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Abstract. We investigate simulation hemi-metrics between certain forms of turn-based $2\frac{1}{2}$ -player games played on infinite topological spaces. They have the desirable property of bounding the difference in payoffs obtained by starting from one state or another. All constructions are described as the special case of a unique one, which we call the Hutchinson hemi-metric on various spaces of continuous previsions. We show a directed form of the Kantorovich-Rubinstein theorem, stating that the Hutchinson hemi-metric on spaces of continuous probability valuations coincides with a notion of trans-shipment hemi-metric. We also identify the class of so-called sym-compact spaces as the right class of topological spaces, where the theory works out as nicely as possible.

1 Introduction

Given two (stochastic) transition systems, or two states in the same transition system, we may evaluate whether they have the same behavior by testing whether they are bisimilar. A finer measure of closeness is obtained by computing *distances* between states, so that two states are at distance 0 if and only if they are bisimilar—so-called *bisimulation distances*. This was pioneered for (infinite-state) labeled Markov processes (LMP) by Desharnais *et al.* [5]. One may see LMPs as turn-based stochastic $1\frac{1}{2}$ -player games, where, at each state, one player chooses a probability distribution p on states by its label, and the half-player picks the next state by drawing at random along p .

In [9] we explored so-called ludic transition systems, which are essentially LMPs, where the probability p is replaced by a so-called *continuous game* ν . When ν is a belief function, this naturally models a form of turn-based $2\frac{1}{2}$ -player games, where player 1 chooses a continuous game ν , then some state is drawn at random along ν —the latter step being the same thing as a half-player picking at random some set from which player 2 picks non-deterministically, demonically. In such transition systems, a simple modal logic (that of Desharnais *et al.* [4], plus binary disjunction) was shown to characterize similarity. It is a natural question to extend the notion of bisimulation distance to “simulation metrics”. This is what we do here, in the general case of (infinite-state) topological spaces and slightly more general *prevision* transition systems.

Related Work. Bisimulation metrics were explored by Desharnais *et al.* [4], and were given an elegant treatment by Ferns *et al.* [6]. The latter works also on infinite (measurable) state spaces. We use the topological setting of [9, 10], which, as we hope

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to demonstrate again here, has a certain elegance. Notions of simulation rather than bisimulation are more natural here, and correspondingly, we shall develop hemi-metrics rather than metrics. Hemi-metrics were also considered recently by de Alfaro *et al.* [3]. Their paper is both more general than ours (we only consider certain forms of turn-based games) and less general (they only consider finite state spaces). Our last theorem (Theorem 7, non trivial) is that our hemi-metric coincides, in the finite case, with their a posteriori and a priori metrics in the demonic, resp. angelic case, respectively.

The main object of study of this paper is a hemi-metric we call the *Hutchinson hemi-metric* d_H . It turns out to have many good properties, including a duality theorem à la Kantorovich-Rubinstein, and the fact that it generalizes several variants of the Hausdorff metric. The closest notion we know of is due to Baddeley [2], whose generalized Hausdorff metric generalizes the Levy-Prohorov metric on spaces of measures instead of the Hutchinson (or Kantorovich) metric. Also, Baddeley only considers T_2 topological spaces, despite admitting that it appears that the T_0 case is more interesting (which this paper should confirm).

Outline. We need quite a lot of preliminaries. Well-known facts are recapped in Section 2, while the basic theory of hemi-metric spaces is laid out in Section 3. We develop Hausdorff-like hemi-metrics on classical powerdomains for demonic, angelic, and chaotic non-determinism in Section 4, so as to appreciate how d_H will generalize them later on. Section 5 introduces prevision transition systems, a natural generalization of our ludic transition systems [9], then defines d_H , and shows how a hemi-metric computed from d_H bounds errors in payoff evaluations. Section 6 is the core of this paper, and develops the mathematical theory of d_H . We conclude in Section 7.

2 Preliminaries

We work on general topological spaces X , and let $\mathcal{O}(X)$ be the lattice of open subsets of X . The notion of *continuous valuation* is a natural alternative to the more well-known notion of measure [13]. Taking the conventions of [9], a *capacity* ν on X is a map from $\mathcal{O}(X)$ to $\mathbb{R}^+$ such that $\nu(\emptyset) = 0$; ν is a *game* iff it is additionally *monotonic*: $\nu(U) \leq \nu(V)$ whenever $U \subseteq V$; ν is *modular* (resp., *convex*, resp. *concave*) iff $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ (resp. $\geq$, resp. $\leq$) for all opens U, V . A modular game is called a *valuation*. A game ν is *continuous* iff $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$ for every directed family $(U_i)_{i \in I}$ of opens. A (sub)probability valuation ν is additionally such that ν is (sub)normalized, i.e., that $\nu(X) = 1$ (≤ 1). Continuous valuations extend to measures on the Borel σ -algebra of the topology, under mild assumptions [16], showing that the two notions are close.

Each topological space X has a *specialization quasi-ordering* $\leq$: $x \leq y$ iff every open containing x also contains y . X is T_0 iff $\leq$ is an ordering, i.e., $x \leq y$ and $y \leq x$ imply $x = y$. X is T_2 iff for any two distinct elements x and y , there are disjoint open subsets containing x and y respectively. A subset Q of X is *compact* iff one may extract a finite subcover from every open cover of Q , and is *saturated* iff it is upward-closed.

Each poset can be equipped with the so-called *Scott topology*, whose opens are the upward-closed subsets U such that, for every directed family $(x_i)_{i \in I}$ that has a least upper bound (a.k.a., a *sup*) in U , then $x_i \in U$ for some $i \in I$ already. The specialization

ordering of the Scott topology is always the original ordering $\leq$. A *cpo* is a poset in which every directed family has a sup. A function f from the poset X to the poset Y is Scott-continuous iff it is continuous for the respective Scott topologies, or equivalently, iff f is monotonic and for every directed family $(x_i)_{i \in I}$ of elements of X with a sup $x \in X$, the directed family $(f(x_i))_{i \in I}$ has $f(x)$ as sup. See [1, 8, 19] for background material on domain theory and topology.

Let $\langle X \rightarrow \mathbb{R}^+ \rangle$ be the space of bounded continuous function from X to $\mathbb{R}^+$, with the Scott topology of the pointwise ordering; $\mathbb{R}^+$ is equipped with its Scott-topology, whose non-trivial opens are the open intervals $(r, +\infty)$, $r \in \mathbb{R}^+$. The *Choquet integral* of $f \in \langle X \rightarrow \mathbb{R}^+ \rangle$ along the continuous game ν , which we shall write $\oint_{x \in X} f(x) d\nu$, is the Riemann integral $\int_0^{+\infty} \nu(f^{-1}(t, +\infty)) dt$. There is a more complex formula defining the Choquet integral of functions f taking values in $\mathbb{R}$ rather than $\mathbb{R}^+$ [9], but we shall only note that this can be alternatively defined by $\oint_{x \in X} f(x) d\nu = -a\nu(X) + \oint_{x \in X} (f(x) + a) d\nu$ for some arbitrary $a \geq -\inf_{x \in X} f(x)$. The Choquet integral is Scott-continuous and linear in ν , Scott-continuous and *positively homogeneous* ($\oint_{x \in X} af(x) d\nu = a \oint_{x \in X} f(x) d\nu$ for every $a \in \mathbb{R}^+$) in f . It is not in general additive in f (i.e., $\oint_{x \in X} (f(x) + g(x)) d\nu$ is not in general equal to $\oint_{x \in X} f(x) d\nu + \oint_{x \in X} g(x) d\nu$), unless ν is a valuation. The *Dirac valuation* δ_x maps each open U to 1 if $x \in U$, to 0 otherwise: then $\oint_{x' \in X} f(x') d\delta_x = f(x)$. (See [9].)

For each continuous map $f : X \rightarrow Y$, and game ν on X , the *push-forward* game $f[\nu]$ on Y is defined by $f[\nu](V) = \nu(f^{-1}(V))$ for every $V \in \mathcal{O}(Y)$. Then $f[\nu]$ is convex, concave, linear, continuous respectively as soon as ν is, and $\oint_{y \in Y} g(y) df[\nu] = \oint_{x \in X} g(f(x)) d\nu$. For any game ν on $X \times Y$, call $\pi_1[\nu]$ and $\pi_2[\nu]$ the first and second *marginals* of ν , where π_1 and π_2 are the first and second projections onto X , resp. Y .

A *continuous prevision* F on a topological space (e.g., a cpo) X is a Scott-continuous map from $\langle X \rightarrow \mathbb{R}^+ \rangle$ to $\mathbb{R}^+$ such that $F(af) = aF(f)$ for every $a \in \mathbb{R}^+$ (*positive homogeneity*). A prevision F is *lower* iff $F(h+h') \geq F(h) + F(h')$ for every h, h' , *upper* iff $F(h+h') \leq F(h) + F(h')$ for every h, h' , *linear* iff $F(h+h') = F(h) + F(h')$, *normalized* iff $F(a+h) = a + F(h)$ for every function h and constant $a \in \mathbb{R}^+$, *subnormalized* iff $F(a+h) \leq a + F(h)$ for every h and constant a . The integration functional α_e , defined as $\alpha_e(\nu) = \lambda h \in \langle X \rightarrow \mathbb{R}^+ \rangle \cdot \oint_{x \in X} h(x) d\nu$, maps continuous (resp., and convex, concave, modular) games to continuous previsions (resp., lower, upper, linear). Conversely, define $\gamma_e(F)$ for any prevision F as the game such that $\gamma_e(F)(U) = F(\chi_U)$ for each open U , where χ_U is the (continuous) map sending every $x \in U$ to 1, and every $x \notin U$ to 0. Then α_e and γ_e define an isomorphism between the space $\mathbf{V}_1(X)$ (resp., $\mathbf{V}_{\leq 1}(X)$) of continuous (sub)probability valuations and the space $\mathbf{P}_1^\Delta(X)$ (resp., $\mathbf{P}_{\leq 1}^\Delta(X)$) of continuous (sub)normalized linear previsions, both ordered pointwise. (See [10].) We shall write $\nabla \mathbf{P}_1(X)$, resp. $\Delta \mathbf{P}_1(X)$, the space of all normalized continuous lower (resp., upper) previsions on X . We write $\mathbf{F}_1(X)$ the space of all normalized *forks*, where a *fork* is a pair (F^-, F^+) of a continuous lower prevision F^- and a continuous upper prevision F^+ satisfying *Walley's condition* $F^-(h+h') \leq F^-(h) + F^+(h') \leq F^+(h+h')$, for every $h, h' \in \langle X \rightarrow \mathbb{R}^+ \rangle$; a fork is *normalized* iff both F^- and F^+ are. While $\nabla \mathbf{P}_1(X)$ (resp., $\Delta \mathbf{P}_1(X)$) is an adequate model of mixed probabilistic and demonic (resp., angelic) non-deterministic choice, $\mathbf{F}_1(X)$ is one of probabilistic and chaotic non-deterministic choice [10, 12].

A *hemi-metric* d on X is a function from $X \times X$ to $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$ such that $d(x, x) = 0$ for every $x \in X$, and satisfying the *triangular inequality*: $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$. A *metric* also satisfies $d(x, y) = 0 \Rightarrow x = y$, and enjoys symmetry: $d(x, y) = d(y, x)$. Hemi-metrics (not taking the value $+\infty$) are called directed metrics by de Alfaro *et al.* [3], and just metrics by Lawvere [18]. While most works on hemi-metrics consist in characterizing notions of limits or completions (see, e.g., [17] and references therein), we shall concentrate on the topology. (In the absence of a definitive textbook on hemi-metrics, we prove these properties in Appendix A.) Every hemi-metric d induces a topology $\mathcal{O}^d$, the smallest containing all *open balls* $B_{x, < \epsilon}^d = \{y \in X \mid d(x, y) < \epsilon\}$. As in the metric case, a subset U is open in $\mathcal{O}^d$ iff for every $x \in U$, some open ball $B_{x, < \epsilon}^d$ is contained in U . The specialization quasi-ordering of $\mathcal{O}^d$ is such that $x \leq y$ iff $d(x, y) = 0$. So $\mathcal{O}^d$ is a T_0 topology iff $d(x, y) = d(y, x) = 0$ implies $x = y$; and if d is a metric, then $\mathcal{O}^d$ is T_2 . The canonical hemi-metric on $\mathbb{R}$, or $\overline{\mathbb{R}}^+$, is $d_{\mathbb{R}}$ defined by $d_{\mathbb{R}}(s, t) = \max(s - t, 0)$ (on $\overline{\mathbb{R}}^+$, we agree that $d_{\mathbb{R}}(+\infty, +\infty) = 0$).

For any two hemi-metric spaces X, Y with hemi-metrics d and d' respectively, and any function $f : X \rightarrow Y$, f is *continuous* at $x \in X$ iff for every $\epsilon > 0$, there is an $\eta > 0$ such that for any $x' \in X$ such that $d(x, x') < \eta$, $d'(f(x), f(x')) < \epsilon$. It is equivalent to require that f is continuous, in the usual topological sense (the inverse image of every open is open), or to require that f be continuous at every element x of X . For any $c \in \overline{\mathbb{R}}^+$, say that f is *c-Lipschitz* iff $d(f(x), f(x')) \leq cd(x, x')$ for all $x, x' \in X$. 1-Lipschitz functions are sometimes called *non-expansive*. We say that f is *Lipschitz* iff f is *c-Lipschitz* for some c . Every Lipschitz function is continuous.

The hemi-metric d is *bounded* iff there is fixed real a such that $d(x, x') \leq a$ for all $x, x' \in X$. Every hemi-metric is *topologically equivalent* to a bounded one, i.e., the two hemi-metrics generate the same topology. The *opposite* hemi-metric d^{op} is defined by $d^{op}(x, y) = d(y, x)$, and the *symmetrized* hemi-metric d^{sym} is defined by $d^{sym}(x, y) = \max(d(x, y), d(y, x))$. The specialization quasi-ordering of d^{op} is then opposite $\geq$ of that, $\leq$, of d ($x \geq y$ iff $y \leq x$). We shall write X^{op} , X^{sym} the space X equipped with d^{op} , resp. d^{sym} . The topology of X^{sym} is finer (has at least as many opens as) those of X and X^{op} . A hemi-metric space X is *totally bounded* [17] iff for every $\epsilon > 0$, there are finitely many elements $x_1, \dots, x_n$ of X such that $X = \bigcup_{i=1}^n B_{x_i, < \epsilon}^{d^{sym}}$. (Beware: d^{sym} , not d .) We call X *sym-compact* iff X is T_0 and X^{sym} is compact. Clearly, every sym-compact space is totally bounded. The converse fails.

3 More on Hemi-Metric Spaces

There is an obvious duality in hemi-metric spaces: replace d by d^{op} , getting X^{op} . Then $(X^{op})^{op} = X$. Another well-known duality, this time at the topological level, is given by Nachbin's (1948) theory of stably compact spaces and compact pospaces (see Jung [14] for an excellent introduction). We shall show that these two dualities match, in a precise sense, on sym-compact spaces.

First recall the theory of stably compact spaces [14]. A topological space X is *stably compact* iff X is T_0 , *well-filtered* (for every filtered family $(Q_i)_{i \in I}$ of compact saturated subsets, for every open U , if $\bigcap_{i \in I} Q_i \subseteq U$ then $Q_i \subseteq U$ already for some $i \in I$),

locally compact (whenever $x \in U$ with U open, there is a compact saturated subset Q such that $x \in \text{int}(Q) \subseteq Q \subseteq U$, where $\text{int}(Q)$ denotes the interior of Q), *coherent* (the intersection of any two compact saturated subsets is again so) and *compact*. When X is stably compact, the *de Groot dual* X^d of X is just X , only with the so-called *cocompact topology*, whose opens are the cocompacts, i.e., subsets of the form $X \setminus Q$, Q compact saturated subset of X . Well-filteredness, coherence, and compactness imply that this is indeed a topology. Then X^d is again stably compact, and $X^{dd} = X$. The theory relies on the study of the *patch space* X^{patch} , which is X with the *patch topology* (the least collection of opens containing both the topology of X and that of X^d). If X is stably compact, then X^{patch} is compact, T_2 , and the graph of $\leq$ is closed in $X \times X$, where $\leq$ is the specialization quasi-ordering of X : $(X^{\text{patch}}, \leq)$ is a so-called *compact pospace*. Conversely, whenever $(X', \leq)$ is a compact pospace, the *upper space* X of X' , defined as X' with the topology consisting of just the $\leq$ -upward-closed open subsets of X' , is stably compact, has $\leq$ as specialization ordering, and $X^{\text{patch}} = X'$. Also, X^d is exactly the *lower space* of X' , i.e., X' with the topology consisting of $\leq$ -downward-closed opens of X' . We show that this duality extends to the hemi-metric case:

Theorem 1. *Let X be a T_0 hemi-metric space. Then X is sym-compact iff: (*) X , qua topological space, is stably compact, and X^{op} , qua topological space, coincides with X^d . In this case, $(X^{\text{sym}}, \leq)$ is exactly the patch space of X , where $\leq$ is the specialization ordering of X .*

Proof. Let us show that (*) implies that X is sym-compact. Since $X^{op} = X^d$ (qua topological spaces), the open subset $B_{x, < \epsilon}^{d^{op}}$ of X^{op} is cocompact in X . So the symmetric open ball $B_{x, < \epsilon}^{d^{sym}} = B_{x, < \epsilon}^d \cap B_{x, < \epsilon}^{d^{op}}$ is open in X^{patch} . It follows that every open of X^{sym} is open in X^{patch} : the patch topology is finer than that of X^{sym} . However, the topology of X^{sym} is T_2 , the patch topology is compact, and it is well-known that any compact topology finer than a T_2 topology coincides with it, so $X^{\text{sym}} = X^{\text{patch}}$ is compact.

Conversely, we use Proposition 2.10 of Jung [14]. This states that, if X^o is a stably compact space, with specialization ordering $\leq$, $\mathcal{B}$ is a family of opens of X , $\mathcal{C}$ is a family of cocompacts of X , such that for any $x, y \in X$ with $x \not\leq y$, there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ with $x \in B$, $y \in C$, and $B \cap C = \emptyset$, then $\mathcal{B}$ is a subbasis for the topology of X^o . (I.e., the topology of X^o is the least one containing $\mathcal{B}$.) In our case, assume X^{sym} compact, let $\leq$ be the specialization ordering of X , and let X^o be the upper space of $(X^{\text{sym}}, \leq)$. Let $\mathcal{B}$ denote the topology of X . Clearly, $\mathcal{B}$ is a family of opens of X^o .

Let $\mathcal{C}$ the set of open balls of the form $B_{x, < \epsilon}^{d^{op}}$, $x \in X$, $\epsilon > 0$. Every element of $\mathcal{C}$ is cocompact in X^o . Indeed, $B_{x, < \epsilon}^{d^{op}}$ is open in X^{sym} (since its topology is finer than that of X^{op}), its complement Q is therefore closed in the compact space X^{sym} , hence compact in X^{sym} . Any open cover of Q in X is also an open cover in X^{sym} , so Q is compact in X , too. Also, Q is upward-closed in $\leq$ because $B_{x, < \epsilon}^{d^{op}}$ is upward-closed in the specialization ordering $\geq$ of d^{op} . So $B_{x, < \epsilon}^{d^{op}}$ is cocompact in X .

To use Jung's Proposition 2.10, fix $x, y \in X$, with $x \not\leq y$, then: (†) there is an $\epsilon > 0$ such that $B_{x, < \epsilon}^d \cap B_{y, < \epsilon}^{d^{op}} = \emptyset$. (Take $\epsilon = d(x, y)/2$. Since $x \not\leq y$, $d(x, y) \neq 0$, so $\epsilon > 0$.)

By (†), the graph of $\leq$ is closed in $X^{\text{sym}} \times X^{\text{sym}}$, i.e., $(X^{\text{sym}}, \leq)$ is a compact pospace. Indeed, for any (x, y) in the complement W of this graph, by definition $x \not\leq y$ and using (†), for some $\epsilon > 0$, $B_{x, < \epsilon}^d \times B_{y, < \epsilon}^{d^{op}}$ is an open in $X^{\text{sym}} \times X^{\text{sym}}$ containing

(x, y) . It is also contained in W : if $(x', y') \in B_{x, < \epsilon}^d \times B_{y, < \epsilon}^{d^{op}}$ was not in W , i.e., $x' \leq y'$, then y' would be $B_{x, < \epsilon}^d$, since the latter is upward-closed, contradiction. So W is open.

So X^o is stably compact, and has $\leq$ as specialization ordering. By $(\dagger)$ again, Jung's Proposition 2.10 applies with $B = B_{x, < \epsilon}^d$ and $C = B_{y, < \epsilon}^{d^{op}}$. So the least topology containing $\mathcal{B}$ is that of X . Recall that $\mathcal{B}$ is the topology of X , so $X = X^o$. In particular, X is stably compact. Exchanging the roles of d and d^{op} , X and X^{op} , $\leq$ and $\geq$, X^{op} is stably compact. Moreover, the topology X^{op} consists of the $\geq$ -upward-closed, i.e., the $\leq$ -downward-closed opens of X^{sym} . These are exactly the opens of $X^{od} = X^d$. $\square$

We shall also use the function $d^2 : X \times X \rightarrow \overline{\mathbb{R}}^+$ defined by $d^2((x, y), (x', y')) = d(x, x') + d(y', y)$, where d is a hemi-metric on X . It is easy to check that d^2 is a hemi-metric on $X \times X$. Let $X^{(2)}$ be $X \times X$, equipped with d^2 . It is also easy to see that the hemi-metric d is a 1-Lipschitz function from $X^{(2)}$ to $\overline{\mathbb{R}}^+$. However, the topology of $X^{(2)}$ is *not* the product topology of $X \times X$ in general (see Appendix B):

Lemma 1. *Let X be equipped with a bounded hemi-metric d . The topology of $X^{(2)}$ is that of the topological product $X \times X^{op}$. Its specialization ordering is $\leq \times \geq$.*

Let $d(x, A)$, the distance of x to a subset A of X be $\inf_{y \in A} d(x, y)$, taking this to be $+\infty$ when A is empty. The function $x \mapsto d(x, A)$ is 1-Lipschitz in the sense that $d(x, A) \leq d(x, y) + d(y, A)$, and $d(x, A) = 0$ iff x is in the topological closure $cl(A)$ of A . (See Appendix C for proofs of this and the following.) Using this, we may define the *thinning* $U^{d, -(\epsilon)} = \{x \in X \mid d(x, X \setminus U) > \epsilon\}$ of the open set U by ϵ : $U^{d, -(\epsilon)}$ is open, contained in U , grows larger as ϵ decreases, and for any family of non-negative reals $(\epsilon_i)_{i \in I}$ having 0 as $\inf$, $U^{d, -(\epsilon_i)}$ is a directed family of opens whose union is U . We may also define the *thickening* of a subset A of X as $A^{d, +(\epsilon)} = \bigcup_{x \in A} B_{x, < \epsilon}^d$. This is always open. Moreover, $(U^{d, -(\epsilon)})^{d, +(\epsilon)} \subseteq U$ for every open U and $\epsilon > 0$.

4 Hemi-Metrics on Powerdomains

It is standard to model non-determinism in domain theory through the use of *powerdomains* [1, Section 6.2]. The *Smyth powerdomain* $\mathcal{Q}(X)$ is the set of all non-empty compact saturated subsets Q of X , ordered by reverse inclusion $\supseteq$. This models *demonic* non-determinism, $Q \in \mathcal{Q}(X)$ denoting the set of all possible choices of elements $x \in Q$. (The theory of [9] probably enlightens what “demonic” means, in the sense that the choice of $x \in Q$ is resolved by an *adversary* who picks some $x \in Q$ that pleases you least. Mathematically, Q gives rise to the so-called unanimity game u_Q , which maps each U containing Q to 1, every other to 0; assuming you get $f(x)$ dollars if you pick x , your expected payoff will be the Choquet integral of $f(x)$ along u_Q , which is exactly $\min_{x \in Q} f(x)$ —i.e., you will get the least possible amount of money.) The *Hoare powerdomain* $\mathcal{H}(X)$ is the set of all non-empty closed subsets F of X , ordered by ordinary inclusion $\subseteq$, and models *angelic* non-determinism. (Each such F gives rise to the *example game* e_F , mapping each open U that meets F to 1, every other open to 0; it is an easy exercise that the Choquet integral of $f(x)$ along e_F equals $\sup_{x \in F} f(x)$ —i.e., you get the most money you can.) The *Plotkin powerdomain* $\mathcal{P}\ell(X)$ is the set of all *lenses*

L , where a lens is the non-empty intersection of a compact saturated subset Q (which we can take equal to the upward-closure $\uparrow L$ of L) and a closed subset F (which we can take equal to the topological closure $cl(L)$ of L) of X , ordered by the topological Egli-Milner ordering $\sqsubseteq_{EM}$, defined by $L \sqsubseteq_{EM} L'$ iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$.

These powerdomains are endowed with their Scott topology. More relevant topologies when X is a general topological space, not just a cpo, are the *Vietoris* topologies. Let $\mathcal{Q}_V(X)$ be $\mathcal{Q}(X)$ with the smallest topology containing the basic opens $\square U = \{Q \in \mathcal{Q}(X) \mid Q \subseteq U\}$, U open in X , $\mathcal{H}_V(X)$ be $\mathcal{H}(X)$ with the smallest topology containing the subbasic opens $\diamond U = \{F \in \mathcal{H}(X) \mid F \cap U \neq \emptyset\}$, and $\mathcal{P}\ell_V(X)$ be $\mathcal{P}\ell(X)$ with the smallest topology containing both $\square U = \{L \mid L \subseteq U\}$ and $\diamond U = \{L \mid L \cap U \neq \emptyset\}$. When X is well-filtered and locally compact, $\mathcal{Q}_V(X) = \mathcal{Q}(X)$. When X is a continuous cpo, $\mathcal{H}_V(X) = \mathcal{H}(X)$, and when X is also coherent, then $\mathcal{P}\ell_V(X) = \mathcal{P}\ell(X)$.

