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Research Report LSV–99–2, Feb. 1999

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http://www.lsv.ens-cachan.fr/Publis/
Research Report LSV-99-2, Lab. Spécification et Vérification, CNRS & ENS de Cachan, France, Feb. 1999

A Well-Structured Framework for Analysing Petri Net Extensions

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February 23, 1999

(Classification: Logic and verification of computer programs, theory of parallel and distributed computation, algorithms, decidability)

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Proposed running head: A Well-Structured Framework

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Abstract

Transition systems defined from recursive functions $\mathbb{N}^p \to \mathbb{N}^p$ are introduced and named WSN, or *well-structured nets*. Such "minimally Petri net-like" transition systems sit conveniently between Petri net extensions and general transition systems. In a first part, we study decidability properties of WSN classes obtained by imposing natural restrictions to their defining functions, with respect to termination, to coverability, and to four variants of the boundedness problem. We are able to answer optimally almost all the questions which arise, thus gaining much insight into old and new generalized Petri net decidability results. In a second part, we specialize our analysis to WSN defined from affine functions. Such affine WSN elegantly encompass most Petri net extensions studied in the litterature. Again, we study decidability properties of natural classes of affine WSN with respect to six computational problems. In particular, we develop a nontrivial algorithm computing limits of iterated nonnegative affine functions, in order to decide the *path-place* variant of the boundedness problem for the relevant affine WSN. Undecidability results are scattered throughout the paper, concerning, for example, extending the domain of a recursive function from $\mathbb{N}^p$ to $(\mathbb{N} \cup \{\omega\})^p$.

1 Introduction

A *transition system* is a set S endowed with a transition relation "$\rightarrow$". It is a WSTS, i.e. a *well-structured* transition system, if a reflexive and transitive well-quasi-ordering on S exists which is compatible with "$\rightarrow$" ([11, 1, 13], or see Section 3 below). We deal in this paper with infinite state systems, with WSTS, and with Petri net extensions.

Our motivating theme is that Petri net extensions can naturally and fruitfully be defined and studied from the perspective of WSTS. This theme is only marginally new: WSTS were introduced as abstractions of Petri nets in the first place, as a means of identifying the essential properties of Petri net problems, algorithms, and extensions. But WSTS in fact capture many other infinite state systems [13], and WSTS are so general that (1) only rather coarse Petri net extension decidability results, concerning boundedness detection for example, are inherited from WSTS, and (2) the classification of Petri net extensions afforded by WSTS is not as complete as one might have hoped.

In this paper we respond to both weaknesses of the WSTS with the introduction of the WSN, or *well-structured net*. A WSN is a WSTS in which the set S is $\mathbb{N}^p$ and the transition relation "$\rightarrow$" is given by a finite set of recursive functions with upward-closed domains (see Section 3). Thus a WSN is a "minimally Petri net-like" WSTS. We convincingly establish the suitability of the WSN model for a systematic study of Petri net extensions and their decidability properties.

Our paper is formed of two parts. The first part deals with WSN in general. The second part deals with WSN defined from affine functions.

In the first part, we define progressively stronger properties of functions: nondecreasing, increasing, ρ-increasing, strongly increasing, each with or without the *ω-recursivity* hypothesis that the extension of the function to the domain $(\mathbb{N} \cup \{\omega\})^p$ is recursive (see Section 3 for formal definitions). These properties give rise to two groups of four WSN classes. We exhaustively study these classes with respect to six decision problems. These problems are termination, coverability, and four variants of the boundedness problem, of which the path-unbounded-witness and the path-place-boundedness are new and are introduced here to demonstrate the subtleties of boundedness detection. Our results are summarized on Figure 2: only four of the 48 relevant decidability questions (some of whose answers were implicitly or explicitly known) remain open. We also claim new results on ω-recursive functions. We single out, from the first part of our paper, the following:

- Coverability for ω-WSN, i.e. WSN whose functions satisfy the ω-recursivity hypothesis, is shown decidable, while coverability is shown undecidable for our most restrictive WSN class in the absence of the ω-recursivity hypothesis. The decidability proof uses the "effective pred basis" criterion [13] for decidability of coverability for WSTS (also applied in [9] to prove decidability of coverability for Reset Petri nets), coupled with a characterization of effective upward-closed subsets of $\mathbb{N}^p$.

- The decidability status of the four boundedness problems for WSN is resolved. This provably separates three of our WSN classes, and the fourth class is distinguished from the others when viewed as a ω-WSN class.

- Boundedness is shown decidable for increasing WSN but undecidable for ω-WSN.

- Path-unbounded-witness is shown decidable for ρ-increasing WSN but undecidable for increasing WSN.

- Path-place-boundedness, known to be decidable for reset Petri nets (and for transfer Petri nets), is shown undecidable for strongly increasing WSN and for ω-WSN.

- Place-boundedness is shown decidable for strongly increasing ω-WSN. This is proved by generalizing both the coverability tree of Karp and Miller [20] and the minimal coverability tree [12], noticing that the exact computation of the limit of an infinite repetitive sequence can be replaced by the computation of a good enough approximation. This construction and this result are shown to extend neither to strongly increasing WSN nor to ρ-increasing ω-WSN.

In the second part of our paper, we specialize our study to affine WSN, i.e. to WSN defined from affine functions. We do this because testing for nontrivial properties of general recursive functions is mostly undecidable (see Section 7.1), and because, in the concrete cases likely to be encountered in "practice", functions are either affine, or their properties are known a priori, or these are easy to test. On Figure 3, we record some easy (but appealing) connections between affine WSN, Petri net extensions, and general WSN. For example, generalized transfer Petri nets can be seen as WSN defined from affine functions $f(X) = AX + B$ satisfying $A \geq \mathbf{0}$ with every column of the matrix A different from $\mathbf{0}$. Affine WSN thus capture a wealth of known Petri net extensions, including Petri nets themselves, Reset Petri nets, Post SM nets, Double Petri nets, and they suggest new extensions of independent interest.

We then develop a nontrivial polynomial time algorithm to solve the following problem: given a nonnegative $n \times n$-matrix A and two nonnegative n-vectors B and X, compute $d \geq 1$ such that $\lim_{n\to\omega} f^{n^d}(X)$ exists, where f is the affine function $f(X) = AX + B$, and compute this limit. We use this algorithm to solve the path-place-boundedness problem for affine WSN defined from nonnegative matrices and vectors.

Finally, we record the decidability status of our six computational problems, for five prominent classes of affine WSN, including Petri nets and Self-Modifying nets [26, 27]: only four out of the 30 resulting questions (whose answers were known for Petri nets and Self-Modifying nets) remain open. See Figure 4.

Section 2 in this paper introduces our various functions (excluding ρ-increasing functions), the notion of limit, ω-recursivity, and proves relevant undecidability results. Section 3 formally defines well-structured nets and our computational problems. Section 4 examines decidability of termination and coverability for WSN. Section 5 does the same for the four boundedness problems, after defining ρ-increasing functions. Section 6 summarizes our decidability results for WSN. Section 7 discusses affine WSN, their relationship with WSN and extended Petri nets, and their decidability properties. Section 8 concludes.

2 Notation and preliminaries

Sets. When X is a set, a *word* $v \in X^*$ is a possibly empty sequence of elements of X; the empty word is denoted λ; we write $|v|$ for the length of v. We write $[p]$ for $\{1, 2, \ldots, p\}$, $\mathbb{N}^+$ for $\mathbb{N} \setminus \{0\}$, and $\mathbb{N}_\omega$ for the classical completion $\mathbb{N} \cup \{\omega\}$ of the set $\mathbb{N}$ of nonnegative integers (where $n < \omega$ for any $n \in \mathbb{N}$).

Upward-closed sets. Fix any integer $p \geq 1$. Let $\leq$ be the usual ordering on $\mathbb{N}_\omega$. We also unambiguously write $\leq$ for the ordering on $\mathbb{N}_\omega^p$ defined by: for every $u, v \in \mathbb{N}_\omega^p$, $u \leq v$ iff for every $i \in [p]$, $u(i) \leq v(i)$. We write $u < v$ if $u \leq v$ and $u \neq v$. Addition in $\mathbb{N}_\omega$ and in $\mathbb{N}_\omega^p$ is defined as expected, with $x + \omega = \omega + x = \omega$ for any $x \in \mathbb{N}_\omega$. Let $U + V$ denote the *sum* of two subsets $U, V \subseteq \mathbb{N}_\omega^p$: $U + V = \{u + v : u \in U, v \in V\}$. We say that a set $K \subseteq \mathbb{N}^p$ is *upward-closed* (for $\leq$) if $K = K + \mathbb{N}^p$. A *basis* of an upward-closed set K is a set $b(K)$ such that $K = b(K) + \mathbb{N}^p$. We define the *downward-closure* of $u \in \mathbb{N}_\omega^p$, written $\downarrow u$, as $\downarrow u = \{v \in \mathbb{N}^p : v \leq u\}$. The two well-known facts which follow can be found in [14], or in [13] where their proofs are briefly recalled:

1. Every upward-closed set $K \subseteq \mathbb{N}^p$ has a finite basis.

2. Any strictly increasing sequence $K_0 \subset K_1 \subset K_2 \cdots$ of upward-closed sets in $\mathbb{N}^p$ is finite.

