \times X'), |X \times X'|, e_0 \times e_0)$ where $\Box(X \times X')$ is naturally isomorphic to $\Box X \times \Box X'$ (through the F and G morphisms used in the proof of Proposition 53), that this isomorphism carries $e_0 \times e_0$ to e_0, and that $|X \times X'| = |X| \times |X'|$. So $\Box((X, E, p) \times (X', E', p')) = \Box(X, E, p) \times \Box(X', E', p')$.

Finally, it is easy to check that the strict monoidal comonad equations (n)–(y) are satisfied, knowing that they already hold on the first component of morphisms, by Proposition 53. We have already checked (n), (r), (v), (w); (o), (x), (y) are obvious.

Let us check (p): $\mathbf{d} \circ \Box(f, r) = (f, r) \circ \mathbf{d}$: letting (f, r) be a morphism from (X, E, p) to (X', E', p'), this requires us to show that $(\mathbf{d}, |p'|) \circ (\Box f, |f|) = (f, r) \circ (\mathbf{d}, |p|)$, namely that $\mathbf{d} \circ \Box f = f \circ \mathbf{d}$ in **CGHaus** (by Proposition 53) and that for every $x \in X$, $p'(f(x)) = r(p(x))$: this is exactly the condition stating that (f, r) is a morphism from (X, E, p) to (X', E', p').

Let us check (q): $\mathbf{s} \circ \Box(f, r) = \Box^2(f, r) \circ \mathbf{s}$. Letting (f, r) be a morphism from (X, E, p) to (X', E', p'), the left-hand side is $(\mathbf{s}, \mathbf{k}) \circ (\Box f, |f|)$, and the right-hand side is $(\Box^2 f, |\Box f|) \circ (\mathbf{s}, \mathbf{k})$. So it remains to show that $\mathbf{k} \circ |f| = |\Box f| \circ \mathbf{k}$, that is, that for every $x \in X$, the function $t \mapsto f(x)$ equals $(\alpha \mapsto f \circ \alpha)(t \mapsto x)$. This is clear.

Let us check (s): $\mathbf{d} \circ \mathbf{s}$ as a morphism from $\Box(X, E, p)$ to $\Box(X, E, p)$ in the scone is $(\mathbf{d}, |e_0|) \circ (\mathbf{s}, \mathbf{k}) = (\mathbf{d} \circ \mathbf{s}, e_0 \circ \mathbf{k})$. But $e_0 \circ \mathbf{k}$ maps every $x \in X$ to $(t \mapsto x)(0) = x$, so this is indeed the identity map.

Let us check (t): $\Box\mathbf{d} \circ \mathbf{s}$ as a morphism from $\Box(X, E, p)$ to $\Box(X, E, p)$ in the scone is $(\Box\mathbf{d}, |\mathbf{d}|) \circ (\mathbf{s}, \mathbf{k}) = (\Box\mathbf{d} \circ \mathbf{s}, |\mathbf{d}| \circ \mathbf{k})$. But $|\mathbf{d}| \circ \mathbf{k}$ maps every $x \in X$ to $(t \mapsto x)(1) = x$, so this is again the identity.

Let us check (u): $\Box\mathbf{s} \circ \mathbf{s}$ as a morphism from $\Box(X, E, p)$ to $\Box^3(X, E, p)$ in the scone is $(\Box\mathbf{s}, |\mathbf{s}|) \circ (\mathbf{s}, \mathbf{k})$, while $\mathbf{s} \circ \mathbf{s}$ is $(\mathbf{s}, \mathbf{k}) \circ (\mathbf{s}, \mathbf{k})$. They are equal since $|\mathbf{s}| \circ \mathbf{k}$ is the function mapping each $x \in X$ to the function $t', t'' \mapsto (t \mapsto x)(t't'')$, namely the constant function of two arguments returning x, and this is also what $\mathbf{k} \circ \mathbf{k}$ is. $\Box$

A more categorical definition of $\Box$ is as the following pullback:

$$\begin{tikzcd}
\Box X \arrow[r, "i"] \arrow[d, "e_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X) \arrow[d, "App_0"] \\
\mathbf{D}(|X|) \arrow[r, "\subseteq"'] & X
\end{tikzcd} \tag{40}$$

where $\subseteq$ is the canonical inclusion of $\mathbf{D}(|X|)$ inside X (equivalently, the counit of the adjunction $\mathbf{D} \dashv |\_|$) and App_0 maps every continuous function $f : [0,1] \to X$ to $f(0)$. Similarly, for every morphism (f, r) from (X, E, p) to (X', E', p'), $\Box(f, r)$ is defined by the following diagram:

$$\begin{tikzcd}
\Box X \arrow[rr, "i"] \arrow[dr, dashed, "\Box f"'] \arrow[dd, "e_0"'] & & \mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X) \arrow[dr, "\Lambda(f \circ App)"] \arrow[dd, "App_0"' near start] & \\
& \Box X' \arrow[rr, "i'" near start] \arrow[dd, "e_0"' near start] & & \mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X') \arrow[dd, "App_0"] \\
\mathbf{D}(|X|) \arrow[rr, "\subseteq"' near end] \arrow[dr, "\Box|f|"'] & & X \arrow[dr, "f"] & \\
& \mathbf{D}(|X'|) \arrow[rr, "\subseteq"'] & & X'
\end{tikzcd} \tag{41}$$

where the back square is the definition of $\Box X$, the front square that of $\Box X'$, the bottom square is trivial, the right square is immediate. Then the left square says precisely that $(\Box f, |f|)$ is a morphism in the scone. We let the reader check that this is equivalent to Definition 77.

Similarly to Proposition 54, we have:

Proposition 79. *The $\Box$ functor has a left-adjoint $\blacklozenge$ in $\mathbf{CGHaus} \downarrow \mathbf{D}$.*

Proof. Given a labeled space (X, E, p), define $\blacklozenge_{E,p}X$ as the pushout of the maps $X \xrightarrow{p} \mathbf{D}(E)$ and $X \xrightarrow{(\_,0)} X \times [0, 1]$ in **CGHaus**, where $(\_, 0)$ is the map $x \mapsto (x, 0)$ (recall that **CGHaus** is cocomplete). Informally, $\blacklozenge_{E,p}X$ consists of formal barycentric combinations $ax + be$, $a \geqslant 0$, $b \geqslant 0$, $a + b = 1$, with $p(x) = e$. We may equate this with e when $a = 0$, $b = 1$, except that we do have an element $e \in \blacklozenge_{E,p}X$ even when $p(x) = e$ for no $x \in X$. In other words, $\blacklozenge_{E,p}X$ is the sum of the cones of each $p^{-1}(e)$, $e \in E$ (in the usual sense) plus isolated points corresponding to those labels that are not in the range of p. This is also known as the *mapping cylinder* of the labeling map p.

We now let $\blacklozenge(X, E, p)$ be $(\blacklozenge_{E,p}X, E, \hat{p})$, where $\hat{p}$ is the unique map making the following diagram commute (the pushout square being the definition of $\blacklozenge_{E,p}X$):

$$
\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "{(\_,0)}"'] & \mathbf{D}(E) \arrow[d, "k"] \arrow[ddr, "\mathrm{id}"] & \\
X \times [0,1] \arrow[r, "q"'] \arrow[drr, "p \circ \pi_1"'] & \blacklozenge_{E,p}X \arrow[dr, "\hat{p}"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \mathbf{D}(E)
\end{tikzcd}
\tag{42}
$$

For every morphism (f, r) from (X, E, p) to (X', E', p'), $\blacklozenge(f, r)$ is given as $(\blacklozenge f, r)$, where $\blacklozenge f$ is the unique continuous map making the following diagram commute:

$$
\begin{tikzcd}
X \arrow[rr, "p"] \arrow[dr, "f"] \arrow[dd, "{(\_,0)}"'] & & \mathbf{D}(E) \arrow[dr, "r"] \arrow[dd, "k"] & \\
& X' \arrow[rr, "p'"] \arrow[dd, "{(\_,0)}"'] & & \mathbf{D}(E') \arrow[dd, "k'"] \\
X \times [0,1] \arrow[rr, "q"'] \arrow[dr, "f+\mathrm{id}"'] & & \blacklozenge_{E,p}X \arrow[dr, dashed, "\blacklozenge f"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& X' \times [0,1] \arrow[rr, "q'"'] & & \blacklozenge_{E',p'}X'
\end{tikzcd}
\tag{43}
$$

(The back square is the definition of $\blacklozenge_{E,p}$, the front square is that of $\blacklozenge_{E',p'}X'$, the top square is the fact that (f, r) is a morphism in the scone, the left square commutes trivially.) Now the following two diagrams commute:

$$
\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "{(\_,0)}"'] & \mathbf{D}(E) \arrow[d, "k"] \arrow[ddr, "r"] & \\
X \times [0,1] \arrow[r, "q"'] \arrow[drr, "r \circ p \circ \pi_1"'] \arrow[ur, phantom, "\ulcorner"] & \blacklozenge_{E,p} X \arrow[dr, "r \circ \hat{p}"] & \\
& & \mathbf{D}(E')
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "{(\_,0)}"'] & \mathbf{D}(E) \arrow[d, "k"] \arrow[ddr, "r"] & \\
X \times [0,1] \arrow[r, "q"'] \arrow[drr, "r \circ p \circ \pi_1"'] \arrow[ur, phantom, "\ulcorner"] & \blacklozenge_{E,p} X \arrow[dr, "\hat{p}' \circ \blacklozenge f"] & \\
& & \mathbf{D}(E')
\end{tikzcd}
$$

(The left one because of Diagram (42), composed with r on the left, the right one because: $\hat{p}' \circ \blacklozenge f \circ q = \hat{p}' \circ q' \circ (\mathrm{id} \times f)$ [using the bottom face of Diagram (43)] $= p' \circ \pi_1 \circ (\mathrm{id} \times f)$ [by Diagram (42)] $= p' \circ f \circ \pi_1 = r \circ p \circ \pi_1$ [since (f, r) is a morphism], and $\hat{p}' \circ \blacklozenge f \circ k = \hat{p}' \circ k' \circ r$ [using the right face of Diagram (43)] $= r$ [by Diagram (42)].) By the universal property of pushouts, $\hat{p}' \circ \blacklozenge f = r \circ \hat{p}$, therefore $(\blacklozenge f, r)$ is a morphism.

Because $\blacklozenge$ is given by universal constructions, $\blacklozenge$ yields a functor from $\mathbf{CGHaus} \downarrow \mathbf{D}$ to $\mathbf{CGHaus} \downarrow \mathbf{D}$.

We claim that $\blacklozenge \dashv \Box$. To show this, define the unit $\eta = (\eta^1, \eta^2)$ of the adjunction by the pullback property of $\Box$:

$$
\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, dashed, "\eta^1_X"] \arrow[drr, "\Lambda(q)"] & & \\
\mathbf{D}(E) \arrow[dr, "\mathbf{D}(|k|)"'] & \Box \blacklozenge_{E,p} X \arrow[r, "i"] \arrow[d, "e_0"] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Hom}_{\mathbf{CGHaus}}([0,1], \blacklozenge_{E,p} X) \arrow[d, "App_0"] \\
& \mathbf{D}(|\blacklozenge_{E,p} X|) \arrow[r, "\subseteq"'] & \blacklozenge_{E,p} X
\end{tikzcd}
$$

The outer diagram indeed commutes, since $App_0 \circ \Lambda(q)(x) = q(x, 0) = k(p(x))$ by Diagram (42). Letting $\eta^2_X \hat{=} |k|$ in the above diagram, it is immediate that η_X is a morphism in the scone (this is the leftmost parallelogram in the above diagram). Since η is defined by a universal construction, it is clearly natural in X.