We observe that, when X is a (nice enough) hemi-metric space, all these spaces can be equipped with hemi-metrics that generate the Vietoris topology. These hemi-metrics will be asymmetric variants of the well-known Hausdorff metric on T_2 spaces. We shall see later that these are special cases of the Hutchinson hemi-metric, to be defined later. Proofs can be found in Appendix D.

Proposition 1. *Let X be equipped with the hemi-metric d . The Hausdorff-Smyth hemi-metric $d_{\mathcal{Q}}$ on $\mathcal{Q}(X)$ is defined by $d_{\mathcal{Q}}(Q, Q') = \sup_{x' \in Q'} \inf_{x \in Q} d(x, x')$. The bounds are attained, i.e., $d_{\mathcal{Q}}(Q, Q') = \max_{x' \in Q'} \min_{x \in Q} d(x, x')$. This defines a hemi-metric, whose topology is exactly the Vietoris topology of $\mathcal{Q}_V(X)$.*

Observe also that the function $\eta_{\mathcal{Q}} : X \rightarrow \mathcal{Q}_V(X)$ mapping x to $\uparrow x = \{y \in X \mid x \leq y\}$, is an *isometric embedding*, namely $d_{\mathcal{Q}}(\eta_{\mathcal{Q}}(x), \eta_{\mathcal{Q}}(x')) = d(x, x')$.

The case of the Hoare powerdomain requires us to assume d to be totally bounded.

Proposition 2. *Let X be equipped with the hemi-metric d . The Hausdorff-Hoare hemi-metric $d_{\mathcal{H}}$ on $\mathcal{H}(X)$ is defined by $d_{\mathcal{H}}(F, F') = \sup_{x \in F} \inf_{x' \in F'} d(x, x')$. This defines a hemi-metric, whose topology is finer than the Vietoris topology of $\mathcal{H}_V(X)$, and coincides with it as soon as d is totally bounded.*

Proposition 3. *Let X be equipped with the hemi-metric d . The Hausdorff hemi-metric $d_{\mathcal{P}\ell}$ on $\mathcal{P}\ell(X)$ is defined by $d_{\mathcal{P}\ell}(L, L') = \max(\sup_{x' \in L'} \inf_{x \in L} d(x, x'), \sup_{x \in L} \inf_{x' \in L'} d(x, x'))$. Then $d_{\mathcal{P}\ell}(L, L') = \max(d_{\mathcal{Q}}(\uparrow L, \uparrow L'), d_{\mathcal{H}}(cl(L), cl(L')))$, $d_{\mathcal{P}\ell}$ is a hemi-metric, its topology is finer than the Vietoris topology of $\mathcal{P}\ell_V(X)$, and coincides with it as soon as d is totally bounded.*

Then $\eta_{\mathcal{H}} : X \rightarrow \mathcal{H}_V(X)$, which sends x to $\downarrow x = cl(\{x\}) = \{y \in X \mid y \leq x\}$, and $\eta_{\mathcal{P}\ell} : X \rightarrow \mathcal{P}\ell_V(X)$, which sends x to $(\uparrow x, \downarrow x)$, are isometric embeddings.

5 Prevision Transition Systems

In [9], we defined ludic transition systems on the state space X by analogy with LMPs, as collections σ of maps σ_{ℓ} , where ℓ ranges over a finite set $\mathcal{L}$ of labels, where $\sigma_{\ell} : X \rightarrow \mathbf{J}_{\leq 1}^{wk}(X)$, and where $\mathbf{J}_{\leq 1}^{wk}(X)$ is the space of all continuous subnormalized games over the topological space X , with the weak topology. Intuitively, the system

evolves from state $x \in X$ by letting one player P pick a label $\ell \in \mathcal{L}$, then the other player draws a next state at random along the continuous game $\sigma_\ell(x)$. When $\sigma_\ell(x)$ is a belief function, results from [9] imply that this second player can be thought of as one half-player picking some $Q \in \mathcal{Q}(X)$ at random, along some subprobability distribution $(\sigma_\ell(x))^*$, then the second player C picking the next state in a (demonically) non-deterministic way out of Q .

Since any continuous game ν can be seen as a continuous prevision, namely $\alpha_C(\nu)$, we only define a larger class of transition systems by considering *prevision transition systems* (PrTS), which are collections π of maps $\pi_\ell : X \rightarrow \mathbf{P}_{\leq 1 \text{ wk}}(X)$, where $\mathbf{P}_{\leq 1 \text{ wk}}(X)$ is the space of all continuous subnormalized previsions on X , with the weak topology. This lends itself to a sleeker mathematical treatment. Call a PrTS lower, upper, normalized, when $\pi_\ell(x)$ is so, for every $\ell \in \mathcal{L}$ and $x \in X$. When $\pi_\ell(x)$ is normalized, for every $h \in \langle X \rightarrow \mathbb{R} \rangle$ (i.e., with values in $\mathbb{R}$, not $\mathbb{R}^+$), write $\hat{\pi}_\ell(x)(h) = \pi_\ell(x)(h + a) - a \in \mathbb{R}$, for any constant $a \geq -\inf_{x \in X} h(x)$. Then $\hat{\pi}_\ell$ is monotonic, positively homogeneous, normalized, Scott-continuous, and lower, resp. upper when π_ℓ is [12]. Note the similarity with the Choquet integral of functions in $\langle X \rightarrow \mathbb{R} \rangle$.

As in MDP theory and in [9], we may add rewards and discounts to PrTSes. Imagine P plays according to a finite-state program Π , i.e., an automaton with *internal states* q, q' and transitions $q \xrightarrow{\ell} q'$. Let $r_{q \xrightarrow{\ell} q'} : X \rightarrow \mathbb{R}$ be a family of bounded continuous *reward* functions: we may think that $r_{q \xrightarrow{\ell} q'}(x)$ is the amount of money P gains if she fires her internal transition $q \xrightarrow{\ell} q'$, drawing the next state y at random along $\pi_\ell(x)$. Let $\gamma_{q \xrightarrow{\ell} q'} \in (0, 1]$ be a family of so-called *discounts*. Define the average payoff, starting from state x when P is in its internal state q , by:

$$V_q(x) = \sup_{\ell, q' / q \xrightarrow{\ell} q'} \left[r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \hat{\pi}_\ell(x)(V_{q'}) \right] \quad (1)$$

This is obtained from formula (4) of [9] by replacing the Choquet integral $\oint_{y \in X} V_{q'}(y) d\sigma_\ell(x)$, i.e., $\alpha_C(\sigma_\ell(x))(V_{q'})$, by the more general formula $\hat{\pi}_\ell(x)(V_{q'})$. When π is lower and normalized, using [10, Theorem 5], $\pi_\ell(x)(h) = \min_{G \in CCoeur_1(\pi_\ell(x))} G(h) = \min_{p \in \mathbf{V}_1(X)/\alpha_C(p) \geq \pi_\ell(x)} \oint_{y \in X} h(y) dp$ for all $h \in \langle X \rightarrow \mathbb{R}^+ \rangle$, from which $\hat{\pi}_\ell(x)(V_{q'}) = \min_{p \in \mathbf{V}_1(X)/\alpha_C(p) \geq \pi_\ell(x)} \oint_{y \in X} V_{q'}(y) dp$. Intuitively, P plays first, maximizing her gains, then C picks a distribution p (from some set) to minimize gains, from which a next state y is chosen at random. So C plays before random choice is effected, contrarily to the case of ludic transition systems, but as in other proposals [20, 23]. As in [9], we have:

Theorem 2. Assume π is standard, i.e., $\pi_\ell(x)(\chi_X)$ is always either 0 or 1, and the set $\{x \in X | \pi_\ell(x)(\chi_X) = 0\}$ of deadlock states is open; or that $r_{q \xrightarrow{\ell} q'}(x) \geq 0$ for all $q, \ell, q', x \in X$. Assume also that the rewards are uniformly bounded, i.e., there are $a, b \in \mathbb{R}$ with $a \leq r_{q \xrightarrow{\ell} q'}(x), \gamma_{q \xrightarrow{\ell} q'} \leq b$ for all $q, \ell, q', x \in X$. Then (1) has a unique uniformly bounded solution in any of the following two cases:

[Finite Horizon] If all paths in Π have bounded length.

[Discount] If there is a constant $\gamma \in (0, 1)$ such that $\gamma_{q \xrightarrow{\ell} q'} \leq \gamma$ for every q, ℓ, q' .

Proof. (Sketch.) This is by induction on the length of paths in the Finite Horizon case. In the Discount case, let $\mathcal{V}$ be the operator that maps the family $V = (V_q)_q$ of functions

to the family of functions of x indexed by q defined as the right-hand side of (1). We let this operator work on families that are uniformly bounded in q , i.e., such that there is a unique interval $[a', b']$ such that $V_q(x) \in [a', b']$ for all $x \in X$, and internal state q . Then $\mathcal{V}$ is Scott-continuous, and γ -Lipschitz in the sense that $\mathcal{V}(\mathbf{V})_q(x) - \mathcal{V}(\mathbf{V}')_q(x) \leq \gamma \cdot \sup_{\ell, q' / q \xrightarrow{\ell} q'} \sup_{y \in X} d_{\mathbb{R}}(V_{q'}(y), V'_{q'}(y)) \leq \gamma \cdot \sup_{q', y \in X} d_{\mathbb{R}}(V_{q'}(y), V'_{q'}(y))$. Since $\gamma < 1$, this implies that any two uniformly bounded solutions $\mathbf{V}$ and $\mathbf{V}'$ to (1) must be such that $z \leq \gamma z$, where $z = \sup_{q', y \in X} d_{\mathbb{R}}(V_{q'}(y), V'_{q'}(y))$, so $z = 0$, whence $V_{q'} \leq V'_{q'}$ for all q' , and by symmetry, $V_{q'} = V'_{q'}$ for all q' . Then start with $V_q(x)$ defined as some constant $\alpha \leq 0$ for all q , sufficiently low that $\mathcal{V}(\mathbf{V})_q(x) \geq V_q(x)$ for all q and $x \in X$, i.e., $\alpha \leq \frac{\min(a, 0)}{(1-\gamma)}$. Then $(\mathcal{V}^n(\mathbf{V}))_{n \in \mathbb{N}}$ is an increasing chain, and converges to some family, since $\mathcal{V}$ is Scott-continuous and γ -Lipschitz. Since rewards are uniformly bounded, it is easy to see that each iterate, as well as the limit, is, too. $\square$

It is interesting to check whether two states x, y are close to each other in some metric that would inform us about the difference between the payoffs $V_q(x)$ and $V_q(y)$. Ferns *et al.* [6] introduce a nice bisimulation metric, which is computed as a fixed point. Probabilistic distributions are compared in the so-called Hutchinson (or Kantorovich) metric, while sets of non-deterministic choices are compared in the Hausdorff metric.

We introduce a similar construction, with the following differences. First, we work with hemi-metrics instead of metrics, and this will provide us a measure of *simulation*, not bisimulation. Second, we generalize the Hutchinson (hemi-)metric to work not just on probabilities, but on games and even on previsions. This way, the generalized notion will actually encompass both the Hutchinson and the Hausdorff (hemi-)metrics, as used in [6]; this encompassment is the first part of our Theorem 7 (see later).

For any two continuous games ν and ν' on X , define the *Hutchinson hemi-metric* $d_H(\nu, \nu')$ as $\sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} d_{\mathbb{R}}(\int_{x \in X} f(x) d\nu, \int_{x \in X} f(x) d\nu')$, where $\langle X \rightarrow \mathbb{R}^+ \rangle_1$ is the space of all bounded 1-Lipschitz functions from X to $\mathbb{R}^+$. The classical Hutchinson *metric* uses $d_{\mathbb{R}}^{sym}$ instead of $d_{\mathbb{R}}$, i.e., would be the sup of $|\int_{x \in X} f(x) d\nu - \int_{x \in X} f(x) d\nu'|$. Note that our definition is valid for general continuous games, not just valuations.

We can, in turn, extend this to continuous previsions by $d_H(F, F') = \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} d_{\mathbb{R}}(F(f), F'(f))$, so that $d_H(\alpha_{\mathbb{C}}(\nu), \alpha_{\mathbb{C}}(\nu')) = d_H(\nu, \nu')$ for every continuous games ν and ν' ; and to forks by $d_H((F^-, F^+), (F'^-, F'^+)) = \max(d_H(F^-, F'^-), d_H(F^+, F'^+))$.

Let $\mathcal{M}(X)$ be the space of all hemi-metrics on X . This is a complete lattice, ordered pointwise. Define the operator $\mathcal{D} : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ that maps d to $\mathcal{D}(d)$ defined as $\mathcal{D}(d)(x, y) = \sup_{q, \ell, q' / q \xrightarrow{\ell} q'} (d_{\mathbb{R}}(r_{q \xrightarrow{\ell} q'}(x), r_{q \xrightarrow{\ell} q'}(y)) + \gamma_{q \xrightarrow{\ell} q'} d_H(\pi_{\ell}(x), \pi_{\ell}(y)))$. Let d_{π} be the least fixed point of $\mathcal{D}$ on $\mathcal{M}(x)$. We check that d_{π} bounds the discrepancy of payoffs $V_q(x), V_q(y)$, starting from states x and y .

Proposition 4. *Under the assumptions of Theorem 2, let $\mathbf{V} = (V_q)_q$ be the unique solution of (1). Then $V_q(x) - V_q(y) \leq d_{\pi}(x, y)$ for every $x, y \in X$.*

Proof. By induction on the length of paths in the Finite Horizon case. We deal with the more interesting Discount case. When $\mathbf{V}$ consists of constant functions, the inequality (*) $V_q(x) - V_q(y) \leq d_{\pi}(x, y)$ for all x, y is clear. If (*) holds of $\mathbf{V}$, then we claim it holds for $\mathcal{V}(\mathbf{V})$. Let $\mathbf{V}' = \mathcal{V}(\mathbf{V})$. Since by (*) V_q is 1-Lipschitz wrt. d_{π} , by definition of d_H we obtain $\hat{\pi}_{\ell}(x)(V_q) - \hat{\pi}_{\ell}(y)(V_q) \leq (d_{\pi})_H(\pi_{\ell}(x), \pi_{\ell}(y))$.

Also, $V'_q(x) = \sup_{\ell, q'/q \xrightarrow{\ell} q'} [r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \hat{\pi}_\ell(x)(V_{q'})]$. Now $r_{q \xrightarrow{\ell} q'}(x) + \gamma_{q \xrightarrow{\ell} q'} \hat{\pi}_\ell(x)(V_{q'}) \leq r_{q \xrightarrow{\ell} q'}(y) + d_{\mathbb{R}}(r_{q \xrightarrow{\ell} q'}(x), r_{q \xrightarrow{\ell} q'}(y)) + \gamma_{q \xrightarrow{\ell} q'} \hat{\pi}_\ell(y)(V_{q'}) + \gamma_{q \xrightarrow{\ell} q'}(d_\pi)_H(\pi_\ell(x), \pi_\ell(y)) \leq r_{q \xrightarrow{\ell} q'}(y) + \gamma_{q \xrightarrow{\ell} q'} \hat{\pi}_\ell(y)(V_{q'}) + \mathcal{D}(d_\pi)(x, y)$. Taking sups over ℓ and q' , and since $\mathcal{D}(d_\pi) = d_\pi, (*)$ holds for $\mathbf{V}'$. So it holds of $\mathcal{V}^n(\mathbf{V})$ where $\mathbf{V}$ is constant, by induction on n . Hence it holds for the unique solution of (1). $\square$

In other words, the payoff $V_q(x)$ at q cannot exceed $V_q(y)$ by more than $d_\pi(x, y)$. The corresponding symmetrized distance d_π^{sym} measures, as usual, the absolute value of the difference: $|V_q(x) - V_q(y)| \leq d_\pi^{sym}(x, y)$.

6 The Hutchinson Hemi-Metric on Games, on Previsions

While our notion of Hutchinson hemi-metric extends a classical notion on measures, we said earlier that the Hutchinson hemi-metric would also extend hemi-metrics on powerdomains (see Appendix E for a proof):

Proposition 5. *For all $Q, Q' \in \mathcal{Q}(X)$, $d_H(u_Q, u_{Q'}) = d_Q(Q, Q')$. For all $F, F' \in \mathcal{H}(X)$, $d_H(\mathbf{e}_F, \mathbf{e}_{F'}) = d_{\mathcal{H}}(F, F')$. For every $L \in \mathcal{P}\ell(X)$, $F_L = (\alpha_{\mathcal{C}}(u_{\uparrow L}), \alpha_{\mathcal{C}}(\mathbf{e}_{cl(L)}))$ is a normalized fork, and for all $L, L' \in \mathcal{P}\ell(X)$, $d_H(F_L, F_{L'}) = d_{\mathcal{P}\ell}(L, L')$.*

It follows that the maps $Q \in \mathcal{Q}(X) \mapsto \alpha_{\mathcal{C}}(u_Q) \in \nabla \mathbf{P}_1(X)$, $F \in \mathcal{H}(X) \mapsto \alpha_{\mathcal{C}}(\mathbf{e}_F) \in \Delta \mathbf{P}_1(X)$, and $L \in \mathcal{P}\ell(X) \mapsto F_L \in \mathbf{F}_1(X)$ are isometric embeddings, where the powerdomains are equipped with their respective variant of the Hausdorff hemi-metric, and spaces of previsions or forks are equipped with the Hutchinson hemi-metric.

On spaces of previsions, the *weak topology* plays an important role [10, 12]. On any space of previsions Y over X , this is the least one containing the subbasic opens $[f > r] = \{F \in Y \mid F(f) > r\}$, $f \in \langle X \rightarrow \mathbb{R}^+ \rangle$, $r \in \mathbb{R}$. Similarly, the weak topology on spaces Y of games on X [9] has as subbasic open sets $[f > r] = \{\nu \in Y \mid \oint_{x \in X} f(x) d\nu > r\}$. This coincides with the product topology, whose subbasic open sets are $[U > r] = [\chi_U > r] = \{\nu \in Y \mid \nu(U) > r\}$, for each open subset U of X . We show that the Hutchinson hemi-metric defines the weak topology on spaces of continuous normalized games—including $\mathbf{V}_1(X)$, the space of all continuous probability valuations. This is similar to a famous theorem in measure theory stating that the Hutchinson metric on the space of all probability measures over a Polish space has the weak topology (sometimes called the weak* topology), defined as above.

Theorem 3. *Let X be equipped with a hemi-metric d , Y a space of continuous games over X . The topology of d_H on Y is finer than the weak topology on Y , and coincides with it when Y is a space of normalized games and d is bounded and totally bounded.*

We omit the proof (see Appendix F). This requires defining Lipschitz approximations to the indicator functions χ_U : for every $\epsilon > 0$, let $\chi_U^\epsilon(x) = \min(\epsilon, d(x, X \setminus U))$. Then $(\frac{1}{\epsilon} \chi_U^\epsilon)_{\epsilon > 0}$ is a directed family of Lipschitz functions with $\chi_{U^{d, -(\epsilon)}} \leq \frac{1}{\epsilon} \chi_U^\epsilon \leq \chi_U$ —in particular, has χ_U as sup. The proof also relies on another hemi-metric, the Levy-Prohorov hemi-metric d_{LP} (imitated from the classical Levy-Prohorov metric in measure theory), defined by $d_{LP}(\nu, \nu') = \inf\{\epsilon > 0 \mid \forall U \in \mathcal{O}(X) \cdot \nu(U) \leq \nu'(U^{d, +(\epsilon)}) +$

$\epsilon\}$. That d_{LP} is indeed a hemi-metric depends on the fact that $(U^{d,+(\epsilon)})^{d,+(\epsilon')} \subseteq U^{d,+(\epsilon+\epsilon')}$ for every open U . We then show that whenever a is an upper bound of d , then $d_H(\nu, \nu') \leq (a+1)d_{LP}(\nu, \nu')$, which implies that the topology of d_{LP} is finer than that of d_H , and finally, that when d is totally bounded, the weak topology is even finer than d_{LP} . So all topologies coincide when d is bounded and totally bounded.

Note also that the map $\eta_V : X \rightarrow V_1(X)$ sending x to δ_x is (again) an isometric embedding, where $V_1(X)$ is equipped with d_H .

It is well-known that, on Polish spaces, the Hutchinson metric between two probability measures p, p' coincides with the so-called *trans-shipment distance*, defined as the least value of $\oint_{(x,y) \in X \times X} d(x,y) dp^2$, where p^2 ranges over all probability measures on $X \times X$ having p as first marginal and p' as second marginal. This is the so-called *Kantorovich-Rubinstein Theorem* (on compact metric spaces, this was first proved in [15]). When X is finite, p can be written $\sum_{x \in X} a_x \delta_x$, p' can be written $\sum_{y \in X} a'_y \delta_y$, and $p^2 = \sum_{x,y \in X} b_{xy} \delta_{(x,y)}$. Interpreting a_x (resp. a'_x) as some mass located at x , saying that p^2 should have p and p' as marginals means that we can distribute the mass a_x at x into small chunks b_{xy} ($a_x = \sum_{y \in X} b_{xy}$) that will each contribute to give total mass a'_y at y ($a'_y = \sum_{x \in X} b_{xy}$). Then $\oint_{(x,y) \in X \times X} d(x,y) dp^2$ is the total amount of work $\sum_{x,y \in X} b_{xy} d(x,y)$ needed to move these chunks, where moving b_{xy} units of mass from x to y costs $d(x,y)$. The finite case is an easy case of duality in linear programming (see e.g., van Breugel and Worrell [26]; the subject has a long history [22]).

We prove a similar theorem for continuous probability valuations, in a non- T_2 setting. The main difficulty is that the hemi-metric d is continuous, not from $X \times X$ to $\mathbb{R}^+$, but from $X^{(2)} = X \times X^{op}$ to $\mathbb{R}^+$, so that the integral $\oint_{(x,y) \in X \times X} d(x,y) dp^2$ is in fact *meaningless*. So p^2 should be a valuation on $X \times X^{op}$, but its second marginal would then be a valuation on X^{op} , while p' is on X .

We solve the difficulty in two steps. First, we replace p^2 by a linear prevision on $X^{(2)}$ —or rather something looking like it, but taking Lipschitz functions instead of continuous functions as arguments, which we call *LL-previsions* (for Linear, Lipschitz). Formally, an LL-prevision over X is a map k from the space $\langle X \rightarrow \mathbb{R} \rangle_L$ of bounded Lipschitz functions from X to $\mathbb{R}$ that is additive, positively homogeneous, monotonic, and *positive* (i.e., if $\varphi(x) \geq 0$ for every x , then $k(\varphi) \geq 0$). It is *normalized* iff $k(\chi_X) = 1$. Next, we shall observe that, in sym-compact spaces, any normalized LL-prevision on $X^{(2)}$ actually defines a continuous probability valuation p^2 on $X \times X^{op}$ with the desired properties. We shall require that the problematic second marginal of p equal the *convex-concave dual* $p'^\perp$ of p' , see below.

For the first step, let $f \ominus g$ abbreviate the function mapping $(x,y) \in X^{(2)}$ to $f(x) - g(y)$. If f and g are Lipschitz, then so is $f \ominus g$ (with the hemi-metric d^2 on $X^{(2)}$). Abusing the concept slightly, we shall say that the *first marginal* of an LL-prevision k on $X^{(2)}$ is the valuation p on X iff $k(f \ominus 0) = \oint_{x \in X} f(x) dp$ for every bounded 1-Lipschitz function f from X to $\mathbb{R}^+$. (Note that, if k were obtained by integrating along p^2 , and if we allowed for more general continuous functions f , this would define p as $\pi_1[p^2]$.) We say that the *second marginal* of k is the valuation p' on X iff $k(0 \ominus g) = - \oint_{y \in X} g(y) dp'$ for every bounded 1-Lipschitz function g from X to $\mathbb{R}^+$. The minus sign copes for

the problem with d mentioned above. Let $\mathbf{K}(p, p')$ be the set of all normalized LL-previsions on $X^{(2)}$ whose first marginal is p , and whose second marginal is p' :

Theorem 4. *Let X be equipped with a bounded hemi-metric d . For every normalized probability valuations p and p' on X , $d_H(p, p') = \min_{\mathbf{k} \in \mathbf{K}(p, p')} \mathbf{k}(d)$.*

Proof. (Sketch. See Appendix G.) It is easy to see that $d_H(p, p') \leq \mathbf{k}(d)$ for every $\mathbf{k} \in \mathbf{K}(p, p')$. Conversely, we must build some $\mathbf{k} \in \mathbf{K}(p, p')$ such that $\mathbf{k}(d) = d_H(p, p')$. We use an extension theorem on ordered cones akin to the Hahn-Banach Theorem on normed vector spaces, and derived from Roth's Sandwich Theorem [21, 25], to extend the monotonic linear functional $\mathbf{f} : Z \rightarrow \mathbb{R}^+$, defined by $\mathbf{f}(f \odot g) = \oint_{x \in X} f(x) dp - \oint_{y \in X} g(y) dp'$ on the convex subset $Z = \{f \odot g \mid f, g \in \langle X \rightarrow \mathbb{R}^+ \rangle_L, f \odot g \geq 0\}$ of the ordered cone $C = \langle X^{(2)} \rightarrow \mathbb{R}^+ \rangle$, to a monotonic linear functional $\mathbf{k}_0$ on the whole of C . Then we let $\mathbf{k}(\varphi) = \mathbf{k}_0(b + \varphi) - b$ for any $b \geq -\inf_{(x, y) \in X^{(2)}} \varphi(x, y)$. $\square$

For the second step, for any continuous probability valuation p' on a stably compact space X , define its *dual* $p'^\perp$ on X^d by $p'^\perp(X \setminus Q) = 1 - p'^\dagger(Q)$, for any co-compact $X \setminus Q$ of X , where $p'^\dagger(Q) = \inf_{U \in \mathcal{O}(X), U \supseteq Q} p'(U)$. Using a result by Tix [24, Satz 3.4], $p'^\perp$ is a continuous probability valuation on X^d . By [12, Section 5], in particular Definition 1 and the fact that $\alpha_C(p'^\perp) = \alpha_C(p')^\perp$, (see also [11, Section 6.2.2]), $\oint_{y \in X^{op}} g(y) dp'^\perp = \alpha_C(p')^\perp(g) = -\inf_{f \text{ step function}, f \geq -g} \alpha_C(p')(f) = -\inf_{f \text{ step function}, f \geq -g} \oint_{x \in X} f(x) dp'$, where step functions are those continuous maps that only take finitely many values. The formula simplifies in the case of Lipschitz maps and sym-compact spaces, observing that $-h$ is Lipschitz from X^{op} to $\mathbb{R}^+$ whenever h is Lipschitz from X to $\mathbb{R}^+$ (see Appendix H):

Lemma 2. *Let X be sym-compact, and p' a continuous probability valuation on X . For any bounded Lipschitz map h from X to $\mathbb{R}$, $-\oint_{x \in X} h(x) dp' = \oint_{x \in X^{op}} -h(x) dp'^\perp$.*

Note that this makes sense: the right-hand side, notably, integrates $-f$, which is Lipschitz from X^{op} to $\mathbb{R}$, while $p'^\perp$ is defined on X^d . But $X^{op} = X^d$ by Theorem 1.