We will require another result from Valk and Jantzen:

Theorem 2.1 *[29] A finite basis of an upward-closed set $K \subseteq \mathbb{N}^p$ is effectively computable iff for any vector $u \in \mathbb{N}^p_\omega$ the predicate $\downarrow u \cap K \neq \emptyset$ is decidable.*

Transition systems. A *transition system* (TS) is a structure $\mathcal{S} = \langle S, \rightarrow, \ldots \rangle$ where $S = \{s, t, \ldots\}$ is a set of *states*, and $\rightarrow \subseteq S \times S$ is any set of *transitions*. We write $Succ(s)$ (resp. $Pred(s)$) for the set $\{t \in S : s \rightarrow t\}$ of immediate *successors* of s (resp. for the set $\{t \in S : t \rightarrow s\}$ of immediate *predecessors* of s). We write $s \stackrel{+}{\rightarrow} t$ to mean that there exists $k > 0$ and a sequence of states $s_0 = s, s_1, \ldots, s_k = t$ such that $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_k$. We write $s \stackrel{*}{\rightarrow} t$ to mean that $s \stackrel{+}{\rightarrow} t$ or $s = t$. We say that $\mathcal{S}$ is *finitely branching* if $Succ(s)$ is finite for each $s \in S$; the usefulness of this property is tied to König's lemma [21] stating that an infinite tree, finitely branching, has an infinite path.
Reachability. A state t of a TS $\mathcal{S}$ is *reachable* from a state s if $s \stackrel{*}{\rightarrow} t$. The reachability tree $RT(\mathcal{S}, s_0)$ of $\mathcal{S}$ with initial state s_0 is a directed unordered tree where nodes are labelled by states of S. The root node n_0 is labelled by s_0. A node labelled by s has a son labelled by t if and only if $t \in \mathrm{Succ}(s)$. The *reachability set* of $\mathcal{S}$ from s_0 is denoted $\mathrm{RS}(\mathcal{S}, s_0)$ and is defined as the set of states reachable from s_0.
Orders and limits. Consider an infinite nondecreasing sequence $\{u_n\}_{n \geq 1}$ of natural numbers. This sequence is *bounded* iff there exists $v \in \mathbb{N}$ such that $u_n \leq v$ holds for every $n \in \mathbb{N}$. We define

$$\lim u_n = \begin{cases} \omega & \text{if } \{u_n\}_{n \geq 1} \text{ is not bounded,} \\ \max\{u_n\} & \text{otherwise.} \end{cases}$$

Now consider a nondecreasing infinite sequence $\{u_n\}_{n \geq 1}$ of p-tuples $u_n \in \mathbb{N}^p$. Then we have :

$$\lim u_n = (\lim u_n(1), \lim u_n(2), \ldots, \lim u_n(p)).$$

For our purpose, we shall need the following orders : for any nonempty subset $Q \subseteq [p]$, let $<_Q$ be the order relation defined on $\mathbb{N}^p_\omega$ by:

$$u <_Q v \qquad \text{iff} \qquad (\forall i \in Q)[u(i) < v(i)] \text{ and } (\forall i \notin Q)[u(i) \leq v(i)].$$

Note that $u <_Q v$ implies $u <_{Q'} v$ for any $\emptyset \neq Q' \subseteq Q$. It is well-known that the ordering $\leq$ on $\mathbb{N}^p_\omega$ is a well-ordering [8]: for any infinite sequence $u_0, u_1, \ldots \in \mathbb{N}^p_\omega$, there exist two indices $i < j$ with $u_i \leq u_j$. Using $<_Q$, we can be more precise:

Lemma 2.2 *For any infinite strictly increasing sequence $u_0 < u_1 < u_2 \ldots$ in $\mathbb{N}^p_\omega$, there exists a nonempty maximal set $Q \subseteq [p]$ such that one can extract an infinite strictly increasing subsequence $u_{i_0} <_Q u_{i_1} <_Q u_{i_2} \ldots$ (with $i_0 < i_1 < i_2 \ldots$).*

Proof. $Q = \{k \in [p] \mid \forall i, u_i(k) \neq \omega \wedge \lim_{i \to \omega} u_i(k) = \omega\}$. ■

2.1 Transitions as functions

Throughout this paper, we shall consider transitions systems whose set of states is a subset of $\mathbb{N}^p$, for some p in $\mathbb{N}$. Transitions are considered as recursive functions on $\mathbb{N}^p$. Thus, functions are not necessarily total. Consider a function $f : X \to Y$. We write $Dom\ f$ for $\{x \in X \mid f(x) \text{ is defined}\}$. We adopt the convention that we compose functions from right to left, i.e. $gf(n) = g(f(n))$. A function $f : \mathbb{N}^p \to \mathbb{N}^p$ is

(type 1) *nondecreasing* if $u \leq v \Rightarrow f(u) \leq f(v)$ for all $u, v \in Dom\ f$,

(type 2) *increasing* if $u < v \Rightarrow f(u) < f(v)$ for all $u, v \in Dom\ f$,

(type 3) *strongly increasing* if

$$(\forall Q \subseteq [p], Q \neq \emptyset)(\forall u, v \in Dom\ f)[u <_Q v \Longrightarrow f(u) <_Q f(v)].$$

(*ρ-increasing* functions, intermediate between increasing functions and strongly increasing functions, will be defined and motivated in Section 5.2.)

Proposition 2.3 *Let f and g be two functions from $\mathbb{N}^p$ to $\mathbb{N}^p$.*
(1) If f and g are of type t above, $1 \leq t \leq 3$, then gf is again of type t.
(2) If f is nondecreasing and both f and g have upward-closed domains, then the domain of gf is upward-closed.
(3) If f and g are recursive, then gf is recursive.

Proof. Straightforward. ■

Extension of a nondecreasing function with upward-closed domain. For any $v \in \mathbb{N}^p_\omega$, define the sequence $\{v[\omega/n]\}_{n\geq 0}$ by setting $v[\omega/n](i) = v(i)$ if $v(i) \in \mathbb{N}$, and $v[\omega/n](i) = n$ otherwise. Then $\{v[\omega/n]\}_{n\geq 0}$ is a nondecreasing sequence, and $\lim v[\omega/n] = v$. The *extension* $\overline{f} : \mathbb{N}^p_\omega \to \mathbb{N}^p_\omega$ of a nondecreasing function $f : \mathbb{N}^p \to \mathbb{N}^p$ with upward-closed domain is defined by setting $Dom\ \bar{f} = Dom\ f + \mathbb{N}^p_\omega$ and, for any $v \in Dom\ \bar{f}$,

$$\overline{f}(v) = \lim_{n\to\omega} f(v[\omega/n]).$$

Since f is nondecreasing, $\{f(v[\omega/n])\}_{n\geq 0}$ is a nondecreasing sequence and it admits a limit in $\mathbb{N}^p_\omega$. Note that if $v \in Dom\ f$ then $\bar{f}(v) = f(v)$.

ω-recursive functions. Let f be a nondecreasing function $f : \mathbb{N}^p \to \mathbb{N}^p$ with upward-closed domain. We say f is *ω-recursive* if its extension $\bar{f}$ is recursive.

We will see in Section 4 that computing $\bar{f}$ plays a crucial role in deciding the coverability problem and the place-boundedness problem (defined below). Here we note that the computability of $\bar{f}$ is a nonvacuous hypothesis in a strong sense. Indeed, although $\overline{f}$ is recursive for any fixed nondecreasing recursive function $f : \mathbb{N} \to \mathbb{N}$ (because $\overline{f}$ agrees with f on $\mathbb{N}$, and because $\overline{f}(\omega)$ can be hardwired into a machine, i.e. there exists a machine computing $\bar{f}$ even though we may not know which one), even for a fixed two-argument function whose limit exists, it may not be possible to compute $\bar{f}$:

Proposition 2.4 *There is a nondecreasing total recursive function $g : \mathbb{N}^2 \to \mathbb{N}^2$ which is not ω-recursive.*

Proof. Let $g(m, n) = (0,\ |\{j \leq m \ : \text{TM } j \text{ halts on input } j \text{ in at most } n \text{ steps}\}|\ )$. Then g is a nondecreasing total recursive function, and $\overline{g}$ is not computable because

$$\overline{g}(m, \omega) - \overline{g}(m-1, \omega) = \begin{cases} (0,1) & \text{if the } m\text{th Turing machine halts on input } m, \\ (0,0) & \text{otherwise.} \end{cases}$$

■

In a similar vein, we have

Proposition 2.5 *The language of all $m \in I\!N$ such the m-th Turing machine computes a function $f : I\!N \to I\!N$ for which $\bar{f}(\omega)$ is defined and different from ω, is undecidable.*

Proof. The (well-defined) extensions $\bar{f}_m$ of the recursive functions f_m, defined as

$$f_m(n) = \begin{cases} t & \text{if } t \leq n \text{ and the TM } m \text{ halts on } m \text{ in exactly } t \text{ steps,} \\ n & \text{otherwise,} \end{cases}$$

have the property that the TM m halts on m iff $\bar{f}_m(\omega) \neq \omega$. ■

Proposition 2.5 further implies that any function $e : \mathbb{N} \to \mathbb{N}$ for which $e(m)$ is a Turing machine computing $\bar{f}$ whenever m is a Turing machine computing a nondecreasing total function $f : \mathbb{N} \to \mathbb{N}$, is not recursive. A similar statement holds for *strongly increasing* functions, when we allow two-argument functions f:

Proposition 2.6 *Any function $e : I\!N \to I\!N$ for which $e(m)$ is a Turing machine computing $\bar{f}$ whenever m is a Turing machine computing a strongly increasing total function $f : I\!N \times I\!N \to I\!N \times I\!N$, is not recursive.*