Now define the counit $\epsilon = (\epsilon^1, \epsilon^2)$ of the adjunction by the pushout property of

$\blacklozenge$:

$$\begin{tikzcd}
\square X \arrow[r, "e_0"] \arrow[d, "{(_,0)}"'] & \mathbf{D}(|X|) \arrow[d, "k'"] \arrow[ddr, "\subseteq"] & \\
\square X \times [0,1] \arrow[r, "q'"'] \arrow[dr, "i \times id"'] & \blacklozenge_{|X|,e_0}\square X \arrow[dr, dashed, "\epsilon^1_X"] & \\
& \mathbf{Hom}_{\mathbf{CGHaus}}([0,1],X) \times [0,1] \arrow[r, "App"'] & X
\end{tikzcd}$$

The outer square commutes because both paths from $\square X$ to X are $\alpha \mapsto \alpha(0)$. Then ϵ^2_X is defined to be $|p| : |X| \to E$. Check that ϵ_X is a morphism, i.e., that $\mathbf{D}(|p|) \circ \hat{e}_0 = p \circ \epsilon^1_X$. Composing the diagram above with p, we get the diagram below, left.

$$\begin{tikzcd}
\square X \arrow[r, "e_0"] \arrow[d, "{(_,0)}"'] & \mathbf{D}(|X|) \arrow[d, "k'"] \arrow[ddr, "p\circ \subseteq"] & \\
\square X \times [0,1] \arrow[r, "q'"'] \arrow[drr, "p \circ App \circ (i \times id)"'] & \blacklozenge_{|X|,e_0}\square X \arrow[dr, "p \circ \epsilon^1_X"] & \\
& & \mathbf{D}(E)
\end{tikzcd}
\qquad
\begin{tikzcd}
\square X \arrow[r, "e_0"] \arrow[d, "{(_,0)}"'] & \mathbf{D}(|X|) \arrow[d, "k'"] \arrow[ddr, "p\circ \subseteq"] & \\
\square X \times [0,1] \arrow[r, "q'"'] \arrow[drr, "p \circ App \circ (i \times id)"'] & \blacklozenge_{|X|,e_0}\square X \arrow[dr, "\mathbf{D}(|p|) \circ \hat{e}_0"] & \\
& & \mathbf{D}(E)
\end{tikzcd}$$

Now notice that $\mathbf{D}(|p|)$ is exactly $p\circ \subseteq$. Also, (a) $\hat{e}_0 \circ q' = e_0 \circ \pi_1$ and (b) $\hat{e}_0 \circ k' = \mathrm{id}$, by the commutativity of the defining square for $\blacklozenge_{|X|,e_0}\square X$. Then, (c) $\subseteq \circ e_0 = App_0 \circ i$ by the commutativity of the pullback square defining $\square X$. Finally, (d) $p \circ App \circ (i \times \mathrm{id}) = p \circ App_0 \circ i \circ \pi_1$; indeed the left-hand side maps (α, t) to $p(i(\alpha)(t))$, the right-hand side maps it to $p(i(\alpha)(0))$: since p maps connected components to connected components (one-element sets in the discrete topology), $p(i(\alpha)(t)) = p(i(\alpha)(0))$. It follows that the diagram on the right above commutes: $\mathbf{D}(|p|) \circ \hat{e}_0 \circ q' = p\circ \subseteq \circ \hat{e}_0 \circ q' = p\circ \subseteq \circ e_0 \circ \pi_1$ (by (a)) $= p \circ App_0 \circ i \circ \pi_1$ (by (c)) $= p \circ App \circ (i \times \mathrm{id})$ (by (d)), and $\mathbf{D}(|p|) \circ \hat{e}_0 \circ k' = p\circ \subseteq \circ \hat{e}_0 \circ k' = p\circ \subseteq$ (by (b)). By the universality property of pushouts, considering the two diagrams above, $\mathbf{D}(|p|) \circ \hat{e}_0 = p \circ \epsilon^1_X$.

Again, since ϵ is defined by universal constructions, it is natural.

Now check that $\epsilon \circ \blacklozenge\eta = \mathrm{id}$. First observe that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "p"] \arrow[d, "(_,0)"'] & & \mathbf{D}(E) \arrow[d, "k"] \arrow[rr, "\mathbf{D}(|k|)"] & & \mathbf{D}(|\blacklozenge_{E,p}X|) \arrow[d, "k'"] \arrow[ddr, hook, "\subseteq"] & \\
X\times[0,1] \arrow[rr, "q"'] \arrow[drr, "\eta^1_X\times id"'] & & \blacklozenge_{E,p}X \arrow[drr, "\blacklozenge\eta^1_X"] & (a) & & \\
& (b) & \Box\blacklozenge_{E,p}X \arrow[rr, "q'"] \arrow[drr, "i\times id"'] & & \blacklozenge_{|\blacklozenge_{E,p}X|,e_0}\Box\blacklozenge_{E,p}X \arrow[dr, "\epsilon^1_{\Box\blacklozenge_{E,p}X}"] & (c) \\
& & & (d) & \mathbf{Hom_{CGHaus}}([0,1],X)\times[0,1] \arrow[r, "App"'] & \blacklozenge_{E,p}X
\end{tikzcd}$$

Indeed, (a) is by the commutativity of the right square in the definition of $\blacklozenge\eta_X$ (see Diagram (43)), (b) is by the bottom square, (c) and (d) are by definition of $\epsilon^1_{\blacklozenge_{E,p}X}$. Since the morphism from $X \times [0,1]$ to $\blacklozenge_{E,p}X$ in this diagram is $App \circ (i \times \mathrm{id}) \circ (\eta^1_X \times \mathrm{id}) = App \circ ((i \circ \eta^1_X) \times \mathrm{id}) = App(\Lambda(q) \times \mathrm{id})$ (by the diagram defining $\eta^1_X) = q$, and since the morphism from $\mathbf{D}(E)$ to $\blacklozenge_{E,p}X$ in this diagram is $\subseteq \circ\, \mathbf{D}(|k'|) = k'$, it follows by the universal property of the pushout that $\epsilon^1_X \circ \blacklozenge\eta^1_X = \mathrm{id}$.

Therefore, $(\epsilon \circ \blacklozenge\eta)_X = (\epsilon^1_{\blacklozenge_{E,p}X}, \epsilon^2_{\blacklozenge_{E,p}X}) \circ \blacklozenge(\eta^1_X, \eta^2_X) = (\epsilon^1_{\blacklozenge_{E,p}X}, |\hat{p}|) \circ (\blacklozenge\eta^1_X, |k'|) = (\epsilon^1_{\blacklozenge_{E,p}X} \circ \blacklozenge\eta^1_X, |\hat{p}| \circ |k'|)$ is the identity by the above considerations, and by the fact that $\hat{p} \circ k = \mathrm{id}$, using Diagram (42).

It remains to check that $\Box\epsilon \circ \eta = \mathrm{id}$. Again, first observe that the following diagram commutes:

$$\begin{tikzcd}
\Box X \arrow[d, "e_0"] \arrow[drr, "\eta^1_{\Box X}"] \arrow[drrr, "\Lambda(q')", "(b)"'] & & & & \\
\mathbf{D}(|X|) \arrow[drr, "\mathbf{D}(|k'|)"'] & (a) & \Box\blacklozenge_{|X|,e_0}\Box X \arrow[r, "i'"] \arrow[d, "e_0"] \arrow[dr, "\Box\epsilon^1_X"] & \mathbf{Hom_{CGHaus}}([0,1], \blacklozenge_{|X|,e_0}\Box X) \arrow[dr, "\Lambda(\epsilon^1_X\circ App)"] & \\
& & \mathbf{D}(|\blacklozenge_{|X|,e_0}\Box X|) \arrow[dr, "\mathbf{D}(|\epsilon^1_X|)"'] & \Box X \arrow[r, "i"] \arrow[d, "e_0"] & \mathbf{Hom_{CGHaus}}([0,1],X) \arrow[d, "App_0"] \\
& & (c) & \mathbf{D}(|X|) \arrow[r, "\subseteq"'] & X
\end{tikzcd}$$

Indeed (a) and (b) are by definition of $\eta^1_{\Box X}$, (c) is the left square of Diagram (41),

(d) is the top square of Diagram (41). Also, $\mathbf{D}(\epsilon_X^1) \circ \mathbf{D}(|k'|) \circ e_0 = \mathbf{D}(\subseteq) \circ e_0$ (by definition of ϵ_X^1) $= e_0$, and $\Lambda(\epsilon_X^1 \circ App) \circ \Lambda(q') = \Lambda(\epsilon_X^1 \circ App \circ (\Lambda(q') \times \mathrm{id}))$ (by (k')) $= \Lambda(\epsilon_X^1 \circ q')$ (by (l'')) $= \Lambda(App \circ (i \times \mathrm{id}))$ (by definition of ϵ_X^1) $= i$ (by (m')). By the universal property of the pushout, $\Box\epsilon_X^1 \circ \eta_{\Box X}^1 = \mathrm{id}$.

It follows that $(\Box\epsilon \circ \eta)_X = (\Box\epsilon_X^1, |\epsilon_X^1|) \circ (\eta_{\Box X}^1, |k'|) = (\Box\epsilon_X^1 \circ \eta_{\Box X}^1, |\epsilon_X^1| \circ |k'|) = \mathrm{id}$ by the above discussion, and because $|\epsilon_X^1| \circ |k'| = |\subseteq| = \mathrm{id}$ by the definition of ϵ_X^1. $\Box$

The interested reader may check that $\blacklozenge$ is the base functor for a strong monad $(\blacklozenge, \mathfrak{e}, \mathfrak{m})$, where $\mathfrak{e} \hat{=} \mathbf{d} \circ \eta$ and $\mathfrak{m} \hat{=} \mathrm{lad}(\mathrm{lad}(\mathbf{s} \circ \eta)) = \epsilon \circ \blacklozenge(\epsilon \circ \blacklozenge(\mathbf{s} \circ \eta))$. Concretely, $\mathfrak{e} : (X, E, p) \to \blacklozenge(X, E, p)$ maps $x \in X$ to the formal sum $1.x + 0.p(x) \in \blacklozenge_{E,p} X$ and $e \in E$ to itself; and $\mathfrak{m} : \blacklozenge^2(X, E, p) \to \blacklozenge(X, E, p)$ maps the formal sum $a'(ax + be) + b'e \in \blacklozenge_{E,\hat{p}} \blacklozenge_{E,p} X$ to $(a'a)x + (a'b + b')e \in \blacklozenge_{E,p} X$, and $e \in E$ to itself.

Corollary 80. $\Box$ *preserves pullbacks.*

Proof. As a right-adjoint, $\Box$ preserves all limits. $\Box$

Proposition 81. *A morphism (f, r) in* **CGHaus** $\downarrow$ **D** *is monic if and only if both f and r are one-to-one.*

Proof. Consider any monic (f, r) from (X, E, p) to (X', E', p').

In particular, for any pair of maps (x, e), (x', e') from the initial object $(\mathbf{1}, \mathbf{1}, !)$ to (X, E, p) (hence x, x' are points of X, e, e' are elements of E such that $p(x) = e$, $p(x') = e'$), $f(x) = f(x')$ and $r(e) = r'(e')$ implies $x = x'$ and $e = e'$. So for any $x, x' \in X$, letting $e \hat{=} p(x)$, $e' \hat{=} p(x')$, if $f(x) = f(x')$, then $r(e) = r(p(x)) = p'(f(x)) = p'(f(x')) = r(p(x')) = r(e')$, so $x = x'$: f is one-to-one.

Similarly, consider the labeled Kelley space $(\emptyset, \mathbf{1}, !)$ (where ! denotes the unique map from $\emptyset$ to any set). Then for any pair of maps $(!, e)$, $(!, e')$ from $(\emptyset, \mathbf{1}, !)$ to (X, E, p) (i.e., any pair or elements e, e' of E), if $f \circ ! = f \circ !$ (always true) and $r(e) = r(e')$, then $e = e'$. So r is one-to-one.

Conversely, it is clear that if f and r are one-to-one, then (f, r) is monic. $\Box$

Corollary 82. $\Box$ *preserves monics.*

Proof. If (f, r) is monic, then f and r are one-to-one. In particular, $\Box f$ is one-to-one: if $\Box f(\alpha) = \Box f(\alpha')$, then for every $t \in [0, 1]$, $f(\alpha(t)) = f(\alpha'(t))$, so $\alpha(t) = \alpha'(t)$; i.e., $\alpha = \alpha'$. It follows that $\Box(f, r) = (\Box f, |f|)$ is monic. $\Box$

6.2. Singular Simplices, Geometric Realization

We now come to the construction we are after:

Definition 83. *Let* **Sing** : **CGHaus** $\downarrow$ **D** $\to \widehat{\Delta}$ *be the* singular simplex functor *defined as follows.*

For every labeled space (X, E, p), $K \hat{=} \mathbf{Sing}(X, E, p)$ is defined by: $K_{-1} \hat{=} E$, K_q is the set of all continuous functions from Δ_q to X (the singular q-simplices *of X). Faces and degeneracies are given by $\partial_0^0 x \hat{=} p(x(1))$, $\partial_q^i x \hat{=} x \circ \delta_q^i$ ($0 \leqslant i \leqslant q$, $q \geqslant 1$), $s_q^i x \hat{=} x \circ \sigma_q^i$ ($0 \leqslant i \leqslant q$).*

On morphisms (f, r), let $\mathbf{Sing}(f, r)_{-1} \hat{=} r$, $\mathbf{Sing}(f, r)_q(x) \hat{=} f \circ x$ for $q \geqslant 0$.

Once restricted to non-negative dimensions, **Sing** coincides with the ordinary singular simplex functor $\mathbf{Sing}^0$, mapping topological spaces to (non-augmented) simplicial sets [**39**].