Theorem 5. *Let X be a sym-compact space, with bounded hemi-metric d . For all continuous probability valuations p, p' on X , $d_H(p, p') = \min_{p^2} \oint_{(x, y) \in X^{(2)}} d(x, y) dp^2$, where p^2 ranges over the elements of $\mathbf{V}_1(X^{(2)})$ with $\pi_1[p^2] = p$, $\pi_2[p^2] = p'$.*

Proof. (Sketch. See Appendix I.) We find p^2 from the $\mathbf{k}$ gotten in Theorem 4. It would be tempting to define $p^2(W) = \mathbf{k}(\chi_W)$ for all opens W of $X^{(2)}$, however χ_W is continuous but not Lipschitz. Instead, we approximate it by the directed family of Lipschitz functions $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$, $\epsilon > 0$, i.e., we define $\nu^2(W)$ as $\sup_{\epsilon > 0} \mathbf{k}(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}})$. The key to the continuity of ν^2 is the fact that, not only we can write every open subset U as the directed union of all opens $V \subseteq U$, where $V \subseteq U$ means that there is a saturated compact Q such that $V \subseteq Q \subseteq U$ (as in every locally compact space [8]), but in fact that we can choose V of the special form $U^{d, -(\epsilon)}$, $\epsilon > 0$. (That $U^{d, -(\epsilon)} \subseteq U$ relies on Theorem 1: we let $Q = \{x \in X \mid d(x, X \setminus U) \geq \epsilon/2\}$, and realize this is open in $X^{op} = X^d$.) It is then enough to check that $\nu^2(\bigcup_{\epsilon > 0} U^{d, -(\epsilon)}) = \sup_{\epsilon > 0} \nu^2(U^{d, -(\epsilon)})$. The hardest part is to show that ν^2 is modular. Note that $\frac{1}{\epsilon} \chi_{(U \cap V)^{d, -(\epsilon)}} = \min(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}, \frac{1}{\epsilon} \chi_{V^{d, -(\epsilon)}})$, so

that $\nu(U \cap V) = \sup_{\epsilon > 0} \mathbf{k}(\min(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon))$. For unions, the best we can say in general is that $\frac{1}{\epsilon} \chi_{(U \cup V)^{d,-(\epsilon)}}^\epsilon \geq \max(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon)$. However, because X is sym-compact, we can show that for every $\epsilon > 0$, there is an $\epsilon' > 0$ such that for every $\epsilon'' > 0$, $\frac{1}{\epsilon''} \chi_{(U^{d,-(\epsilon)} \cup V^{d,-(\epsilon)})^{d,-(\epsilon'')}}^{\epsilon''} \leq \max(\frac{1}{\epsilon'} \chi_{U^{d,-(\epsilon')}}^{\epsilon'}, \frac{1}{\epsilon'} \chi_{V^{d,-(\epsilon')}}^{\epsilon'})$, from which we conclude that $\nu(U \cup V) = \sup_{\epsilon > 0} \mathbf{k}(\max(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon))$. That ν is modular then follows from the linearity of $\mathbf{f}$ and the fact that $\min(a, b) + \max(a, b) = a + b$. We use Lemma 2 to show that the second marginal of p^2 is $p'^\perp$. $\square$

To characterize d_H over spaces of continuous previsions (including other brands of continuous games), we need a form of the so-called *minimax theorem*, whose proof, inspired from Frenk and Kassay [7], can be found in Appendix J. Say that $f : X \times Y \rightarrow \mathbb{R}$ is *convex in X* (in the sense of Ky Fan) iff for every $\alpha \in (0, 1)$, for every x_1, x_2 in X , there is an $x_0 \in X$ such that for every $y \in Y$, $f(x_0, y) \leq \alpha f(x_1, y) + (1 - \alpha)f(x_2, y)$. Say that f is *concave in Y* iff for every $\alpha \in (0, 1)$, for every $y_1, y_2 \in Y$, there is a $y_0 \in Y$ such that for all $x \in X$, $f(x, y_0) \geq \alpha f(x, y_1) + (1 - \alpha)f(x, y_2)$.

Theorem 6 (Minimax). *Let X be a non-empty compact space, Y a set. Let f be any map from $X \times Y$ to $\mathbb{R}$, such that $\lambda x \in X \cdot f(x, y)$ is continuous for every $y \in Y$. If f is convex in X and concave in Y , then $\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y)$. Moreover, the inf on the right is attained.*

For disambiguation purposes, write d_H^Y the Hutchinson distance on Y , where Y is any space of continuous previsions. By [10], each $F \in \nabla \mathbf{P}_1(X)$ has a *heart* $CCoeur_1(F) = \{G \in \mathbf{P}_1^\Delta(X) | G \geq F\} \in \mathcal{Q}(\mathbf{P}_{1\,wk}^\Delta(X))$, where $\mathbf{P}_{1\,wk}^\Delta(X)$ is $\mathbf{P}_1^\Delta(X)$ with the weak topology; each $F \in \Delta \mathbf{P}_1(X)$ has a *skin* $CPeau_1(F) = \{G \in \mathbf{P}_1^\Delta(X) | G \leq F\} \in \mathcal{H}(\mathbf{P}_{1\,wk}^\Delta(X))$; and each $F = (F^-, F^+) \in \mathbf{F}_1(X)$ has a *body* $CCorps_1(F) = CCoeur_1(F^-) \cap CPeau_1(F^+) \in \mathcal{P}(\mathbf{P}_{1\,wk}^\Delta(X))$. Then:

Theorem 7. *Let X be a sym-compact hemi-metric space, with bounded hemi-metric d . Then $CCoeur_1$ is an isometric embedding of $\nabla \mathbf{P}_1(X)$ (with $d_H^{\nabla \mathbf{P}_1(X)}$) into $\mathcal{Q}(\mathbf{P}_{1\,wk}^\Delta(X))$ (with $(d_H^{\mathbf{P}_1^\Delta(X)})_\mathcal{Q}$); $CPeau_1$ is an isometric embedding of $\Delta \mathbf{P}_1(X)$ (with $d_H^{\Delta \mathbf{P}_1(X)}$) into $\mathcal{Q}(\mathbf{P}_{1\,wk}^\Delta(X))$ (with $(d_H^{\mathbf{P}_1^\Delta(X)})_\mathcal{H}$); and $CCorps_1$ is an isometric embedding of $\mathbf{F}_1(X)$ (with $d_H^{\mathbf{F}_1(X)}$) into $\mathcal{P}(\mathbf{P}_{1\,wk}^\Delta(X))$ (with $(d_H^{\mathbf{P}_1^\Delta(X)})_{\mathcal{P}\ell}$).*

Proof. By [10, Theorem 5], for every continuous normalized lower prevision F on X , $F(f) = \min_{G \in \mathbf{P}_1^\Delta(X), G \geq F} G(f)$. Let now $F, F' \in \mathbf{P}_1^\Delta(X)$. For short, we let f range implicitly over $\langle X \rightarrow \mathbb{R}^+ \rangle_1$, G and G' over $\mathbf{P}_1^\Delta(X)$. Note that $(d_H^{\mathbf{P}_1^\Delta(X)})_\mathcal{Q}(CCoeur_1(F), CCoeur_1(F')) = \max_{G' \geq F'} \min_{G \geq F} \sup_f \max(G(f) - G'(f), 0)$. If $F \leq F'$, then $d_H(F, F') = 0$ (using Theorem 3 and the easy fact that the specialization ordering of the weak topology is $\leq$), while every $G' \geq F'$ is also above F , so $(d_H^{\mathbf{P}_1^\Delta(X)})_\mathcal{Q}(CCoeur_1(F), CCoeur_1(F')) = 0$. Otherwise, $d_H^{\nabla \mathbf{P}_1(X)}(F, F') = \sup_f (F(f) - F'(f)) = \sup_f (\min_{G \geq F} G(f) - \min_{G' \geq F'} G'(f)) = \sup_{G' \geq F'} \sup_f \min_{G \geq F} (G(f) - G'(f)) = \sup_{G' \geq F'} \min_{G \geq F} \sup_f (G(f) - G'(f))$ (by Theorem 6) $= (d_H^{\mathbf{P}_1^\Delta(X)})_\mathcal{Q}(CCoeur_1(F),$

$CCoeur_1(F')$). To check that Theorem 6 applies, we first realize that $\mathbf{P}_1^\Delta(X)$ equipped with the Hutchinson hemi-metric has the weak topology (Theorem 3), which is (stably) compact because X is stably compact (see Proof of Proposition 4, Appendix, in [10, Long version]), and $\lambda G, f \cdot G(f) - G'(f)$ is continuous in G for each f (since the inverse image of $(t, +\infty)$ is the weak open $[f > G'(f) + t]$), convex in $\langle X \rightarrow \mathbb{R}^+ \rangle$ (i.e., in f ; this is because G and G' are in fact linear), and is concave in $\mathbf{P}_1^\Delta(X)$ (i.e., in G ; this is in fact again linear). The argument is similar for $d_H^\Delta(\mathbf{P}_1(X))$ and $(d_H^{\mathbf{P}_1^\Delta(X)})_{\mathcal{H}}$, where by [10, Concluding remarks], for any continuous normalized upper previsions F and F' , when $F \not\leq F'$, $d_H^{\Delta \mathbf{P}_1(X)}(F, F') = \sup_f (F(f) - F'(f)) = \sup_f (\sup_{G \leq F} G(f) - \sup_{G' \leq F'} G'(f)) = \sup_{G \leq F} \sup_f \inf_{G' \leq F'} (G(f) - G'(f))$, and $(d_H^{\mathbf{P}_1^\Delta(X)})_{\mathcal{H}}(CPeau_1(F), CPeau_1(F')) = \sup_{G \leq F} \inf_{G' \leq F'} \sup_f \max(G(f) - G'(f), 0)$ and we again conclude by Theorem 6. Finally, the $d_H^{\mathbf{F}_1(X)}$ and $(d_H^{\mathbf{P}_1^\Delta(X)})_{\mathcal{P}\ell}$ case is a consequence of the first two cases, and of Proposition 3. $\square$

7 Conclusion

We contend that our Hutchinson hemi-metric d_H is not only a natural way of building simulation hemi-metrics through a fixpoint construction à la Ferns *et al.* [6], allowing one to bound the error in payoffs when starting from different states; but also that d_H has an elegant mathematical theory: it generalizes all variants of the Hausdorff hemi-metric on spaces of demonic, angelic, and chaotic non-determinism, and admits a (directed) Kantorovich-Rubinstein theorem in the probability case. Theorem 7 shows that d_H is actually a composition of a Hausdorff-like hemi-metric with a Kantorovich-like hemi-metric $d_H^{\mathbf{P}_1^\Delta(X)}$, as in Ferns *et al.* [6]. Finally, Theorem 7 exhibits $d_H^{\nabla \mathbf{P}_1(X)}$ as the $\sup \inf \sup$ formula $d_H^{\nabla \mathbf{P}_1(X)}(F, F') = \sup_{G' \geq F'} \min_{G \geq F} \sup_f d_{\mathbb{R}}(G(f), G'(f'))$. This is akin to the *a posteriori* directed metric on the finite-state games of de Alfaro *et al.* [3, Equation (5)] in the turn-based case. On the other hand, $d_H^{\Delta \mathbf{P}_1(X)}$ is defined as the $\sup \sup \inf$ formula $\sup_f \sup_{G \leq F} \inf_{G' \leq F'} d_{\mathbb{R}}(G(f), G'(f'))$, which matches exactly the definition of the *a priori* directed metric of de Alfaro *et al.* [3, Equation (6)]. So the difference between the two seems to be one between demonic and angelic choice. Further work should identify whether the corresponding simulation hemi-metrics can be characterized by logics similar to that of op.cit., in the topological setting of PrTSes.

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A Fundamental Theorems on Hemi-Metric Spaces

We start by proving hemi-metric analogues of well-known results on metrics. The only difficulty is to be aware that hemi-metrics d are not symmetric.

Lemma 3. *Let d be a hemi-metric on X , and $B_{x_i, < \epsilon_i}^d$, $1 \leq i \leq n$, a finite family of open balls. For every $x \in \bigcap_{i=1}^n B_{x_i, < \epsilon_i}^d$, there is an $\epsilon > 0$ such that $B_{x, < \epsilon}^d$ is contained in $\bigcap_{i=1}^n B_{x_i, < \epsilon_i}^d$.*

Proof. Let $\epsilon = \min_{1 \leq i \leq n} (\epsilon_i - d(x_i, x))$. For every i , $d(x_i, x) \leq \epsilon_i - \epsilon$. For every $y \in B_{x, < \epsilon}^d$, $d(x, y) < \epsilon$, so by the triangular inequality $d(x_i, y) < (\epsilon_i - \epsilon) + \epsilon = \epsilon_i$. Hence $B_{x, < \epsilon}^d \subseteq B_{x_i, < \epsilon_i}^d$. $\square$

Corollary 1. *Let d be a hemi-metric on X . Every open U in $\mathcal{O}^d$ is a union of open balls $B_{x, < \epsilon_x}^d$, $x \in U$.*

Proof. U can be written as a union of finite intersections $\bigcup_{i \in I} \bigcap_{j=1}^{n_i} B_{x_{ij}, < \epsilon_{ij}}^d$. For every $x \in U$, x is in one of the intersections, say $\bigcap_{j=1}^{n_i} B_{x_{ij}, < \epsilon_{ij}}^d$. By Lemma 3, there is an $\epsilon_x > 0$ such that this intersection contains the open ball $B_{x, < \epsilon_x}^d$. So U contains the union of all $B_{x, < \epsilon_x}^d$, $x \in U$. The converse inclusion is obvious. $\square$

Corollary 2. *Let d be a hemi-metric on X . U is open in $\mathcal{O}^d$ iff for every $x \in U$, there is an $\epsilon > 0$ with $B_{x, < \epsilon}^d \subseteq U$.*

Proof. If for every $x \in U$, there is an $\epsilon_x > 0$ such that $B_{x, < \epsilon_x}^d \subseteq U$, then U is the union of all $B_{x, < \epsilon_x}^d$, and is therefore in $\mathcal{O}^d$. The converse direction is by Corollary 1. $\square$

Proposition 6. *Let d be a hemi-metric on X , and $\leq$ the specialization quasi-ordering of $\mathcal{O}^d$. Then $x \leq y$ iff $d(x, y) = 0$.*

Proof. First, $x \leq y$ iff for every $z \in X$, for every $\epsilon > 0$, $x \in B_{z, < \epsilon}^d$ implies $y \in B_{z, < \epsilon}^d$, iff for every $z \in X$, for every $\epsilon > 0$, $d(z, x) < \epsilon$ implies $d(z, y) < \epsilon$, iff for every $z \in X$, $d(z, y) \leq d(z, x)$. The latter implies $d(x, y) \leq d(x, x) = 0$, so $d(x, y) = 0$. Conversely, if $d(x, y) = 0$ then $d(z, y) \leq d(z, x) + d(x, y) = d(z, x)$. $\square$

Lemma 4. *Let d be a hemi-metric on X . If d is symmetric and T_0 , then $\mathcal{O}^d$ is T_2 .*

Proof. Assume d is symmetric and T_0 , and $x \neq y$. Let $\epsilon = d(x, y)/2$: ϵ is positive, otherwise $d(x, y) = 0$, hence also $d(y, x) = 0$ by symmetry, so $x = y$ since d is T_0 . Clearly, $x \in B_{x, < \epsilon}^d$ and $y \in B_{y, < \epsilon}^d$. Moreover $B_{x, < \epsilon}^d$ and $B_{y, < \epsilon}^d$ do not meet: if z is in the intersection then $d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + d(y, z) < 2\epsilon \leq d(x, y)$, a contradiction. $\square$

Lemma 5. *The topology $\mathcal{O}^{d_{\mathbb{R}}}$ of the hemi-metric $d_{\mathbb{R}}$ is the Scott topology on $\mathbb{R}$.*

Proof. Note that $B^{d_{\mathbb{R}}}(s, \epsilon) = (s - \epsilon, +\infty)$ is Scott-open. Conversely, every non-trivial Scott-open $(t, +\infty)$ is an open ball, say $B^{d_{\mathbb{R}}}(t + \epsilon, \epsilon)$ ($\epsilon > 0$). $\square$

Lemma 6. *Let f be a function from the hemi-metric space X to the hemi-metric space Y . Then f is continuous from X to Y , relatively to the topologies of their hemi-metrics d , resp. d' , iff for every $x \in X$:*

$$\text{for every } \epsilon > 0, \text{ there is an } \eta > 0 \text{ such that for all } x' \in X \text{ such that } d(x, x') < \eta, \\ d'(f(x), f(x')) < \epsilon.$$

Proof. The stated property is equivalent to: (*) for every $\epsilon > 0$, there is an $\eta > 0$ such that $B_{x, < \eta}^d \subseteq f^{-1}(B_{f(x), < \epsilon}^{d'})$. If f is continuous, then this is a consequence of Corollary 2. Conversely, assume (*) satisfied. Fix an open set V . For every $x \in f^{-1}(V)$, by Corollary 2 there is an $\epsilon > 0$ such that $B_{f(x), < \epsilon}^{d'} \subseteq V$. So for some $\eta > 0$, $B_{x, < \eta}^d$ is contained in $f^{-1}(B_{f(x), < \epsilon}^{d'}) \subseteq f^{-1}(V)$. This implies that $f^{-1}(V)$ is open, using Corollary 2 on $U = f^{-1}(V)$. $\square$

Lemma 7. *Call a function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ strict iff $\varphi(0) = 0$, and sub-additive iff $\varphi(s + t) \leq \varphi(s) + \varphi(t)$ for every $s, t \in \mathbb{R}^+$.*

Let d be a hemi-metric on X , φ a strict, monotonic, sub-additive function from $\mathbb{R}^+$ to $\mathbb{R}^+$, and assume that there is a closed interval $[0, a]$, $a > 0$, such that φ is a homeomorphism of $[0, a]$ onto $[0, \varphi(a)]$ (these intervals being equipped with $d_{\mathbb{R}}$). The map $\varphi \circ d$ is a hemi-metric on X , and is topologically equivalent to d .

Proof. Let $d' = \varphi \circ d$. For every $x \in X$, $d'(x, x) = \varphi(d(x, x)) = \varphi(0) = 0$ since φ is strict. For every x, y, z , $d'(x, y) = \varphi(d(x, y)) \leq \varphi(d(x, z) + d(z, y))$ (since d obeys the triangular inequality, and φ is monotonic) $\leq \varphi(d(x, z)) + \varphi(d(z, y))$ (since φ is sub-additive) $= d'(x, z) + d'(z, y)$. Write X' the space X , only with the hemi-metric d' .

The identity map from X to X' is continuous. Indeed, by Lemma 6, and since φ is continuous at 0, for every $\epsilon > 0$, there is an $\eta > 0$ such that for all $s \in [0, a]$, $s < \eta$ implies $\varphi(s) - \varphi(0) < \epsilon$, and therefore also $\varphi(s) < \epsilon$, since $\varphi(0) = 0$. In particular, for every $x \in X$, for every $\epsilon > 0$, there is an $\eta > 0$ such that for all $y \in X$, if $d(x, y) < \eta$ if $d'(x, y) < \epsilon$.

Conversely, using $\varphi^{-1} : [0, \varphi(a)] \rightarrow [0, a]$, the identity map from X' to X is also continue. So X and X' are homeomorphic. $\square$

Corollary 3. *Let d be a hemi-metric on X . For every $a > 0$, there is a hemi-metric d' on X which is topologically equivalent to d , and such that $d'(x, y) \leq a$ for all $x, y \in X$. In particular, d' is bounded.*

Proof. Let $\varphi(t) = \min(t, a)$, $\varphi(t) = at/(1 + t)$, or $\varphi(t) = 2a/\pi \arctan t$ for example. $\square$

Lemma 8. *For any hemi-metric d on X , the specialization quasi-ordering of d^{op} is the opposite $\geq$ of the quasi-ordering $\leq$ of d .*

Proof. By Proposition 6, the specialization quasi-ordering $\preceq$ of d^{op} is defined by $x \preceq y$ iff $d^{op}(x, y) = 0$, iff $d(y, x) = 0$, i.e., $y \leq x$. $\square$

Lemma 9. *The topology of X^{sym} is finer than that of X and X^{op} .*

Proof. Let us show that the topology of X^{sym} is finer than that of X , the case of X^{op} being symmetric. We only have to show that every open ball $B_{x, < \epsilon}^d$ is open in X^{sym} . Let $y \in B_{x, < \epsilon}^d$, and let $a = d(x, y) < \epsilon$. For every $z \in B_{y, < \epsilon - a}^{d^{sym}}$, $d(y, z) \leq d^{sym}(y, z) < \epsilon - a$, so $d(x, z) \leq d(x, y) + d(y, z) < \epsilon$. So $B_{y, < \epsilon - a}^{d^{sym}} \subseteq B_{x, < \epsilon}^d$. It follows that $B_{x, < \epsilon}^d$ is open in X^{sym} by Corollary 2 applied to d^{sym} . $\square$

B Proof of Lemma 1

Lemma 1. *Let X be equipped with a bounded hemi-metric d . The topology of $X^{(2)}$ is that of the topological product $X \times X^{op}$. Its specialization ordering is $\leq \times \geq$.*

Proof. The open ball $B^{d^2}((x, y), \epsilon)$ is the set of all pairs (x', y') such that $d(x, x') + d(y', y) < \epsilon$. This is the union, when ϵ_1 and ϵ_2 range over non-negative reals with $\epsilon_1 + \epsilon_2 = \epsilon$, of all sets $\{(x', y') \in X \times X \mid d(x, x') < \epsilon_1, d(y', y) < \epsilon_2\} = (B_{x, < \epsilon_1}^d \times X) \cap (X \times B_{y, < \epsilon_2}^{d^{op}})$. The latter are open in $X \times X^{op}$. So $B^{d^2}((x, y), \epsilon)$ is open in $X \times X^{op}$. Therefore the topology of the latter is finer than that of $X^{(2)}$.

Conversely, consider the first projection $\pi_1 : X^{(2)} \rightarrow X$, sending (x, y) to x . It is easy to see that π_1 is 1-Lipschitz: $d(x, x') \leq d^2((x, y), (x', y'))$. Similarly, the second projection $\pi_2 : X^{(2)} \rightarrow X^{op}$ is 1-Lipschitz. In particular, π_1 and π_2 are continuous so the identity function id , which is nothing that the function that maps (x, y) to $(\pi_1(x, y), \pi_2(x, y))$, is continuous from $X^{(2)}$ to $X \times X^{op}$. For every open U of $X \times X^{op}$, $\text{id}^{-1}(U)$ is then open in $X^{(2)}$. This implies that the topology of $X^{(2)}$ is finer than that of $X \times X^{op}$.

It immediately follows that the specialization quasi-ordering of d^2 is $\leq \times \geq$. $\square$

C More on Hemi-Metrics, Thinnings, Thickenings

Lemma 10. *Let X be equipped with a hemi-metric d , $A \subseteq X$. The map $x \mapsto d(x, A) \in \overline{\mathbb{R}}^+$ is 1-Lipschitz. $d(x, A) = 0$ iff $x \in cl(A)$.*

Proof. For every $y \in A$, $d(x, y) \leq d(x, x') + d(x', y)$. Take infs on each side, getting $d(x, A) \leq d(x, x') + d(x', A)$.

If $d(x, A) = 0$, let us show that $x \in cl(A)$. It is enough to show that every open U containing x meets A . For some $\epsilon > 0$, $B_{x, < \epsilon}^d \subseteq U$. But since $d(x, A) = 0$, for some $y \in A$, $d(x, y) < \epsilon$. Then $y \in B_{x, < \epsilon}^d$, so $y \in U$.