Proof. For each j, define f_j by :

$$f_j(n,k) = \begin{cases} (n,0) & \text{if } k = 0 \text{ and TM } j \text{ does not halt on } j \text{ within } n \text{ steps} \\ (n,n) & \text{if } k = 0 \text{ and TM } j \text{ halts on } j \text{ within } n \text{ steps} \\ (n,n+k) & \text{otherwise.} \end{cases}$$

For each j, f_j is strongly increasing, recursive, and total. Furthermore, the TM j halts on j iff $\bar{f}_j(\omega,0) \neq (\omega,0)$. Now suppose to the contrary that the recursive function e exists. To determine, given j, whether the TM j halts on j, it suffices to compute the TM m_j which computes f_j, and to run the TM $e(m_j)$ (computing $\bar{f}_j$, since TM m_j computes a strongly increasing total function as promised) on input $(\omega,0)$. The TM $e(m_j)$ will necessarily halt, since $\bar{f}_j(\omega,0)$ is well defined, and will output $(\omega,0)$ iff j runs forever on j. ■

3 Well-structured nets

We recall [13] that a *well-structured transition system* (WSTS) is a TS $\mathcal{S} = (S, \to, \leq)$ equipped with a reflexive and transitive relation $\leq \subseteq S \times S$ fulfilling the two following conditions:
(1) *well-quasi-ordering*: for any infinite sequence $u_0, u_1, \ldots \in S$, there exist two indices $i < j$ with $u_i \leq u_j$.
(2) *compatibility*: $\leq$ is (upward) compatible with $\to$, i.e. for all $s_1 \leq t_1$ and transition $s_1 \to s_2$, there exists a sequence $t_1 \stackrel{*}{\to} t_2$ such that $s_2 \leq t_2$.
For every state s we denote by $\uparrow s$ the set $\{t \in S; t \geq s\}$. A WSTS has *effective Pred-basis* if there exists an algorithm which, given $s \in S$, returns a finite basis of $\uparrow Pred(\uparrow s)$.

Well-Structured Nets. A *Well-Structured Net* (WSN) is a TS $\mathcal{S} = (S, \stackrel{F}{\longrightarrow}, \leq)$ such that

- there exists $p \in \mathbb{N}$ such that $S = \mathbb{N}^p$,
- F is a finite set of nondecreasing recursive functions defined on upward-closed subsets of $\mathbb{N}^p$, and $s \stackrel{F}{\longrightarrow} t$ for $(s,t) \in \mathbb{N}^p \times \mathbb{N}^p$ means that $f(s) = t$ for some $f \in F$, also written $s \stackrel{f}{\to} t$,

- $\leq$ is the usual well-ordering on $\mathbb{N}^p$.

A WSN $\mathcal{S} = (S, \xrightarrow{F}, \leq)$ is *increasing* (resp. *strongly increasing*) if every function in F is increasing (resp. *strongly increasing*). In the context of a WSN, we can speak of the boundedness of a place: given $\mathcal{S}$ and an initial state s_0, the ith component (or the place p_i) is *bounded* iff there exists $k \in \mathbb{N}$ such that for each $s \in RS(\mathcal{S}, s_0)$, $s(i) \leq k$.

Note that a WSN is a WSTS ! Note also that every WSN is finitely branching (in fact $|F|$-ary).

ω-Well-Structured Nets. A ω-WSN $\mathcal{S} = (S, \xrightarrow{F}, \leq)$ is a WSN such that every function in F is ω-recursive.

Problems. We will consider the following computational problems, where in each case, a WSN $\mathcal{S} = (S, \xrightarrow{F}, \leq)$ is given as part of the input:

1. *Termination.* Given a state s_0, is the reachability tree $RT(\mathcal{S}, s_0)$ finite?
2. *Coverability.* Given states s_0 and s, does there exist a state s' in $RS(\mathcal{S}, s_0)$ which covers s, i.e. which satisfies $s' \geq s$?
3. *Boundedness.* Given a state s_0, is the reachability set $RS(\mathcal{S}, s_0)$ finite?
4. *Path-unbounded-witness* [1]: Given a state s_0 and a finite sequence $f_1, f_2, \ldots, f_k$ in F, can we find an unbounded place on the possibly infinite path from s_0 obtained by iterating the sequence $f_k f_{k-1} \ldots f_1$?
5. *Path-place-boundedness*: Given a state s_0, a finite sequence $f_1, f_2, \ldots, f_k \in F$, and a place i, is the place i bounded on the infinite path from s_0 obtained by iterating the sequence $f_k f_{k-1} \ldots f_1$?
6. *Place-boundedness*: Given an initial state s_0 and a place i, is i bounded?

In this list, we have introduced two new problems related to the boundedness problem :

- The *Path-place-boundedness* can be seen as the place-boundedness problem restricted to regular paths of the reachability tree. It may be decidable when the boundedness problem is not (e.g. reset Petri Nets [9]). Note that decidability of the path-place-boundedness problem implies decidability of the path-unbounded-witness problem.

We do not mention the usual *Reachability problem* (given two states s and s', does $s \xrightarrow{*} s'$ hold ?), as it is irrelevant to our paper, being undecidable in all the classes we shall study.

4 When are termination and coverability decidable?

Theorem 4.1 *Termination is decidable for well-structured nets (WSN).*

Proof. A WSN $\mathcal{S} = (\mathbb{N}^p, \xrightarrow{F}, \leq)$ is a WSTS. If $s_1 \xrightarrow{f} s_2$ for a function $f \in F$ and $s_1 \leq t_1$, then because f is nondecreasing and $Dom\ f = \uparrow Dom\ f$, we have $t_1 \xrightarrow{f} t_2$ and $s_2 \leq t_2$. The latter property (called strong compatibility) and the recursivity of f allow us to apply Theorem 4.6 from [13] and then to decide termination. ■

[1] Formally, decidability of this problem, for a given class of WSNs, will mean that there is a Turing machine which, given a WSN and a function sequence and a state s_0, is guaranteed to halt with the name of an unbounded place found along the path iterating the sequence from s_0, *whenever* the WSN is indeed from the class and iterating the sequence from s_0 indeed yields at least one unbounded place.

Theorem 4.2 *Coverability is decidable for ω-well-structured nets (ω-WSN).*

Proof. We will show that it is possible to compute a finite basis of $\uparrow Pred(\uparrow s)$ for any $s \in \mathbb{N}^p$. Then by using Theorem 3.6 from [13], we will conclude that coverability is decidable.

Recall Theorem 2.1, namely Valk and Jantzen's result, which we now apply to $K = \uparrow Pred(\uparrow s)$ for $s \in \mathbb{N}^p$. We have $K = \uparrow K$ and, defining $Pred_f(\uparrow s)$ as $\{t \in \mathbb{N}^p \mid f(t) \geq s\}$,

$$K = \uparrow \bigcup_{f \in F} Pred_f(\uparrow s) = \bigcup_{f \in F} \uparrow Pred_f(\uparrow s) = \bigcup_{f \in F} Pred_f(\uparrow s),$$

where the rightmost equality holds because $Dom\ f = \uparrow Dom\ f$ and f is nondecreasing. Because F is finite, it suffices to be able to compute $Pred_f(\uparrow s)$ for $f \in F$. The Valk and Jantzen result yields that $Pred_f(\uparrow s)$ is computable iff the predicate $\downarrow t \cap Pred_f(\uparrow s) \neq \emptyset$ is computable for all $t \in \mathbb{N}^p_\omega$. Two cases arise:

- $t \in \mathbb{N}^p$. Then $\downarrow t \cap Pred_f(\uparrow s) \neq \emptyset$ iff $f(t) \geq s$. Since f is recursive, this is obviously decidable.
- $t \in \mathbb{N}^p_\omega \setminus \mathbb{N}^p$. Then $\downarrow t \cap Pred_f(\uparrow s) \neq \emptyset$ iff $(\exists t' \in \mathbb{N}^p,\ t' \leq t)[f(t') \geq s]$ iff $\bar{f}(t) \geq s$, which is decidable because $\bar{f}$ is recursive.

■

The computability of $\bar{f}$ is a hypothesis required to decide coverability, since we have as a counterpart to theorem 4.2 that our most restrictive class of WS nets has an undecidable coverability when this hypothesis is absent:

Theorem 4.3 *The coverability problem for strongly increasing well-structured nets is undecidable* [2].