Using the fact that $\mathbf{Sing}^0$ has a left-adjoint $\mathbf{Geom}^0$, we get the following Lemma. Recall that $\mathbf{Geom}^0(K)$ is the topological space built as the quotient $\amalg_{q\geqslant 0} K_q\cdot\Delta_q/\sim$, where $\sim$ is the smallest equivalence relation such that $(\partial_q^i y, v) \sim (y, \delta_q^i v)$ for all $0 \leqslant i \leqslant q$, $q \geqslant 1$, and $(s_q^i y, v) \sim (y, \sigma_q^i v)$ for all $0 \leqslant i \leqslant q$. Write $|y, v|$ the equivalence class of (y, v). On morphisms, $\mathbf{Geom}^0(f)$ maps $|y, v|$ to $|f_q(y), v|$, for every y of dimension q. The unit $\eta^0 : K \to \mathbf{Sing}^0\,\mathbf{Geom}^0(K)$ is the function mapping $y \in K_q$ to the singular simplex mapping $v \in \Delta_q$ to $|y, v|$, while the counit $\epsilon^0 : \mathbf{Geom}^0\,\mathbf{Sing}^0(X) \to X$ is the continuous function mapping $|\alpha, v|$ to $\alpha(v)$ [**39**].

Lemma 84. **Sing** *has a left-adjoint* **Geom**, *defined as* $\mathbf{Geom}(K)\hat{=}(\mathbf{Geom}^0(K), K_{-1}, (|y, v| \mapsto \pi_0(y)))$, *where* $\pi_0(y)\hat{=}\partial_0^0\partial_1^0 \ldots \partial_q^0 y$ *for every* $y \in K_q$. *On morphisms* f, $\mathbf{Geom}(f)\hat{=}(\mathbf{Geom}^0(f), f_{-1})$.

Proof. The function $|y, v| \mapsto \pi_0(y)$ is well-defined: by induction on the length of a given derivation of $(y, v) \sim (y', v')$, it is easy to check that $\pi_0(y) = \pi_0(y')$.

Recall [**39**] that any simplicial map $f\hat{=}(f_q)_{q\geqslant 0}$ gives rise to a continuous map $\mathbf{Geom}^0(f)$ mapping $|y, v|$ to $|f_q(y), v|$, where $y \in K_q$. Note that K_{-1} can be seen as a simplicial set K_{-1}^* in the following degenerate way: its set of q-dimensional simplices is K_{-1}, for every $q \geqslant 0$, its faces and degeneracies are all the identity function. Recall also that every class $|y, v|$ is characterized by the unique non-degenerate point (y, v) in it (y not a degenerate simplex, $v = (t_0, \ldots, t_q)$ with no t_i equal to zero). In K_{-1}^* every simplex of dimension $\geqslant 1$ is degenerate, so $\mathbf{Geom}^0(K_{-1}^*)$ is the set of all $|y, (1)|$, $y \in K_{-1}$, with the discrete topology. In other words, $\mathbf{Geom}^0(K_{-1}^*) = \mathbf{D}(K_{-1})$, up to isomorphism. Since π_0 yields a simplicial map from $(K_q)_{q\geqslant 0}$ to K_{-1}^*, $\mathbf{Geom}^0(\pi_0)$ is a continuous function from $\mathbf{Geom}^0(K)$ to $\mathbf{D}(K_{-1})$; this is $|y, v| \mapsto |\pi_0(y), v| = \pi_0(y)$. So $\mathbf{Geom}(K)$ is a labeled topological space. It is Kelley because $\mathbf{Geom}^0(K)$ is a CW-complex [**39**], hence Kelley.

Write η^0 and ϵ^0 for the unit and the counit of the adjunction $\mathbf{Geom}^0 \dashv \mathbf{Sing}^0$, respectively. Define the unit η as η^0 in non-negative dimensions and id in dimension -1; the counit ϵ as the pair $(\epsilon^0, \mathrm{id})$. These are clearly natural transformations. Also, η^0 is a.s.: the only thing to check is that for every $y \in K_0$, $\partial_0^0\eta^0(y) = \mathrm{id}(\partial_0^0 y)$ (the left-hand side maps () to $p(\eta^0(y)(1))$, where p maps $|y, v|$ to $\pi_0(y)$; so it maps () to $\pi_0(y) = \partial_0^0 y$ since $y \in K_0$). And ϵ^0 is a morphism of labeled topological spaces: we need to check that for every labeled Kelley space (X, E, p), every $|\alpha, v| \in \mathbf{Geom}^0\,\mathbf{Sing}^0(X)$ ($\alpha : \Delta_q \to X$, $v \in \Delta_q$, $q \geqslant 0$), $p(\epsilon^0|\alpha, v|) = \mathrm{id}(p'|\alpha, v|)$, where $p'|\alpha', v'|\hat{=}\pi_0(\alpha') = \partial_0^0\partial_1^0 \ldots \partial_q^0\alpha' = \partial_0^0((1) \mapsto \alpha'(0, 0, \ldots, 0, 1)) = p(\alpha'(0, \ldots, 0, 1))$. This is equivalent to $p(\alpha(v)) = p(\alpha(0, \ldots, 0, 1))$, which is clear since p is continuous with values in a discrete topological space.

Checking that $\epsilon \circ \mathbf{Geom}(\eta) = \mathrm{id} : \mathbf{Geom}(K) \to \mathbf{Geom}(K)$ and $\mathbf{Sing}(\epsilon) \circ \eta = \mathrm{id} : \mathbf{Sing}(X) \to \mathbf{Sing}(X)$ is straightforward. For the first equation, $\epsilon \circ \mathbf{Geom}(\eta) = (\epsilon^0 \circ \mathbf{Geom}^0(\eta^0), \mathrm{id} \circ \mathrm{id}_{-1}) = (\mathrm{id}, \mathrm{id}) = \mathrm{id}$ (using $\mathbf{Geom}^0 \dashv \mathbf{Sing}^0$). For the second equation, $\mathbf{Sing}(\epsilon) \circ \eta$ is $\mathbf{Sing}^0(\epsilon^0) \circ \eta^0 = \mathrm{id}$ in every non-negative dimension, and is $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$ in dimension -1. $\square$

Lemma 85. **Geom** *preserves all finite products.*

Proof. $\mathbf{Geom}^0(\mathbf{1})$ has only one non-degenerate point, $|*, (1)|$, where $*$ is the unique non-degenerate simplex of the simplicial set $\mathbf{1}$. It follows that $\mathbf{Geom}(\mathbf{1}) = (\mathbf{1}, \mathbf{1}, !)$, where ! is the unique function from $\mathbf{1}$ to $|\mathbf{1}|$: this is the terminal object in $\mathbf{CGHaus} \downarrow \mathbf{D}$.

It is well-known that $\mathbf{Geom}_0$ preserves all binary products ([**39**], Theorem 14.3 and final note to Chapter III, showing that this holds for Kelley spaces). So, given any two a.s. sets K and L, $\mathbf{Geom}(K \times L) = (\mathbf{Geom}^0(K \times L), K_{-1} \times L_{-1}, (|(y, y'), v| \mapsto \pi_0(y, y')))$ is the product of $\mathbf{Geom}(K) \times \mathbf{Geom}(L)$. (The natural isomorphism between $\mathbf{Geom}^0(K \times L)$ and $\mathbf{Geom}^0(K) \times \mathbf{Geom}^0(L)$ maps $|(y, y'), v|$ to some pair $(|y, v_1|, |y, v_2|)$, as shown in [**39**], Theorem 14.3, hence the map $|(y, y'), v| \mapsto \pi_0(y, y')$ is indeed the product map $\pi_0 \times \pi_0$.) □

Lemma 86. *For every a.s. set K,* **Geom** *is faithful on morphisms from $\mathbf{1}_{\hat{\Delta}}$ to K.*

Proof. Let $f \hat{=} (f_q)_{q \geqslant -1}$ and $g \hat{=} (g_q)_{q \geqslant -1}$ be two a.s. maps from $\mathbf{1}$ to K such that $\mathbf{Geom}(f) = \mathbf{Geom}(g)$, i.e., $(\mathbf{Geom}^0(f), f_{-1}) = (\mathbf{Geom}^0(g), g_{-1})$. In particular $f_{-1} = g_{-1}$. On the other hand $\mathbf{Geom}^0(f)$ maps every non-degenerate point $|y, v|$ in $\mathbf{1}_q$, ($y \in \mathbf{1}_q$, $v \in \Delta_q$, $q \geqslant 0$) to $|f_q(y), v|$. Since y is non-degenerate if and only if $q = 0$, $\mathbf{Geom}^0(f)$ maps $|*, (1)|$ to $|f_0(*), (1)|$, where $*$ denotes the unique element of $\mathbf{1}_0$. Now $f_0(*) \in K_0$ is non-degenerate. By the uniqueness of representations of points in $\mathbf{Geom}_0(K)$ as non-degenerate pairs, $\mathbf{Geom}^0(f) = \mathbf{Geom}^0(g)$ implies $f_0(*) = g_0(*)$. So $f_0 = g_0$. For every $q \geqslant 1$ now, $f_q(*) = s^0_{q-1} \dots s^0_0 f_0(*)$ because f is a.s., and $* = s^0_{q-1} \dots s^0_0 *$, and similarly for g; so $f_q(*) = g_q(*)$, therefore $f_q = g_q$. □

We now define a $(\mathbf{Sing}, \Box)$-cocommutator $\mathfrak{S} : \Box\,\mathbf{Sing} \to \mathbf{Sing}\,\Box$ (see Proposition 71). Given a labeled space (X, E, p), $\Box\,\mathbf{Sing}(X, E, p)$ has as q-simplices all paths $\alpha : \Delta_{q+1} \to X$. On the other hand, the q-simplices of $\mathbf{Sing}(\Box(X, E, p))$ are the functions from Δ_q to $\Box X$ if $q \geqslant 0$, the points of X if $q = -1$; and ∂^0_0 maps $\alpha : \Delta_0 \to \Box X$ to $\alpha(1)(0)$.

Definition 87. *Let $\mathfrak{S}_{(X,E,p)} : \Box\,\mathbf{Sing}(X, E, p) \to \mathbf{Sing}(\Box(X, E, p))$ be the a.s. map such that:*

$$\mathfrak{S}_{(X,E,p),q}(\alpha) = \begin{cases} (t_0, \dots, t_q) \in \Delta_q \mapsto t \in [0,1] \mapsto \alpha(1-t, tt_0, \dots, tt_q) & (q \geqslant 0) \\ \alpha(1) & (q = -1) \end{cases}$$

for every $\alpha : \Delta_{q+1} \to X$.

Proposition 88. *$\mathfrak{S}_{(X,E,p)}$ is well-defined, and defines a $(\mathbf{Sing}, \Box)$-cocommutator $\mathfrak{S} : \Box\,\mathbf{Sing} \to \mathbf{Sing}\,\Box$.*

Proof. First we show that, when $q \geqslant 0$, $\mathfrak{S}_{(X,E,p),q}(\alpha)$ is a continuous function from Δ_q to $\Box X$. Using Claim A, since Δ_q is connected, we have to show that $\mathfrak{S}_{(X,E,p),q}(\alpha)(t_0, \dots, t_q)(0)$ is independent of $(t_0, \dots, t_q)$: indeed $\mathfrak{S}_{(X,E,p),q}(\alpha)(t_0, \dots, t_q)(0) = \alpha(1, 0, \dots, 0)$. As a function from Δ_q to $\mathbf{Hom}_{\mathbf{CGHaus}}([0,1], X)$, it is continuous since it is the curried version of the function $((t_0, \dots, t_q), t) \in \Delta_q \times [0,1] \mapsto \alpha(1-t, tt_0, \dots, tt_q)$, which is continuous.

Next we show that $\mathfrak{S}_{(X,E,p)}$ is augmented simplicial. When $q \geqslant 1$, $\partial_q^i \mathfrak{S}_{(X,E,p),q}(\alpha) = (t_0, \ldots, t_{q-1}) \mapsto t \mapsto \alpha(1-t, tt_0, \ldots, tt_{i-1}, t.0, tt_i, \ldots, tt_q) = \mathfrak{S}_{(X,E,p),q-1}(\partial_q^i \alpha)$. Similarly for s_q^i, $q \geqslant 0$. When $q = 0$, $\partial_0^0 \mathfrak{S}_{(X,E,p),0}(\alpha) = \mathfrak{S}_{(X,E,p),0}(\alpha)(1)(0) = \alpha(1,0)$, while $\mathfrak{S}_{(X,E,p),-1}(\partial_0^0 \alpha) = \partial_0^0 \alpha(1) = \alpha(1,0)$.