Conversely, if $x \in cl(A)$, then for each $\epsilon > 0$, the open ball $B_{x, < \epsilon}^d$ meets A , say at y . By definition, $d(x, y) < \epsilon$, so $d(x, A) < \epsilon$. As ϵ is arbitrary, $d(x, A) = 0$. $\square$

Lemma 11. *Let X be a space, with a hemi-metric d . For every open subset U of X , for every $\epsilon \geq 0$, $U^{d, -(\epsilon)}$ is an open subset of U , which coincides with U if $\epsilon = 0$.*

Proof. By Lemma 10, the map $x \mapsto d(x, X \setminus U) \in \overline{\mathbb{R}}^+$ is 1-Lipschitz, hence continuous. Then $U^{d, -(\epsilon)}$ is the inverse image of the open set $]\epsilon, +\infty]$ by this function, and is therefore open. For every $x \in U^{d, -(\epsilon)}$, $d(x, X \setminus U) > \epsilon$. If x did not belong to U , it would be in $X \setminus U$, so $d(x, X \setminus U)$ would be zero; so $x \in U$. It follows that

$U^{d,-(\epsilon)} \subseteq U$. If $\epsilon = 0$, $x \in U^{d,-(\epsilon)}$ iff $d(x, X \setminus U) \neq 0$, iff $x \notin X \setminus U$, by Lemma 10 and the fact that $X \setminus U$ is closed. So $x \in U^{d,(0)}$ iff $x \in U$. $\square$

Lemma 12. *Let X be equipped with a hemi-metric d , U an open subset of X . If $\epsilon \leq \epsilon'$, then $U^{d,-(\epsilon)} \supseteq U^{d,-(\epsilon')}$. For every family of non-negative reals $(\epsilon_i)_{i \in I}$, having 0 as inf, the family $U^{d,-(\epsilon_i)}$, $i \in I$, is directed, and its union is U .*

Proof. If $\epsilon \leq \epsilon'$, then for every $x \in U^{d,-(\epsilon')}$, $d(x, X \setminus U) > \epsilon'$, so $d(x, X \setminus U) > \epsilon$, so $x \in U^{d,-(\epsilon)}$. Since the family $(\epsilon_i)_{i \in I}$ is totally order, it is in particular filtered, therefore $U^{d,-(\epsilon_i)}$, $i \in I$, is directed. Its union is contained in U by Lemma 11. Conversely, if $x \in U$, for some $\epsilon > 0$, $B_{x,<\epsilon}^d \subseteq U$. So $d(x, X \setminus U) \geq \epsilon$, otherwise for some $y \notin U$, $d(x, y) < \epsilon$, so $y \in B_{x,<\epsilon}^d$. Since $\inf_{i \in I} \epsilon_i = 0$, for some $i \in I$, $\epsilon_i < \epsilon$. So $d(x, X \setminus U) > \epsilon_i$, i.e., $x \in U^{d,-(\epsilon_i)}$. $\square$

Lemma 13. *Let X be equipped with a hemi-metric d , U an open subset of X , and $\epsilon > 0$. Then:*

$$(U^{d,-(\epsilon)})^{d,+(\epsilon)} \subseteq U$$

Proof. For every $z \in (U^{d,-(\epsilon)})^{d,+(\epsilon)}$, there is a $y \in U^{d,-(\epsilon)}$ such that $d(y, z) < \epsilon$. If z did not belong to U , z would be in $X \setminus U$, so $d(y, X \setminus U) < \epsilon$. But this contradicts the fact that $y \in U^{d,-(\epsilon)}$. So $z \in U$. $\square$

D Proofs of Theorems on Hausdorff Hemi-Metrics on Powerdomains

Proposition 1. *Let X be equipped with the hemi-metric d . The Hausdorff-Smyth hemi-metric d_Q on $\mathcal{Q}(X)$ is defined by $d_Q(Q, Q') = \sup_{x' \in Q'} \inf_{x \in Q} d(x, x')$. The bounds are attained, i.e., $d_Q(Q, Q') = \max_{x' \in Q'} \min_{x \in Q} d(x, x')$. This defines a hemi-metric, whose topology is exactly the Vietoris topology of $\mathcal{Q}_V(X)$.*

Proof. Let us show that the bounds are attained. For every $x' \in Q'$, the map $d(_, x')$ sending x to $d(x, x')$ is 1-Lipschitz, using the triangular inequality, hence continuous. So the direct image K of Q by $d(_, x')$ is compact. Since every open is upward-closed, the upward-closure $\uparrow K$ is compact. But the compact saturated subsets of $\mathbb{R}^+$ are all of the form $[r, +\infty]$: so $\uparrow K$ has a least element r . Necessarily, r is also the least element of K . By definition, there is an element $x_0 \in Q$ such that $r = d(x_0, x')$, and $r = \inf_{x \in Q} d(x, x')$. So r is the minimum $\min_{x \in Q} d(x, x')$. The map that sends x' to $-\min_{x \in Q} d(x, x')$ is also 1-Lipschitz: for every x', x'' , and each $x \in Q$, the triangular inequality gives us $d(x, x'') \leq d(x, x') + d(x', x'')$, so $\min_{x \in Q} d(x, x'') \leq \min_{x \in Q} d(x, x') + d(x', x'')$, hence $-\min_{x \in Q} d(x, x'') - (-\min_{x \in Q} d(x, x')) \leq d(x', x'')$. By the same argument as above, the minimum value $\min_{x' \in Q'} (-\min_{x \in Q} d(x, x'))$ is attained: this is $-\max_{x' \in Q'} \min_{x \in Q} d(x, x') = -d_Q(Q, Q')$.

Let us show that d_Q is a hemi-metric. First, if $Q \supseteq Q'$, for every $x' \in Q'$ there is an $x \in Q$, namely $x = x'$, such that $d(x, x') = 0$. So $d_Q(Q, Q') = 0$. In particular, $d_Q(Q, Q) = 0$. Next, fix $Q, Q', Q'' \in \mathcal{Q}(X)$. For every $x \in Q$, $x' \in Q'$,

$x'' \in Q''$, we have $d(x, x'') \leq d(x, x') + d(x', x'')$. Take mins over $x \in Q$, getting $\min_{x \in Q} d(x, x'') \leq \min_{x \in Q} d(x, x') + d(x', x'')$. Since this holds for every $x' \in Q'$:

$$\begin{aligned} \min_{x \in Q} d(x, x'') &\leq \inf_{x' \in Q'} (\min_{x \in Q} d(x, x') + d(x', x'')) \\ &\leq \inf_{x' \in Q'} (\max_{x' \in Q'} \min_{x \in Q} d(x, x') + d(x', x'')) = \inf_{x' \in Q'} (d_Q(Q, Q') + d(x', x'')) \\ &= d_Q(Q, Q') + \min_{x' \in Q'} d(x', x'') \end{aligned}$$

Take maxima over $x'' \in Q''$: $d_Q(Q, Q'') \leq d_Q(Q, Q') + d_Q(Q', Q'')$.

Finally, let us show that the topology of d_Q is the Vietoris topology. First note that:

$$\begin{aligned} B_{Q, < \epsilon}^{d_Q} &= \{Q' \in \mathcal{Q}(X) \mid d_Q(Q, Q') < \epsilon\} = \{Q' \in \mathcal{Q}(X) \mid \max_{x' \in Q'} \min_{x \in Q} d(x, x') < \epsilon\} \\ &= \{Q' \in \mathcal{Q}(X) \mid \forall x' \in Q' \cdot \exists x \in Q \cdot d(x, x') < \epsilon\} = \square Q^{d, +(\epsilon)} \end{aligned}$$

Letting $\mathcal{O}$ the smallest topology containing the opens of the form $\square Q^{d, +(\epsilon)}$, the topology of d_Q is therefore just $\mathcal{O}$. Let $\mathcal{O}'$ be the topology of $\mathcal{Q}_V(X)$. $\mathcal{O}'$ is finer than $\mathcal{O}$.

Conversely, for every open U of X , let us show that $\square U$ is open in $\mathcal{O}$. Fix $Q \in \square U$, i.e., $Q \subseteq U$. U is the union of the directed family of all $U^{d, -(\epsilon)}$, $\epsilon > 0$. Since Q is compact and contained in U , there is an $\epsilon > 0$ such that $Q \subseteq U^{d, -(\epsilon)}$. Then $Q^{d, +(\epsilon)} \subseteq (U^{d, -(\epsilon)})^{d, +(\epsilon)} \subseteq U$. For every $Q' \in B_{Q, < \epsilon}^{d_Q} = \square Q^{d, +(\epsilon)}$, we then obtain $Q' \subseteq U$: so $B_{Q, < \epsilon}^{d_Q} \subseteq \square U$. It follows that $\square U$ is open in the topology $\mathcal{O}$ of d_Q . As U is arbitrary, $\mathcal{O}$ is finer than $\mathcal{O}'$. So $\mathcal{O} = \mathcal{O}'$. $\square$

Proposition 2. *Let X be equipped with the hemi-metric d . The Hausdorff-Hoare hemi-metric $d_{\mathcal{H}}$ on $\mathcal{H}(X)$ is defined by $d_{\mathcal{H}}(F, F') = \sup_{x \in F} \inf_{x' \in F'} d(x, x')$. This defines a hemi-metric, whose topology is finer than the Vietoris topology of $\mathcal{H}_V(X)$, and coincides with it as soon as d is totally bounded.*

Proof. Let us first show that if $F \subseteq F'$, then $d_{\mathcal{H}}(F, F') = 0$. For every $x \in F$, there is an $x' \in F'$ such that $d(x, x') = 0$, namely $x' = x$. So $\inf_{x' \in F'} d(x, x') = 0$. So $d_{\mathcal{H}}(F, F') = 0$.

In particular, $d_{\mathcal{H}}(F, F) = 0$. Let us show the triangular inequality. Fix $F, F', F'' \in \mathcal{H}(X)$, and let $a = d_{\mathcal{H}}(F, F')$, $a' = d_{\mathcal{H}}(F', F'')$. Let $\epsilon > 0$ be arbitrary. By definition of a , for every $x \in F$, there is an $x' \in F'$ such that $d(x, x') < a + \epsilon/2$. Similarly, for each of these $x' \in F'$, there is an $x'' \in F''$ such that $d(x', x'') < a' + \epsilon/2$. So, for each $x \in F$, there is an $x'' \in F''$ such that $d(x, x'') \leq d(x, x') + d(x', x'') < a + a' + \epsilon$. This entails $d_{\mathcal{H}}(F, F'') \leq a + a' + \epsilon$. As ϵ is arbitrary, $d_{\mathcal{H}}(F, F'') \leq a + a' = d_{\mathcal{H}}(F, F') + d_{\mathcal{H}}(F', F'')$.

Let us show that the topology of $d_{\mathcal{H}}$ is finer than that of $\mathcal{H}_V(X)$. Fix an open U of X , and $F \in \diamond U$. We show that there is a real number $\epsilon > 0$ such that $B_{F, < \epsilon}^{d_{\mathcal{H}}} \subseteq \diamond U$. Since $F \in \diamond U$, there is an x in $F \cap U$. Since U is open, for some $\epsilon > 0$, $B_{x, < \epsilon}^d \subseteq U$. For every $F' \in B_{F, < \epsilon}^{d_{\mathcal{H}}}$, since $d_{\mathcal{H}}(F, F') < \epsilon$, we obtain $\inf_{x' \in F'} d(x, x') \leq d_{\mathcal{H}}(F, F') < \epsilon$. So there is an $x' \in F'$ such that $d(x, x') < \epsilon$. This means that $x' \in B_{x, < \epsilon}^d$, so $x' \in U$ since $B_{x, < \epsilon}^d \subseteq U$. Since $x' \in F'$, F' meets U , i.e., $F' \in \diamond U$. As F' is arbitrary, $B_{F, < \epsilon}^{d_{\mathcal{H}}} \subseteq \diamond U$. Therefore $\diamond U$ is open in the topology of $d_{\mathcal{H}}$.

Let us show the converse, assuming d totally bounded. Fix $F' \in B_{F, < \epsilon}^{d_{\mathcal{H}}}$, and let us show that there is an open subset $\mathcal{U}$ of $\mathcal{H}_V(X)$ such that $F' \in \mathcal{U} \subseteq B_{F, < \epsilon}^{d_{\mathcal{H}}}$. It will follow that $B_{F, < \epsilon}^{d_{\mathcal{H}}}$ is open in $\mathcal{H}_V(X)$. So the topology of $\mathcal{H}_V(X)$ will be finer than that of $d_{\mathcal{H}}$.

Let $a = d_{\mathcal{H}}(F, F') < \epsilon$, and $\eta = (\epsilon - a)/4$. Since X is totally bounded, there is a finite set E such that every element of X is at d^{sym} -distance strictly less than η to an element of E . Let $\mathcal{U} = \bigcap_{x \in F^{d, +(\eta)} \cap E} \diamond B_{x, < a+2\eta}^d$. This is a finite intersection of opens of the form $\diamond U$, and is therefore open in $\mathcal{H}_V(X)$. This is where we need d to be totally bounded.

First show that $F' \in \mathcal{U}$. For every $x \in F^{d, +(\eta)}$, by definition there is a $z \in F$ such that $d(x, z) < \eta$. Since $a = d_{\mathcal{H}}(F, F')$, $\inf_{y \in F'} d(z, y) \leq a$, so there is a $y \in F'$ such that $d(z, y) < a + \eta$. It follows that $d(x, y) \leq d(x, z) + d(z, y) < a + 2\eta$, so $y \in B_{x, < a+2\eta}^d$. Since $y \in F'$, $F' \in \diamond B_{x, < a+2\eta}^d$. Since x is arbitrary, $F' \in \mathcal{U}$.

Then show that $\mathcal{U} \subseteq B_{F, < \epsilon}^{d_{\mathcal{H}}}$. Let $F'' \in \mathcal{U}$ be arbitrary. For every $x \in F$, there is an element y of E such that $d^{sym}(x, y) < \eta$. Since $x \in F$, it follows that $y \in F^{d, +(\eta)}$, so $y \in F^{d, +(\eta)} \cap E$. Now use the fact that $F'' \in \mathcal{U}$, in particular $F'' \in \diamond B_{y, < a+2\eta}^d$. So there is an element z of F'' such that $d(y, z) < a + 2\eta$. Then $d(x, z) \leq d(x, y) + d(y, z) < a + 3\eta = \epsilon - \eta$. In particular, $\inf_{z \in F''} d(x, z) \leq \epsilon - \eta$. Since $x \in F$ is arbitrary, $d_{\mathcal{H}}(F, F'') = \sup_{x \in F} \inf_{z \in F''} d(x, z) \leq \epsilon - \eta < \epsilon$. So $F'' \in B_{F, < \epsilon}^{d_{\mathcal{H}}}$. Since F'' is arbitrary, $\mathcal{U} \subseteq B_{F, < \epsilon}^{d_{\mathcal{H}}}$. $\square$

Lemma 14. *Let $X_1, \dots, X_n$ be n hemi-metric spaces, X_i coming with the hemi-metric d_i . Define $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max(d_1(x_1, y_1), \dots, d_n(x_n, y_n))$ (this being 0 if $n = 0$). The topology of d is exactly that of the topological product $X_1 \times \dots \times X_n$.*

Proof. The open ball $B_{(x_1, \dots, x_n), < \epsilon}^d$ equals the product $B_{x_1, < \epsilon}^{d_1} \times \dots \times B_{x_n, < \epsilon}^{d_n}$, and is therefore open in the topological product. Conversely, the product topology is the smallest containing products of open sets, say $U_1 \times \dots \times U_n$, where for some i , $U_i = B_{x_i, < \epsilon}^{d_i}$ and for all $j \neq i$, $U_j = X_j$. For every $(y_1, \dots, y_n) \in U_1 \times \dots \times U_n$, we have $d(x_i, y_i) < \epsilon$. Then the open ball $B_{(y_1, \dots, y_n), < \epsilon - d(x_i, y_i)}^d$ is contained in $U_1 \times \dots \times U_n$, so $U_1 \times \dots \times U_n$ is open in the topology of d . $\square$

Proposition 3. *Let X be equipped with the hemi-metric d . The Hausdorff hemi-metric $d_{\mathcal{H}}$ on $\mathcal{P}_\ell(X)$ is defined by $d_{\mathcal{H}}(L, L') = \max(\sup_{x' \in L'} \inf_{x \in L} d(x, x'), \sup_{x \in L} \inf_{x' \in L'} d(x, x'))$. Then $d_{\mathcal{H}}(L, L') = \max(d_Q(\uparrow L, \uparrow L'), d_{\mathcal{H}}(cl(L), cl(L')))$, $d_{\mathcal{H}}$ is a hemi-metric, its topology is finer than the Vietoris topology of $\mathcal{P}_V(X)$, and coincides with it as soon as d is totally bounded.*

Proof. By the triangular inequality, the map that sends x to $d(x, x')$, where x' is fixed, is 1-Lipschitz, hence continuous, hence monotonic for the respective specialization orderings. Similarly, the function mapping x' to $-d(x, x')$, where x is fixed, is 1-Lipschitz, hence continuous, hence monotonic.

For every x' , since the function mapping x to $d(x, x')$ is monotonic, we obtain $\inf_{x \in \uparrow L} d(x, x') = \inf_{x \in L} d(x, x')$. The function mapping x' to $\inf_{x \in \uparrow L} d(x, x')$ is antitonic on the other hand, so by a similar argument $\sup_{x' \in \uparrow L'} d(x, x')$

$\inf_{x \in \uparrow L} d(x, x') = \sup_{x' \in L'} \inf_{x \in \uparrow L} d(x, x') = \sup_{x' \in L'} \inf_{x \in L} d(x, x')$, i.e., $\sup_{x' \in L'} \inf_{x \in L} d(x, x') = d_Q(\uparrow L, \uparrow L')$.

For every x , let us show that $\inf_{x' \in L'} d(x, x') = \inf_{x' \in cl(L')} d(x, x')$. For every real a , $\sup_{x' \in cl(L')} -d(x, x') > a$ iff there is an $x' \in cl(L')$ such that $-d(x, x') > a$, iff $cl(L')$ intersects the inverse image U_a of $(a, +\infty]$ by the function mapping x' to $-d(x, x')$. Since this is continuous, U_a is open. So U_a meets $cl(L')$ iff U_a meets L' . By a similar argument, U_a meets L' iff $\sup_{x' \in L'} -d(x, x') > a$. We have just shown that, for every a , $\sup_{x' \in L'} -d(x, x') > a$ iff $\sup_{x' \in cl(L')} -d(x, x') > a$. So $\sup_{x' \in L'} -d(x, x') = \sup_{x' \in cl(L')} -d(x, x')$, whence $\inf_{x' \in L'} d(x, x') = \inf_{x' \in cl(L')} d(x, x')$. Let f be the function mapping x to $\inf_{x' \in L'} d(x, x')$; f is again 1-Lipschitz, hence continuous. For every real b , $\sup_{x \in cl(L)} f(x) > b$ iff $cl(L)$ meets the open $f^{-1}]b, +\infty]$, iff L meets $f^{-1}]b, +\infty]$, iff $\sup_{x \in L} f(x) > b$. So $\sup_{x \in cl(L)} f(x) = \sup_{x \in L} f(x)$, i.e., $d_{\mathcal{H}}(cl(L), cl(L')) = \sup_{x \in L} \inf_{x' \in L'} d(x, x')$.

From all this, we deduce $d_{\mathcal{P}\ell}(L, L') = \max(d_Q(\uparrow L, \uparrow L'), d_{\mathcal{H}}(cl(L), cl(L')))$. Since d_Q and $d_{\mathcal{H}}$ are hemi-metrics, so is $d_{\mathcal{P}\ell}$. Moreover, $d_{\mathcal{P}\ell}(L, L') = 0$ iff $d_Q(\uparrow L, \uparrow L') = 0$ and $d_{\mathcal{H}}(cl(L), cl(L')) = 0$, iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$, i.e., to $L \sqsubseteq_{\text{EM}} L'$.

Let us show that the topology of $d_{\mathcal{P}\ell}$ is finer than the topology of $\mathcal{P}\ell_V(X)$. Let i be the function mapping $L \in \mathcal{P}\ell_V(X)$ to $(\uparrow L, cl(L)) \in \mathcal{Q}_V(X) \times \mathcal{H}_V(X)$. Clearly, since $L = \uparrow L \cap cl(L)$, i is one-to-one from $\mathcal{P}\ell_V(X)$ to the subspace of $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$ consisting of pairs (Q, F) such that $Q \cap F \neq \emptyset$, $Q = \uparrow(Q \cap F)$, and $F = cl(Q \cap F)$. We note that i is an topological embedding of $\mathcal{P}\ell_V(X)$ into $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$, i.e., a homeomorphism onto its image: i is continuous, since the inverse image of the subbasic open set $\square U \times \diamond V$ of $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$ is the open set $\square U \cap \diamond V$ of $\mathcal{P}\ell_V(X)$; since the direct image of the subbasic open $\square U$ of $\mathcal{P}\ell_V(X)$ is the intersection of the image of i with the open subset $\square U \times \mathcal{H}_V(X)$ of $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$, and the direct image of the subbasic open set $\diamond V$ of $\mathcal{P}\ell_V(X)$ is the intersection of the image of i with the open subset $\mathcal{Q}_V(X) \times \diamond V$ of $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$, so i^{-1} is continuous from its image to $\mathcal{P}\ell_V(X)$.

Similarly, i is an isometric embedding of $\mathcal{P}\ell_V(X)$, with the hemi-metric $d_{\mathcal{P}\ell}$, into the product of $\mathcal{Q}(X)$ (with d_Q) and $\mathcal{H}(X)$ (with $d_{\mathcal{H}}$): this is the formula $d_{\mathcal{P}\ell}(L, L') = \max(d_Q(\uparrow L, \uparrow L'), d_{\mathcal{H}}(cl(L), cl(L')))$, using the definition of product hemi-metric from Lemma 14.

Any isometric embedding is clearly also a topological embedding, so, up to homeomorphism, $\mathcal{P}\ell_V(X)$ is a topological subspace of $\mathcal{Q}_V(X) \times \mathcal{H}_V(X)$, whether equipped with their Vietoris topologies, or with the topologies of their hemi-metrics. By Proposition 1 the topology of d_Q is that of $\mathcal{Q}_V(X)$, and by Proposition 2 the topology of $d_{\mathcal{H}}$ is finer than that of $\mathcal{H}_V(X)$, and coincides with it if X is totally bounded. So the same can be said for $d_{\mathcal{P}\ell}$ and $\mathcal{P}\ell_V(X)$. $\square$

E Proof of Proposition 5

Proposition 5. *For all $Q, Q' \in \mathcal{Q}(X)$, $d_{\mathcal{H}}(u_Q, u_{Q'}) = d_Q(Q, Q')$. For all $F, F' \in \mathcal{H}(X)$, $d_{\mathcal{H}}(e_F, e_{F'}) = d_{\mathcal{H}}(F, F')$. For every $L \in \mathcal{P}\ell(X)$, $F_L = (\alpha_{\mathcal{Q}}(u_{\uparrow L}), \alpha_{\mathcal{H}}(e_{cl(L)}))$ is a normalized fork, and for all $L, L' \in \mathcal{P}\ell(X)$, $d_{\mathcal{H}}(F_L, F_{L'}) = d_{\mathcal{P}\ell}(L, L')$.*

Proof. Recalling that $\oint_{x \in X} f(x) du_Q = \min_{x \in Q} f(x)$,

$$\begin{aligned}
d_H(u_Q, u_{Q'}) &= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} d_{\mathbb{R}}(\min_{x \in Q} f(x), \min_{x' \in Q'} f(x')) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\min_{x \in Q} f(x) + \max_{x' \in Q'} -f(x'), 0) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\max_{x' \in Q'} (\min_{x \in Q} f(x)) - f(x'), 0) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\max_{x' \in Q'} \min_{x \in Q} (f(x) - f(x')), 0) \\
&\leq \max_{x' \in Q'} \min_{x \in Q} d(x, x') \quad \text{since } f \text{ is 1-Lipschitz} \\
&= d_Q(Q, Q')
\end{aligned}$$

Conversely, there is an element $x_0 \in Q$ and an element $x'_0 \in Q'$ such that $d_Q(Q, Q') = \min_{x \in Q} d(x, x'_0) = d(x_0, x'_0)$. Let $f(x) = d(x, x'_0)$: f is 1-Lipschitz by the triangular inequality, $\min_{x \in Q} f(x) = d_Q(Q, Q')$, and $\min_{x' \in Q'} f(x') \leq f(x'_0) = 0$, so $d_H(u_Q, u_{Q'}) \geq d_{\mathbb{R}}(\min_{x \in Q} f(x), \min_{x' \in Q'} f(x')) = d_Q(Q, Q')$.

For angelic non-determinism, using the fact that $\oint_{x \in X} f(x) d\epsilon_F = \sup_{x \in X} f(x)$,

$$\begin{aligned}
d_H(\epsilon_F, \epsilon_{F'}) &= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} d_{\mathbb{R}}(\sup_{x \in F} f(x), \sup_{x' \in F'} f(x')) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\sup_{x \in F} f(x) + \inf_{x' \in F'} -f(x'), 0) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\sup_{x \in F} (f(x) + \inf_{x' \in F'} -f(x')), 0) \\
&= \sup_{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1} \max(\sup_{x \in F} \inf_{x' \in F'} (f(x) - f(x')), 0) \\
&\leq \sup_{x \in F} \inf_{x' \in F'} d(x, x') \quad \text{since } f \text{ is 1-Lipschitz} \\
&= d_{\mathcal{H}}(F, F')
\end{aligned}$$

Conversely, let $f(x) = d(x, F')$: by Lemma 10, f is 1-Lipschitz. Moreover, $\sup_{x \in F} f(x) = d_{\mathcal{H}}(F, F')$, $\sup_{x' \in F'} f(x') = 0$, so $d_H(\epsilon_F, \epsilon_{F'}) \geq d_{\mathbb{R}}(\sup_{x \in F} f(x), \sup_{x' \in F'} f(x')) = d_{\mathcal{H}}(F, F')$.