Proof. Consider the family $\{f_j : \mathbb{N}^2 \to \mathbb{N}^2\}$ of strongly increasing recursive functions

$$f_j(n,k) = \begin{cases} (n,0) & \text{if } k = 0 \text{ and TM } j \text{ does not halt on } j \text{ within } n \text{ steps} \\ (n,n) & \text{if } k = 0 \text{ and TM } j \text{ halts on } j \text{ within } n \text{ steps} \\ (n,n+k) & \text{otherwise,} \end{cases}$$

and the strongly increasing function $g : \mathbb{N}^2 \to \mathbb{N}^2$ defined by $g(n,k) = (n+1,k)$. The strongly increasing WSN $\mathcal{S}_j = (\mathbb{N}^2, \{f_j, g\}, \leq)$ with initial state $(0,0)$ have the property that the state $(1,1)$ is coverable iff the TM j halts on j. Hence there is no Turing machine which correctly determines coverability and halts whenever its input is a strongly increasing WSN. ■

5 When are boundedness problems decidable ?

5.1 The boundedness problem

An increasing WSN is a well-structured transition system with strict strong compatibility, i.e. $s_1 \to s_2$ and $t_1 > s_1$ imply $t_1 \to t_2$ with $t_2 > s_2$, while a WSN only enjoys strong compatibility. Given strict strong compatibility, boundedness can be decided by searching for a sequence $s_0 \xrightarrow{*} s_1 \xrightarrow{\sigma} s_2$ with $s_1 < s_2$. Then, σ can be iterated to produce an increasing

[2] The precise technical meaning of this statement is that there is no Turing machine which correctly decides whether some state in $RS(\mathcal{S}, s_0)$ covers s, and halts, whenever its input $\mathcal{S}$ is indeed a strongly increasing WSN (with its defining functions given by Turing machines) together with s_0 and s.

sequence of reachable states $s_0 \stackrel{*}{\rightarrow} s_1 \stackrel{\sigma}{\rightarrow} s_2 \stackrel{\sigma}{\rightarrow} s_3 \ldots$ with $s_1 < s_2 < s_3 \ldots$, and the system is unbounded. Because any WSN by definition has a recursive Succ (i.e. Succ(s) is computable, for every s in S), Theorem 4.11 from [13] yields:

Theorem 5.1 *Boundedness is decidable for increasing well-structured nets.*

We note that extending a recursive function f to $\bar{f}$ was never needed in Theorem 5.1. We note further that Theorem 5.1 is tight:

Theorem 5.2 *Boundedness is undecidable for ω-well-structured nets.*

Proof. Boundedness is undecidable for reset Petri nets [9], and reset Petri nets are ω-WSN. ■

5.2 The path-unbounded-witness problem

In this section, we show how to detect at least one unbounded place in some unbounded systems. To guarantee that this is possible, we must restrict increasing WSN to use only ρ-increasing functions, which we now define.

Growth graph. We define the *growth graph* $G(f) = (N, A)$ of an arbitrary function $f : \mathbb{N}^p \rightarrow \mathbb{N}^p$ as follows:

- N is the set of all *nonempty* subsets of $[p]$,
- for each $Q \in N$, consider the unique (possibly empty) largest subset R of $[p]$ such that

$$(\forall u, v \in Dom\ f)[u <_Q v \Rightarrow f(u) <_R f(v)];$$

  the arc (Q, f, R) is present iff $R \neq \emptyset$.

Note that $G(f)$ is generally not computable (even when f is recursive, see Proposition 5.3), that $0 \leq |A| \leq |N|$ by construction, and that each node Q in $G(f)$ has at most one outgoing arc.

ρ-increasing function. A function $f : \mathbb{N}^p \rightarrow \mathbb{N}^p$ is *ρ-increasing* if $|A| = |N|$ in its growth graph $G(f) = (N, A)$.

Observe that the outdegree of each node in the graph of a ρ-increasing function f is precisely one. (The name ρ-increasing was chosen to reflect the fact that the unique path starting from any node in the graph leads into a cycle, albeit possibly of length 1, and resembles the Greek letter ρ.)

Proposition 5.3 *There is no Turing machine which, given m, computes the growth graph $G(f)$ whenever $f : \mathbb{N}^p \rightarrow \mathbb{N}^p$ is a ρ-increasing function and m is a Turing machine computing f.*

Proof. For each $m \in \mathbb{N}$, define $f_m : \mathbb{N}^3 \rightarrow \mathbb{N}^3$ as follows:

$$f_m(j,k,n) = \begin{cases} (j+k+n, k, n+1) & \text{if the } m\text{th TM halts on } m \text{ within } k-1 \text{ steps} \\ (j+k+n, k+1, n) & \text{otherwise.} \end{cases}$$

Every f_m is recursive, and also ρ-increasing because

$$(\forall Q \subseteq [p], Q \neq \emptyset)(\forall u, v \in \mathbb{N}^3)[u <_Q v \Longrightarrow f(u) <_{\{1\}} f(v)].$$

Now we claim that the mth TM halts on m iff $G(f_m)$ possesses an arc $(\{2\}, R)$ with $2 \notin R$. Indeed, if the mth TM halts on m in some number $k-1$ of steps, then $(0, k-1, 0) <_{\{2\}} (0, k, 0)$, while $f_m(0, k-1, 0) = (k-1, k, 0)$ and $f_m(0, k, 0) = (k, k, 1)$, so that the (unique) arc out of node $\{2\}$ points to a node R for some R not containing 2. Conversely, if the mth TM runs forever on m, $G(f_m)$ clearly contains the arc $(\{2\}, \{1, 2\})$. ■

Clearly, any strongly increasing function is ρ-increasing. And since $u < v$ obviously implies the existence of a Q such that $u <_Q v$, any ρ-increasing function is increasing. The converse is false however: the total (hence upward-closed) increasing function $f : \mathbb{N}^2 \to \mathbb{N}^2$ given by

$$f(i,j) = \begin{cases} (0,0) & \text{if } i = j = 0, \\ (i, i-1) & \text{if } j = 0 \text{ and } i \text{ is odd}, \\ (i-1, i) & \text{if } j = 0 \text{ and } i > 0 \text{ is even}, \\ (i+1, i+j) & \text{if } j > 0 \end{cases}$$

has the property that no restriction of f to an upward-closed domain is ρ-increasing (this is because, for each $n > 0$, $(2n, 0) <_{\{1\}} (2n+1, 0) <_{\{1\}} (2n+2, 0)$ and applying f to these pairs yields $(2n-1, 2n) <_{\{1\}} (2n+1, 2n) <_{\{2\}} (2n+1, 2n+2)$, so that no nonempty set $R \subseteq \{1, 2\}$ fulfills the condition that, for each $u <_{\{1\}} v$ in any upward-closed subset of $\mathbb{N} \times \mathbb{N}$, $f(u) <_R f(v)$; on the other hand, the reader can check by case analysis that f is indeed increasing).

Consider a *set* F of ρ-increasing functions $f : \mathbb{N}^p \to \mathbb{N}^p$. We denote by $G(F)$ the labelled graph $(\{Q \subseteq [p] : Q \neq \emptyset\} , \{(Q, f, R) : (Q, R) \text{ is an arc in } G(f)\})$. For a node Q in $G(F)$ and f in F, we denote by Q^f the unique node such that (Q, Q^f) is an arc of $G(f)$ and more generally, for σ in F^*, we denote by Q^σ the unique node reached in $G(f)$ by starting at node Q and by following the unique path labelled σ. Formally :

$$Q^\sigma = \begin{cases} Q & \text{if } |\sigma| = 0, \\ (Q^f)^{\sigma'} & \text{if } \sigma = \sigma' f \text{ where } f \in F. \end{cases}$$

Proposition 5.4 *The composition of two ρ-increasing functions from $\mathbb{N}^p$ to $\mathbb{N}^p$ is ρ-increasing.*

Because a ρ-increasing WSN is increasing, boundedness is also decidable for ρ-increasing WSN. Moreover we have:

Theorem 5.5 *There is a TM which is guaranteed to halt with on its tape a nonempty subset of unbounded places, whenever the TM input is a ρ-increasing well-structured net $\mathcal{S}$ and a state s_0 from which $\mathcal{S}$ is unbounded.*

Proof. We first compute the finite reachability tree $\mathrm{FRT}(\mathcal{S}, s_0)$ of $\mathcal{S}$ from s_0, as introduced in [13]. We recall that a leaf node in $\mathrm{FRT}(\mathcal{S}, s_0)$ is either (1) a node whose state label is without successor or (2) a node n (labelled s) such that there exists a path from the root n_0 (labelled s_0) to n which contains another node n' (labelled s') such that $s' \leq s$. Because $\mathcal{S}$ is unbounded from s_0 and increasing, there exists a path $n_0(s_0) \to n'(s') \to n(s)$ in $\mathrm{FRT}(\mathcal{S}, s_0)$ with $s' < s$. Let $\sigma \in F^+$ be the sequence of functions which lead from s' to s.

Dom σ is upward-closed (from Proposition 2.3), and since σ is a sequence of the ρ-increasing functions, we claim that

$$(\forall Q \subseteq [p], Q \neq \emptyset)(\forall u, v \in \mathit{Dom}\ \sigma)[u <_Q v \Rightarrow \sigma(u) <_{Q^\sigma} \sigma(v)].$$

Let us prove the claim for a sequence $\sigma = gf$ of length 2. Consider $\emptyset \neq Q \subseteq [p]$. Pick $u, v \in \mathit{Dom}\ gf$. Then $u, v \in \mathit{Dom}\ f$ and $f(u), f(v) \in \mathit{Dom}\ g$. Hence by the ρ-increase

properties of f and g,

$$u <_Q v \Longrightarrow f(u) <_{Q^f} f(v) \Longrightarrow g(f(u)) <_{(Q^f)^g} g(f(v)).$$

Now $(Q^f)^g = Q^{gf}$ is independent of u and v, proving the claim.