$\mathfrak{S}_{(X,E,p)}$ is natural in (X,E,p). Indeed, let (f,r) be any morphism from (X,E,p) to (X',E',p') (in particular $p' \circ f = r \circ p$). $\mathfrak{S}_{(X',E',p')} \circ \square\,\mathbf{Sing}(f,r)$ maps $\alpha \in (\square\,\mathbf{Sing}(X,E,p))_q$, $q \geqslant 0$, to $(t_0, \ldots, t_q) \in \Delta_q \mapsto t \in [0,1] \mapsto f(\alpha(1-t, tt_0, \ldots, tt_q))$, and $\alpha \in (\square\,\mathbf{Sing}(X,E,p))_{-1}$ to $f(\alpha(1))$. On the other hand, $\mathbf{Sing}(\square(f,r)) \circ \mathfrak{S}_{(X,E,p)} = \mathbf{Sing}(f,|f|) \circ \mathfrak{S}_{(X,E,p)}$ maps $\alpha \in (\square\,\mathbf{Sing}(X,E,p))_q$, $q \geqslant 0$, to $(t_0, \ldots, t_q) \in \Delta_q \mapsto t \in [0,1] \mapsto f(\alpha(1-t, tt_0, \ldots, tt_q))$, and $\alpha \in (\square\,\mathbf{Sing}(X,E,p))_{-1}$ to $|f|(\alpha(1))$. These are equal, so $\mathfrak{S}$ is indeed a natural transformation.

To show that it is a $(\mathbf{Sing}, \square)$-cocommutator, we have to check:

(27) $\mathbf{Sing}(\mathbf{d}) \circ \mathfrak{S}_{(X,E,p)} = \mathbf{d}$, i.e., $\mathbf{Sing}(\beta \mapsto \beta(1), |p|)_q \circ \mathfrak{S}_{(X,E,p),q} = \partial_q^0$. Indeed, when $q \geqslant 0$, the left-hand side maps $\alpha : \Delta_{q+1} \to X$ to $(\beta \mapsto \beta(1)) \circ \mathfrak{S}_{(X,E,p),q}(\alpha) = (t_0, \ldots, t_q) \mapsto \mathfrak{S}_{(X,E,p),q}(\alpha)(t_0, \ldots, t_q)(1) = (t_0, \ldots, t_q) \mapsto \alpha(0, t_0, \ldots, t_q) = \partial_q^0 \alpha$. When $q = -1$, the left-hand side maps $\alpha \in: \Delta_0 \to X$ to $|p|(\mathfrak{S}_{(X,E,p),-1}(\alpha)) = p(\alpha(1))$: this is the right-hand side, as ∂_0^0 maps $\alpha \in \mathbf{Sing}(X,E,p)_0 = \Delta_0 \to X$ to $p(\alpha(1)) \in \mathbf{Sing}(X,E,p)_{-1} = E$.

(28) $\mathbf{Sing}(\mathbf{s}) \circ \mathfrak{S}_{(X,E,p)} = \mathfrak{S}_{\square(X,E,p)} \circ \square\mathfrak{S}_{(X,E,p)} \circ \mathbf{s}$, i.e., $\mathbf{Sing}((\beta \mapsto t, t' \mapsto \beta(tt')), \mathbf{k})_q \circ \mathfrak{S}_{(X,E,p),q} = \mathfrak{S}_{\square(X,E,p),q} \circ \mathfrak{S}_{(X,E,p),q+1} \circ s_q^0$. Indeed, when $q \geqslant 0$, the left-hand side maps $\alpha \in \Delta_{q+1} \to X$ to $(\beta \mapsto t \mapsto t' \mapsto \beta(tt')) \circ \mathfrak{S}_{(X,E,p),q}(\alpha) = (t_0, \ldots, t_q) \mapsto t \mapsto t' \mapsto \mathfrak{S}_{(X,E,p),q}(\alpha)(t_0, \ldots, t_q)(tt') = (t_0, \ldots, t_q) \mapsto t \mapsto t' \mapsto \alpha(1-tt', tt't_0, \ldots, tt't_q)$; while the right-hand side maps α to $(t_0, \ldots, t_q) \mapsto t \mapsto \mathfrak{S}_{(X,E,p),q+1}(s_q^0(\alpha))(1-t, tt_0, \ldots, tt_q) = (t_0, \ldots, t_q) \mapsto t \mapsto t' \mapsto s_q^0(\alpha)(1-t', (1-t)t', tt't_0, \ldots, tt't_q) = (t_0, \ldots, t_q) \mapsto t \mapsto t' \mapsto \alpha(1-t'+(1-t)t', tt't_0, \ldots, tt't_q) = (t_0, \ldots, t_q) \mapsto t \mapsto t' \mapsto \alpha(1-tt', tt't_0, \ldots, tt't_q)$. And if $q = -1$, the left-hand side maps $\alpha \in \Delta_0 \to X$ to $\mathbf{k}(\alpha(1)) = t \mapsto \alpha(1)$, while the right-hand side maps it to $\mathfrak{S}_{(X,E,p),q+1}(s_q^0(\alpha))(1) = t \mapsto s_q^0(\alpha)(1-t, t) = t \mapsto \alpha(1-t+t) = t \mapsto \alpha(1)$.

$\square$

By Proposition 71, this allows us to build a corresponding $(\mathbf{Geom}, \square)$-commutator $\mathfrak{I}$ as $\mathrm{lad}(\mathfrak{S} \circ \square)$. Expanding the definitions, we get the definition of Proposition 89 below. The construction is a bit subtle, because of the shift in dimensions involved in $\square K$. Letting (X,E,p) be $\mathbf{Geom}(\square K)$, the points of X are equivalence classes $|y,v|'$ where $y \in (\square K)_q = K_{q+1}$ and $v \in \Delta_q$, where equivalence classes (denoted with a prime so as not to confuse them with points of $\mathbf{Geom}(K)$) are with respect to the smallest equivalence relation $\sim'$ such that $(\partial_{q+1}^{i+1} y, v) \sim' (y, \delta_q^i v)$ $(0 \leqslant i \leqslant q,$ $q \geqslant 1)$ and $(s_{q+1}^{i+1} y, v) \sim' (y, \sigma_q^i v)$ $(0 \leqslant i \leqslant q)$. Then E is K_0, and p maps $|y,v|'$ to $\partial_1^1 \ldots \partial_{q+1}^1 y$.

Proposition 89. *Let* $\mathfrak{I}_K : \mathbf{Geom}(\square K) \to \square\,\mathbf{Geom}(K)$*, for each a.s. set* K*, be* (f,r) *where* f *maps* $|y,(t_0, \ldots, t_q)|'$ $(y \in K_{q+1}$*,* $(t_0, \ldots, t_q) \in \Delta_q)$ *to the path* $t \in [0,1] \mapsto |y, (1-t, tt_0, \ldots, tt_q)|$*, and* r *maps* $y \in K_0$ *to* $|y,(1)| \in |\mathbf{Geom}(K)|$.

Then $\mathfrak{I}$ *is a* $(\mathbf{Geom}, \square)$*-commutator.*

Proposition 90. *If $K_0 \neq \emptyset$, then there is a morphism $\mathfrak{R}_K : \Box\,\mathbf{Geom}(K) \to \mathbf{Geom}(\Box K)$ of labeled Kelley spaces such that $\mathfrak{R}_K \circ \mathfrak{I}_K = \mathrm{id}$.*

Proof. Since $K_0 \neq \emptyset$, fix $y_0 \in K_0$.

Let $\mathfrak{R}_K$ be (f', r'), where f' maps every path $\alpha \in \Box\,\mathbf{Geom}(K)$ to

$$\begin{array}{ll} |y,(t_0,\ldots,t_q)|' & \text{if } \alpha(0) = |y,(1,t_0,\ldots,t_q)| \\ & \text{for some } q \geqslant 0, y \in K_{q+1} \text{ non-degenerate}, t_0 \neq 0,\ldots,t_q \neq 0 \\ |s_0^0 y_0,(1)|' & \text{otherwise} \end{array}$$

Recall that every point in $\mathbf{Geom}^0(K)$ can be written in a unique way as $|y,v|$ with $y \in K_n$ non-degenerate, $v \in \Delta_n$ non-degenerate (no zero coordinate). In the first case, we require $n \geqslant 1$ and the first coordinate of v equal 1. Note that f' is continuous, because it factors as a composite

$$\begin{array}{ccccc} \Box\,\mathbf{Geom}^0(K) & \xrightarrow{e_0} & \mathbf{D}(|\mathbf{Geom}^0(K)|) & \longrightarrow & \mathbf{Geom}^0(\Box K) \\ \alpha & \longmapsto & \alpha(0) & \longmapsto & f'(\alpha) \end{array}$$

where the leftmost morphism e_0 is continuous by definition of $\Box$ in labeled Kelley spaces, the rightmost one is continuous because any function from a discrete topological space is continuous.

On the other hand, r' maps $x \in |\mathbf{Geom}(K)|$ to

$$\begin{array}{ll} \partial_1^1 \ldots \partial_{q+1}^1 y & \text{if } x = |y,(1,t_0,\ldots,t_q)| \\ & \text{for some } q \geqslant 0, y \in K_{q+1} \text{ non-degenerate}, t_0 \neq 0,\ldots,t_q \neq 0 \\ y_0 & \text{otherwise} \end{array}$$

Recall that the labeling function in $\Box\,\mathbf{Geom}(K)$ is $\alpha \mapsto \alpha(0)$, while in $\mathbf{Geom}(\Box K)$ it is $|y,v|' \mapsto \partial_1^1 \ldots \partial_{q+1}^1 y$ ($y \in K_{q+1}$, $v \in \Delta_q$). It is then straightforward to check that (f', r') is a morphism in $\mathbf{CGHaus} \downarrow \mathbf{D}$.

Finally, it is clear from Proposition 89 that $\mathfrak{R}_K \circ \mathfrak{I}_K = \mathrm{id}$. □

6.3. Can Function Spaces be Retracted onto Syntactic Ones?

It remains to show that $\mathbf{Geom}(\mathcal{S4}_\Sigma \llbracket F \supset G \rrbracket (\subseteq))$ is a strong retract of the internal hom $\mathbf{Hom}_{\mathbf{CGHaus}\downarrow\mathbf{D}}(\mathbf{Geom}(\mathcal{S4}_\Sigma \llbracket F \rrbracket (\subseteq)), \mathbf{Geom}(\mathcal{S4}_\Sigma \llbracket G \rrbracket (\subseteq)))$, for every formulae F and G. One standard line of attack is to use Whitehead's Theorem (if a map $f : X \to Y$ between connected CW-complexes is a weak homotopy equivalence, i.e., if f induces isomorphisms of all homotopy groups, then f is a homotopy equivalence; and if f is an embedding of CW-complexes, then X is a deformation retract of Y [**25**]). This runs into a number of problems, including minor ones such as the fact that we are dealing with labeled spaces instead of topological spaces, that our spaces are not connected in general, and that, to our knowledge, it is not known whether internal homs between CW-complexes are CW-complexes (although every continuous map between CW-complexes can be homotoped to a cellular map). The major problem with this approach, however, is to show that the homotopy groups of the given spaces are isomorphic. General theorems showing this for CW-complexes obtained as geometric realizations (e.g., [**39**], Theorem 16.6) only obtain when the underlying simplicial sets are Kan. But we have seen that the spaces of interest to us are definitely not Kan (Proposition 21).

Instead we should be able to use the fact that our a.s. sets are extremely non-Kan, in the following sense:

Definition 91. *An a.s. set* $K\hat{=}(K_q)_{q\geqslant -1}$ *is called* ordered *if and only if the simplicial set* $(K_q)_{q\geqslant 0}$ *is the nerve of a partial order* $(E,\preceq)$*. It is a (complete)* latticed a.s. set *if and only if, in addition, for every* $x\in K_{-1}$*, the set of all those* $y\in K_0$ *such that* $\partial_0^0(y)=x$ *is a (complete) lattice.*

The order $(E,\preceq)$ is defined uniquely from the ordered a.s. set K. Indeed, by definition its carrier E must be K_0, while $x\preceq y$ if and only if there is a 1-simplex z with $\partial_1^1 z=x$ and $\partial_1^0 z=y$. So the ordering $\preceq$ on K_0 must be the contiguity relation $\twoheadrightarrow^=$. Note that, by Theorem 30 and Proposition 31, and the fact that $\mathcal{S}4_\Sigma\,[\![F]\!]\,(\subseteq)_0$ is never empty, each $\mathcal{S}4_\Sigma\,[\![F]\!]\,(\subseteq)$ is a complete latticed a.s. set.

We conjecture the following:

Conjecture 92. *Let* K *and* L *be complete latticed a.s. sets. Then there is a strong retraction* $\mathfrak{R}_{K\to L}$ *of* $\mathbf{Hom}_{\mathbf{CGHaus}\downarrow\mathbf{D}}(\mathbf{Geom}(K),\mathbf{Geom}(L))$ *onto the space* $\mathbf{Geom}(\mathbf{Hom}_{\widehat{\Delta}}(K,L))$*, meaning that* $\mathfrak{R}_{K\to L}\circ\Lambda(\mathbf{Geom}(App))=\mathrm{id}$.