For any lens L , write F_L as (F_L^-, F_L^+) . Clearly, $F_L^-(h) = \min_{x \in Q} h(x)$ and $F_L^+(h') = \sup_{y \in F} h'(y)$, where $Q = \uparrow L$ and $F = cl(L)$. Recall that, then, $L = Q \cap F$. We must check Walley's condition. To this end, we realize that $F_L^-(h) = \min_{x \in L} h(x)$: certainly, $F_L^-(h) \leq \min_{x \in L} h(x)$, since $L \subseteq Q$, and conversely, every element of Q is above some element of L , and h is monotonic (since continuous), so $\min_{x \in L} h(x) \leq F_L^-(h)$. We also realize that $F_L^+(h') = \sup_{y \in L} h'(y)$. Indeed, the inequality $\geq$ follows from $L \subseteq F$. Conversely, letting $a = \sup_{y \in F} h'(y)$, for any $\epsilon > 0$, there is a $y \in F$ such that $h'(y) > a - \epsilon$, so $F = cl(L)$ meets the open $h'^{-1}(a - \epsilon, +\infty)$. It follows that L also meets $h'^{-1}(a - \epsilon, +\infty)$, i.e., $\sup_{y \in L} h'(y) > a - \epsilon$. We conclude since ϵ is arbitrary. Now $F_L^-(h + h') = \min_{x \in L} (h(x) + h'(x)) \leq \min_{x \in L} (h(x) + \sup_{y \in L} h'(y)) = \min_{x \in L} h(x) + \sup_{y \in L} h'(y) = F_L^-(h) + F_L^+(h')$. The inequality $F_L^-(h) + F_L^+(h') \leq F_L^+(h + h')$ is proved similarly.

Finally, letting $Q = \uparrow L$, $Q' = \uparrow L'$, $F = cl(L)$, $F' = cl(L')$, $d_H(F_L, F_{L'}) = \max(d_H(F_L^-, F_{L'}^-), d_H(F_L^+, F_{L'}^+)) = \max(d_H(u_Q, u_{Q'}), d_H(e_F, e_{F'})) = \max(d_Q(Q, Q'), d_{\mathcal{H}}(F, F')) = d_{\mathcal{P}\ell}(L, L')$, using the definition first, the first two parts of the Proposition next, and Proposition 3 to conclude. $\square$

F Proof of Theorem 3

Lemma 15. *The function χ_U^ϵ is bounded and 1-Lipschitz. The function $\frac{1}{\epsilon}\chi_U^\epsilon$ takes its values in $[0, 1]$, is 0 on $X \setminus U$, 1 on $U^{d, -(\epsilon)}$, is monotonic in U and antitonic in $d\epsilon$. The family $(\frac{1}{\epsilon}\chi_U^\epsilon)_{\epsilon > 0}$ is directed, and has χ_U as sup. More precisely:*

$$\chi_{U^{d, -(\epsilon)}} \leq \frac{1}{\epsilon}\chi_U^\epsilon \leq \chi_U$$

Proof. First, χ_U^ϵ is clearly bounded. By Lemma 10, $\chi_U^\epsilon(x) \leq \min(\epsilon, d(x, x') + d(x', X \setminus U)) \leq \min(d(x, x') + \epsilon, d(x, x') + d(x', X \setminus U)) = d(x, x') + \chi_U^\epsilon(x')$, so χ_U^ϵ is 1-Lipschitz. It is clear that $\frac{1}{\epsilon}\chi_U^\epsilon(x)$ equals 0 for each $x \in X \setminus U$, and that $\frac{1}{\epsilon}\chi_U^\epsilon(x)$ is between 0 and 1. It follows that $\frac{1}{\epsilon}\chi_U^\epsilon \leq \chi_U$. Note also that $\frac{1}{\epsilon}\chi_U^\epsilon(x)$ equals 1 whenever $d(x, X \setminus U) \geq \epsilon$, in particular when $x \in U^{d, -(\epsilon)}$. So $\chi_{U^{d, -(\epsilon)}} \leq \frac{1}{\epsilon}\chi_U^\epsilon$.

To show that χ_U^ϵ is monotonic in U , we show that $d(x, X \setminus U)$ is monotonic in U , i.e., that $d(x, F)$ is antitonic in the closed subset F : if $F \supseteq F'$, then $d(x, F) = \inf_{y \in F} d(x, y) \leq \inf_{y \in F'} d(x, y) = d(x, F')$.

To show that χ_U^ϵ is antitonic in ϵ , assume $\epsilon \leq \epsilon'$, then $\frac{1}{\epsilon}\chi_U^\epsilon(x) = \min(1, \frac{1}{\epsilon}d(x, X \setminus U)) \leq \min(1, \frac{1}{\epsilon'}d(x, X \setminus U)) = \frac{1}{\epsilon'}\chi_U^{\epsilon'}(x)$. In particular, the family $(\frac{1}{\epsilon}\chi_U^\epsilon)_{\epsilon > 0}$ is directed. For every $x \in U$, $x \in U^{d, -(\epsilon)}$ for $\epsilon > 0$ small enough. Then $\frac{1}{\epsilon}\chi_U^\epsilon(x) = 1$. So $\sup_{\epsilon > 0} \frac{1}{\epsilon}\chi_U^\epsilon(x) = 1$ for every $x \in U$. Since on the other hand $\frac{1}{\epsilon}\chi_U^\epsilon(x) = 0$ for every $x \notin U$, it follows that $\sup_{\epsilon > 0} \frac{1}{\epsilon}\chi_U^\epsilon = \chi_U$. $\square$

Lemma 16. *On any space Y of continuous games, d_H is a hemi-metric, and has the pointwise ordering $\leq$ as specialization ordering.*

Proof. First, if $\nu \leq \nu'$, then $\oint_{x \in X} f(x) d\nu - \oint_{x \in X} f(x) d\nu' \leq 0$ for every 1-Lipschitz map f . So $d_H(\nu, \nu') = 0$. In particular, $d_H(\nu, \nu) = 0$ for every game ν . The triangular inequality is obvious.

Finally, we show that if $d_H(\nu, \nu') = 0$ then $\nu \leq \nu'$. Fix an open subset U of X , and let us show that $\nu(U) \leq \nu'(U)$. Since $d_H(\nu, \nu') = 0$, $\oint_{x \in X} f(x) d\nu \leq \oint_{x \in X} f(x) d\nu'$ for every bounded 1-Lipschitz map f . Let $f = \chi_U^\epsilon$, with $\epsilon > 0$ arbitrary. Multiplying by $\frac{1}{\epsilon}$, and since Choquet integration is positively homogeneous, $\oint_{x \in X} \frac{1}{\epsilon}\chi_U^\epsilon(x) d\nu \leq \oint_{x \in X} \frac{1}{\epsilon}\chi_U^\epsilon(x) d\nu'$. The sup over $\epsilon > 0$ of the directed family of all $\frac{1}{\epsilon}\chi_U^\epsilon$ is χ_U . When ν is a continuous game, Choquet integration is continuous in the function argument, so the integral of χ_U along ν is at most that of χ_U along ν' . That is, $\alpha_e(\nu)(\chi_U) \leq \alpha_e(\nu')(\chi_U)$, i.e., $\nu(U) \leq \nu'(U)$. $\square$

Lemma 17. *Let X be equipped with a hemi-metric d , Y a space of continuous games on X . The topology of d_H is finer than the weak topology on Y .*

Proof. It is enough to show that, for every open subset U of X , for every $r \in \mathbb{R}$, for every $\nu \in [U > r]$, there is an $\epsilon > 0$ such that $B_{\nu, < \epsilon}^{d_H} \subseteq [U > r]$. Let $a = \nu(U)$. By Lemma 15, and since ν is continuous, the sup of all $\oint_{x \in X} \frac{1}{\eta} \chi_U^\eta d\nu$, $\eta > 0$, is $\oint_{x \in X} \chi_U d\nu = a$. Since $a > r$, we also get $a > (a + r)/2$. For η small enough, then, $\oint_{x \in X} \frac{1}{\eta} \chi_U^\eta(x) d\nu \geq (a + r)/2$. Fix $\epsilon > 0$ arbitrary with $\epsilon < \eta(a - r)/2$. For every $\nu' \in B_{\nu, < \epsilon}^{d_H}$, for every bounded 1-Lipschitz map f ,

$$\oint_{x \in X} f(x) d\nu \leq \oint_{x \in X} f(x) d\nu' + \epsilon \quad (2)$$

Take $f = \chi_U^\eta$, therefore:

$$\begin{aligned} \nu'(U) &= \oint_{x \in X} \chi_U(x) d\nu' \\ &\geq \oint_{x \in X} \frac{1}{\eta} \chi_U^\eta(x) d\nu' \quad \text{by Lemma 15} \\ &\geq \oint_{x \in X} \frac{1}{\eta} \chi_U^\eta(x) d\nu - \frac{\epsilon}{\eta} \quad \text{by (2) and positive homogeneity} \\ &\geq \frac{a + r}{2} - \frac{\epsilon}{\eta} \\ &> r \end{aligned}$$

since $\epsilon < \eta(a - r)/2$. So $\nu' \in [U > r]$. $\square$

Lemma 18. *Let X be equipped with a hemi-metric d , U an open subset of X . For every $\epsilon, \epsilon' \in \mathbb{R}^+$, $(U^{d, +(\epsilon)})^{d, +(\epsilon')} \subseteq U^{d, +(\epsilon + \epsilon')}$.*

Proof. For every $z \in (U^{d, +(\epsilon)})^{d, +(\epsilon')}$, by definition there is a $y \in U^{d, +(\epsilon)}$ such that $d(y, z) < \epsilon'$. By definition of $U^{d, +(\epsilon)}$, there is an $x \in U$ such that $d(x, y) < \epsilon$. So $d(x, z) \leq d(x, y) + d(y, z) < \epsilon + \epsilon'$, i.e., $z \in B_{x, < \epsilon + \epsilon'}^d$. Since $x \in U$, $z \in U^{d, +(\epsilon + \epsilon')}$. $\square$

Proposition 7. *d_{LP} is a hemi-metric. If $\nu \leq \nu'$ then $d_{LP}(\nu, \nu') = 0$.*

Proof. If $\nu \leq \nu'$, then for every $\epsilon > 0$, $\nu(U) \leq \nu'(U) \leq \nu'(U^{d, +(\epsilon)}) + \epsilon$ for every open subset U , since $U \subseteq U^{d, +(\epsilon)}$. So $d_{LP}(\nu, \nu') \leq \epsilon$. Since ϵ is arbitrary, $d_{LP}(\nu, \nu') = 0$. In particular, $d_{LP}(\nu, \nu) = 0$.

Fix three games ν , ν' , and ν'' . Let $a = d_{LP}(\nu, \nu'')$, $a' = d_{LP}(\nu'', \nu')$. For every $\epsilon > 0$, there is an $\eta \in \mathbb{R}^+$ such that $a \leq \eta \leq a + \epsilon/2$, and $\nu(U) \leq \nu''(U^{d, +(\eta)}) + \eta$ for every open subset U ; there is also an $\eta' \in \mathbb{R}^+$, such that $a' \leq \eta' \leq a' + \epsilon/2$, and $\nu''(V) \leq \nu'(V^{d, +(\eta')}) + \eta'$ for every open subset V . Letting $V = U^{d, +(\eta)}$, we obtain $\nu(U) \leq \nu'((U^{d, +(\eta)})^{d, +(\eta')}) + \eta + \eta' \leq \nu'(U^{d, +(\eta + \eta')}) + \eta + \eta'$ by Lemma 18. Since U is arbitrary, $d_{LP}(\nu, \nu') \leq \eta + \eta' \leq a + a' + \epsilon$. Since ϵ is arbitrary, $d_{LP}(\nu, \nu') \leq a + a' = d_{LP}(\nu, \nu'') + d_{LP}(\nu'', \nu')$. So d_{LP} is a hemi-metric. $\square$

Before we can show that the topology of d_{LP} is finer than that of d_H , we need some technical adjustments. First, while Choquet integration is in general not additive, additivity works provided one of the integrated functions is a constant. (A more general version of this states that it is enough that the two functions are comonotonic, but this is out of scope here.)

Lemma 19. *For any game ν on X , for every $a \in \mathbb{R}^+$, for every $f \in \langle X \rightarrow \mathbb{R}^+ \rangle$, $\oint_{x \in X} (f(x) + a) d\nu = a\nu(X) + \oint_{x \in X} f(x) d\nu$.*

Proof.

$$\begin{aligned}
\oint_{x \in X} (f(x) + a) d\nu &= \int_0^{+\infty} \nu\{x \in X \mid f(x) + a > t\} dt \\
&= \int_0^a \nu\{x \in X \mid f(x) + a > t\} dt + \int_a^{+\infty} \nu\{x \in X \mid f(x) + a > t\} dt \\
&= \int_0^a \nu(X) dt + \int_a^{+\infty} \nu\{x \in X \mid f(x) + a > t\} dt \\
&= \int_0^a \nu(X) dt + \int_0^{+\infty} \nu\{x \in X \mid f(x) > t'\} dt' \\
&= a\nu(X) + \oint_{x \in X} f(x) d\nu
\end{aligned}$$

using the change of variables $t' = t - a$. □

Lemma 20. *Let X be equipped with a bounded hemi-metric d : $d(x, y) \leq a$ for all $x, y \in X$. For every 1-Lipschitz map f from X to $\mathbb{R}^+$, there is a number $b \in \mathbb{R}^+$ such f takes its values in $[b, b + a]$.*

Proof. Let $b = \inf_{x \in X} f(x)$. If the result was false, there would be an $y \in X$ with $f(y) > b + a$. Let c be such that $f(y) > c > b + a$. Fix $\epsilon > 0$ such that $\epsilon < c - b - a$. Since $b = \inf_{x \in X} f(x)$, there is an $x \in X$ such that $f(x) < b + \epsilon$. So $f(y) - f(x) > c - b - \epsilon > a$, since $\epsilon < c - b - a$. But $a \geq d(y, x)$, contradicting the fact that f is 1-Lipschitz. □

We can then require that f be upper bounded by a in the definition of d_H .

Corollary 4. *Let X be equipped with a bounded hemi-metric d : $d(x, y) \leq a$ for all $x, y \in X$. For every normalized games ν, ν' on X ,*

$$d_H(\nu, \nu') = \sup_{\substack{f \in \langle X \rightarrow \mathbb{R}^+ \rangle_1 \\ \sup_{x \in X} f(x) \leq a}} d_{\mathbb{R}} \left(\oint_{x \in X} f(x) d\nu, \oint_{x \in X} f(x) d\nu' \right)$$

Proof. The right-hand side is clearly less than or equal to $d_H(\nu, \nu')$. Conversely, for every (bounded) 1-Lipschitz map f , build $[b, b + a]$ using Lemma 20. Then $f - b$ is

bounded, 1-Lipschitz, upper bounded by a , and:

$$\begin{aligned}
& d_{\mathbb{R}} \left(\oint_{x \in X} (f(x) - b) d\nu, \oint_{x \in X} (f(x) - b) d\nu' \right) \\
&= \max \left(\oint_{x \in X} (f(x) - b) d\nu - \oint_{x \in X} (f(x) - b) d\nu', 0 \right) \\
&= \max \left(\oint_{x \in X} f(x) d\nu - b\nu(X) - \oint_{x \in X} f(x) d\nu' + b\nu'(X), 0 \right) \quad \text{by Lemma 19} \\
&= d_{\mathbb{R}} \left(\oint_{x \in X} f(x) d\nu, \oint_{x \in X} f(x) d\nu' \right)
\end{aligned}$$

since $\nu(X) = \nu'(X) = 1$. Now take sups over all f . $\square$

Lemma 21. *Let X be equipped with a hemi-metric d . For every 1-Lipschitz map f from X to $\mathbb{R}$, for every $t \in \mathbb{R}$, for every $\epsilon \in \mathbb{R}^+$, $(f^{-1}(t + \epsilon, +\infty))^{d, +(\epsilon)} \subseteq f^{-1}(t, +\infty)$.*

Proof. For every $x \in (f^{-1}(t + \epsilon, +\infty))^{d, +(\epsilon)}$, there is a $y \in f^{-1}(t + \epsilon, +\infty)$ with $d(y, x) < \epsilon$. If we had $f(x) \leq t$, since f is 1-Lipschitz, we would have $f(y) \leq f(x) + d(y, x) < t + \epsilon$, contradicting $y \in f^{-1}(t + \epsilon, +\infty)$. So $f(x) > t$, i.e., $x \in f^{-1}(t, +\infty)$. $\square$

Proposition 8. *Let X be equipped with a bounded hemi-metric d , i.e., assume that $d(x, y) \leq a$ for all $x, y \in X$. For all normalized games ν, ν' on X , $d_H(\nu, \nu') \leq (a + 1)d_{LP}(\nu, \nu')$.*

Given any space Y of continuous games on X , the topology of d_{LP} on Y is finer than that of d_H .

Proof. Let $\epsilon \in \mathbb{R}^+$ be such that $\nu(U) \leq \nu'(U^{d, +(\epsilon)}) + \epsilon$ for every open U . Fix a bounded 1-Lipschitz map f from X to $\mathbb{R}^+$, upper bounded by a . (This is legitimate by Corollary 4.) If $a \geq \epsilon$, then:

$$\begin{aligned}
\oint_{x \in X} f(x) d\nu &= \int_0^a \nu(f^{-1}(t, +\infty)) dt \\
&\leq \int_0^a \left[\nu'(f^{-1}(t, +\infty))^{d, +(\epsilon)} + \epsilon \right] dt \quad \text{taking } U = f^{-1}(t, +\infty) \\
&\leq \int_0^\epsilon \nu'(f^{-1}(t, +\infty))^{d, +(\epsilon)} dt + \int_\epsilon^a \nu'(f^{-1}(t, +\infty))^{d, +(\epsilon)} dt + a\epsilon \\
&\leq \epsilon\nu'(X) + a\epsilon + \int_0^{a-\epsilon} \nu'(f^{-1}(t + \epsilon, +\infty))^{d, +(\epsilon)} dt \\
&\leq \epsilon\nu'(X) + a\epsilon + \int_0^{a-\epsilon} \nu'(f^{-1}(t, +\infty)) dt \quad \text{by Lemma 21} \\
&(\leq \epsilon\nu'(X) + a\epsilon + \oint_{x \in X} f(x) d\nu')
\end{aligned}$$

If $a < \epsilon$, a similar argument shows that the same inequality holds (in fact, we get $\oint_{x \in X} f(x) d\nu \leq \epsilon\nu'(X) + a\epsilon$). By Corollary 4, and since $\nu'(X) = 1$, we obtain $d_H(\nu, \nu') \leq (a + 1)\epsilon$. So $d_H(\nu, \nu') \leq (a + 1)d_{LP}(\nu, \nu')$.

The identity map from Y with the hemi-metric d_{LP} , to Y with the hemi-metric d_H , is then $(a + 1)$ -Lipschitz, hence continuous. It follows that every d_H -open is d_{LP} open. $\square$

Next we show that the weak topology is even finer than that of d_{LP} , when d is totally bounded. This depends on the following bizarre characterization of total boundedness.

Lemma 22. *Let X be equipped with a hemi-metric d . Then d is totally bounded iff for every $\epsilon > 0$, there is a finite subset E of X such that, for every open subset U of X , $U \subseteq (U^{d,+(\epsilon)} \cap E)^{d,+(\epsilon)}$.*

Proof. Assume d totally bounded, and let $E = \{x_1, \dots, x_n\}$ be such that every element of X is at d^{sym} -distance strictly less than ϵ of some element of E . For every open subset U of X , let $x \in U$. For some i , $d^{sym}(x_i, x) < \epsilon$, so $d(x_i, x) < \epsilon$ and $d(x, x_i) < \epsilon$. Since $d(x, x_i) < \epsilon$, x_i is in $U^{d,+(\epsilon)}$, so in $U^{d,+(\epsilon)} \cap E$. Since $d(x_i, x) < \epsilon$, it follows that $x \in (U^{d,+(\epsilon)} \cap E)^{d,+(\epsilon)}$. As x is arbitrary, $U \subseteq (U^{d,+(\epsilon)} \cap E)^{d,+(\epsilon)}$.

Conversely, assume there is a finite subset $E = \{x_1, \dots, x_n\}$ of X such that $U \subseteq (U^{d,+(\epsilon/2)} \cap E)^{d,+(\epsilon/2)}$ for every open subset U of X . For every $x \in X$, consider the open $U = B_{x, < \epsilon/2}^d$. There is an x_i in $U^{d,+(\epsilon/2)} \cap E$ such that $d(x_i, x) < \epsilon/2 < \epsilon$. On the other hand, since $x_i \in U^{d,+(\epsilon/2)}$, there is a $y \in U$ such that $d(y, x_i) < \epsilon/2$, so $d(x, x_i) \leq d(x, y) + d(y, x_i) < \epsilon$. As ϵ is arbitrary, X is totally bounded. $\square$

Proposition 9. *Let X be equipped with a totally bounded hemi-metric d , Y any space of continuous games on X . Then the weak topology on Y is finer than that of d_{LP} .*

Proof. Fix $\nu \in Y$, $\epsilon > 0$, and let ν' be an arbitrary element of $B_{\nu, < \epsilon}^{d_{LP}}$. By definition there is an $\eta > 0$ such that $\eta < \epsilon$ and $\nu(U) \leq \nu'(U^{d,+(\eta)}) + \eta$ for every open subset U of X . Let $\epsilon' = \epsilon - \eta$. Since X is totally bounded, using Lemma 22, there is a finite subset E of X such that: (*) $U \subseteq (U^{d,+(\epsilon'/3)} \cap E)^{d,+(\epsilon'/3)}$ for every open subset U of X . For each (finite) subset A of E , let us define:

$$r_A = \sup_{\substack{U \text{ open in } X \\ U^{d,+(\eta+\epsilon'/3)} \cap E \subseteq A}} \nu(U) - \eta - \epsilon'/3$$

$$\mathcal{U} = \bigcap_{A \subseteq E} [A^{d,+(\epsilon'/3)} > r_A]$$

Note that $A^{d,+(\epsilon'/3)}$ is open in X , so $[A^{d,+(\epsilon'/3)} > r_A]$ is open in the weak topology. Hence $\mathcal{U}$ is also open in the weak topology, as a finite intersection of opens. (This is where we need d to be totally bounded.)

We first show that $\mathcal{U} \subseteq B_{\nu, < \epsilon}^{d_{LP}}$. For every game $\nu'' \in \mathcal{U}$, for every open subset U of X , using $A = U^{d,+(\eta+\epsilon'/3)} \cap E$, we obtain $r_A \geq \nu(U) - \eta - \epsilon'/3$ by definition of r_A ,

so:

$$\begin{aligned}
\nu(U) &\leq r_A + \eta + \epsilon'/3 \\
&< \nu''(A^{d,+(\epsilon'/3)}) + \eta + \epsilon'/3 \quad \text{since } \nu'' \in \mathcal{U} \subseteq [A^{d,+(\eta+\epsilon'/3)} > r_A] \\
&\leq \nu''((U^{d,+(\eta+\epsilon'/3)})^{d,+(\epsilon'/3)}) + \eta + \epsilon'/3 \quad \text{since } A \subseteq U^{d,+(\epsilon'/3)} \\
&\leq \nu''(U^{d,+(\eta+2\epsilon'/3)}) + \eta + \epsilon'/3 \quad \text{by Lemma 18} \\
&\leq \nu''(U^{d,+(\eta+2\epsilon'/3)}) + \eta + 2\epsilon'/3
\end{aligned}$$

So $d_{\text{LP}}(\nu, \nu'') \leq \eta + 2\epsilon'/3 < \epsilon$, i.e., $\nu'' \in B_{\nu, < \epsilon}^{d_{\text{LP}}}$.

Next, we show that $\nu' \in \mathcal{U}$. Fix an arbitrary subset A of E , and an arbitrary open subset U of X such that $U^{d,+(\eta+\epsilon'/3)} \cap E \subseteq A$. Then:

$$\begin{aligned}
\nu'(A^{d,+(\epsilon'/3)}) &\geq \nu'((U^{d,+(\eta+\epsilon'/3)} \cap E)^{d,+(\epsilon'/3)}) \\
&\geq \nu'(((U^{d,+(\eta)})^{d,+(\epsilon'/3)} \cap E)^{d,+(\epsilon'/3)}) \quad \text{by Lemma 18} \\
&\geq \nu'(U^{d,+(\eta)}) \quad \text{by } (*) \\
&\geq \nu(U) - \eta \quad \text{by definition of } \eta
\end{aligned}$$

Now take sups when U varies: $\nu'(A^{d,+(\epsilon'/3)}) \geq r_A + \epsilon'/3$. In particular, $\nu'(A^{d,+(\epsilon'/3)}) > r_A$, so $\nu' \in [A^{d,+(\epsilon'/3)} > r_A]$. Since A is arbitrary, $\nu' \in \mathcal{U}$.

To sum up, for every $\nu' \in B_{\nu, < \epsilon}^{d_{\text{LP}}}$, there is an open subset $\mathcal{U}$ of Y in the weak topology such that $\nu' \in \mathcal{U} \subseteq B_{\nu, < \epsilon}^{d_{\text{LP}}}$. That is, $B_{\nu, < \epsilon}^{d_{\text{LP}}}$ is open in the weak topology. The claim follows. $\square$

Theorem 3. *Let X be equipped with a hemi-metric d , Y a space of continuous games over X . The topology of d_{H} on Y is finer than the weak topology on Y , and coincides with it when Y is a space of normalized games and d is bounded and totally bounded.*

Proof. The first part is by Lemma 17. The second part follows by using Proposition 8 and Proposition 9. $\square$

G Proof of Theorem 4

First start with the easy direction.