Now we show that it is possible to find some unbounded places along the infinite path obtained by iterating σ. Let Q be the largest subset of $[p]$ verifying $s' <_Q s$ and let $k, n \geq 1$ verify $Q^{\sigma^n} = Q^{\sigma^{n+k}}$. An inductive application of the claim proved above yields

$$s = v_1 <_{Q^\sigma} v_2 <_{Q^{\sigma^2}} v_3 <_{Q^{\sigma^3}} \cdots$$

where $v_1 \xrightarrow{\sigma} v_2 \xrightarrow{\sigma} v_3 \cdots$. Define $u_i = v_{n+ik}$ for each $i \geq 0$. Then for each $i \geq 0$,

$$u_i = v_{n+ik} <_{Q^{\sigma^{n+ik}}} v_{n+ik+1} <_{Q^{\sigma^{n+ik+1}}} \cdots <_{Q^{\sigma^{n+(i+1)k-1}}} v_{n+(i+1)k} = u_{i+1}.$$

But it follows from $Q^{\sigma^n} = Q^{\sigma^{n+k}}$ that $Q^{\sigma^{n+j}} = Q^{\sigma^{n+ik+j}}$ holds for each $i, j \geq 0$, so that

$$u_i = v_{n+ik} <_{Q^{\sigma^n}} v_{n+ik+1} <_{Q^{\sigma^{n+1}}} v_{n+ik+2} <_{Q^{\sigma^{n+2}}} \cdots <_{Q^{\sigma^{n+k-1}}} v_{n+(i+1)k} = u_{i+1}.$$

This precisely implies $u_i <_R u_{i+1}$ for $R = \bigcup_{j=1}^{k} Q^{\sigma^{n+j}} \neq \emptyset$. It follows that all places in R are unbounded along the infinite path $s_0 \xrightarrow{*} s' \xrightarrow{\sigma} s \xrightarrow{\sigma} \cdots$. This concludes the proof. ■

Corollary 5.6 *The path-unbounded-witness problem is decidable for ρ-increasing well-structured nets.*

Theorem 5.5 is tight. It provably extends neither to the guaranteed identification of at least one unbounded place in increasing well-structured nets (nor to the decidability of path-place-boundedness in ρ-increasing WSN, cf. next section), as we now show.

Theorem 5.7 *The path-unbounded-witness problem is not decidable for increasing well-structured nets.*

Proof. For each m, we define $f'_m : \mathbb{N}^2 \to \mathbb{N}^2$ by neglecting the first component of the function $f_m(j,k,n)$ defined in proving proposition 5.3:

$$f'_m(k,n) = \begin{cases} (k, n+1) & \text{if the } m\text{th TM halts on } m \text{ within } k-1 \text{ steps} \\ (k+1, n) & \text{otherwise.} \end{cases}$$

Every f'_m is total (hence upward-closed), recursive, and increasing[3]. Now, from any initial state $(k,n) \in \mathbb{N}^2$, the increasing WS net $\mathcal{S}_m = (\mathbb{N}^2, u \xrightarrow{\{f'_m\}} v, \leq)$ possesses one and only one unbounded place, which is the second place iff the mth TM halts. This proves that the path-unbounded-witness problem is undecidable for increasing WSN. ■

5.3 The path-place-boundedness problem

Theorem 5.8 *Path-place-boundedness is undecidable for strongly increasing well-structured nets and for ω-well-structured nets.*

Proof.

[3] We note that f'_m is ρ-increasing iff the mth TM does not halt. Similar observations are systematically grouped together in Section 7.

- Consider the strongly increasing WSN $S_j = (\mathbb{N}^2, \{f_j, g\}, \leq)$ from the proof of theorem 4.3, with initial state $(0,0)$ and the sequence $\sigma = gf_j$. Then, place 2 is unbounded along the infinite path $(0,0) \xrightarrow{\sigma}\xrightarrow{\sigma}\xrightarrow{\sigma}$ ... if and only if the jth TM halts on j.

- Consider the family of functions $g_j : \mathbb{N} \to \mathbb{N}$ defined by

$$g_j(n) = \begin{cases} n & \text{if the } j\text{th TM halts on } j \text{ within } n \text{ steps} \\ n+1 & \text{otherwise.} \end{cases}$$

  For every $j \geq 1$, the function g_j is nondecreasing, total and ω-recursive ($\bar{g}_j(\omega) = \omega$). Let $T_j = (\mathbb{N}, \{g_j\}, \leq)$ with initial state 0. The only place is bounded along the infinite path $0 \xrightarrow{g_j} ... \xrightarrow{g_j}$ ... if and only if the jth TM halts on j.

■

The following proposition essentially justifies our interest in the path-place boundedness problem. Although boundedness in undecidable for Reset Petri nets [9], path-place boundedness is decidable.

Proposition 5.9 *Path-place-boundedness is decidable for reset Petri nets.*

Proof. Given a reset Petri net N, a marking m, a place p and a sequence σ of transitions, it is easy to see that p is unbounded along the (possibly) infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$ iff $m_2 \geq m_1 \geq m$ and $m_2(p) > m_1(p) > m(p)$. ■

5.4 The place-boundedness problem

Let us recall the construction of the coverability tree of a Petri net, which is used to solve the place-boundedness problem. The strategy of the construction is to develop every path of the reachability tree until one meets two markings m and m' such that $m_0 \xrightarrow{*} m \xrightarrow{\sigma} m'$ with $m <_Q m'$ and Q a nonempty subset of places. When one meets two such markings m and m', one replaces m' by the limit of the infinite sequence of markings obtained from m by iterating σ. This limit is computable for Petri nets (and for transfer and reset Petri nets as well):

$$\lim_{n\to\omega} \sigma^n(m)(i) = \begin{cases} \omega & \text{if } i \in Q \\ m(i) & \text{if } i \notin Q. \end{cases}$$

Then one continues using the same strategy with all transitions extended from $\mathbb{N}^p$ to $\mathbb{N}^p_\omega$. Termination is guaranteed because $\leq$ is still a well-ordering on $\mathbb{N}^p_\omega$.

To apply the above strategy to construct a coverability tree for a WSN, it is first necessary to be able to effectively extend functions to $\mathbb{N}^p_\omega$. This means that functions have to be ω-recursive. Moreover, either $\lim_{n\to\omega} \sigma^n(u)$ must be computable, or a good enough approximation $\bar{u}$ must be computable such that $u < \bar{u} \leq \lim_{n\to\omega} \sigma^n(u)$ and $\bar{u}$ has at least one more ω-component than u. (Note that in general, $\bar{u}$ can satisfy $\bar{u} < \lim_{n\to\omega} \sigma^n(u)$, for example when $f(m,n) = (m+n, n+1)$ and $u = (0,0)$, in which case $f(u) = (0,1)$ and $\bar{u}$ can be chosen as $(0,\omega)$, while $\lim_{n\to\omega} f^n(u) = (\omega,\omega)$.) Furthermore, the extension $\bar{f}$ of any nondecreasing function f is nondecreasing, but any type i-property of f may not be preserved on $\bar{f}$. In particular, strongly increasing functions extended to $\mathbb{N}^p_\omega$ may no longer be strongly increasing (for example, when f is defined as $f(m,n) = (m, m+n)$). Nevertheless, any strongly increasing WSN $(S, \xrightarrow{F}, \leq)$ verifies the interesting property:

Proposition 5.10 *Let u, v be in $\mathbb{N}^p_\omega$. If there exists $f_1, \ldots, f_k$ in F such that $u \xrightarrow{\sigma} v$ for $\sigma = \bar{f}_k \bar{f}_{k-1} \ldots \bar{f}_1$, and there exists a nonempty Q such that $u <_Q v$, then $\lim_{n\to\omega} \sigma^n(u) \geq \bar{u} > u$, where $\bar{u}$ is defined by $\bar{u}(i) = \omega$ if $i \in Q$, and $\bar{u}(i) = u(i)$ otherwise.*

Proof. As $\{\sigma^n(u)\}$ is a nondecreasing sequence, $\lim_{n\to\omega}\sigma^n(u)$ exists. As a composition of strongly increasing functions, σ is strongly increasing. So $u \xrightarrow{\sigma} v$ and $u <_Q v$ imply $\sigma^n(u) <_Q \sigma^{n+1}(u)$, for any positive integer n. So for i in Q, $\lim_{n\to\omega}\sigma^n(u)(i) = \omega$ and for i not in Q, we do not know if this limit is finite or not, but at least we have $\lim_{n\to\omega}\sigma^n(u)(i) \geq u(i)$. Note that $\bar{u}$ contains at least $|Q|$ more ω than u. ■

We thus obtain :

Theorem 5.11 *For a strongly increasing ω-well-structured net $(S, \xrightarrow{F}, \leq)$, place-boundedness is decidable.*

Proof sketch. With the strongly increasing hypothesis, it is possible to generalize the Karp-Miller coverability tree [20] and the minimal coverability tree [12] algorithms (MCT). Finkel [11, 12] constructs a *(minimal) coverability tree* for well-structured transition systems explicitely gives the algorithm computing this set for standard Petri nets. The algorithm generalisation to WSN consists in appealing to a function "**approximate-limit** (s, σ)" that computes, possibly in clever ways, the limit (or a good enough approximation to the limit) of a sequence $\{s_n\}$. For standard Petri nets, the function **approximate-limit** (s, σ) reduces to a trivial computation performed on each place p separately: for standard Petri nets, **approximate-limit** in the algorithm spelled out below consists of the instruction *if* $s_1(p) < s(p)$, *then* $s(p) \leftarrow \omega$.