This is technical. In particular, this seems to require a fine understanding of what $\Lambda(\mathbf{Geom}(App))$ actually computes. Looking at the simplicial side only, remember [**39**] that a q-simplex of the internal hom (in the category of simplicial sets) $\mathbf{Hom}(K,L)$ is a simplicial map $f:K\times\Delta[q]\to L$, where $\Delta[q]$ has as p-simplices all monotonic maps from $[p]$ to $[q]$; faces and degeneracies are given by $\partial_q^i f=f\circ(\mathrm{id}\times\delta_q^i)$, $s_q^i f=f\circ(\mathrm{id}\times\sigma_q^i)$. App in the category of simplicial sets maps every $f:K\times\Delta[q]\to L$ and every $y\in K_q$ to $f_q(y,\mathrm{id})\in L_q$. Now $\mathbf{Geom}^0(App)$ is the continuous function from $\mathbf{Geom}^0(\mathbf{Hom}(K,L)\times K)$ to $\mathbf{Geom}^0(L)$ that maps $|(f,y),v|$ (f simplicial map $:K\times\Delta[q]\to L$, $y\in K_q$, $v\in\Delta_q$) to $|f_q(y,\mathrm{id}),v|$. Taking into account the rather complex isomorphism showing that $\mathbf{Geom}^0$ preserves products [**39**] is done as follows: for every $v\in\Delta_q$, $v'\in\Delta_{q'}$, there is a simplex $w\in\Delta_{q''}$ for some $q''\geqslant q,q'$ such that both v and v' can be written as codegeneracies of w: $v=\sigma_q^{i_1}\ldots\sigma_{q+k-1}^{i_k}w$ (with $q+k=q''$) and $v'=\sigma_{q'}^{i'_1}\ldots\sigma_{q'+k'-1}^{i'_{k'}}w$ (with $q'+k'=q''$). For example, if $v=(1/3,1/4,5/12)$ and $v'=(1/4,1/3,1/12,1/3)$, we may take $w=(1/4,1/12,1/4,1/12,1/3)$ since $v=(1/4+1/12,1/4,1/12+1/3)$ and $v'=(1/4,1/12+1/4,1/12,1/3)$. Then $\Lambda(\mathbf{Geom}^0(App))$ maps $|f,v|\in\mathbf{Geom}^0(\mathbf{Hom}(K,L))$ (f simplicial $:K\times\Delta[q]\to L$, $v\in\Delta_q$) to the continuous function mapping $|y,v'|\in\mathbf{Geom}^0(K)$ ($y\in K_{q'}$, $v'\in\Delta_{q'}$) to $|f_{q''}(s_{q'+k'-1}^{i'_{k'}}\ldots s_{q'}^{i'_1}y,\sigma_q^{i_1}\circ\ldots\circ\sigma_{q+k-1}^{i_k}),w|\in\mathbf{Geom}^0(L)$. We have not found yet a suitable left-inverse to this function—the main difficulty being in showing that the obtained left-inverse is continuous.

Theorem 93. *Assuming the validity of Conjecture 92, there is a valuation* $\rho_2:\Sigma\to\mathbf{CGHaus}\downarrow\mathbf{D}$ *such that, for every* λ_{S4}*-terms* M *and* N *of type* F *under* Γ*,* $M\approx N$ *if and only if* $(\mathbf{CGHaus}\downarrow\mathbf{D})\,[\![M]\!]\,\rho_2=(\mathbf{CGHaus}\downarrow\mathbf{D})\,[\![N]\!]\,\rho_2$.

Proof. By Theorem 74, using Proposition 78, Lemma 85, Corollary 82, Corollary 80, Conjecture 92, Proposition 90, and finally Lemma 86. □

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Appendix A. Validity and Provability

Recall that the typing system of Figure 1 is also a natural deduction system for intuitionistic S4. Let us say that a sequent $F_1, \ldots, F_n \vdash F$ is *provable* if and only if there is a λ_{S4}-term M such that $x_1 : F_1, \ldots, x_n : F_n \vdash M : F$ is typable in the system of Figure 1. It is *valid* if and only if, in every Kripke frame $(\mathcal{W}, \trianglerighteq, \geqslant)$, for every $w \in \mathcal{W}$ and every valuation ρ, $w, \rho \models F$ as soon as $w, \rho \models F_i$ for every i, $1 \leqslant i \leqslant n$. In the sequel, we write Γ ambiguously for contexts $x_1 : F_1, \ldots, x_n : F_n$, and for lists $F_1, \ldots, F_n$.

Lemma 94 (Soundness). *If $\Gamma \vdash F$ is provable, then $\Gamma \vdash F$ is valid.*

Proof. First observe that the set of worlds at which some formula holds in a given Kripke frame, given some valuation ρ, is an upper set. In other words, (†) if $w, \rho \models F$

and $w' \geqslant w$, then $w', \rho \models F$. (In the case of boxed formulae, this uses the condition $\geqslant \subseteq \trianglerighteq$.)

Fix a Kripke frame and a valuation ρ. We show that, if π is a derivation of $\Gamma \vdash F$, then F holds at every world at which Γ holds. (We say that F holds at w if and only if $w, \rho \models F$. We say that Γ holds at w if and only if every formula of Γ holds at w.) This is by structural induction on π.

If π ends in (Ax), then this is clear.

If π ends in $(\supset I)$, i.e., F is of the form $F_1 \supset F_2$, and by induction hypothesis F_2 holds at every world where both Γ and F_1 hold, then we claim that $F_1 \supset F_2$ holds at every world w where Γ holds. Indeed, assume that Γ holds at w, take $w' \geqslant w$, and assume that F_1 holds at w'. By (†) Γ holds at w', so by assumption F_2 holds at w'.

If π ends in $(\supset E)$, by induction hypothesis we know that some formula G holds at every world w at which Γ holds, and $G \supset F$ also holds at every such world. In particular, if G holds at w (which it does), then F holds at w.

If π ends in $(\Box I)$, then F is of the form $\Box G$, and there are formulae $\Box F_1, \ldots, \Box F_n$ which hold at every world at which Γ holds, and such that G holds at every world at which all of $\Box F_1, \ldots, \Box F_n$ hold, by induction hypothesis. Take any world w at which Γ holds, and show that $\Box G$ holds at w. That is, take any world $w' \trianglerighteq w$, and show that G holds at w'. Since Γ holds at w, $\Box F_1, \ldots, \Box F_n$ all hold at w, too. So each F_i holds at every $w'' \trianglerighteq w$, in particular at every $w'' \trianglerighteq w'$. (We use the transitivity of $\trianglerighteq$ here.) So each $\Box F_i$ holds at w'. It follows that G holds at w', by assumption.

If π ends in $(\Box E)$, then by induction hypothesis $\Box F$ holds at every world w at which Γ holds. In particular, since $\trianglerighteq$ is reflexive, F holds at w. $\Box$

We now show the converse of Lemma 58. To do this, consider an unprovable sequent $\Gamma \vdash F$, and build a Kripke frame and a valuation such that Γ holds at some world but F does not.

A *signed formula* is a pair of a *sign*, $+$ or $-$, and a formula F; this pair is written $+F$ or $-F$, respectively. A set S of signed formulas is *consistent* if and only if for every $+F_1, \ldots, +F_n, -G \in S$, $F_1, \ldots, F_n \vdash G$ is not provable. Equivalently, if $+F_1, \ldots, +F_n \in S$ such that $F_1, \ldots, F_n \vdash G$ is provable, then $-G$ is not in S. S is *maximally consistent* if and only if S is consistent and there is no proper consistent superset of S. By Zorn's Lemma, every consistent set is a subset of some maximally consistent set.

Define our Kripke frame $(\mathcal{W}, \geqslant, \trianglerighteq)$ by letting the worlds of $\mathcal{W}$ be maximally consistent sets w of signed formulae, and defining:

$$\begin{array}{lcl} w' \geqslant w & \hat{=} & \text{for every } +(F \supset G) \in w, \text{ if } +F \in w' \text{ then } +G \in w' \\ w' \trianglerighteq w & \hat{=} & \text{for every } +\Box F \in w, +F \in w' \end{array}$$

Lemma 95. *For any formula F, $+F$ and $-F$ cannot both be in the same world w.*

Proof. This would contradict the consistency of w, since $F \vdash F$ is provable, by (Ax). $\Box$

Lemma 96. *For every formula F, for every world w, either $+F$ or $-F$ is in w.*

Proof. Assume by contradiction that neither $+F$ nor $-F$ is in w, and build $w_0 \hat{=} w \cup \{-F\}$, $w_1 \hat{=} w \cup \{+F\}$. We claim that w_0 or w_1 is consistent.

If w_0 is inconsistent, there are signed formulae $+F_1, \ldots, +F_n, -G$ in w_0 such that $F_1, \ldots, F_n \vdash G$ is provable. Since w is consistent, and $+F_1, \ldots, +F_n$ are in w, $-G$ must lie outside w So G is F. Therefore: (a) $F_1, \ldots, F_n \vdash F$ is provable.

If w_1 is inconsistent, there are signed formulae $+F'_1, \ldots, +F'_{n'}, -G'$ in w_1 such that $F'_1, \ldots, F'_{n'} \vdash G'$ is provable. Since $-G'$ is in w, and w is consistent, we can partition $+F'_1, \ldots, +F'_{n'}$ into signed formulas in w and signed formulas out of w (i.e., $+F$). By standard arguments, the rules of exchange and contraction are admissible in intuitionistic S4; that is, we may consider left-hand sides of sequents as sets of formulas. So, without loss of generality, we may assume that the only index i such that $F'_i = F$ is 1. To sum up: (b) $F, F'_2, \ldots, F'_{n'} \vdash G$, where $+F'_2, \ldots, +F'_{n'}, -G$ are in w.

So, if both w_0 and w_1 are inconsistent, using (a) and (b), plus weakening and $(\supset I)$ and $(\supset E)$, we infer that $F_1, \ldots, F_n, F'_2, \ldots, F'_{n'} \vdash G$ is provable, where $+F_1, \ldots, +F_n, +F'_2, \ldots, +F'_{n'}$ and $-G$ are in w, contradicting the consistency of w. So one of w_0 or w_1 is consistent; but this contradicts the maximality of w. $\square$

Lemma 97. *If $+F_1, \ldots, +F_n$ are in a world w and $F_1, \ldots, F_n \vdash F$ is provable, then $+F$ is in w, too.*

Proof. Otherwise $-F$ would be in w, by Lemma 96, contradicting the consistency of w. $\square$

Lemma 98. *If $+F$ is in a world w, then $+F$ is in every world $w' \geqslant w$.*

Proof. Assume $+F \in w$. $F \vdash (G \supset G) \supset F$ is provable, so $(G \supset G) \supset F$ is in w, by Lemma 97. Now let $w' \geqslant w$. By definition of $\geqslant$, this implies that if $+(G \supset G)$ is in w' then $+F$ is in w'. However by Lemma 97 again, $+(G \supset G)$ is in w'. So $+F$ is in w'. $\square$

Lemma 99. *The $\geqslant$ relation is a preordering.*

Proof. Reflexivity. Assume $+(F \supset G) \in w$ and $+F \in w$, then $+G \in w$ by Lemma 97. So $w \geqslant w$.

Transitivity. Assume $w'' \geqslant w'$, $w' \geqslant w$, and let $+(F \supset G)$ be in w'', $+F$ be in w. By Lemma 98, $+(F \supset G)$ is in w', so by definition of $\geqslant$ $+G$ is in w. So $w'' \geqslant w$. $\square$

Lemma 100. *The $\trianglerighteq$ relation is a preordering.*

Proof. Reflexivity. Assume $+\Box F \in w$, then by $(\supset E)$ and Lemma 97, $+F$ is in w.

Transitivity. Assume $w'' \trianglerighteq w'$, $w' \trianglerighteq w$, and let $+\Box F$ be in w''. Then, because of the proof:

$$\dfrac{\dfrac{}{\Box F \vdash \Box F}\,(Ax) \qquad \dfrac{}{\Box F \vdash \Box F}\,(Ax)}{\Box F \vdash \Box\Box F}\,(\Box I)$$

and Lemma 97, $+\Box\Box F$ is in w'', so $\Box F$ is in w', so F is in w. $\square$

Lemma 101. *It holds: $\geqslant \subseteq \trianglerighteq$.*

Proof. Assume $w_1 \geqslant w_2$, and show $w_1 \trianglerighteq w_2$. For any $+\Box F$ in w_1, $+\Box F$ is in w_2 by Lemma 98. Using Lemma 97 and rule $(\Box E)$, $+F$ is in w_2, too. So $w_1 \trianglerighteq w_2$. $\Box$

Lemma 102. *If $+F$ is in every $w' \trianglerighteq w$, then $+\Box F$ is in w.*

Proof. Consider the set $S \hat{=} \{+G \mid +\Box G \in w\} \cup \{-F\}$. If S were consistent, then by Zorn's Lemma there would be a world $w_0 \supseteq S$. By construction $w_0 \trianglerighteq w$, since every $+G$ such that $+\Box G \in w$ is in S, hence in w_0. By assumption, $+F$ must therefore be in w_0. However by construction $-F$ is in S, hence in w_0, therefore contradicting the consistency of w_0.