Lemma 23. *For every $\mathbf{k} \in \mathbf{K}(p, p')$, $d_{\text{H}}(p, p') \leq \mathbf{k}(d)$, where d is the map $(x, y) \in X^{(2)} \mapsto d(x, y)$.*

Proof. For every bounded 1-Lipschitz map f from X to $\mathbb{R}^+$,

$$\begin{aligned}
\oint_{x \in X} f(x) dp - \oint_{y \in X} f(y) dp' &= \mathbf{k}(f \ominus 0) + \mathbf{k}(0 \ominus f) \\
&= \mathbf{k}(f \ominus f)
\end{aligned} \tag{3}$$

using the additivity of $\mathbf{k}$. Now $f \ominus f$ maps $(x, y) \in X^{(2)}$ to $f(x) - f(y)$. This is at most $d(x, y)$, since f is 1-Lipschitz, so the quantity (3) is at most $\mathbf{k}(d)$, as $\mathbf{k}$ is monotonic. $\square$

The converse direction is much harder, and relies on an extension theorem akin to the Hahn-Banach Theorem on normed vector spaces. However, $\langle X \rightarrow \mathbb{R} \rangle_L$ is not a vector space, because the opposite $-f$ of an element f of $\langle X \rightarrow \mathbb{R} \rangle_L$ is not in $\langle X \rightarrow \mathbb{R} \rangle_L$ in general (it is in $\langle X^{op} \rightarrow \mathbb{R} \rangle_L$). We must replace vector spaces by *cones*, and rely on Roth's Sandwich Theorem [21], as in [25, 10, 12].

A *cone* is a set C , together with a binary operation $+$ turning it into a commutative monoid and a scalar multiplication $\cdot$ from $\mathbb{R}^+ \times C$ to C , such that $1 \cdot a = a$, $0 \cdot a = 0$, $(rs) \cdot a = r \cdot (s \cdot a)$, $r \cdot (a + b) = r \cdot a + r \cdot b$, and $(r + s) \cdot a = r \cdot a + s \cdot a$. An *ordered cone* is equipped in addition with a partial ordering $\leq$ making $+$ and $\cdot$ monotonic. The function q is *positively homogeneous* iff $q(s \cdot a) = sq(a)$ for all $s \in \mathbb{R}^+$, *super-additive* iff $q(a + b) \geq q(a) + q(b)$, *sub-additive* iff $q(a + b) \leq q(a) + q(b)$, and *linear* iff it has all three properties.

Roth's Sandwich Theorem states that on every ordered cone C , for every positively homogeneous super-additive function $q : C \rightarrow \overline{\mathbb{R}}^+$ and every positively homogeneous sub-additive function $p : C \rightarrow \overline{\mathbb{R}}^+$ such that $a \leq b$ implies $q(a) \leq p(b)$ (e.g., when $q \leq p$ and either q or p is monotonic), then there is a monotonic linear function $f : C \rightarrow \overline{\mathbb{R}}^+$ such that $q \leq f \leq p$.

The following extension theorem is a consequence of it. (The closest we know in the literature is [25, Theorem 3.16].) We shall use it with $C = \langle X \rightarrow \mathbb{R}^+ \rangle_L$ (with pointwise $+$, $\cdot$, and $\leq$). A subset Z of a cone C is called *convex* iff the *barycenter* $r \cdot x + (1 - r) \cdot y$ ($r \in [0, 1]$) of any $x, y \in Z$ is again in Z .

Theorem 8 (Extension). *Let C be an ordered cone, Z a convex subset of C . Let f be a monotonic map from Z to $\overline{\mathbb{R}}^+$ that preserves barycenters, i.e., $f(r \cdot x + (1 - r) \cdot y) = rf(x) + (1 - r)f(y)$ for every $r \in [0, 1]$, $x, y \in Z$. Let finally p be a sub-additive, positively homogeneous map from C to $\overline{\mathbb{R}}^+$, such that for every $x, y \in Z$, $c \in C$, and $\lambda > 0$ with $x \leq \lambda \cdot y + c$, then $f(x) \leq \lambda f(y) + p(c)$.*

Then there is a monotonic linear function g from C to $\overline{\mathbb{R}}^+$ such that $g(x) = f(x)$ for every $x \in Z$, and $g \leq p$.

Proof. If Z is empty, this is clear: take $g(x) = 0$ pour tout $x \in C$. So assume Z non-empty, and let:

$$p'(x) = \inf_{\substack{y \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda \cdot y + c}} \lambda f(y) + p(c) \quad q'(y) = \sup_{\substack{x \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda \cdot y + c}} \frac{f(x) - p(c)}{\lambda}$$

where the inf is taken to be $+\infty$ (the sup to be 0) if they are computed over no element at all. First note that p' and q' take their values in $\overline{\mathbb{R}}^+$. For q' , this is because whenever $x \in Z$, $c \in C$, and $\lambda > 0$ satisfy $x \leq \lambda \cdot y + c$, then also $x \leq \lambda' \cdot y + c$ for every $\lambda' \geq \lambda$, so $q'(y) \geq (f(x) - p(c))/\lambda'$. When λ' tends to $+\infty$, it follows that $q'(y) \geq 0$. When there are no such x, c and λ , then by definition $q'(y) = 0$.

Now p' is clearly monotonic. We claim p' is positively homogeneous. First, we show that $p'(0) = 0$. Fix $y \in Z$. For every $\epsilon > 0$, pick $\lambda > 0$ such that $\lambda \leq \epsilon/f(y)$ if $f(y) \neq 0$, arbitrary otherwise, let $c = 0$, then $p'(0) \leq \lambda f(y) + p(c) \leq \epsilon$. Since ϵ is

arbitrary, $p'(0) = 0$. Next, for every $r > 0$,

$$\begin{aligned}
p'(r \cdot x) &= \inf_{\substack{y \in Z, c \in C \\ \lambda > 0 \\ r \cdot x \leq \lambda \cdot y + c}} \lambda f(y) + p(c) = \inf_{\substack{y \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda/r \cdot y + 1/r \cdot c}} \lambda f(y) + p(c) \\
&= \inf_{\substack{y \in Z, c' \in C \\ \lambda' > 0 \\ x \leq \lambda' \cdot y + c'}} r \lambda' f(y) + p(r \cdot c') = r p'(x)
\end{aligned}$$

So p' is indeed positively homogeneous. Moreover,

$$\begin{aligned}
p'(x) + p'(x') &= \inf_{\substack{y \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda \cdot y + c}} (\lambda f(y) + p(c)) + \inf_{\substack{y' \in Z, c' \in C \\ \lambda' > 0 \\ x' \leq \lambda' \cdot y' + c'}} (\lambda' f(y') + p(c')) \\
&= \inf_{\substack{y, y' \in Z \\ \lambda, \lambda' > 0 \\ x \leq \lambda \cdot y + c \quad x' \leq \lambda' \cdot y' + c'}}_{c, c' \in C} [\lambda f(y) + \lambda' f(y') + p(c) + p(c')] \\
&\geq \inf_{\substack{y, y' \in Z \\ \lambda, \lambda' > 0 \\ x \leq \lambda \cdot y + c \quad x' \leq \lambda' \cdot y' + c'}}_{c, c' \in C} \left[(\lambda + \lambda') f \left(\frac{\lambda}{\lambda + \lambda'} \cdot y + \frac{\lambda'}{\lambda + \lambda'} \cdot y' \right) + p(c + c') \right] \\
&\quad \text{since } f \text{ preserves barycenters and } p \text{ is sub-additive} \\
&\geq \inf_{\substack{y'' \in Z \\ \lambda'' > 0 \\ x + x' \leq \lambda'' \cdot y'' + c''}}_{c'' \in C} \lambda'' f(y'') + p(c'') = p'(x + x')
\end{aligned}$$

since $y'' = \lambda/(\lambda + \lambda') \cdot y + \lambda'/(\lambda + \lambda') \cdot y'$, $\lambda'' = \lambda + \lambda'$, $c'' = c + c'$ satisfy $\lambda'' > 0$ and $x + x' \leq \lambda'' \cdot y'' + c''$ whenever $\lambda, \lambda' > 0$, and $x \leq \lambda \cdot y + c$, $x' \leq \lambda' \cdot y' + c'$. So p' is sub-additive.

Now on to q' . Clearly q' is monotonic. Choosing $y = 0$, for every $x \in Z$, for every $c \in C$ and every $\lambda > 0$ with $x \leq \lambda \cdot y + c$, we have $x \leq c$, so also $x \leq \lambda' \cdot x + c$ for any $\lambda' > 0$. By assumption, it follows that $f(x) \leq \lambda' f(x) + p(c)$. So $(f(x) - p(c))/\lambda \leq \lambda' f(x)/\lambda$. Let λ' tend 0: $(f(x) - p(c))/\lambda \leq 0$. Take sups when x, λ and c varies, obtaining $q'(0) \leq 0$. Since q' takes its values in $\overline{\mathbb{R}}^+$, $q'(0) = 0$. To show that q' is positively homogeneous, we have to show that $q'(r \cdot y) = r q'(y)$ for every $r > 0$:

$$q'(r \cdot y) = \sup_{\substack{x \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda r \cdot y + c}} \frac{f(x) - p(c)}{\lambda} = \sup_{\substack{x \in Z, c \in C \\ \lambda' > 0 \\ x \leq \lambda' \cdot y + c}} r \frac{f(x) - p(c)}{\lambda'} = r q'(y)$$

by taking $\lambda' = \lambda r$. We now claim that q' is super-additive:

$$\begin{aligned}
q'(y) + q'(y') &= \sup_{\substack{x \in Z, c \in C \\ \lambda > 0 \\ x \leq \lambda \cdot y + c}} \frac{f(x) - p(c)}{\lambda} + \sup_{\substack{x' \in Z, c' \in C \\ \lambda' > 0 \\ x' \leq \lambda' \cdot y' + c'}} \frac{f(x') - p(c')}{\lambda'} \\
&= \sup_{\substack{x, x' \in Z \\ \lambda, \lambda' > 0 \\ x \leq \lambda \cdot y + c \quad x' \leq \lambda' \cdot y' + c'}} \frac{\lambda + \lambda'}{\lambda \lambda'} \left[\frac{\lambda'}{\lambda + \lambda'} (f(x) - p(c)) + \frac{\lambda}{\lambda + \lambda'} (f(x') - p(c')) \right] \\
&= \sup_{\substack{x, x' \in Z \\ \lambda, \lambda' > 0 \\ x \leq \lambda \cdot y + c \quad x' \leq \lambda' \cdot y' + c'}} \frac{\lambda + \lambda'}{\lambda \lambda'} \left[f \left(\frac{\lambda'}{\lambda + \lambda'} \cdot x + \frac{\lambda}{\lambda + \lambda'} \cdot x' \right) - p \left(\frac{\lambda'}{\lambda + \lambda'} \cdot c + \frac{\lambda}{\lambda + \lambda'} \cdot c' \right) \right] \\
&\quad \text{since } f \text{ preserves barycenters and } p \text{ is sub-additive} \\
&\leq \sup_{\substack{x'' \in Z, c'' \in C \\ \lambda'' > 0 \\ x'' \leq \lambda'' \cdot (y + y') + c''}} \frac{f(x'') - p(c'')}{\lambda''} = q'(y + y')
\end{aligned}$$

by taking $x'' = \lambda' / (\lambda + \lambda') \cdot x + \lambda / (\lambda + \lambda') \cdot x'$, $c'' = \lambda' / (\lambda + \lambda') \cdot c + \lambda / (\lambda + \lambda') \cdot c'$, $\lambda'' = \lambda \lambda' / (\lambda + \lambda')$.

We now claim that $q'(z) \leq p'(z)$ for every $z \in C$. It is enough to show that $(f(x) - p(c)) / \lambda \leq \lambda' f(y) + p(c')$ for every $\lambda, \lambda' > 0$, $x, y \in Z$, $c, c' \in C$ such that $x \leq \lambda \cdot z + c$ et $z \leq \lambda' \cdot y + c'$. Indeed, $x \leq \lambda \lambda' \cdot y + (c + \lambda \cdot c')$, so by assumption $f(x) \leq \lambda \lambda' f(y) + p(c + \lambda \cdot c') \leq \lambda \lambda' f(y) + p(c) + \lambda p(c')$ (since p is sub-additive). The desired inequality follows.

We can therefore applies Roth's Sandwich Theorem: there is a monotonic linear map g from C to $\mathbb{R}^+$ such that $q' \leq g \leq p'$.

We claim that $g \leq p$. For every $x \in C$, by definition $p'(x) \leq \lambda f(y) + p(c)$ for every $y \in Z$, every $c \in C$ and every $\lambda > 0$ with $x \leq \lambda \cdot y + c$. Fix an arbitrary $y \in Z$ (using the fact that Z is non-empty), and $c = x$. Then $p'(x) \leq \lambda f(y) + p(x)$. Since λ is arbitrary, $p'(x) \leq p(x)$. So $g \leq p$.

Finally, for every $x \in Z$, we may take $y = x$, $\lambda = 1$ and $c = 0$ and the definition of p' , whence $p'(x) \leq f(x)$; similarly, for every $y \in Z$, we may take $x = y$, $c = 0$, and $\lambda = 1$ in the definition of q' , whence $q'(y) \geq f(y)$. So for every $x \in Z$, $q'(x) \geq f(x) \geq p'(x) \geq q'(x)$, therefore $q'(x) = f(x) = p'(x)$. Since $q' \leq g \leq p'$, it follows that $q'(x) = g(x) = p'(x)$, so $g(x) = f(x)$. $\square$

We shall also need to use a slightly looser characterization of d_H , where instead of taking sups over 1-Lipschitz maps, we take sups over pairs of maps f, g that are not too far. Also, f and g are just required to be continuous, not Lipschitz, but this will not be essential in the sequel.

Lemma 24. For all games ν, ν' on X ,

$$d_H(\nu, \nu') = \sup_{\substack{f, g \in (X \rightarrow \mathbb{R}^+) \\ \forall x, y \in X: f(x) - g(y) \leq d(x, y)}} d_{\mathbb{R}} \left(\oint_{x \in X} f(x) d\nu, \oint_{y \in X} g(y) d\nu' \right)$$

Proof. Let $d'_H(\nu, \nu')$ be the right-hand side of the equality. It is clear that $d'_H(\nu, \nu') \geq d_H(\nu, \nu')$ (restricting the sup to the case where $f = g$ and f is 1-Lipschitz). For the converse inequality, let f and g two bounded continuous maps such that $f(x) - g(y) \leq d(x, y)$ for all $x, y \in X$. Let $h(x) = \inf_{y \in Y} (d(x, y) + g(y))$.

First, h is bounded, since $h(x) \leq d(x, x) + g(x) = g(x)$ (taking $y = x$), and because g is bounded. Next, h is 1-Lipschitz. To show this, note that for every $x, x', y \in X$, $[d(x, y) + g(y)] \leq d(x, x') + [d(x', y) + g(y)]$ by the triangular inequality. Taking infs over y , $h(x) \leq d(x, x') + h(x')$.

We have already seen that $h \leq g$. On the other hand, $f \leq h$ since $f(x) \leq g(y) + d(x, y)$ for every y . So $\oint_{x \in X} f(x) d\nu - \oint_{y \in X} g(y) d\nu' \leq \oint_{x \in X} h(x) d\nu - \oint_{y \in X} h(y) d\nu' \leq d_H(\nu, \nu')$. Taking sups over f and g , $d'_H(\nu, \nu') \leq d_H(\nu, \nu')$. $\square$

Proposition 10. Let X be equipped with a bounded hemi-metric d , and p and p' two normalized valuations on X . Then $\mathbf{k}(d) = d_H(p, p')$ for some $\mathbf{k} \in \mathbf{K}(p, p')$.

Proof. First eliminate the degenerate case where $d(x, y) = 0$ for all $x, y \in X$. In this case, $x \leq y$ $\forall x, y \in X$. The only Lipschitz maps are the constant maps, so $d_H(p, p') = 0$, and we can define $\mathbf{k}(\varphi) = \oint_{(x, y) \in X^{(2)}} \varphi(x, y) dp^2$ for any probability valuation p^2 on $X^{(2)}$, for instance $\delta_{(x, x)}$ for a fixed element of X . (Note that X cannot be empty, since $p(X) = 1$ as p is normalized, and $p(\emptyset) = 0$.)

In the non-degenerate case, we use the Extension Theorem 8 on the ordered cone $C = \langle X^{(2)} \rightarrow \mathbb{R}^+ \rangle_L$ of all Lipschitz maps, using for Z the subspace of all maps of the form $f \ominus g$, where $f, g \in \langle X \rightarrow \mathbb{R}^+ \rangle_L$ are such that $f(x) \geq g(y)$ for all $x, y \in X$ (i.e., $f \ominus g \geq 0$). Observe that Z is convex.

Let f, f', g, g' be four bounded 1-Lipschitz maps from X to $\mathbb{R}^+$, with $f(x) - g(y) \leq f'(x) - g'(y)$ for all $x, y \in X$. Then $f(x) - f'(x) \leq g(y) - g'(y)$ for all $x, y \in X$. In particular, $a = \sup_{x \in X} (f(x) - f'(x))$, $b = \inf_{y \in X} (g(y) - g'(y))$ are well-defined, and for every $x, y \in X$, $f(x) - f(x') \leq a \leq b \leq g(y) - g'(y)$. In particular, $f(x) \leq f(x') + a$, and $g(y) \geq g'(y) + b$, so:

$$\begin{aligned} \oint_{x \in X} f(x) dp - \oint_{y \in X} g(y) dp' &\leq \oint_{x \in X} [f'(x) + a] dp - \oint_{y \in X} [g'(y) + b] dp' \\ &= \oint_{x \in X} f'(x) dp - \oint_{y \in X} g'(y) dp' + a - b \\ &\quad \text{by Lemma 19, since } p \text{ and } p' \text{ are normalized} \\ &\leq \oint_{x \in X} f'(x) dp - \oint_{y \in X} g'(y) dp' \end{aligned}$$

because $a \leq b$. Hence the functional:

$$\mathbf{f}(f \ominus g) = \oint_{x \in X} f(x) dp - \oint_{y \in X} g(y) dp'$$

is well-defined and monotonic from Z to $\mathbb{R}$. Moreover, it takes its values in $\mathbb{R}^+$: for every element $f \ominus g$ of Z , $f(x) \geq g(y)$, so $f \ominus g \geq 0$. Since $\mathbf{f}$ is monotonic, $\mathbf{f}(f \ominus g) \geq \mathbf{f}(0) = 0$. We then note that $\mathbf{f}$ preserves barycenters: for every $r \in [0, 1]$,

$$\begin{aligned} & \mathbf{f}(r \cdot (f \ominus g) + (1-r) \cdot (f' \ominus g')) \\ &= \oint_{x \in X} [rf(x) + (1-r)f'(x)]dp - \oint_{y \in X} [rg(y) + (1-r)g'(y)]dp' \\ &= r \oint_{x \in X} f(x)dp + (1-r) \oint_{x \in X} f'(x)dp - r \oint_{y \in X} g(y)dp - (1-r) \oint_{y \in X} g'(y)dp' \\ &= r\mathbf{f}(f \ominus g) + (1-r)\mathbf{f}(f' \ominus g') \end{aligned}$$

since Choquet integration is linear in the integrated function, provided we integrate along valuations.

For every Lipschitz map φ from $X^{(2)}$ to $\mathbb{R}^+$, let:

$$\mathbf{p}_1(\varphi) = \inf_{\substack{\lambda > 0 \\ \forall x, y \in X \cdot \lambda \varphi(x, y) \leq d(x, y)}} \frac{1}{\lambda}$$

The map $\mathbf{p}_1$ takes its values in $\overline{\mathbb{R}}^+$, and the value $+\infty$ is attained when there is no $\lambda > 0$ such that $\lambda \varphi(x, y) \leq d(x, y)$ for every $x, y \in X$. Since for every $\lambda > 0$, $\lambda \cdot 0 \leq d(x, y)$, we obtain $\mathbf{p}_1(0) = 0$. For every $r > 0$,

$$\begin{aligned} \mathbf{p}_1(r \cdot \varphi) &= \inf_{\substack{\lambda > 0 \\ \forall x, y \in X \cdot \lambda r \varphi(x, y) \leq d(x, y)}} \frac{1}{\lambda} \\ &= \inf_{\substack{\lambda' > 0 \\ \forall x, y \in X \cdot \lambda' \varphi(x, y) \leq d(x, y)}} \frac{r}{\lambda'} = r\mathbf{p}_1(\varphi) \end{aligned}$$

by letting $\lambda' = \lambda r$. So $\mathbf{p}_1$ is positively homogeneous. Now observe that $\mathbf{p}_1$ is sub-additive:

$$\begin{aligned} \mathbf{p}_1(\varphi) + \mathbf{p}_1(\varphi') &= \inf_{\substack{\lambda > 0 \\ \forall x, y \in X \cdot \lambda \varphi(x, y) \leq d(x, y)}} \frac{1}{\lambda} + \inf_{\substack{\lambda' > 0 \\ \forall x, y \in X \cdot \lambda' \varphi'(x, y) \leq d(x, y)}} \frac{1}{\lambda'} \\ &= \inf_{\substack{\lambda, \lambda' > 0 \\ \forall x, y \in X \cdot \lambda \varphi(x, y) \leq d(x, y), \lambda' \varphi'(x, y) \leq d(x, y)}} \frac{\lambda + \lambda'}{\lambda \lambda'} \end{aligned}$$

Since $\lambda \varphi(x, y) \leq d(x, y)$ and $\lambda' \varphi'(x, y) \leq d(x, y)$ imply $\lambda \lambda' / (\lambda + \lambda') (\varphi(x, y) + \varphi'(x, y)) \leq d(x, y)$, by letting $\lambda'' = \lambda \lambda' / (\lambda + \lambda')$ we obtain:

$$\mathbf{p}_1(\varphi) + \mathbf{p}_1(\varphi') \leq \inf_{\substack{\lambda'' > 0 \\ \forall x, y \in X \cdot \lambda'' (\varphi(x, y) + \varphi'(x, y)) \leq d(x, y)}} \frac{1}{\lambda''} = \mathbf{p}_1(\varphi + \varphi')$$

Furthermore, $\mathbf{p}_1$ is monotonic. So the map $\mathbf{p}$:

$$\mathbf{p}(\varphi) = d_H(p, p') \mathbf{p}_1(\varphi)$$

is also positively homogeneous, sub-additive, and monotonic from C to $\overline{\mathbb{R}}^+$.

Let us now show that whenever $f \ominus g \in Z$, $f' \ominus g' \in Z$, $\lambda > 0$, and $f \ominus g \leq \lambda(f' \ominus g') + \varphi$, then $\mathbf{f}(f \ominus g) \leq \lambda \mathbf{f}(f' \ominus g') + \mathbf{p}(\varphi)$. This is the last condition we need to apply Theorem 8, but also the most complex. When $\mathbf{p}(\varphi) = +\infty$, this is obvious. Otherwise, fix an arbitrary $\epsilon > 0$. There must be a $\lambda' > 0$ such that $\lambda' \varphi(x, y) \leq d(x, y)$ for all $x, y \in X$ and $\mathbf{p}(\varphi) \geq d_H(p, p')/\lambda' - \epsilon$. Fix c such that φ is c -Lipschitz. Then define:

$$\begin{aligned} f''(x) &= \sup_{y \in X} (f(y) - \lambda f'(y) - c d(y, x)) \\ g''(y) &= \inf_{x \in X} (g(x) - \lambda g'(x) + c d(y, x)) \end{aligned}$$

These functions are well-defined, and take their values in $\mathbb{R}$. In particular, none is either $-\infty$ or $+\infty$. Since f, f', g, g' , and d are bounded, in fact f'' and g'' are bounded. Moreover, they are c -Lipschitz. In the case of f'' , this is because the triangular inequality entails $(f(y) - \lambda f'(y) - c d(y, x)) \leq (f(y) - \lambda f'(y) - c d(y, x')) + c d(x, x')$; taking the sup over all $y \in X$, we obtain $f''(x) \leq f''(x') + c d(x, x')$. The argument is similar for g'' . Next, we observe that: (a) $f''(x) \geq f(x) - \lambda f'(x)$ for every $x \in X$ (take $y = x$ in the definition of f''); (b) $g''(y) \leq g(y) - \lambda g'(y)$ for all $y \in X$ (take $x = y$ in the definition of g''). For all $x, x', y, y' \in X$, we use the fact that $f \ominus g \leq \lambda(f' \ominus g') + \varphi$ and the fact that φ is c -Lipschitz from $X^{(2)}$ to $\mathbb{R}$ to conclude that:

$$\begin{aligned} f(y') - \lambda f'(y') &\leq g(x') - \lambda g'(x') + \varphi(y', x') \\ \varphi(y', x') &\leq \varphi(x, y) + c d(y', x) + c d(y, x') \end{aligned}$$

So:

$$f(y') - \lambda f'(y') - c d(y', x) \leq [g(x') - \lambda g'(x') + c d(y, x')] + \varphi(x, y)$$

Takings sups over $y' \in X$ and infs over $x' \in X$, $f''(x) \leq g''(y) + \varphi(x, y)$, i.e.: (c) $f'' \ominus g'' \leq \varphi$. Let a be such that $a \leq \inf_{x \in X} f''(x)$ and $a \leq \inf_{y \in X} g''(y)$. Let $f_1 = \lambda'(f'' - a)$, $g_1 = \lambda'(g'' - a)$: f_1 and g_1 take their values in $\mathbb{R}^+$. By (c) and the fact that $\lambda' \varphi(x, y) \leq d(x, y)$ for all $x, y \in X$, $f_1(x) - g_1(y) = \lambda'(f''(x) - g''(y)) \leq \lambda' \varphi(x, y) \leq d(x, y)$. Now f_1 and g_1 are Lipschitz, hence continuous. By Lemma 24, it follows:

$$d_H(p, p') \geq \max \left(\oint_{x \in X} f_1(x) dp - \oint_{y \in X} g_1(y) dp', 0 \right)$$

Since $\mathbf{p}(\varphi) \geq d_H(p, p')/\lambda' - \epsilon$, therefore:

$$\begin{aligned} \mathbf{p}(\varphi) &\geq \max \left(\oint_{x \in X} f_1(x)/\lambda' dp - \oint_{y \in X} g_1(y)/\lambda' dp', 0 \right) - \epsilon \\ &= \max \left(\oint_{x \in X} f''(x) dp - \oint_{y \in X} g''(y) dp', 0 \right) - \epsilon \\ &\geq \oint_{x \in X} f''(x) dp - \oint_{y \in X} g''(y) dp' - \epsilon \end{aligned}$$

using Lemma 19. So:

$$\begin{aligned}
\lambda \mathbf{f}(f' \ominus g') + \mathbf{p}(\varphi) &\geq \oint_{x \in X} \lambda f'(x) dp - \oint_{y \in X} \lambda g'(y) dp' + \oint_{x \in X} f''(x) dp - \oint_{y \in X} g''(y) dp' - \epsilon \\
&= \oint_{x \in X} [\lambda f'(x) + f''(x)] dp - \oint_{y \in X} [\lambda g'(y) + g''(y)] dp' - \epsilon \\
&\quad \text{since Choquet integration along valuations is linear} \\
&\geq \oint_{x \in X} f(x) dp - \oint_{y \in X} g(y) dp' - \epsilon = \mathbf{f}(f \ominus g) - \epsilon
\end{aligned}$$

using remarks (a) and (b) above. Since ϵ is arbitrary, we obtain $\mathbf{f}(f \ominus g) \leq \mathbf{f}(f' \ominus g') + \mathbf{p}(\varphi)$, as announced.