The generalized minimal coverability tree (MCT) algorithm as it applies to a strongly increasing ω-WSN is given next, followed by a paragraph explaining its operation:

```
MCT ← tree with single node (1, s_0) {* "(NODE NUMBER, MARKING)" *}
repeat
.  (n, s) ← next unprocessed node in MCT;
.  {* "PROCESSED" MEANS THAT THE case STATEMENT HAS BEEN EXECUTED ON IT
*}
.  case [1..4] of
.    1:a processed node (n_1, s) exists in MCT: {* DO NOTHING *}
.    2:a processed node (n_1, s_1) with s_1 > s exists in MCT: remove (n, s) from MCT;
.    3:a processed node (n_1, s_1) with s_1 < s exists in MCT:
.        if an ancestor (n_1, s_1) of (n, s) exists such that s_1 < s then
.            pick among these the ancestor (n_1, s_1) closest to the root
.            write ("The reachability tree is infinite")
.            let σ(s_1) = s
.            if σ is increasing then write ("The reachability set is infinite")
.            (n_1, s_1) ← (n_1, approximate-limit(s_1, σ))
.            (n_1, s_1) becomes unprocessed
.            delete any node subtended by (n_1, s_1)
.            (n, s) becomes unprocessed
.            for each processed node (n_1, s_1) in MCT for which s_1 < s do
.                    remove (n_1, s_1) and any node subtended by (n_1, s_1) from MCT
.    4:otherwise:
.        for each function f ∈ F such that f(s) = s' do
.            n' ← new node number
.            add node (n', s') with arc (n, f, n') to MCT
until all nodes in MCT have been processed
```

The `case` statement is such that upon completion of case **i**, control exits the `case` statement. The algorithm initially executes case 4. Whenever it is activated, case 4 replaces a leaf

marked s with a depth-one subtree whose leaves are *unprocessed* and arise each from a function defined at s. An *unprocessed* leaf is also created in case 3 when a node marked s selected for processing has an ancestor marked $s_1 < s$: then the subtree rooted at s_1 is replaced with an *unprocessed* leaf marked with a limit resulting from iterating the path σ that led from s_1 to s. When the node selected for processing is marked s, case 3 also takes care of deleting any subtree not containing s and rooted at a node marked $s_1 < s$ (this deletion process is actually postponed until a subsequent iteration, when an ancestor of s marked smaller than s is found upon entering case 3). We point out finally that case 2 can only arise when s_1 is the immediate ancestor of (leaf) s.

Then we define the function **approximate-limit** as follows:

function **approximate-limit**(m_1, σ)
  `for` each p such that $m_1(p) < m(p)$ `do` $m(p) \leftarrow \omega$
  approximate-limit $\leftarrow m$

The function trivially terminates. It computes a good enough approximation to $\lim m_n$ ("good enough" in the sense that it computes $\bar{u}$ fulfilling the hypothesis of Proposition 5.10) having at least one new ω component. Now, we show that the MCT algorithm terminates. Assume the contrary. Then the algorithm computes an infinite set of markings. Let us denote by P_i the set of tree markings at the end of the ith iteration of the `repeat` loop. The sequence (P_i) is infinite. For every i, (i) P_i is a finite set of pairwise incomparable elements, (ii) for any $m \in P_i$ there exists $m' \in P_{i+1}$ such that $m \leq m'$, and (iii) for any $m \in P_i$ there exists $m' \in P_{i+1}$ such that $\neg(m' \leq m)$. Hence, for each i, $P_i \neq P_{i+1}$. Because $\leq$ is a well-ordering on $\mathbb{N}_\omega^p$, we can extract from the sequence $(P_i)_{i \geq 1}$ an infinite subsequence (P_{i_j}) such that, for each j, the i_jth iteration involved case 3. For every j, there exists $m \in P_{i_j}$, there exists $m' \in P_{i_{j+1}}$, such that $m < m'$. Intuitively, every execution of case (3) in the MCT algorithm inserts a "new real" ω in some place of some vector. After a finite number of executions of case (3), no "new ω" can be inserted. Hence case (3) is executed a finite number of times, yielding a contradiction. Hence the algorithm terminates. The formal correctness proof is the same as that of the minimal coverability tree algorithm [12]. ■

Theorem 5.11 is optimal in the following sense :

Theorem 5.12 *Place boundedness is undecidable for both strongly increasing well-structured nets and for ρ-increasing ω-well-structured nets.*

Proof. Undecidability of place-boundedness holds for ρ-increasing ω-WSN because transfer Petri nets [9] are ρ-increasing ω-WSN and undecidability of place-boundedness holds for those. For proving the undecidability for strongly increasing WSN, we again use the family $\{S_j\}$ with initial state $(0,0)$ as in Theorem 4.3. The place 2 is unbounded if and only if the jth TM halts on j. ■

6 Summary

Consider the shorthand notation for our nets defined on Figure 1:

Theorem 6.1 *The decidability status of the six problems defined in Section 3 of this paper, for the eight classes depicted in Figure 1, is summarized on Figure 2.*

S_1	WSN (well-structured nets)
S_2	increasing WSN
S_3	ρ-increasing WSN
S_4	strongly increasing WSN
$\bar{S}_1$	ω-WSN
$\bar{S}_2$	increasing ω-WSN
$\bar{S}_3$	ρ-increasing ω-WSN
$\bar{S}_4$	strongly increasing ω-WSN

Figure 1: Abbreviations

	S_1	S_2	S_3	S_4	$\bar{S}_1$	$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_4$
Termination	D	D	D	D	D	D	D	D
Coverability	U	U	U	U	D	D	D	D
Boundedness	U	D	D	D	U	D	D	D
Path-unbounded-witness	U	U	D	D	?	?	D	D
Path-place boundedness	U	U	U	U	U	?	?	D
Place-boundedness	U	U	U	U	U	U	U	D

Figure 2: Decidability (D) or undecidability (U) for well-structured nets

7 Affine realizations of well-structured nets

7.1 On the difficulty of testing general recursive functions

Given a set F of functions $\mathbb{N}^p \to \mathbb{N}^p$ prescribed by their respective Turing machines, deciding whether F gives rise to a WSN is undecidable, as is shown in Proposition 7.1:

Proposition 7.1 *For each type (t) of function chosen from (1) nondecreasing, (2) increasing, (3) strongly increasing, and (4) ρ-increasing, the following languages*
(a) $\{k \mid$ the kth TM computes $f : I\!N \to I\!N$ of type (t)$\}$
(b) $\{k \mid$ the kth TM computes $f : I\!N \to I\!N$ of type (t) whose domain is upward-closed$\}$
(c) $\{k \mid$ the kth TM computes $f : I\!N \to I\!N$ whose domain is upward-closed$\}$
are undecidable.

Proof. In all cases, the idea is to reduce the complement of the halting problem to the relevant language. The following reduction takes care of the eight cases in (a) and (b). For each $m \in \mathbb{N}$, define the function $f_m : \mathbb{N} \to \mathbb{N}$ as

$$f_m(n) = \begin{cases} n-2 & \text{if the } m\text{th TM, on input } m, \text{ halts in } \textit{exactly } n-2 \text{ steps,} \\ n & \text{otherwise.} \end{cases}$$

If the mth Turing machine halts on m, it does so in a certain number, n, of steps. For this n, $f_m(n+1) = n+1 > f_m(n+2) = n$, so that f_m is not strongly increasing (and hence possesses none of the properties listed in cases (a) and (b)). Conversely, if the mth

Turing machine runs forever on m, then f_m is the identity function, which possesses all the properties listed in cases (a) and (b).

To take care of case (c) dealing with the upward-closure property alone, we modify f_m as follows:

$$f_m(n) = \begin{cases} \text{undefined} & \text{if the } m\text{th TM, on input } m, \text{ halts in } \textit{exactly } n-1 \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

■

Nevertheless, in concrete cases, it is generally known or easy to decide in which class the given set of functions lies. In many Petri net extensions, this given set is in fact contained in the set of affine functions. Many connections between affine functions and Petri net extensions are not so well-known. For example, (generalized) transfer Petri nets are precisely those WSN obtained using affine functions $f(X) = AX + B$ with A a matrix with non negative coefficients and no null column [9]. Section 7.2 describes WSN for which deciding the properties of relevant functions is easy. Section 7.4 summarizes the decidability status of problems involving affine functions.

7.2 Petri net extensions and affine well-structured nets

An affine function $f : \mathbb{N}^p \to \mathbb{N}^p$ is given by $f(X) = AX + B$ for some $A = (A_{i,j}) \in \mathbb{N}^{p \times p}$ and $B \in \mathbb{Z}^p$. We denote by A_j the jth column of the matrix A, for any j in $\{1, ..., p\}$. We write $A \geq \mathbf{0}$ to mean that $A_{i,j} \geq 0$, for all i, j. The following holds:

1. f is nondecreasing iff $A \geq \mathbf{0}$,
2. f is increasing iff f is ρ-increasing iff $A \geq \mathbf{0}$ and $A_j \neq \mathbf{0}$, for every j in $\{1, ..., p\}$
3. f is strongly increasing iff $A \geq Id$ (i.e. for all i, j, $A_{ii} = 1$ and $A_{ij} \geq 0$).

Hence, given an affine function f, it is very simple to test its "increasing" property. Moreover, $Dom\ f = \{X \in \mathbb{N}^p : AX + B \geq \mathbf{0}\}$ is easy to compute, and it is easy to see that $Dom\ f$ is upward-closed whenever f is nondecreasing ($A \geq \mathbf{0}$).

The composition of two affine functions is affine and every affine function is recursive. Furthermore, the extension of a nondecreasing affine function f is recursive as the extension can also be defined by its affine form $AX+B$ on $X \in \mathbb{N}_\omega^p$ with the usual conventions $0\omega = 0$ and $m\omega = \omega$, for all $m \in \mathbb{N}$ such that $m > 0$. Hence every affine WSN is a ω-WSN.