So S is inconsistent. This means that there are $+G_1, \ldots, +G_n$ in S (therefore $+\Box G_1, \ldots, +\Box G_n$ are in w) such that $G_1, \ldots, G_n \vdash F$ is provable. We can then infer:

$$\dfrac{\overbrace{\dfrac{\dfrac{}{\Box G_1, \ldots, \Box G_n \vdash \Box G_i}\,(Ax)}{\Box G_1, \ldots, \Box G_n \vdash G_i}\,(\Box E)}^{1 \leqslant i \leqslant n} \qquad G_1, \ldots, G_n \vdash F}{\Box G_1, \ldots, \Box G_n \vdash \Box F}\,(\Box I)$$

By Lemma 97 it follows that $+\Box F$ is in w, as desired. $\Box$

Lemma 103. *If for every $w' \geqslant w$ such that $+F \in w'$, $+G \in w'$, then $+(F \supset G)$ is in w.*

Proof. Let S be $\{+H \mid +H \in w\} \cup \{+F, -G\}$. If S were consistent, then there would be a world $w' \supseteq S$ by Zorn's Lemma. We claim that $w' \geqslant w$. Indeed, take any $+(H_1 \supset H_2)$ in w such that $+H_1 \in w'$: by construction $+(H_1 \supset H_2)$ is in S, hence in w', so by Lemma 97 $+H_2$ is in w'. Since $w' \geqslant w$, and $+F$ is in w' by construction, by assumption $+G$ is in w', too. However $-G$ is in w' by construction, contradicting the consistency of w'.

So S is inconsistent. This entails that $H_1, \ldots, H_n \vdash F \supset G$ is provable for some $+H_1, \ldots, +H_n$ in w. In particular, $-(F \supset G)$ cannot be in w, otherwise w would be inconsistent. So $+(F \supset G)$ is in w, by Lemma 96. $\Box$

Lemma 104. *Let ρ be the valuation mapping each variable A to the set of all worlds w containing $+A$. Then for every world w, for every formula F, $w, \rho \models F$ if and only if $+F \in w$.*

Proof. First, ρ is indeed a valuation because the set of all worlds containing $+A$ is an upper set, by Lemma 98. The claim is then proved by structural induction on F. This is obvious if F is a variable.

If F is of the form $F_1 \supset F_2$. If $w, \rho \models F$ then by definition for every $w' \geqslant w$, if $w', \rho \models F_1$ then $w', \rho \models F_2$. By induction hypothesis, this means that if $+F_1 \in w'$ then $+F_2 \in w'$ for every $w' \geqslant w$. By Lemma 103 $+F$ is in w. Conversely, if $+F$ is in w, by definition of $\geqslant$, for every $w' \geqslant w$, if $+F_1$ is in w' then $+F_2$ is in w'. By induction hypothesis, if $w', \rho \models F_1$ then $w', \rho \models F_2$. So $w, \rho \models F$.

If F is of the form $\Box F_1$. If $w, \rho \models F$ then for every $w' \trianglerighteq w$ $w', \rho \models F_1$. So by induction hypothesis $+F_1 \in w'$, for every $w' \trianglerighteq w$. By Lemma 102 $+F$ is in w.

Conversely, if $+F$ is in w then $+F_1$ is in every $w' \trianglerighteq w$, so $w', \rho \models F_1$ for every $w' \trianglerighteq w$ by induction hypothesis, therefore $w, \rho \models F$. $\square$

Theorem 105 (Completeness). *If* $\Gamma \vdash F$ *is valid, then* $\Gamma \vdash F$ *is provable.*

Proof. Write Γ as $F_1, \ldots, F_n$. If $\Gamma \vdash F$ is not provable, then $\{+F_1, \ldots, +F_n, -F\}$ is consistent. Let w_0 be a world containing this set, by Zorn's Lemma. Since w_0 contains each $+F_i$, $1 \leqslant i \leqslant n$, by Lemma 104 $w_0, \rho \models F_i$, therefore Γ holds at w_0. Also since $-F \in w_0$, by consistency $+F$ is not in w_0. By Lemma 104, $w_0, \rho \not\models F$. This contradicts the fact that $\Gamma \vdash F$ is valid. $\square$

Appendix B. Critical Pairs for λ_{S4}

The critical pairs are as follows:

- Between (β) and some rule: then this rule must be (η). This happens for terms of the form $(\lambda x \cdot Mx)N$, with x not free in M, which may rewrite to $(Mx)\{x := N\} = MN$ by (β) or to MN by (η).
- Between (d) and (d): no critical pair.
- Between (d) and (gc): if x is not free in M, then $d\left(\boxed{M} \cdot (\theta, \{x := N\})\right)$ rewrites either to $M(\theta, \{x := N\}) = M\theta$ by (d) or to $d\left(\boxed{M} \cdot \theta\right)$ by (gc). The latter rewrites to the former by (d).
- Between (d) and (ctr): $d\left(\boxed{M} \cdot (\theta, \{x := N, y := N\})\right)$ may rewrite either to $d\left(\boxed{M\{y := x\}} \cdot (\theta, \{x := N\})\right)$ by (ctr) or to $M(\theta, \{x := N, y := N\})$ by (d). The former rewrites to $M\{y := x\}(\theta, \{x := N\}) = M(\theta, \{x := N, y := N\})$, because of the variable naming convention.
- Between (d) and $(\boxed{\square})$: $d\left(\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})\right)$ rewrites to the λ_{S4}-term $d\left(\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta')\right)$ by $(\boxed{\square})$ and to $M(\theta, \{x := \boxed{N} \cdot \theta'\})$ by (d). Notice that the latter rewrites in zero, one or more (gc) steps to $M(\theta, \{x := \boxed{N}\theta'\})$ (beware of the presence or absence of $\cdot$ after a box; recall that $\boxed{N}$ is $\boxed{N} \cdot \{x_1 := x_1, \ldots, x_n := x_n\}$ where $x_1, \ldots, x_n$ are the free variables of N, so $\boxed{N}\theta = \boxed{N} \cdot \{x_1 := x_1\theta, \ldots, x_n := x_n\theta\}$ may be distinct from $\boxed{N} \cdot \theta$, although it can always be obtained from it by series of (gc) steps), while the former rewrites to $M\{x := \boxed{N}\}(\theta, \theta')$ by (d), and this is exactly $M(\theta, \{x := \boxed{N}\theta'\})$ by the variable naming convention.
- Between (d) and (η): no critical pair.
- Between (d) and $(\eta\square)$: $d\left(\boxed{dx} \cdot \{x := N\}\right)$ rewrites by (d) to dN, and by $(\eta\square)$ to the same term. There is no other critical pair between these rules, as the term $\boxed{dM} \cdot \theta$ is an $(\eta\square)$ redex only when M is a variable.

- Between (gc) and itself. If neither x nor y is free in the term M, then $\boxed{M} \cdot (\theta, \{x := N, y := P\})$ may rewrite to $\boxed{M} \cdot (\theta, \{x := N\})$, or to the term $\boxed{M} \cdot (\theta, \{y := P\})$. Both rewrite to $\boxed{M} \cdot \theta$ by (gc).
- Between (gc) and (ctr): when x is not free in M, $\boxed{M} \cdot (\theta, \{x := N, y := N\})$ may rewrite to $\boxed{M} \cdot (\theta, \{y := N\})$ by (gc) or to $\boxed{M\{y := x\}} \cdot (\theta, \{x := N\})$ by (ctr). By α-renaming, the latter is also $\boxed{M\{y := x\}\{x := y\}} \cdot (\theta, \{y := N\})$, and this is equal to $\boxed{M} \cdot (\theta, \{y := N\})$ because, since x is not free in M, $M\{y := x\}\{x := y\} = M$.
  There is a symmetric case where y instead of x is not free in M. Then $\boxed{M} \cdot (\theta, \{x := N, y := N\})$ may rewrite to $\boxed{M} \cdot (\theta, \{x := N\})$ by (gc) or to $\boxed{M\{y := x\}} \cdot (\theta, \{x := N\})$ by (ctr). Since y is not free in M, both terms are equal.
- Between (gc) and ($\boxed{\square}$). If $y \notin \text{fv}(M)$, $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta', y := P\})$ rewrites to $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})$ by (gc) and to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta', \{y := P\})$ by ($\boxed{\square}$). The former rewrites to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta')$ by ($\boxed{\square}$), and the latter rewrites to the same term by (gc).
  There is another critical pair where x is not free in M: $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})$ rewrites to $\boxed{M} \cdot \theta$ by (gc), and to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta')$ by ($\boxed{\square}$). The latter is exactly $\boxed{M} \cdot (\theta, \theta')$, but by the variable naming convention no variable in the domain of θ' is free in M, so this rewrites by (gc) to $\boxed{M} \cdot \theta$.
- Between (gc) and (η): no critical pair.
- Between (gc) and ($\eta\square$): no critical pair.
- Between (ctr) and itself: $\boxed{M} \cdot (\theta, \{x := N, y := N, z := N\})$ rewrites either to $\boxed{M\{y := x\}} \cdot (\theta, \{x := N, z := N\})$ or to $\boxed{M\{z := x\}} \cdot (\theta, \{x := N, z := N\})$. They rewrite respectively to the term $\boxed{M\{y := x\}\{z := x\}} \cdot (\theta, \{x := N\})$ and to $\boxed{M\{z := x\}\{y := x\}} \cdot (\theta, \{x := N\})$ by (ctr), and these are identical terms since x, y, z are distinct variables.
  Another critical pair starts with $\boxed{M} \cdot (\theta, \{x := N, y := N, x' := N', y' := N'\})$, which is easily seen to be joinable.
- Between (ctr) and ($\boxed{\square}$). These critical pairs occur as $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})$ with the explicit substitution part mapping two distinct variables to the same term. If these two variables are in the domain of θ, then the critical pair is easily seen to be joinable.
  Otherwise, consider $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta', y := \boxed{N} \cdot \theta'\})$, which rewrites to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta', y := \boxed{N} \cdot \theta'\})$ by ($\boxed{\square}$), or to the term $\boxed{M\{y := x\}} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})$. The former rewrites to $\boxed{M\{x := \boxed{N}\}\{y := \boxed{N}\}} \cdot (\theta, \theta')$

using $(\boxed{\Box})$ and zero, one or more (ctr) steps, while the latter rewrites to $\boxed{M\{y := x\}\{x := \boxed{N}\}} \cdot (\theta, \theta')$ by $(\boxed{\Box})$. These are the same terms, by the variable naming convention.
There is a symmetric critical pair where $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta', y := \boxed{N} \cdot \theta'\})$ rewrites to $\boxed{M\{y := \boxed{N}\}} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\}, \theta')$ by $(\boxed{\Box})$ or to $\boxed{M\{y := x\}} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\})$. The former rewrites to $\boxed{M\{y := \boxed{N}\}\{x := \boxed{N}\}} \cdot (\theta, \theta')$ by $(\boxed{\Box})$ and zero, one or more instances of (ctr).

- Between (ctr) and (η): no critical pair.
- Between (ctr) and $(\eta\Box)$: no critical pair.
- Between $(\boxed{\Box})$ and itself. Then $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot \theta', y := \boxed{P} \cdot \theta''\})$ rewrites either to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta', \{y := \boxed{P} \cdot \theta''\})$, or to the term $\boxed{M\{y := \boxed{P}\}} \cdot (\theta, \{x := \boxed{N} \cdot \theta'\}, \theta'')$. These two terms rewrite to $\boxed{M\{x := \boxed{N}, y := \boxed{P}\}} \cdot (\theta, \theta', \theta'')$.
There is one other critical pair: $\boxed{M} \cdot (\theta, \{x := \boxed{N} \cdot (\theta', \{y := \boxed{P} \cdot \theta''\})\})$ rewrites either to $\boxed{M\{x := \boxed{N}\}} \cdot (\theta, \theta', \{y := \boxed{P} \cdot \theta''\})$, or to the term $\boxed{M} \cdot (\theta, \{x := \boxed{N\{y := \boxed{P}\}} \cdot (\theta', \theta'')\})$.
The former rewrites to $M_1 \hat{=} \boxed{M\{x := \boxed{N}\}\{y := \boxed{P}\}} \cdot (\theta, \theta', \theta'')$ by $(\boxed{\Box})$, and the latter rewrites to $M_2 \hat{=} \boxed{M\{x := \boxed{N\{y := \boxed{P}\}} \cdot \}} \cdot (\theta, \theta', \theta'')$. By the variable naming convention, y is only free in N, not in M, so $M\{x := \boxed{N}\}\{y := \boxed{P}\} = M\{x := \boxed{N}\{y := \boxed{P}\}\}$. As $\boxed{N}\{y := \boxed{P}\}$ rewrites to $\boxed{N\{y := \boxed{P}\}}$ by $(\boxed{\Box})$, M_1 rewrites to M_2.
- Between $(\boxed{\Box})$ and (η): no critical pair.
- Between $(\boxed{\Box})$ and $(\eta\Box)$: $\boxed{dx} \cdot \{x := \boxed{M} \cdot \theta\}$ rewrites to $\boxed{M} \cdot \theta$ by $(\eta\Box)$ and to $\boxed{d\boxed{M}} \cdot \theta$ by $(\boxed{\Box})$. The latter rewrites to $\boxed{M} \cdot \theta$ by (d).
- Between (η) and itself: no critical pair.
- Between (η) and $(\eta\Box)$: no critical pair.
- Between $(\eta\Box)$ and itself: no critical pair.