So we can apply the Extension Theorem 8: there is functional $\mathbf{k}_0$ from C to $\mathbb{R}^+$ that coincides with $\mathbf{f}$ on Z , and such that $\mathbf{k}_0 \leq \mathbf{p}$.

Note that: (*) $\mathbf{k}_0(\chi_{X^{(2)}}) = 1$. Indeed, $\mathbf{k}_0(\chi_{X^{(2)}}) = \mathbf{k}_0(\chi_X \ominus 0) = \mathbf{f}(\chi_X \ominus 0) = p(X) - 0 = 1$.

For every $\varphi \in \langle X^{(2)} \rightarrow \mathbb{R} \rangle_L$, define $\mathbf{k}(\varphi) = \mathbf{k}_0(\varphi + b) - b$, where b is any real such that $-b \leq \varphi(x, y)$ for all $(x, y) \in X^{(2)}$. This does not depend on the actual value of b , because $\mathbf{k}_0$ is linear, and using (*).

It is also clear that $\mathbf{k}$ is normalized, by (*), that $\mathbf{k}$ is additive. As for positive homogeneity, for every $\varphi \in \langle X^{(2)} \rightarrow \mathbb{R} \rangle_L$, for every $r \geq 0$, letting b be large enough, $\mathbf{k}(r \cdot \varphi) = \mathbf{k}_0(r \cdot \varphi + rb) - rb = r\mathbf{k}_0(\varphi + b) - rb = r\mathbf{k}(\varphi)$, using the fact that $\mathbf{k}_0$ is positively homogeneous.

When $\varphi \leq \varphi'$, for b large enough, $\mathbf{k}(\varphi) = \mathbf{k}_0(\varphi + b) - b \leq \mathbf{k}_0(\varphi' + b) - b = \mathbf{k}(\varphi')$ since $\mathbf{k}_0$ is monotonic. So $\mathbf{k}$ is monotonic. In particular, $\mathbf{k}$ is positive: if $\varphi \geq 0$, then $\mathbf{k}(\varphi) \geq \mathbf{k}(0) = 0$.

Now check marginals. For every bounded 1-Lipschitz map f from X to $\mathbb{R}^+$, $\mathbf{k}(f \ominus 0) = \mathbf{k}_0(f \ominus 0)$ (i.e., take $b = 0$) $= \mathbf{f}(f \ominus 0) = \oint_{x \in X} f(x) dp$, since $\mathbf{k}_0$ coincides with $\mathbf{f}$ on Z . Similarly, with b large enough, $\mathbf{k}(0 \ominus g) = \mathbf{k}_0((0 \ominus g) + b) - b = \mathbf{k}_0(b\chi_X \ominus g) = \mathbf{f}(b\chi_X \ominus g) - b = \oint_{x \in X} b dp - \oint_{y \in Y} g(y) dp' - b = -\oint_{y \in Y} g(y) dp'$, since p is normalized. So $\mathbf{k} \in \mathbf{K}(p, p')$.

Finally, we claim that $\mathbf{k}(d) = d_H(p, p')$. By Lemma 23, $\mathbf{k}(d) \geq d_H(p, p')$. Conversely, remember that $\mathbf{k}(\varphi) \leq \mathbf{p}(\varphi)$ for every bounded Lipschitz map φ , in particular $\varphi = d$. So $\mathbf{k}(d) = \mathbf{k}_0(d) \leq \mathbf{p}(d) = d_H(p, p')\mathbf{p}_1(d)$. Since we have excluded the degenerate case $d = 0$, there are two elements $x, y \in X$ such that $d(x, y) \neq 0$. Then $\mathbf{p}_1(d) = \inf_{\lambda > 0, \forall x, y \in X \cdot \lambda d(x, y) \leq d(x, y)} 1/\lambda = 1$, so $\mathbf{k}(d) \leq d_H(p, p')$. $\square$

H Proof of Lemma 2

Lemma 2. *Let X be sym-compact, and p' a continuous probability valuation on X . For any bounded Lipschitz map h from X to $\mathbb{R}$, $-\oint_{x \in X} h(x) dp' = \oint_{x \in X^{op}} -h(x) dp'^{\perp}$.*

Proof. First $(\geq)$, $\oint_{y \in X^{op}} -h(y) dp'^{\perp} \leq -\inf_{f \in \langle X \rightarrow \mathbb{R} \rangle, f \geq h} \oint_{x \in X} f(x) dp'$ (i.e., where f ranges over all bounded continuous functions, not just the step functions) $= -\oint_{x \in X} h(x) dp'$ (taking $f = h$, which is allowed since every Lipschitz map is continuous).

Conversely ($\leq$), it suffices to show that, for every $\epsilon > 0$, there is a step function f from X to $\mathbb{R}$ with $f \geq h$ such that $-\int_{x \in X} h(x) dp' + \int_{x \in X} f(x) dp' \leq \epsilon$. Observe that (see e.g., [24]) any element h of $\langle X \rightarrow \mathbb{R} \rangle$ is the sup of a directed family of step functions, namely $h_K = a + \frac{1}{2^K} \sum_{k=1}^N \chi_{U_k}$, $K \in \mathbb{N}$, where $U_k = h^{-1}(a + \frac{k}{2^K}, +\infty)$, $N = \lfloor (b-a)2^K \rfloor$, a is any lower bound for h and b is any upper bound for h . Moreover, $h(x) - \frac{1}{2^K} \leq h_K(x) \leq h(x)$ for all $x \in X$. Now take $f = h_K + \frac{1}{2^K}$. Clearly $f \geq h$, f is a step function, and $\int_{x \in X} f(x) dp' \leq \int_{x \in X} h_K(x) dp' + \frac{1}{2^K}$, so $-\int_{x \in X} h(x) dp' + \int_{x \in X} f(x) dp' \leq \frac{1}{2^K}$. Now take K large enough that $\frac{1}{2^K} \leq \epsilon$. $\square$

One may observe that this still holds when p' is replaced by a continuous game ν' , however we do not need the extra generality here.

I Proof of the Kantorovich-Rubinstein Theorem 5

Lemma 25. *Let X be equipped with a hemi-metric d , U an open subset of X . For every $\epsilon, \epsilon' \in \mathbb{R}^+$, $U^{d, -(\epsilon+\epsilon')} \subseteq (U^{d, -(\epsilon)})^{d, -(\epsilon')}$.*

Proof. Let x be outside $(U^{d, -(\epsilon)})^{d, -(\epsilon')}$. Then $d(x, X \setminus U^{d, -(\epsilon)}) \leq \epsilon'$. For every $\epsilon'' > 0$, then, there is an element y of $X \setminus U^{d, -(\epsilon)}$ such that $d(x, y) \leq \epsilon' + \epsilon''$. Since y is not in $U^{d, -(\epsilon)}$, $d(y, X \setminus U) \leq \epsilon$. By Lemma 10, $d(x, X \setminus U) \leq d(x, y) + d(y, X \setminus U) \leq \epsilon + \epsilon' + \epsilon''$. Since $\epsilon'' > 0$ is arbitrary, $d(x, X \setminus U) \leq \epsilon + \epsilon'$, i.e. $x \notin U^{d, -(\epsilon+\epsilon')}$. $\square$

Lemma 26. *Let X be equipped with the hemi-metric d . For every open subset U of X , for every $\epsilon > 0$, the map $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is Lipschitz, monotonic in U and antitonic in ϵ , and:*

$$\chi_{U^{d, -(2\epsilon)}} \leq \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}} \leq \chi_{U^{d, -(\epsilon)}} \leq \chi_U \quad (4)$$

These maps form a directed family whose sup is χ_U .

Proof. That $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is Lipschitz is a direct consequence of the fact that $\frac{1}{\epsilon} \chi_V^\epsilon$ is so for every open V , by Lemma 15. From the same lemma, we obtain:

$$\chi_{(U^{d, -(\epsilon)})^{d, -(\epsilon)}} \leq \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}} \leq \chi_U^{d, -(\epsilon)}$$

That $\chi_U^{d, -(\epsilon)} \leq \chi_U$ follows from $U^{d, -(\epsilon)} \subseteq U$ (Lemma 11). On the other hand, $U^{d, -(2\epsilon)} \subseteq (U^{d, -(\epsilon)})^{d, -(\epsilon)}$ by Lemma 25, whence the leftmost inequality of (4).

We check that $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is monotonic in U : $U \mapsto U^{d, -(\epsilon)}$ is indeed monotonic (easy), and χ_V^ϵ is monotonic in V (Lemma 15.)

Let us show that $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is antitonic in ϵ . Whenever $\epsilon \geq \epsilon'$, $U^{d, -(\epsilon)} \subseteq U^{d, -(\epsilon')}$ by definition of thinning. So $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}} \leq \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon')}} \leq \frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}}^\epsilon$, since χ_V^ϵ is monotonic in V and antitonic in ϵ (Lemma 15).

In particular, the family is directed. Since $\sup_{\epsilon > 0} \chi_{U^{d, -(2\epsilon)}} = \chi_U$ by Lemma 15, the inequalities (4) entail that the sup of the family is exactly χ_U . $\square$

Lemma 27. *Let X be equipped with the hemi-metric d . For all opens U and V in X , for every $\epsilon > 0$:*

$$\begin{aligned}\chi_{U \cap V}^\epsilon &= \min(\chi_U^\epsilon, \chi_V^\epsilon) \\ (U \cap V)^{d, -(\epsilon)} &= U^{d, -(\epsilon)} \cap V^{d, -(\epsilon)} \\ \frac{1}{\epsilon} \chi_{(U \cap V)^{d, -(\epsilon)}}^\epsilon &= \min\left(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d, -(\epsilon)}}^\epsilon\right)\end{aligned}$$

Proof. By definition, $d(x, F \cup F') = \min(d(x, F), d(x, F'))$. Indeed, $d(x, F \cup F') = \inf_{y \in F \cup F'} d(x, y) = \inf(\inf_{y \in F} d(x, y), \inf_{y \in F'} d(x, y)) = \inf(d(x, F), d(x, F'))$. The first inequality follows immediately:

$$\begin{aligned}\chi_{U \cap V}^\epsilon(x) &= \min(\epsilon, d(x, X \setminus (U \cap V))) = \min(\epsilon, d(x, (X \setminus U) \cup (X \setminus V))) \\ &= \min(\epsilon, \min(d(x, X \setminus U), d(x, X \setminus V))) = \min(\chi_U^\epsilon(x), \chi_V^\epsilon(x))\end{aligned}$$

The second, too:

$$\begin{aligned}(U \cap V)^{d, -(\epsilon)} &= \{x \in X \mid d(x, X \setminus (U \cap V)) > \epsilon\} = \{x \in X \mid d(x, (X \setminus U) \cup (X \setminus V)) > \epsilon\} \\ &= \{x \in X \mid \min(d(x, X \setminus U), d(x, X \setminus V)) > \epsilon\} = U^{d, -(\epsilon)} \cap V^{d, -(\epsilon)}\end{aligned}$$

The third follows from the first two. $\square$

Lemma 28. *Let X be equipped with the hemi-metric d . Then:*

$$\begin{aligned}\chi_X^\epsilon &= \epsilon \chi_X \\ X^{d, -(\epsilon)} &= X \\ \frac{1}{\epsilon} \chi_{X^{d, -(\epsilon)}}^\epsilon &= \chi_X\end{aligned}$$

Proof. For every $x \in X$, $d(x, X \setminus X) = d(x, \emptyset) = +\infty$, so $\chi_X^\epsilon(x) = \min(\epsilon, +\infty) = \epsilon$. Similarly, $X^{d, -(\epsilon)} = \{x \in X \mid d(x, \emptyset) > \epsilon\} = X$. The third inequality follows from the first two. $\square$

The case of unions is not so nice in general.

Lemma 29. *Let X be equipped with the hemi-metric d . For all opens U and V in X , for every $\epsilon > 0$,*

$$\begin{aligned}\chi_{U \cup V}^\epsilon &\geq \max(\chi_U^\epsilon, \chi_V^\epsilon) \\ (U \cup V)^{d, -(\epsilon)} &\supseteq U^{d, -(\epsilon)} \cup V^{d, -(\epsilon)} \\ \frac{1}{\epsilon} \chi_{(U \cup V)^{d, -(\epsilon)}}^\epsilon &\geq \max\left(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d, -(\epsilon)}}^\epsilon\right)\end{aligned}$$

Proof. First, $d(x, F \cap F') = \inf_{y \in F \cap F'} d(x, y) \geq \inf_{y \in F} d(x, y) = d(x, F)$, and similarly $d(x, F \cap F') \geq d(x, F')$. So for every $x \in X$:

$$\begin{aligned}\chi_{U \cup V}^\epsilon(x) &= \min(\epsilon, d(x, X \setminus (U \cup V))) = \min(\epsilon, d(x, (X \setminus U) \cap (X \setminus V))) \\ &\geq \min(\epsilon, d(x, X \setminus U)) = \chi_U^\epsilon(x)\end{aligned}$$

and similarly $\chi_{U \cup V}^\epsilon(x) \geq \chi_V^\epsilon(x)$, whence the first inequality.

For every $x \in U^{d, -(\epsilon)}$, $d(x, X \setminus U) > \epsilon$, so $d(x, X \setminus (U \cup V)) \geq d(x, X \setminus U) > \epsilon$, i.e., $x \in (U \cup V)^{d, -(\epsilon)}$. Since x is arbitrary, $U^{d, -(\epsilon)} \subseteq (U \cup V)^{d, -(\epsilon)}$. Similarly, $V^{d, -(\epsilon)} \subseteq (U \cup V)^{d, -(\epsilon)}$, whence the second inequality. The third follows immediately. $\square$

To get a more precise treatment of unions, we require X to be sym-compact. The relation $\subseteq$ is the way-below relation on $\mathcal{O}(X)$, ordered by inclusion; on every locally compact space, $U \subseteq V$ iff $U \subseteq Q \subseteq V$ for some compact saturated set Q , and $\mathcal{O}(X)$ is then a continuous cpo [8], meaning that every open U is the union of the directed family of all opens V with $V \subseteq U$.

Lemma 30. *Let X be a sym-compact space, with hemi-metric d . For every open U of X , for every $\epsilon > 0$, $U^{d, -(\epsilon)} \subseteq U$.*

Proof. Let Q be the set of all $x \in X$ such that $d(x, X \setminus U) \geq \epsilon/2$. Check that $U^{d, -(\epsilon)} \subseteq Q$: for every $x \in U^{d, -(\epsilon)}$, by definition $d(x, X \setminus U) > \epsilon$, so $x \in Q$. We now check that $Q \subseteq U$: if $x \in Q$, then $d(x, X \setminus U) \neq 0$, so x is outside the closure of $X \setminus U$ by Lemma 10; i.e., $x \in U$, since $X \setminus U$ is closed. Finally, let us show that Q is compact and saturated:

$$\begin{aligned} Q &= \{x \in X \mid d(x, X \setminus U) \geq \epsilon/2\} = \{x \in X \mid \forall y \in X \setminus U \cdot d(x, y) \geq \epsilon/2\} \\ &= \bigcap_{y \in X \setminus U} \{x \in X \mid d(x, y) \geq \epsilon/2\} \end{aligned}$$

The complement of this is $\bigcup_{y \in X \setminus U} B_{y, < \epsilon/2}^{d^{op}}$, an open subset of X^{op} . By Theorem 1, this is an open subset of X^d , i.e., Q is compact saturated. $\square$

Proposition 11. *Let X be a sym-compact space, with hemi-metric d , and let $\mathbf{f}$ be a linear monotonic (sub)normalized functional from $\langle X \rightarrow \mathbb{R}^+ \rangle_L$ to $\mathbb{R}^+$. Let $\nu(U) = \sup_{\epsilon > 0} \mathbf{f}(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}})$ for each open subset U of X . Then ν is a continuous (sub)probability valuation on X .*

Proof. First, recall that $(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}})_{\epsilon > 0}$ is a directed family of bounded Lipschitz map, whose sup is χ_U , by Lemma 26. It follows that $\mathbf{f}(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}})$ is well-defined, since $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is Lipschitz. Since $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}} \leq \chi_U \leq \chi_X$ and $\mathbf{f}$ is monotonic, $\mathbf{f}(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}) \leq (\chi_X) \leq 1$. In particular, $\nu(U)$ is in $\mathbb{R}^+$, i.e., is different from $+\infty$. Moreover, $\nu(U) \leq 1$, so ν is subnormalized.

When $U = \emptyset$, $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}} = 0$ for all $\epsilon > 0$, so $\mathbf{f}(\frac{1}{\epsilon} \chi_U) = 0$ since $\mathbf{f}$ is positively homogeneous. So $\nu(\emptyset) = 0$.

On the other hand, $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is monotonic in U by Lemma 26, and since $\mathbf{f}$ is monotonic, so is ν : ν is a game.

Let us show that ν is continuous. Let us start by computing $\sup_{\epsilon > 0} \nu(U^{d, -(\epsilon)})$:

$$\begin{aligned} \sup_{\epsilon > 0} \nu(U^{d, -(\epsilon)}) &= \sup_{\epsilon, \epsilon' > 0} \mathbf{f} \left(\frac{1}{\epsilon'} \chi_{(U^{d, -(\epsilon)})^{d, -(\epsilon')}} \right) \\ &\geq \sup_{\epsilon, \epsilon' > 0} \mathbf{f} \left(\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon + \epsilon')}} \right) \quad \text{by Lemma 25} \\ &\geq \sup_{\epsilon, \epsilon' > 0} \mathbf{f} \left(\frac{1}{\epsilon + \epsilon'} \chi_{U^{d, -(\epsilon + \epsilon')}} \right) \end{aligned}$$

since $\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}$ is antitonic in ϵ by Lemma 26, and $\mathbf{f}$ is monotonic. Now this is clearly greater than or equal to (in fact equal to) $\nu(U)$. So: (*) $\nu(U) = \sup_{\epsilon > 0} \nu(U^{d, -(\epsilon)})$.

It follows that ν is continuous. Indeed, consider a directed family $(U_i)_{i \in I}$ of open subsets of X , and let $U = \bigcup_{i \in I} U_i$. For every $\eta > 0$, by (*) there is an $\epsilon > 0$ such that $\nu(U^{d, -(\epsilon)}) \geq \nu(U) - \eta$. Since $U^{d, -(\epsilon)} \subseteq U$ by Lemma 30, there is an $i \in I$ such that $U^{d, -(\epsilon)} \subseteq U_i$. Since ν is monotonic, $\nu(U_i) \geq \nu(U) - \eta$. Since η is arbitrary, $\sup_{i \in I} \nu(U_i) \geq \nu(U)$. The converse inequality follows from the monotony of ν ; the equality follows.

Let us show finally that ν is a valuation. For all opens U, V ,

$$\begin{aligned} \nu(U \cap V) &= \sup_{\epsilon > 0} \mathbf{f} \left(\frac{1}{\epsilon} \chi_{(U \cap V)^{d, -(\epsilon)}} \right) \\ &= \sup_{\epsilon > 0} \mathbf{f} \left(\min \left(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}, \frac{1}{\epsilon} \chi_{V^{d, -(\epsilon)}} \right) \right) \end{aligned} \quad (5)$$

by Lemma 27. We look for a similar equality for $\nu(U \cup V)$. By Lemma 29 and the fact that $\mathbf{f}$ is monotonic:

$$\begin{aligned} \nu(U \cup V) &= \sup_{\epsilon > 0} \mathbf{f} \left(\frac{1}{\epsilon} \chi_{(U \cup V)^{d, -(\epsilon)}} \right) \\ &\geq \sup_{\epsilon > 0} \mathbf{f} \left(\max \left(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}, \frac{1}{\epsilon} \chi_{V^{d, -(\epsilon)}} \right) \right) \end{aligned} \quad (6)$$

Conversely, fix an arbitrary $\epsilon > 0$. Let $\epsilon' = \epsilon/2$. Then $\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}} \geq \chi_{U^{d, -(\epsilon)}}$ by the leftmost inequality of (4) (Lemma 26), and similarly $\frac{1}{\epsilon'} \chi_{V^{d, -(\epsilon')}} \geq \chi_{V^{d, -(\epsilon)}}$. So $\max(\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}}, \frac{1}{\epsilon'} \chi_{V^{d, -(\epsilon')}}) \geq \max(\chi_{U^{d, -(\epsilon)}}, \chi_{V^{d, -(\epsilon)}}) = \chi_{(U \cup V)^{d, -(\epsilon)}}$. Using the rightmost inequalities of (4), it follows that for every $\epsilon'' > 0$,

$$\frac{1}{\epsilon''} \chi_{(U \cup V)^{d, -(\epsilon'')}} \leq \max \left(\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}}, \frac{1}{\epsilon'} \chi_{V^{d, -(\epsilon')}} \right)$$

Since $\mathbf{f}$ is monotonic,

$$\mathbf{f} \left(\frac{1}{\epsilon''} \chi_{(U \cup V)^{d, -(\epsilon'')}} \right) \leq \mathbf{f} \left(\max \left(\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}}, \frac{1}{\epsilon'} \chi_{V^{d, -(\epsilon')}} \right) \right)$$

Now take sups as $\epsilon'' > 0$ varies:

$$\nu(U^{d, -(\epsilon)} \cup V^{d, -(\epsilon)}) \leq \mathbf{f} \left(\max \left(\frac{1}{\epsilon'} \chi_{U^{d, -(\epsilon')}}, \frac{1}{\epsilon'} \chi_{V^{d, -(\epsilon')}} \right) \right)$$

Now take sups over $\epsilon' > 0$: $\nu(U^{d,-(\epsilon)} \cup V^{d,-(\epsilon)})$ is less than or equal to the right-hand side of (6). Now take sups over $\epsilon > 0$. By Lemma 12, $\bigcup_{\epsilon > 0} (U^{d,-(\epsilon)} \cup V^{d,-(\epsilon)}) = U \cup V$. Since ν is continuous, $\nu(U \cup V)$ is at most the right-hand side of (6), so:

$$\nu(U \cup V) = \sup_{\epsilon > 0} \mathbf{f} \left(\max \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \right) \quad (7)$$

Using (5) and (7), we obtain:

$$\begin{aligned} \nu(U \cup V) + \nu(U \cap V) &= \sup_{\epsilon > 0} \left[\mathbf{f} \left(\min \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \right) \right. \\ &\quad \left. + \mathbf{f} \left(\max \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \right) \right] \\ &\quad \text{since } + \text{ is Scott-continuous} \\ &= \sup_{\epsilon > 0} \mathbf{f} \left(\min \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) + \max \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon, \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \right) \\ &\quad \text{since } \mathbf{f} \text{ is linear} \\ &= \sup_{\epsilon > 0} \mathbf{f} \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon + \frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \\ &\quad \text{by the equality } \min(a, b) + \max(a, b) = a + b \\ &= \sup_{\epsilon > 0} \left[\mathbf{f} \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon \right) + \mathbf{f} \left(\frac{1}{\epsilon} \chi_{V^{d,-(\epsilon)}}^\epsilon \right) \right] \\ &\quad \text{since } \mathbf{f} \text{ is linear} \\ &= \nu(U) + \nu(V) \end{aligned}$$

again using the fact that $+$ is Scott-continuous to distribute sups over sums.

When $\mathbf{f}$ is normalized, i.e., $\mathbf{f}(\chi_X) = 1$, then $\nu(X) = \sup_{\epsilon > 0} \mathbf{f}(\frac{1}{\epsilon} \chi_{X^{d,-(\epsilon)}}^\epsilon) = \mathbf{f}(\chi_X) = 1$, using Lemma 28. So ν is normalized. $\square$

Lemma 31. *Under the assumptions of Proposition 11:*

$$\nu(U^{d,-(2\epsilon)}) \leq \mathbf{f} \left(\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon \right)$$

for every open subset U of X and for every $\epsilon > 0$.

Proof. For every $\epsilon' > 0$, $\frac{1}{\epsilon'} \chi_{(U^{d,-(2\epsilon)})^{d,-(\epsilon')}}^{\epsilon'} \leq \chi_{(U^{d,-(2\epsilon)})}$ by Lemma 26, and this is at most $\frac{1}{\epsilon} \chi_{U^{d,-(\epsilon)}}^\epsilon$, again by Lemma 26. Now apply $\mathbf{f}$ on each side, and take sups over $\epsilon' > 0$. $\square$

Lemma 32. *Under the assumptions of Proposition 11, and if moreover ν is continuous,*

$$\oint_{x \in X} g(x) d\nu \leq \mathbf{f}(g)$$

for every bounded Lipschitz map g from X to $\mathbb{R}$.