Many Petri net extensions are thus surprisingly well parametrized by affine functions. Figures 3 and 4 locate such Petri net extensions within our classes of affine functions. Figure 3 also locates the affine classes within our general framework of well-structured nets.

The Double Petri nets [9] appearing on Figure 3 are extended Petri nets in which some arc from a transition to a place p may double the content of p. Double Petri nets are then affine WSN such that every affine function $f(X) = AX+B$ satisfies $Id \leq A \leq 2Id$. Transfer Petri nets [9] are extended Petri nets in which some transitions may transfer the content of a place p to another place p'. Transfer Petri nets are then affine WSN such that every affine function $f(X) = AX + B$ satisfies $A \geq \mathbf{0}$ and every column A_j satisfies $\Sigma_i A_{ij} = 1$. Generalized Transfer Petri nets may transfer and duplicate, so that they are affine WSN such that $A \geq \mathbf{0}$ and every column A_j is nonnull. Reset Petri nets [3, 9] are extended Petri nets in which an arc may clear a place. This class corresponds to affine WSN such that every affine function $f(X) = AX + B$ satisfies $Id \geq A \geq \mathbf{0}$.

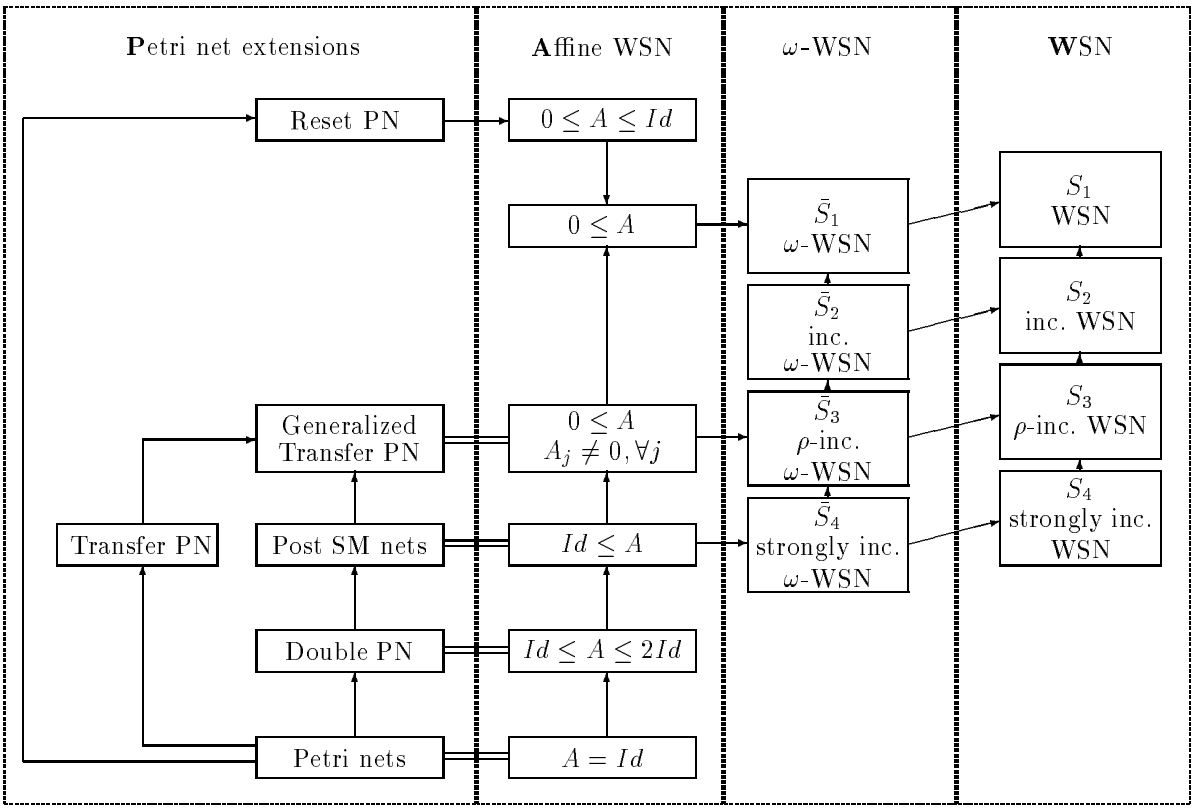

Figure 3: The world of well-structured nets. An arrow $C \to D$ indicates $C \subset D$ and a double line indicates equality.

7.3 A polynomial time algorithm for path-place-boundedness

Definition. A nonnegative affine function is an affine function $f : \mathbb{N}^p \to \mathbb{N}^p$ such that $f(X) = AX + B$ with $B \geq \mathbf{0}$.

A nonnegative affine WSN is a WSN such that every function $f \in F$ is nonnegative affine. We prove here that path-place-boundedness is decidable for nonnegative affine WSN.

For f a nonnegative affine function and X a nonnegative vector in $\mathbb{N}^p$, we prove that the limit of $f^{nd}(X)$ for a suitable d is computable. We can determine precisely which entries in $f^{nd}(X)$ go to infinity. Furthermore, we show that this limit is polynomial time computable by searching a graph. Let $f(X) = AX + B$ be a nonnegative affine function. Then $f^n(X) = A^nX + (A^{n-1} + A^{n-2} + \cdots + A + I)B$ for $n \in \mathbb{N}$.

Note that multiplication is extended by $0\omega = 0$ and $x\omega = \omega$, $\forall x \in \mathbb{N}_\omega - \{0\}$. Let's first remark the following :

Lemma 7.2 *Let $\{A_n\}_{n\geq 0}$ a sequence of non negative matrices having a limit A with possibly ω coefficients. Let X be any non negative vector. Then* $\lim_{n\to\omega} A_nX = AX$.

So we may restrict the problem to the study of the sequences $\{A^n\}_{n\geq 0}$ and $\{I + A + ... + A^n\}_{n\geq 0}$.

Notations. Let M be a matrix. We abusively write $M = 0$ or $M = \omega$ when each coefficient of M is respectively 0 or ω. We write M^ω for $\lim_{n\to\omega} M^n$ when the limit exists.

Let $A = (A_{i,j})_{(i,j)} \in \{1, ..., p\} \times \{1, ..., p\}$ be a matrix with nonnegative integers coefficients. Let's denote $A^n = (A^{(n)}_{i,j})$.

We associate to A the directed graph G_A whose vertices are $1, ..., p$ and whose arcs are : an arc joins vertex i to vertex j if and only if $A_{i,j} \neq 0$; then, the coefficient $A_{i,j}$ is called the *weight* of this arc. The *weight of a path* in the graph is the product of the weigths of its arcs. For all subgraph G of G_A, we shall denote by A_G the submatrix $(A_{i,j})_{(i,j)\in G\times G}$.

Let's recall the *index of imprimitivity* of a graph is the g.c.d. of the lengths of all circuits of the graph.

As we shall see, the behaviour of the sequence $\{A^n\}_{n\geq 0}$ may be completely described by the graph G_A. All the proof is based on the fact that $A^{(n)}_{i,j}$ is the sum of the weights of all pathes of length n from i to j.

Lemma 7.3 *Let C be a strongly connected component of G_A whose index of imprimitivity is equal to* 1. *Then $A_C = 1$ or $A^\omega_C = \omega$.*

Proof. We begin by justify quite well-known facts about primitive strongly connected graphs : Let i, j in C. Note that because C is strongly connected there exists a path from i to j (so there exists t such that $A^{(t)}_{i,j} > 0$) and because C is maximal for this property, all pathes from i to j in G_A are in fact in C. Furthermore, we may find circuits in C with lengths l_1,l_2, ..., l_r such that g.c.d.$(l_1, l_2, ..., l_r) = 1$. So with a path of length q from i to j crossing all these circuits, we can build a path in C of length $q + n_1l_1 + n_2l_2 + ... + n_rl_r$, for any non negative integers $n_1, n_2, ...n_r$. It means there exists a path from i to j of any large enough lentgh, i.e. $A^{(n)}_{i,j} > 0$ for any large enough n.

- If $A^{(n)}_{i,i} = 1$, for sufficiently large integer n, then there is only one circuit in A_C with all edges labelled by 1. So A_C is a permutation matrix and, as its index of imprimitivity is equal to 1, $A_C = 1$.

- Otherwise, there exists s such that $A^{(s)}_{i,i} \geq 2$. Let $(i, i_2), ..., (i_r, j)$ a path from i to j and let q be such that $A^{(n)}_{j,j} > 0$, if $n > q$. Now, if $n > s + q + r$, we may decompose n as $u + r + tp$ with $u \in \{q, ..., q + s - 1\}$ such that $u \equiv n - r$ modulo s and $t > \frac{n-q-s-r}{s}$. Then $m^{(n)}_{i,j} > (m^{(s)}_{i,i})^t m_{i,i_2} ... m_{i_r,j} m^{(q)}_{j,j} \geq 2^t$ which shows $\lim_{n\to\omega} m^{(n)}_{i,j} = \omega$.