Appendix C. The Cone Monad $(\blacklozenge, \mathfrak{e}, \mathfrak{m})$ in $\widehat{\Delta}$

This can also be described more succinctly using categorical language [**46**]. However we expand all constructions to give a more concrete idea of what is going on.

Proposition 106 ($\blacklozenge$). *For every a.s. set K, let $\blacklozenge K$ be the a.s. set whose q-simplices are the pairs (x, j) where $x \in K_{q-1-j}$, $-1 \leqslant j \leqslant q$; $\partial^i_q(x, j)$ is $(x, j-1)$ if $0 \leqslant i \leqslant j$ and $(\partial^{i-1-j}_{q-1-j} x, j)$ if $j < i \leqslant q$; $s^i_q(x, j)$ is $(x, j+1)$ if $0 \leqslant i \leqslant j$ and $(s^{i-1-j}_{q-1-j} x, j)$ if $j < i \leqslant q$.*

Then in $\widehat{\Delta}$, $\blacklozenge \dashv \Box$.

Moreover, $(\blacklozenge, \mathfrak{e}, \mathfrak{m})$ is a strong monad, where $\mathfrak{e}$ maps $x \in K_q$ to $(x, -1) \in (\blacklozenge K)_q$, and $\mathfrak{m}$ maps $((x, j), j') \in (\blacklozenge^2 K)_q$ ($x \in K_{q-2-j-j'}$, $-1 \leqslant j \leqslant q-1-j'$, $-1 \leqslant j' \leqslant q$) to $(x, j+j'+1) \in (\blacklozenge K)_q$.

Proof. We leave to the reader the tedious check that $\blacklozenge K$ is indeed augmented simplicial. Also, $\blacklozenge$ defines a functor by mapping every a.s. map $f \hat{=} (f_q)_{q \geqslant -1}$ to the $\blacklozenge f$ sending every $(x, j) \in (\blacklozenge K)_q$ to $(f_{q-1-j}(x), j)$.

Define the unit η of the adjunction by letting $\eta(x) \hat{=} (x, 0) \in (\Box\blacklozenge K)_q$ for every $x \in K_q$, and the counit ϵ by $\epsilon(x, j) \hat{=} s^0_{q-1} \ldots s^0_{q-j} x \in K_q$, for every $x \in K_{q-j}$, $-1 \leqslant j \leqslant q$.

Check that η is an a.s. map: for every $x \in K_q$, $0 \leqslant i \leqslant q$, $\partial^i_{(\Box\blacklozenge K)_q} \eta(x) = \partial^{i+1}_{q+1}(x, 0) = (\partial^i_q x, 0) = \eta(\partial^i_q x)$, while $s^i_{(\Box\blacklozenge K)_q} \eta(x) = s^{i+1}_{q+1}(x, 0) = (s^i_q x, 0) = \eta(s^i_q x)$.

Check that ϵ is an a.s. map. For every $x \in K_{q-j}$, $-1 \leqslant j \leqslant q$, $\partial^i_q \epsilon(x, j) = \partial^i_q s^0_{q-1} \ldots s^0_{q-j} x = s^0_{q-2} \partial^{i-1}_{q-1} s^0_{q-2} \ldots s^0_{q-j} x = \ldots$; continuing, either $i > j$ and $\partial^i_q \epsilon(x, j) = s^0_{q-2} \ldots s^0_{q-j-1} \partial^{i-j}_{q-j} x = s^0_{q-2} \ldots s^0_{q-j-1} \partial^{i-j-1}_{(\Box K)_{q-j-1}} x = \epsilon(\partial^{i-j-1}_{(\Box K)_{q-j-1}} x, j)$ $= \epsilon(\partial^i_q(x, j))$; or $i \leqslant j$ and $\partial^i_q \epsilon(x, j) = s^0_{q-2} \ldots s^0_{q-i-1} \partial^0_{q-i} s^0_{q-i-1} s^0_{q-i-2} \ldots s^0_{q-j-1} x$ $= s^0_{q-2} \ldots s^0_{q-i-1} s^0_{q-i-2} \ldots s^0_{q-j-1} x = \epsilon(x, j-1) = \epsilon(\partial^i_q(x, j))$.

Similarly, $s^i_q \epsilon(x, j) = s^i_q s^0_{q-1} \ldots s^0_{q-j} x$ is either (if $i > j$) $s^0_q \ldots s^0_{q-j+1} s^{i-j}_{q-j} x = s^0_q \ldots s^0_{q-j+1} s^{i-j-1}_{(\Box K)_{q-j-1}} x = s^0_q \ldots s^0_{q-j+1} s^i_q(x, j) = \epsilon(s^i_q(x, j))$, or (if $i \leqslant j$) this is $s^0_q \ldots s^0_{q-i+1} s^0_{q-i} s^0_{q-i-1} \ldots s^0_{q-j} x = \epsilon(x, j+1) = \epsilon(s^i_q(x, j))$.

Clearly η and ϵ are natural in K. It remains to check that $\epsilon \circ \blacklozenge\eta$ is the identity on $\blacklozenge K$, and that $\Box\epsilon \circ \eta$ is the identity on $\Box K$.

For the first equality, let $(x, j) \in (\blacklozenge K)_q$, that is, $x \in K_{q-1-j}$, $-1 \leqslant j \leqslant q$. Then $\epsilon(\blacklozenge\eta(x, j)) = \epsilon_q(\eta_{q-1-j}(x), j) = \epsilon_q((x, 0), j) = s^0_{q-1} \ldots s^0_{q-j}(x, 0) = s^0_{q-1} \ldots s^0_{q-j+1}$ $(x, 1) = \ldots = (x, j)$.

For the second equality, let $x \in (\Box K)_q = K_{q+1}$. Then $(\Box\epsilon)_q(\eta_q(x)) = (\Box\epsilon)_q(x, 0)$ $= \epsilon_{q+1}(x, 0) = x$.

Let us now check that $(\blacklozenge, \mathfrak{e}, \mathfrak{m})$ is a monad. We again leave the tedious check that $\mathfrak{e}$ and $\mathfrak{m}$ are a.s. to the reader. Clearly $\mathfrak{e}$ and $\mathfrak{m}$ are natural in K. Then check the monad equations:

- $\mathfrak{m} \circ \mathfrak{m} = \mathfrak{m} \circ \blacklozenge\mathfrak{m}$: let $(((x, j), j', j'') \in (\blacklozenge K)_q$ ($x \in K_{q-3-j-j'-j''}$, $-1 \leqslant j \leqslant q-2-j''-j'$, $-1 \leqslant j' \leqslant q-1-j''$, $-1 \leqslant j'' \leqslant q$), then $\mathfrak{m}(\mathfrak{m}(((x, j), j'), j'')) = \mathfrak{m}((x, j), j'+j''+1) = (x, j+j'+j''+2)$, while $\mathfrak{m}(\blacklozenge\mathfrak{m}(((x, j), j'), j'')) = \mathfrak{m}((x, j+j'+1), j'') = (x, j+j'+j''+2)$.
- $\mathfrak{m} \circ \mathfrak{e} = \mathrm{id}$: let $(x, j) \in (\blacklozenge K)_q$ ($x \in K_{q-1-j}$, $-1 \leqslant j \leqslant q$), then $\mathfrak{m}(\mathfrak{e}(x, j)) = \mathfrak{m}((x, j), -1) = (x, j + (-1) + 1) = (x, j)$.

- $\mathfrak{m} \circ \blacklozenge\mathfrak{e} = \mathrm{id}$: let $(x,j) \in (\blacklozenge K)_q$ $(x \in K_{q-1-j},\ -1 \leqslant j \leqslant q)$, then $\mathfrak{m}(\blacklozenge\mathfrak{e}(x,j)) = \mathfrak{m}((x,-1),j) = (x,(-1)+j+1) = (x,j)$.

To show that it is a strong monad, let $\tau^{K,L} : K \times \blacklozenge L \to \blacklozenge(K \times L)$ be the a.s. map sending every q-simplex $(x,(y,k))$, where $x \in K_q$, $y \in L_{q-1-k}$, $-1 \leqslant k \leqslant q$ to the q-simplex $((\partial^0_{q-k} \dots \partial^0_q x, y), k)$.

This is an a.s. map: if $i \leqslant k$, $\tau^{K,L}_{q-1}(\partial^i_q x, \partial^i_q(y,k)) = \tau^{K,L}_{q-1}(\partial^i_q x,(y,k-1)) = ((\partial^0_{q-k} \dots \partial^0_{q-1}\partial^i_q x, y), k-1) = ((\partial^0_{q-k} \dots \partial^0_{q-1}\partial^0_q x, y), k-1) = \partial^i_q \tau^{K,L}_q(x,(y,k))$. On the other hand, if $i > k$, $\tau^{K,L}_{q-1}(\partial^i_q x, \partial^i_q(y,k)) = \tau^{K,L}_{q-1}(\partial^i_q x, (\partial^{i-1-k}_{q-1-k} y, k)) = ((\partial^0_{q-1-k} \dots \partial^0_{q-1}\partial^i_q x, \partial^{i-1-k}_{q-1-k} y), k) = ((\partial^{i-1-k}_{q-1-k}\partial^0_{q-k} \dots \partial^0_q x, \partial^{i-1-k}_{q-1-k} y), k) = \partial^i_q \tau^{K,L}_q (x,(y,k))$. Similarly for s^i.

Then $\tau^{K,L}$ is natural in K, L: whenever $f : K' \to K$, $g : L' \to L$ are a.s. map, $\tau^{K,L} \circ \langle f, \blacklozenge g\rangle$ maps every q-simplex $(x,(y,k))$ to $((\partial^0_{q-k} \dots \partial^0_q f_q(x), g_{q-1-k}(y)), k) = ((f_{q-1-k}(\partial^0_{q-k} \dots \partial^0_q x), g_{q-1-k}(y)), k) = (\blacklozenge\langle f, g\rangle \circ \tau^{K',L'})_q (x,(y,k))$.

Now we must check that: (a) given the a.s. map $\lambda \hat{=} \pi_2 : \mathbf{1} \times \blacklozenge K \to \blacklozenge K$, $\blacklozenge\lambda \circ \tau^{\mathbf{1},K} = \lambda$; (b) $\tau^{K,L} \circ (\mathrm{id} \times \mathfrak{e}) = \mathfrak{e}$; (c) letting $\alpha : (K \times L_1) \times \blacklozenge L_2 \to K \times (L_1 \times \blacklozenge L_2)$ map $((x,y),(z,k))$ to $(x,(y,(z,k)))$, $\tau^{K,L_1 \times L_2} \circ (\mathrm{id} \times \tau^{L_1,L_2}) \circ \alpha = \blacklozenge\alpha \circ \tau^{K \times L_1, L_2}$; and (d) $\mathfrak{m} \circ \blacklozenge\tau^{K,L} \circ \tau^{K,\blacklozenge L} = \tau^{K,L} \circ (\mathrm{id} \times \mathfrak{m}) : K \times \blacklozenge^2 L \to \blacklozenge(K \times L)$. This is easy. $\square$

Appendix D. Proof of Lemma 58

We first check each α-conversion rule; the first one is standard (op. cit.).

For the second let $y_1, \dots, y_n$ be distinct variables, $\Theta \hat{=} x_1 : \Box F_1, \dots, x_n : \Box F_n$, $\Theta' \hat{=} y_{\sigma(1)} : \Box F_1, \dots, y_{\sigma(n)} : \Box F_n$, where σ is a permutation of $1, \dots, n$:

$$\begin{aligned}
&\mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M\{x_1 := y_1, \dots, x_n := y_n\}} \cdot \{y_1 := N_1, \dots, y_n := N_n\} : \Box G\right]\!\!\right]\rho \\
&= \Box\mathcal{C}\,[\![\Theta' \vdash M\{x_1 := y_1, \dots, x_n := y_n\} : G]\!]\,\rho \circ \mathbf{s} \\
&\qquad\qquad \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_{\sigma(1)} : \Box F_{\sigma(1)}]\!]\,\rho, \dots, \mathcal{C}\,[\![\Gamma \vdash N_{\sigma(n)} : \Box F_{\sigma(n)}]\!]\,\rho\rangle \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \circ \langle \pi_2 \circ \pi_1^{n-\sigma^{-1}(1)}, \dots, \pi_2 \circ \pi_1^{n-\sigma^{-1}(n)}\rangle \\
&\qquad\qquad \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_{\sigma(1)} : \Box F_{\sigma(1)}]\!]\,\rho, \dots, \mathcal{C}\,[\![\Gamma \vdash N_{\sigma(n)} : \Box F_{\sigma(n)}]\!]\,\rho\rangle \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \dots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \dots, x_n := N_n\} : \Box G\right]\!\!\right]\rho
\end{aligned}$$

It follows that $M \equiv N$ implies $\mathcal{C}\,[\![\Gamma \vdash M : F]\!]\,\rho = \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho$.