Proof. Observe that (see e.g., [24]) any element g of $\langle X \rightarrow \mathbb{R} \rangle$ is the sup of a directed family of step functions, namely $g_K = a + \frac{1}{2^K} \sum_{k=1}^N \chi_{U_k}$, $K \in \mathbb{N}$, where $N = \lfloor (b-a)2^K \rfloor$, $U_k = g^{-1}(a + \frac{k}{2^K}, +\infty)$, a is any lower bound for g and b is any upper bound for g . For every $\epsilon > 0$, let:

$$g_K^\epsilon = a + 1/2^K \sum_{k=1}^N \frac{1}{\epsilon} \chi_{U_k^{d, -(\epsilon)}} \quad g_K^{(\epsilon)} = a + 1/2^K \sum_{k=1}^N \chi_{U_k^{d, -(\epsilon)}}$$

The function g_K^ϵ is Lipschitz, as sum of Lipschitz map, and $g_K^\epsilon \leq g$.

For every fixed K , $g_K^{(2\epsilon)}$ is antitonic in ϵ , because $U_k^{d, -(2\epsilon)}$ is antitonic in ϵ for each k , by Lemma 12. So the family $(g_K^{(2\epsilon)})_{\epsilon > 0}$ is directed. Its sup is g_K , using Lemma 12 again. The family $(g_K)_{K \in \mathbb{N}}$ is also directed, with sup g . Recall [9] that the Choquet integral of a step function $a + \sum_{i=1}^n a_i \chi_{U_i}$, $U_1 \supseteq U_2 \supseteq \dots \supseteq U_n$, along ν is $a\nu(X) + \sum_{i=1}^n a_i \nu(U_i)$ (even when ν is not a valuation). Since ν is continuous,

$$\begin{aligned} \oint_{x \in X} g(x) d\nu &= \sup_{K \in \mathbb{N}} \oint_{x \in X} g_K(x) d\nu = \sup_{\substack{K \in \mathbb{N} \\ \epsilon > 0}} \oint_{x \in X} g_K^{(2\epsilon)}(x) d\nu \\ &= \sup_{\substack{K \in \mathbb{N} \\ \epsilon > 0}} \left[a\nu(X) + \frac{1}{2^K} \sum_{k=1}^N \nu(U_k^{d, -(2\epsilon)}) \right] \\ &= \sup_{\substack{K \in \mathbb{N} \\ \epsilon > 0}} \left[a\mathbf{f}(\chi_X) + \frac{1}{2^K} \sum_{k=1}^N \nu(U_k^{d, -(2\epsilon)}) \right] \\ &\quad \text{since } \nu(X) = \sup_{\epsilon > 0} \mathbf{f}\left(\frac{1}{\epsilon} \chi_{X^{d, -(\epsilon)}}\right) = \mathbf{f}(\chi_X) \text{ by Lemma 28} \\ &\leq \sup_{\substack{K \in \mathbb{N} \\ \epsilon > 0}} \left[a\mathbf{f}(\chi_X) + \frac{1}{2^K} \sum_{k=1}^N \mathbf{f}\left(\frac{1}{\epsilon} \chi_{U_k^{d, -(\epsilon)}}\right) \right] \quad \text{by Lemma 31} \\ &= \sup_{\substack{K \in \mathbb{N} \\ \epsilon > 0}} \mathbf{f}(g_K^\epsilon) \quad \text{since } \mathbf{f} \text{ is linear} \\ &\leq \mathbf{f}(g) \end{aligned}$$

since $g_K^\epsilon \leq g$ and $\mathbf{f}$ is monotonic. $\square$

Recall that the hemi-metric d^2 on $X^{(2)}$ is defined by $d^2((x, y), (x', y')) = d(x, x') + d(y', y)$.

Lemma 33. *Let X be equipped with the hemi-metric d . For every $\epsilon > 0$, for every open subset U of X , and every open subset V of X^{op} ,*

$$\begin{aligned} \chi_{U \times X^{op}}^\epsilon(x, y) &= \chi_U^\epsilon(x) & \chi_{X \times V}^\epsilon(x, y) &= \chi_V^\epsilon(y) \\ (U \times X^{op})^{d^2, -(\epsilon)} &= U^{d, -(\epsilon)} \times X^{op} & (X \times V)^{d^2, -(\epsilon)} &= X \times V^{d^{op}, -(\epsilon)} \\ \frac{1}{\epsilon} \chi_{(U \times X^{op})^{d^2, -(\epsilon)}}^\epsilon(x, y) &= \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon(x) & \frac{1}{\epsilon} \chi_{(X \times V)^{d^2, -(\epsilon)}}^\epsilon(x, y) &= \frac{1}{\epsilon} \chi_{V^{d^{op}, -(\epsilon)}}^\epsilon(y) \end{aligned}$$

for all $x \in X, y \in Y$.

Proof. First, for all $x \in X$, $y \in Y$, $d^2((x, y), X^{(2)} \setminus (U \times X^{op})) = d^2((x, y), (X \setminus U) \times X^{op}) = \inf_{x' \in X \setminus U, y' \in X^{op}} [d(x, x') + d(y', y)] = \inf_{x' \in X \setminus U} d(x, x') + \inf_{y' \in X^{op}} d(y', y) = d(x, X \setminus U) + 0 = d(x, X \setminus U)$ (take $y' = y$ in the last inf). It follows that $\chi_{U \times X^{op}}^\epsilon(x, y) = \min(\epsilon, d^2((x, y), X^{(2)} \setminus (U \times X^{op}))) = \min(\epsilon, d(x, X \setminus U)) = \chi_U^\epsilon(x)$. It also follows that $(U \times X^{op})^{d^2, -(\epsilon)} = \{(x, y) \in X^{(2)} \mid d^2((x, y), X^{(2)} \setminus (U \times X^{op})) > \epsilon\} = \{(x, y) \in X^{(2)} \mid d(x, X \setminus U) > \epsilon\} = U^{d, -(\epsilon)} \times X^{op}$. The equalities $\chi_{X \times V}^\epsilon(x, y) = \chi_V(y)$ and $(X \times V)^{d^2, -(\epsilon)} = X \times V^{d^{op}, -(\epsilon)}$ are shown similarly. The third row is a consequence of the other two. $\square$

Theorem 5. *Let X be a sym-compact space, with bounded hemi-metric d . For all continuous probability valuations p, p' on X , $d_H(\nu, \nu') = \min_{p^2} \int_{(x,y) \in X^{(2)}} d(x, y) dp^2$, where p^2 ranges over the elements of $\mathbf{V}_1(X^{(2)})$ with $\pi_1[p^2] = p$, $\pi_2[p^2] = p'^\perp$.*

Proof. Let us recap what we know. Since X is sym-compact, by Theorem 1 X is stably compact, X^{op} too and has the same topology as X^d . So the notation $p'^\perp$ is meaningful. Also, $p'^\perp$ is a continuous probability valuation on X^d . The projection maps π_1 and π_2 are continuous from $X \times X^d$ to X and X^d respectively; $X \times X^d$ coincides, as topological space, with $X \times X^{op}$, hence with $X^{(2)}$ by Lemma 1. In particular, $\pi_2[p^2]$ is a continuous probability valuation on X^d : requiring $\pi_2[p^2] = p'^\perp$ is therefore also meaningful.

1. If p^2 is a continuous probability valuation on $X^{(2)}$ such that $\pi_1[p^2] = p$ and $\pi_2[p^2] = p'^\perp$, let us show that $d_H(p, p') \leq \int_{(x,y) \in X^{(2)}} d(x, y) dp^2$. Apply Lemma 23 to the functional $\mathbf{k}$ defined on each Lipschitz map h from $X^{(2)}$ to $\mathbb{R}$ by $\mathbf{k}(h) = \int_{(x,y) \in X^{(2)}} h(x, y) dp^2$: $\mathbf{k}$ is linear because $\alpha_{\mathbb{C}}(p^2)$ is, monotonic, positive since monotonic and linear, normalized ($\mathbf{k}(\chi_{X^{(2)}}) = 1$) because p^2 is normalized. Its first marginal is p , since for every Lipschitz map f from X to $\mathbb{R}$, $\mathbf{k}(f \odot 0) = \int_{(x,y) \in X^{(2)}} f(x) dp^2 = \int_{(x,y) \in X^{(2)}} f(\pi_1(x, y)) dp^2 = \int_{x \in X} f(x) d\pi_1[p^2] = \int_{x \in X} f(x) dp$, since by assumption $\pi_1[p^2] = p$. Its second marginal is $p'^\perp$, since for every Lipschitz map g from X^{op} to $\mathbb{R}$, $\mathbf{k}(0 \odot g) = \int_{(x,y) \in X^{(2)}} -g(y) dp^2 = \int_{(x,y) \in X^{(2)}} -g(\pi_2(x, y)) dp^2 = \int_{y \in X^d} -g(y) dp'^\perp = - \int_{y \in X} g(y) dp'$, using Lemma 2. So we can indeed apply Lemma 23.

2. Conversely, let us show that there is a continuous probability valuation p^2 on $X^{(2)}$ such that $\pi_1[p^2] = p$ and $\pi_2[p^2] = p'^\perp$, and such that $\int_{(x,y) \in X^{(2)}} d(x, y) dp^2 \leq d_H(p, p')$. (Equality will then follow, using item 1.) By Proposition 10, there is a normalized LL-prevision $\mathbf{k} \in \mathbf{K}(p, p')$ such that $\mathbf{k}(d) = d_H(p, p')$. Note that $X^{(2)}$ coincides, qua topological space, with $X \times X^d$; as a product of stably compact spaces, $X^{(2)}$ is stably compact (see [14]). Let $p^2(W) = \sup_{\epsilon > 0} \mathbf{k}(\frac{1}{\epsilon} \chi_{W^{d, -(\epsilon)}})$ for every open subset W of $X^{(2)}$. That is, we get p^2 from $\mathbf{k}$ just as we got p from $\mathbf{f}$ in Proposition 11. So p^2 is a continuous probability valuation on $X^{(2)}$. Moreover, $\int_{(x,y) \in X^{(2)}} \varphi(x, y) dp^2 \leq \mathbf{k}(\varphi)$ for every bounded Lipschitz map φ from $X^{(2)}$ to $\mathbb{R}$. Take $\varphi = d$, then it follows that $\int_{(x,y) \in X^{(2)}} d(x, y) dp^2 \leq \mathbf{k}(d) = d_H(p, p')$.

Next, for every open subset U of X ,

$$\begin{aligned}
\pi_1[p^2](U) &= p^2(U \times X^{op}) = \sup_{\epsilon > 0} \mathbf{k}\left(\frac{1}{\epsilon} \chi_{(U \times X^{op})^{d^2, -(\epsilon)}}^\epsilon\right) \\
&= \sup_{\epsilon > 0} \mathbf{k}\left(\frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon \ominus 0\right) \quad \text{by Lemma 33} \\
&= \sup_{\epsilon > 0} \oint_{x \in X} \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon(x) dp \quad \text{since the first marginal of } \mathbf{k} \text{ is } p \\
&= \oint_{x \in X} \chi_U(x) dp \quad \text{since } \sup_{\epsilon > 0} \frac{1}{\epsilon} \chi_{U^{d, -(\epsilon)}}^\epsilon = \chi_U \text{ by Lemma 26,} \\
&\quad \text{and using Proposition 26, knowing that } p \text{ is continuous} \\
&= p(U)
\end{aligned}$$

Since U is arbitrary, $\pi_1[p^2] = p$. We show $\pi_2[p^2] = p'^\perp$, with a variant due to our need to integrate along $p'^\perp$ instead of p' . For every open subset V of X^{op} :

$$\begin{aligned}
\pi_2[p^2](V) &= p^2(X \times V) = \sup_{\epsilon > 0} \mathbf{k}\left(\frac{1}{\epsilon} \chi_{(X \times V)^{d^2, -(\epsilon)}}^\epsilon\right) \\
&= \sup_{\epsilon > 0} \mathbf{k}\left(0 \ominus \frac{-1}{\epsilon} \chi_{V^{d^{op}, -(\epsilon)}}^\epsilon\right) \quad \text{by Lemma 33} \\
&= \sup_{\epsilon > 0} - \oint_{y \in X} \frac{-1}{\epsilon} \chi_{V^{d^{op}, -(\epsilon)}}^\epsilon(x) dp' \quad \text{since the second marginal of } \mathbf{k} \text{ is } p' \\
&= \sup_{\epsilon > 0} \oint_{y \in X^{op}} \frac{1}{\epsilon} \chi_{V^{d^{op}, -(\epsilon)}}^\epsilon(x) dp'^\perp \quad \text{by Lemma 2} \\
&= \oint_{y \in X^{op}} \chi_V(y) dp'^\perp \quad \text{by Lemma 26 and Proposition 26} \\
&= p'^\perp(V)
\end{aligned}$$

So $\pi_2[p^2] = p'^\perp$. □

J Proof of the Minimax Theorem 6

We require the following consequence of Roth's Sandwich Theorem (see Section G).

Lemma 34. *Let Z be a subset of an ordered cone C , U a convex upward-closed subset of Z , F a convex subset of Z , such that U and F are non-empty and disjoint. Then there is a monotonic linear function $f : C \rightarrow \overline{\mathbb{R}}^+$ such that $f(x) \leq 1$ for every $x \in F$, and $f(y) \geq 1$ and for every $y \in U$.*

Proof. Consider the downward-closure $\downarrow F = \{x \in C \mid \exists y \in F \cdot x \leq y\}$ of F in C , and the upward-closure $\uparrow U = \{y \in C \mid \exists x \in U \cdot x \leq y\}$ of U in C . Then $\uparrow U$ and $\downarrow F$ are disjoint, else there would be $z \in C$, $x \in F$, $y \in U$ such that $y \leq z \leq x$, implying $x \in U$.

Let $q(x) = \sup_{\lambda > 0 / \lambda \cdot x \in \uparrow U} 1/\lambda$, the sup being taken to be 0 if no such λ exists. First, $q(0) = 0$. Indeed, there is no $\lambda > 0$ such that $\lambda \cdot 0 \in \uparrow U$, otherwise 0 would be in $\uparrow U$, hence in U . (In any ordered cone, 0 is the least element.) Since U is upward-closed, we would have $U = Z$, contradicting the fact that F is non-empty and disjoint from U . Then, for every $a > 0$, $q(a \cdot x) = \sup_{\lambda > 0 / \lambda \cdot (a \cdot x) \in \uparrow U} 1/\lambda = \sup_{\lambda' > 0 / \lambda' \cdot x \in \uparrow U} 1/(\lambda'/a)$ (taking $\lambda' = \lambda a$) $= aq(x)$. So q is positively homogeneous. Let us show that q is super-additive. We first observe that $\uparrow U$ is convex: if $a, b \in \uparrow U$, then there are $a', b' \in U$ such that $a' \leq a$ and $b' \leq b$; for every $r \in [0, 1]$, $r \cdot a' + (1 - r) \cdot b' \in U$ since U is convex, and $r \cdot a' + (1 - r) \cdot b' \leq r \cdot a + (1 - r) \cdot b$ since $+$ and $\cdot$ are monotonic. Let $x, x' \in C$. Then:

$$q(x) + q(x') = \sup_{\substack{\lambda, \lambda' > 0 \\ \lambda \cdot x \in \uparrow U \\ \lambda' \cdot x' \in \uparrow U}} \left(\frac{1}{\lambda} + \frac{1}{\lambda'} \right) \leq \sup_{\substack{\lambda, \lambda' > 0 \\ \lambda' / (\lambda + \lambda') \cdot (\lambda \cdot x) + \lambda / (\lambda + \lambda') \cdot (\lambda' \cdot x') \in \uparrow U}} \frac{\lambda + \lambda'}{\lambda \lambda'}$$

since $\uparrow U$ is convex. So $q(x) + q(x') \leq \sup_{\lambda \lambda' / (\lambda + \lambda') \cdot (x + x') \in \uparrow U} (\lambda + \lambda') / (\lambda \lambda') \leq q(x + x')$.

Let then $p(x) = \inf_{\lambda > 0 / \lambda \cdot x \in \downarrow F} 1/\lambda$, this being $+\infty$ if no such λ exists. For every $\lambda > 0$, $\lambda \cdot 0 = 0$ is in $\downarrow F$, since 0 is the least element of C and F is non-empty. So $p(0) = 0$. By similar arguments as above, p is positively homogeneous and sub-linear.

Moreover, for all $a, b \in C$ such that $a \leq b$, for every $\lambda > 0$ such that $\lambda \cdot a \in \uparrow U$ and $\lambda' > 0$ such that $\lambda' \cdot b \in \downarrow F$, necessarily $\lambda > \lambda'$. Otherwise, $\lambda \leq \lambda'$, so $\lambda \cdot a \leq \lambda' \cdot a \leq \lambda' \cdot b$. Since $\lambda \cdot a \in \uparrow U$, $\lambda' \cdot b$ would also be in $\uparrow U$. Since it is already in $\downarrow F$, this would contradict the fact that $\uparrow U$ and $\downarrow F$ are disjoint. So indeed $\lambda > \lambda'$. In particular, $1/\lambda < 1/\lambda'$. Take sups over λ , infs over λ' , getting $q(a) \leq p(b)$, for every $a, b \in C$.

So we can apply Roth's Sandwich Theorem: there is a monotonic linear map f from C to $\mathbb{R}^+$ such that $q \leq f \leq p$. For every $x \in F$, $p(x) \leq 1$ since $\lambda = 1$ satisfies the condition $\lambda \cdot x \in \downarrow F$, hence $f(x) \leq 1$. For every $x \in U$, similarly, $q(x) \geq 1$, so $f(x) \geq 1$. $\square$

A simple probability valuation is any linear combination $\sum_{y \in B} b_y \delta_y$, where B is finite, each b_y is in $\mathbb{R}^+$, and $\sum_{y \in B} b_y = 1$.

Proposition 12. Let X be a non-empty compact space, Y a set. Let f be a function from $X \times Y$ to $\mathbb{R}$, such that $\lambda x \in X \cdot f(x, y)$ is continuous for every $y \in Y$.

Then the $\inf_{x \in X} \sup_{y \in Y} f(x, y)$ is attained: there is an $x \in X$ that minimizes $\sup_{y \in Y} f(x, y)$.

If moreover f is convex in X , then for every $t \in \mathbb{R}$, $\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq t$ iff $\inf_{x \in X} \sum_{y \in B} b_y f(x, y) \leq t$ for every simple probability valuation $\sum_{y \in B} b_y \delta_y$ on Y .

Proof. Let $L(x) = \sup_{y \in Y} f(x, y)$. As a sup of continuous maps, L is continuous: $L^{-1}(t, +\infty) = \{x \in X \mid \sup_{y \in Y} f(x, y) > t\} = \{x \in X \mid \exists y \in Y \cdot f(x, y) > t\} = \bigcup_{y \in Y} (\lambda x \in X \cdot f(x, y))^{-1}(t, +\infty)$ is open for every t .

Since X is compact, the direct image $L(X)$ is compact and non-empty in $\mathbb{R}$, so it has a least element r . Let x_0 be such that $L(x_0) = r$. Then x_0 is the element x that minimizes $L(x)$ that we were looking for.

Let us show the second part of the proposition. For every simple probability valuation $\sum_{y \in B} b_y \delta_y$ on Y , we get $\sum_{y \in B} b_y f(x, y) \leq \sum_{y \in B} b_y L(x) = L(x)$. So $\inf_{x \in X} \sum_{y \in B} b_y f(x, y) \leq \inf_{x \in X} L(x)$, hence if $\inf_{x \in X} L(x) \leq t$, and therefore $\inf_{x \in X} \sum_{y \in B} b_y f(x, y) \leq t$.

Let us show the converse implication. Assume, by contradiction, that $\inf_{x \in X} L(x) > t$. Fix an arbitrary t_2 strictly between t and $\inf_{x \in X} L(x) > t$. Let $t_1 = (t + t_2)/2$, so that $\inf_{x \in X} L(x) > t_2 > t_1 > t$. Note that $\{x \in X \mid L(x) > t_1\}$ is the whole of X , but can also be written as $\{x \in X \mid \exists y \in Y \cdot f(x, y) > t_1\}$, i.e., as $\bigcup_{y \in Y} U_y$, where $U_y = \{x \in X \mid f(x, y) \leq t_1\}$ is open, since $\lambda x \in X \cdot f(x, y)$ is continuous. Since X is compact, there is a finite subset $B = \{y_1, \dots, y_n\}$ of Y such that $\bigcup_{y \in B} U_y = X$. I.e.: (*) for every $x \in X$, there is an i , $1 \leq i \leq n$, such that $f(x, y_i) \leq t_1$.

For each i , $1 \leq i \leq n$, the map $\lambda x \in X \cdot f(x, y_i)$ is continuous. So it reaches its min r_i . Let $r = \min(r_1, \dots, r_n)$. (This is meaningful, as $n \neq 0$: since X is non-empty, fix $x \in X$. By (*), there is an i such that $1 \leq i \leq n$, in particular; so $n \geq 1$.) Let a be a strictly positive real such that $a(r - t_2) \geq -1$. If $r \geq t_2$, pick any $a > 0$. Otherwise, pick $a \leq 1/(t_2 - r)$. Define h by $h(x) = (1 + a(f(x, y_1) - t_2), \dots, 1 + a(f(x, y_n) - t_2))$; h takes its values in $\mathbb{R}^{+n}$: for every $x \in X$, $f(x, y_i) \geq r$, so $a(f(x, y_i) - t_2) \geq a(r - t_2) \geq -1$, for every i , $1 \leq i \leq n$.

Let $U = \uparrow h(X)$. U is non-empty, since X is non-empty. Clearly, U is upward-closed. Let us show that U is convex. For all $z_1, z_2 \in U$, there are $x_1, x_2 \in X$ such that $h(x_1) \leq z_1$ and $h(x_2) \leq z_2$. Fix an arbitrary α in $(0, 1)$. Since f is convex in X , there is an $x_0 \in X$ such that $f(x_0, y_i) \leq \alpha f(x_1, y_i) + (1 - \alpha)f(x_2, y_i)$ for all i , $1 \leq i \leq n$. In particular, $h(x_0) \leq \alpha h(x_1) + (1 - \alpha)h(x_2) \leq \alpha z_1 + (1 - \alpha)z_2$. So $\alpha z_1 + (1 - \alpha)z_2$ is in $\uparrow h(X) = U$.

Let $F = \{\mathbf{1}\}$, where $\mathbf{1}$ is the all ones tuple. F is convex and non-empty. Let us show that U and F are disjoint. Otherwise, there would be an $x \in X$ with $h(x) \leq \mathbf{1}$, i.e., $1 + a(f(x, y_i) - t_2) \leq 1$ for all i , $1 \leq i \leq n$. By (*), for some i , $f(x, y_i) \leq t_1$. So $1 \leq 1 + a(t_1 - t_2)$, which contradicts the fact that $t_1 < t_2$ and a is strictly positive.

By Lemma 34, with $Z = C = \mathbb{R}^{+n}$, there is a monotonic linear map g from $\mathbb{R}^{+n}$ to $\overline{\mathbb{R}}^+$ such that $g(z) \leq 1$ for every $z \in F$ and $g(z) \geq 1$ for every $z \in U$. We may write g as: $g(z) = \sum_{i=1}^n b_i z_i$, where $b_i = g(0, \dots, 0, 1, 0, \dots, 0)$ (the only 1 being at position i). Since $g(z) \leq 1$ for every $z \in F$, $\sum_{i=1}^n b_i \leq 1$, in particular the b_i 's are real (not $+\infty$). Since $g(z) \geq 1$ for every $z \in U$, in particular when $z = h(x)$ ($x \in X$), we obtain $\sum_{i=1}^n b_i (1 + a(f(x, y_i) - t_2)) \geq 1$, so $\sum_{i=1}^n b_i f(x, y_i) \geq 1/a(1 - \sum_{i=1}^n b_i) + t_2 \geq t_2$. In particular, $\inf_{x \in X} \sum_{i=1}^n b_i f(x, y_i) \geq t_2 > t$. $\square$

Lemma 35. *Let X, Y be two sets, and f a map from $X \times Y$ to $\mathbb{R}$ that is concave in Y . For every simple probability valuation $\sum_{y \in B} b_y \delta_y$ on Y , $\inf_{x \in X} \sum_{y \in B} b_y f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y)$.*

Proof. Without loss of generality, assume every b_i to be non-zero. By induction on the cardinality of B , we show that there is a $y_0 \in Y$ such that, for every $x \in X$, $f(x, y_0) \geq \sum_{y \in B} b_y f(x, y)$. If B only contains one element, this is clear. Otherwise, pick $y_1 \in B$, let $B' = B \setminus \{y_1\}$, and for every $y \in B'$, let $b'_y = b_y/(1 - b_{y_1})$. This makes sense, as every b_y is non-zero and B' is non-empty, so $b_{y_1} < 1$. By induction hypothesis, and since $\sum_{y \in B'} b'_y = 1$, there is a $y_2 \in Y$ such that, for every $x \in X$,

$f(x, y_2) \geq \sum_{y \in B'} b'_y f(x, y)$. Since f is concave in Y , taking $\alpha = b_{y_1}$, there is a $y_0 \in Y$ such that, for every $x \in X$, $f(x, y_0) \geq b_{y_1} f(x, y_1) + (1 - b_{y_1}) f(x, y_2) \geq \sum_{y \in B} b_y f(x, y)$. Now take infs over x . $\square$

Theorem 6. *Let X be a non-empty compact space, Y a set. Let f be any map from $X \times Y$ to $\mathbb{R}$, such that $\lambda x \in X \cdot f(x, y)$ is continuous for every $y \in Y$. If f is convex in X and concave in Y , then $\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y)$. Moreover, the inf on the right is attained.*

Proof. The $\leq$ direction is easy. Conversely, let $t = \sup_{y \in Y} \inf_{x \in X} f(x, y)$, so that $\inf_{x \in X} \sum_{y \in B} b_y f(x, y) \leq t$ for every simple probability valuation $\sum_{y \in B} b_y \delta_y$ on Y , by Lemma 35. Then apply Proposition 12. $\square$