■

Proposition 7.4 *Let C be a strongly connected component of G_A with index of imprimitivity equal to c. Let D, E two strongly connected components of $G_{A^c_C}$ and $(i, j) \in D \times E$. Then*

$$(A^c)^{\omega}_{i,j} = \begin{cases} O, & \text{if } D \neq E \\ (1), & \text{if } \{i\} = D = E = \{j\} \text{ and } A^{(c)}_{i,i} = (1) \\ \omega, & \text{otherwise.} \end{cases}$$

Proof. If $D \neq E$, for example, i may not be joined to j by a path of length multiple of c in G_A. As all loops have lengths multiple of c, j may not be joined either to i by a path of length multiple of c. So $\forall n$, $A^{(nc)}_{i,j} = A^{(nc)}_{j,i} = 0$. When $D = E$, we apply lemma 7.3 to $(A^{(c)}_{(i,j)})_{(i,j)\in D\times D}$ which is irreducible with index of imprimitivity equal to 1. ■

The general case :

We compute the strongly connected components of G_A. Grouping nodes by components, we find a permutation matrix S such that SAS^{-1} is a triangular-blocks matrix where each diagonal block is irreducible (corresponding to a strongly connected component of G_A). Let d be the l.c.m. of the indices of imprimitivity of all components of G_A. We already know how to compute in polynomial time the limit of the powers of each irreducible diagonal block of A^d. Let's partition the matrix $B = SA^dS^{-1} = (B_{i,j})_{(i,j)\in\{1,...,r\}\times\{1,...,r\}}$ such that each block $B_{i,i}$ corresponds to a strongly connected component of G_{A^d}. We shall call this component a 0, 1 or ω block according to the nature of its limit in lemma 7.3.

As the matrix B is upper-triangular, each path in G_B is increasing. If $i < j$, we denote by $P_{i,j}$ the set of strictly increasing pathes connecting i to j ; this set contains only pathes of length $< j - i$ so is easily compufigure. All theses sets allow us to express the powers of B :

$$B^{(n)}_{i,j} = \sum_{(i,i_1,...,i_t,j)\in P_{i,j}} \sum_{p_0+...+p_{t+1}=p} B^{p_0}_{i,i} B_{i,i_1} B^{p_1}_{i_1,i_1} \cdots B_{i_t,j} B^{p_{t+1}}_{j,j}.$$

So we get the next result which describe the behaviour of the sequence $\{A^{nd}\}_{n\geq 0}$:

Theorem 7.5 *Let $i < j \leq r$.*

- *If there exists a strictly increasing path in G_B connecting i to j which goes through a node in a ω block or through two nodes in different 1 blocks, then $B^{\omega}_{i,j} = \omega$.*
- *If all strictly increasing pathes in G_B connecting i to j go through nodes in 0 blocks or if there exists no path connecting i to j, then $\lim_{n\to\omega} B^{(n)}_{i,j} = 0$.*
- *Otherwise, all strictly increasing pathes in G_B connecting i to j go through nodes in 0 block except one 1 block $B_{l,l}$. Then $B^{\omega}_{i,j} = \sum_{(i,i_1,...,l,...,i_t,j)\in P_{i,j}} B_{i,i} B_{i,i_1} B_{i_1,i_2} ... B^{\omega}_{l,l} ... B_{i_t,j}$.*

The same considerations allow to compute $L = \lim_{n\to\omega}(A^n d + A^{n-1} d + ... + I)$. If $\lim_{n\to\omega} A^{(nd)}_{i,j} \neq 0$, the i,j-term in L is ω. If $\lim_{n\to\omega} A^{(nd)}_{i,j} = 0$, the number of pathes from i to j is finite ;

in particular, they do not cross any loop and they must be of length $< n$. So the i,j-term in L is $A^d_{i,j} + ... + A^{kd}_{i,j}$ where k is such that $kd < n \leq (k+1)d$ if $i \neq j$ or 1 if $i = j$.

Remark. This result may in fact be extended to matrices with non negative real coefficients (see [25]).

Theorem 7.6 *Let $f(X) = AX + B$ be a nonnegative affine function on $\mathbb{N}^p$. Then there exists an integer $d \geq 1$ such that $\lim_{n\to\omega} f^{nd}(X)$ exists in $\mathbb{N}^p_\omega$. Furthermore, d and $\lim_{n\to\omega} f^{nd}(X)$ are computable in polynomial time.*

Proof. The notation used and the results quoted heavily refer to the Appendix, where we have developed the full theoretical justification for the algorithm.

1. Get the strongly connected components (C) of G_A and their imprimitivity indices (d_C)
2. Compute $C^{d_C\,\omega}$ for each irreducible diagonal block of A, corresponding to a strongly connected component of G_A
   - **i)** Search for connected components (E) of C^{d_C}
   - **ii)** If one of $E = (1)$, then $E^\omega = (1)$ for all E
   - **iii)** Else, $E^\omega = \omega$ for all E
3. Compute the index d and $A^{d\,\omega}$ (*using Theorem 7.5*)
4. Compute $\sum_n A^{nd}$
5. Compute $(\sum_n A^{nd})(X)$ (*using Lemma 7.2*)

■

Corollary 7.7 *Path-place boundedness is decidable in poly time for nonnegative affine WSN.*

Interestingly, although the procedure in Theorem 7.6 computes limits and always terminates, we do not see how to bound the number of limit computations required by the minimal coverability tree strategy from Section 5 (and Appendix).

Gaubert (personal communication, see also [2, 6]) suggests that it may possible to exploit facts developed by the "minmax community", in particular about topical functions, to extend Corollary 7.7 and decide the path-place-boundedness for affine WSN (improving our nonnegative affine WSN results). The resulting algorithm would likely no longer run in polynomial time, however.

7.4 Affine Well-Structured Nets

Theorem 7.8 *The decidability status of the six problems defined in Section 3 of this paper, for five classes of affine ω-WSN, is summarized on Figure 4.*

Proof. First column: Termination and coverability are decidable because affine WSN with $A \geq \mathbf{0}$ are ω-WSN (i.e. in $\bar{S}_1$); then see Figure 2. Boundedness, and hence place-boundedness, are undecidable because boundedness is already undecidable for reset Petri nets [9] and reset Petri nets are affine WSN with $\mathbf{0} \leq A \leq Id$; then see Figure 3.

Second column: path-place-boundedness, hence path-unbounded-witness, is decidable (Corollary 7.7). The case of boundedness is open; boundedness does not reduce to the

	$A \geq \mathbf{0}$	$A \geq \mathbf{0}, B \geq \mathbf{0}$	$A_j > \mathbf{0}, \forall j$	$A \geq Id$	$A = Id$
Termination	D	D	D	D	D
Coverability	D	D	D	D	D
Boundedness	U	?	D	D	D
Path-unbounded-witness	?	D	D	D	D
Path-place boundedness	?	D	D	D	D
Place-boundedness	U	?	U	D	D

Figure 4: Affine ω-well-structured nets

path-place-boundedness problem for $(A \geq \mathbf{0}, B \geq \mathbf{0})$-affine WSN directly because one could imagine detecting a finite path $s \xrightarrow{*} s_1 \xrightarrow{\sigma} s_2$ with $s_1 < s_2$ and $\lim s_n \in \mathbb{N}^p$.

Third column: Boundedness is decidable because affine WSN with each column of each matrix greater than $\mathbf{0}$ are increasing ω-WSN (i.e. in $\bar{S}_2$). Path-place-boundedness, and thus path-unbounded-witness as well, are decidable because p_i is unbounded along the infinite path $s \xrightarrow{\sigma} s_1 \xrightarrow{\sigma} s_2 \cdots$ iff $s_2 > s_1 > s$ and $s_2(i) > s_1(i) > s(i)$. Place-boundedness is undecidable because it is undecidable for transfer Petri nets [9] which are included in $(A_j > \mathbf{0})$-affine WSN.

Fourth column: follows from [26, 27, 28].

Fifth column: affine ω-WSN with $A = Id$ are Petri nets! ■

8 Conclusion

We have answered all but eight decidability questions concerning termination, coverability, boundedness, path-unbounded-witness, path-place boundedness, and place-boundedness, for our eight classes of general WSN, and for five classes of affine WSN. These results are summarized on Figures 2 and 4. We highlight specific contributions:

- The coverability problem (and the termination problem) are shown decidable for every ω-WSN. This uses a characterization of effective upward-closed subsets of $\mathbb{N}^p$.
- To clarify the delicate role of model variations on the difficulty of deciding boundedness, we introduced two new boundedness problems, called the path-unbounded-witness and the path-place-boundedness problems, as companions to the boundedness and the place-boundedness problems classically studied. Our WSN classes provably distinguish the decidability properties of these four boundedness problem variants.
- The place-boundedness problem remains decidable for any class of WSN in which each function is ω-recursive and increases in the strong sense.
- Reset Petri nets, Generalized transfer Petri nets, transfer Petri nets, post-SM nets, double Petri nets, and Petri nets (!) arise as particular cases of WSN in which every function is affine and is defined by matrices satisfying simple conditions: see Figure 3. This point of view is new and puts the decidability results of [28, 9] in a clearer perspective.
- A new polynomial time algorithm of independent interest was developped to compute the limit of a nonnegative affine function (the algorithm extends to nonnegative affine functions with real coefficients).

We leave open the questions for which "?" appears in Figures 2 and 4. It would be interesting to complete the global picture afforded by WSN and affine WSN by answering these questions.

Finally, we have mentioned that well-structured transition systems capture many infinite state systems beyond generalized Petri nets (see for instance [13]). Clearly, any useful such transition system ought to be defined from recursive functions over some domain, so that WSN could in principle represent such systems as well. But there is one limitation of WSN, namely finiteness of their defining sets of functions, which could be lifted to make WSN more expressive. Are there infinite state systems which would require this generality to be modelled conveniently by WSN, and if so, what would become of our Figures 2 and 4 in this more general context?

Acknowledgment

We thank Philippe Schnoebelen for useful comments on two earlier versions of this paper.

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