- (β) This is standard; let Γ be of length n:

$$
\begin{aligned}
& \mathcal{C}\,[\![\Gamma \vdash (\lambda x \cdot M)N : G]\!]\,\rho \\
={}& App \circ \langle \mathcal{C}\,[\![\Gamma \vdash \lambda x \cdot M : F \supset G]\!]\,\rho, \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \\
={}& App \circ \langle \Lambda(\mathcal{C}\,[\![\Gamma, x : F \vdash M : G]\!]\,\rho), \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \\
={}& \mathcal{C}\,[\![\Gamma, x : F \vdash M : G]\!]\,\rho \circ \langle \mathrm{id}, \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \quad \text{(by } (l)\text{)} \\
={}& \mathcal{C}\,[\![\Gamma, x : F \vdash M : G]\!]\,\rho \\
& \quad \circ \langle\langle \ldots \langle\langle \pi_1^{n+1}, \pi_2 \circ \pi_1^n\rangle, \pi_2 \circ \pi_1^{n-1}\rangle \ldots \circ \pi_1, \pi_2\rangle, \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \\
& \qquad\qquad \text{(by (g), (h))} \\
={}& \mathcal{C}\,[\![\Gamma, x : F \vdash M : G]\!]\,\rho \\
& \quad \circ \langle\langle \ldots \langle\langle !, \pi_2 \circ \pi_1^n\rangle, \pi_2 \circ \pi_1^{n-1}\rangle \ldots \circ \pi_1, \pi_2\rangle, \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \quad \text{(by (d))} \\
={}& \mathcal{C}\,[\![\Gamma, x : F \vdash M : G]\!]\,\rho \circ \langle \pi_2 \circ \pi_1^n, \pi_2 \circ \pi_1^{n-1}, \ldots, \pi_2, \mathcal{C}\,[\![\Gamma \vdash N : F]\!]\,\rho\rangle \\
={}& \mathcal{C}\,[\![\Gamma \vdash M\{x_1 := x_1, \ldots, x_n := x_n, x := N\} : G]\!]\,\rho \quad \text{(by (18))} \\
={}& \mathcal{C}\,[\![\Gamma \vdash M\{x := N\} : G]\!]\,\rho
\end{aligned}
$$

- (η) When x is not free in M:

$$
\begin{aligned}
\mathcal{C}\,[\![\Gamma \vdash \lambda x \cdot Mx : F \supset G]\!]\,\rho &= \Lambda(\mathcal{C}\,[\![\Gamma, x : F \vdash Mx : G]\!]\,\rho) \\
&= \Lambda(App \circ \langle \mathcal{C}\,[\![\Gamma, x : F \vdash M : F \supset G]\!]\,\rho, \pi_2\rangle) \\
&= \Lambda(App \circ \langle \mathcal{C}\,[\![\Gamma \vdash M : F \supset G]\!]\,\rho \circ \pi_1, \pi_2\rangle) \\
&= \mathcal{C}\,[\![\Gamma \vdash M : F \supset G]\!]\,\rho
\end{aligned}
$$

- (d) Let $\Theta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$:

$$
\begin{aligned}
& \mathcal{C}\left[\!\!\left[\Gamma \vdash d(\boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\}) : G\right]\!\!\right]\rho \\
={}& \mathbf{d} \circ \mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} : \Box G\right]\!\!\right]\rho \\
={}& \mathbf{d} \circ \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
={}& \mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{d} \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \quad \text{(by (p))} \\
={}& \mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \quad \text{(by (s) and (a))} \\
={}& \mathcal{C}\,[\![\Gamma \vdash M\{x_1 := N_1, \ldots, x_n := N_n\} : G]\!]\,\rho \quad \text{(by (18))}
\end{aligned}
$$

- (gc) When x_n is not free in M:

$$\begin{aligned}
&\mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} : \Box G\right]\!\!\right] \rho \\
&= \Box\mathcal{C}\,[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash M : G]\!]\,\rho \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box(\mathcal{C}\,[\![x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1} \vdash M : G]\!]\,\rho \circ \pi_1) \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathcal{C}\,[\![x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1} \vdash M : G]\!]\,\rho \circ \pi_1 \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathcal{C}\,[\![x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1} \vdash M : G]\!]\,\rho \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho\rangle \\
&= \mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_{n-1} := N_{n-1}\} : \Box G\right]\!\!\right] \rho
\end{aligned}$$

- (ctr) When $N_{n-1} = N_n$, $\Theta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$:

$$\begin{aligned}
&\mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_{n-1} := N_{n-1}, x_n := N_n\} : \Box G\right]\!\!\right] \rho \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \\
&\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \langle \pi_2 \circ \pi_1^{n-2}, \ldots, \pi_2, \pi_2\rangle \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho\rangle \\
&= \Box(\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \langle \pi_2 \circ \pi_1^{n-2}, \ldots, \pi_2, \pi_2\rangle) \circ \mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho\rangle \\
&= \Box(\mathcal{C}\,[\![x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1} \vdash M\{x_n := x_{n-1}\} : G]\!]\,\rho)\mathbf{s} \\
&\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho\rangle \\
&= \mathcal{C}\left[\!\!\left[\begin{array}{c} x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1} \vdash \boxed{M\{x_n := x_{n-1}\}} \cdot \\ \{x_1 := N_1, \ldots, x_{n-1} := N_{n-1}\} : \Box G \end{array}\right]\!\!\right] \rho
\end{aligned}$$

- ($\boxed{\Box}$) Let $\Theta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$, $N_n \hat{=} \boxed{P} \cdot \{y_1 := P_1, \ldots, y_m := P_m\}$, $\Theta' \hat{=} y_1 : \Box F'_1, \ldots, y_m : \Box F'_m$:

$$\begin{aligned}
&\mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{M} \cdot \{x_1 := N_1, \ldots, x_{n-1} := N_{n-1}, x_n := N_n\} : \Box G\right]\!\!\right] \rho \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho, \\
&\qquad\qquad \mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{P} \cdot \{y_1 := P_1, \ldots, y_m := P_m\} : \Box F_n\right]\!\!\right] \rho\rangle \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho, \\
&\qquad\qquad \Box\mathcal{C}\,[\![\Theta' \vdash P : F_n]\!]\,\rho \circ \mathbf{s} \\
&\qquad\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash P_1 : \Box F'_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash P_m : \Box F'_m]\!]\,\rho\rangle\rangle \\
&= \Box\mathcal{C}\,[\![\Theta \vdash M : G]\!]\,\rho \circ \langle \mathbf{s} \circ \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathbf{s} \circ \mathcal{C}\,[\![\Gamma \vdash N_{n-1} : \Box F_{n-1}]\!]\,\rho, \\
&\qquad\qquad \Box^2\mathcal{C}\,[\![\Theta' \vdash P : F_n]\!]\,\rho \circ \mathbf{s} \circ \mathbf{s} \\
&\qquad\qquad\qquad \circ\langle \mathcal{C}\,[\![\Gamma \vdash P_1 : \Box F'_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash P_m : \Box F'_m]\!]\,\rho\rangle\rangle
\end{aligned}$$

whereas:

$$\mathcal{C}\left[\!\!\left[\begin{array}{ll}\Gamma \vdash \boxed{M\{x_n := \boxed{P}\}} \cdot & \{x_1 := N_1, \ldots, x_{n-1} := N_{n-1}, \\ & y_1 := P_1, \ldots, y_m := P_m\} : \Box G\end{array}\right]\!\!\right] \rho$$

$$\begin{aligned}
&= \Box\mathcal{C}\left[\!\!\left[x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1}, \Theta' \vdash M\{x_n := \boxed{P}\} : G\right]\!\!\right] \rho \\
&\qquad \circ \mathbf{s} \circ \langle \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \\
&= \Box\left(\begin{array}{ll}\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ & \langle \pi_2 \circ \pi_1^{n+m-2}, \ldots, \pi_2 \circ \pi_1^{m}, \\ & \mathcal{C}\left[\!\left[x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1}, \Theta' \vdash \boxed{P} : \Box F_n\right]\!\right] \rho \rangle\end{array}\right) \\
&\qquad \circ \mathbf{s} \circ \langle \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad\quad \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \\
&= \begin{array}{ll}\Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ & \langle \pi_2 \circ \pi_1^{n+m-2}, \ldots, \pi_2 \circ \pi_1^{m}, \\ & \Box\mathcal{C}\left[\!\left[x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1}, \Theta' \vdash \boxed{P} : \Box F_n\right]\!\right] \rho \rangle\end{array} \\
&\qquad \circ \mathbf{s} \circ \langle \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad\quad \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \\
&= \begin{array}{ll}\Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ & \langle \pi_2 \circ \pi_1^{n+m-2}, \ldots, \pi_2 \circ \pi_1^{m}, \\ & \Box\mathcal{C}\left[\!\left[x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1}, \Theta' \vdash \boxed{P} : \Box F_n\right]\!\right] \rho \rangle \circ\end{array} \\
&\qquad \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \\
&= \Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \Box\mathcal{C}\left[\!\!\left[x_1 : \Box F_1, \ldots, x_{n-1} : \Box F_{n-1}, \Theta' \vdash \boxed{P} : \Box F_n\right]\!\!\right] \rho \circ \\
&\qquad\quad \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad\quad \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \rangle \\
&= \Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \Box(\Box\mathcal{C}\left[\!\left[\Theta' \vdash P : F_n\right]\!\right] \rho \circ \langle \mathbf{s} \circ \pi_2 \circ \pi_1^{m-1}, \ldots, \mathbf{s} \circ \pi_2 \rangle) \circ \\
&\qquad\quad \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad\quad \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \rangle \\
&= \Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \Box^2\mathcal{C}\left[\!\left[\Theta' \vdash P : F_n\right]\!\right] \rho \circ \langle \mathbf{s} \circ \pi_2 \circ \pi_1^{m-1}, \ldots, \mathbf{s} \circ \pi_2 \rangle \circ \\
&\qquad\quad \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad\quad \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \rangle \\
&= \Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \Box^2\mathcal{C}\left[\!\left[\Theta' \vdash P : F_n\right]\!\right] \\
&\qquad\qquad \circ \langle \mathbf{s} \circ \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \rangle \\
&= \Box\mathcal{C}\left[\!\left[\Theta \vdash M : G\right]\!\right] \rho \circ \langle \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_1 : \Box F_1\right]\!\right] \rho, \ldots, \mathbf{s} \circ \mathcal{C}\left[\!\left[\Gamma \vdash N_{n-1} : \Box F_{n-1}\right]\!\right] \rho, \\
&\qquad \Box^2\mathcal{C}\left[\!\left[\Theta' \vdash P : F_n\right]\!\right] \circ \mathbf{s} \circ \mathbf{s} \circ \langle \mathcal{C}\left[\!\left[\Gamma \vdash P_1 : \Box F_1'\right]\!\right] \rho, \ldots, \mathcal{C}\left[\!\left[\Gamma \vdash P_m : \Box F_m'\right]\!\right] \rho \rangle \rangle
\end{aligned}$$

- ($\eta\Box$) Let $\Theta\hat{=}x_1 : \Box F_1, \ldots, x_n : \Box F_n$, and x be x_i:

$$\begin{aligned}
&\mathcal{C}\left[\!\!\left[\Gamma \vdash \boxed{dx} \cdot \{x_1 := N_1, \ldots, x_n := N_n\} : F_i\right]\!\!\right]\rho \\
&= \Box\mathcal{C}\,[\![\Theta \vdash dx : F_i]\!]\,\rho \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box(\mathbf{d} \circ \pi_2 \circ \pi_1^{n-i}) \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathbf{d} \circ \pi_2 \circ \pi_1^{n-i} \circ \mathbf{s} \circ \langle \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathbf{d} \circ \pi_2 \circ \pi_1^{n-i} \circ \langle \mathbf{s} \circ \mathcal{C}\,[\![\Gamma \vdash N_1 : \Box F_1]\!]\,\rho, \ldots, \mathbf{s} \circ \mathcal{C}\,[\![\Gamma \vdash N_n : \Box F_n]\!]\,\rho\rangle \\
&= \Box\mathbf{d} \circ \mathbf{s} \circ \mathcal{C}\,[\![\Gamma \vdash N_i : \Box F_i]\!]\,\rho = \mathcal{C}\,[\![\Gamma \vdash N_i : \Box F_i]\!]\,\rho \\
&= \mathcal{C}\,[\![\Gamma \vdash x\{x_1 := N_1, \ldots, x_n := N_n\} : F_i]\!]\,\rho
\end{aligned}$$

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