Temporal Logic with Forgettable Past

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We can express this property without F^{-1}:

$$\neg\Big(\ (\neg\,\mathsf{problem})\ \mathsf{U}\ (\mathsf{alarm}\ \wedge\ \neg\,\mathsf{problem}\ )\Big)$$

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The N modality ("from Now on") allows us to forget these past states:

$$\mathsf{G}\Big(\text{reset} \Rightarrow \mathsf{N}\,\mathsf{G}\,(\,\text{alarm} \Rightarrow \mathsf{F}^{-1}\,\text{problem}\,)\Big)$$

Outline

- Definitions
- Expressive power
  - Comparing expressive power of LTL and LTL+Past
  - of LTL+Past and LTL+Past+Now
- Decision procedures
  - Satisfiability, model checking
  - Complexity
  - Model checking a path
- Conclusion

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- **plus** all the standard abbreviations:

$$\mathsf{F}\,\varphi \stackrel{\text{def}}{=} \top\,\mathsf{U}\,\varphi \qquad \mathsf{G}\,\varphi \stackrel{\text{def}}{=} \neg\,\mathsf{F}\,\neg\,\varphi$$
$$\mathsf{F}^{-1}\,\varphi \stackrel{\text{def}}{=} \top\,\mathsf{S}\,\varphi \qquad \mathsf{G}^{-1}\,\varphi \stackrel{\text{def}}{=} \neg\,\mathsf{F}^{-1}\,\neg\,\varphi$$

The N operator

$NLTL$ formulae are interpreted over pairs π, i: a position along a labeled run (π, ξ).

Semantics of N:

Suffix $\pi(i)\pi(i+1)\ldots$

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Basic properties:

$$\mathsf{N}(\varphi \vee \psi) \equiv \mathsf{N}\varphi \vee \mathsf{N}\psi \qquad\qquad \mathsf{N}\neg\varphi \equiv \neg\mathsf{N}\varphi$$

$$\mathsf{N}\,\mathsf{X}^{-1}\varphi \equiv \bot \qquad\qquad \mathsf{N}(\varphi\,\mathsf{S}\,\psi) \equiv \mathsf{N}\psi$$

$$\mathsf{N}\varphi \equiv \varphi \qquad \text{if } \varphi \text{ is pure-future}$$

Expressive power

We use two notions of equivalence:

- (global) equivalence: $\equiv$

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- initial equivalence: $\equiv_i$

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- It is well known that past operators do not add expressive power:

  "Any $PLTL$ formula is *initially equivalent* to an LTL formula."

  where $PLTL$ is $L(\mathsf{U}, \mathsf{X}, \mathsf{S}, \mathsf{X}^{-1})$ and LTL is $L(\mathsf{U}, \mathsf{X})$. [Kam68, Gab89]

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What about succinctness ?

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$$\Phi \stackrel{\text{def}}{=} \mathsf{G}\Big[\Big(\bigwedge_{i=1}^{n}\big(p_i \Leftrightarrow \mathsf{F}^{-1}(\neg \mathsf{X}^{-1}\top \wedge p_i)\big)\Big) \Rightarrow \Big(p_0 \Leftrightarrow \mathsf{F}^{-1}(\neg \mathsf{X}^{-1}\top \wedge p_0)\Big)\Big]$$

states that "any future state that agrees with the initial state on $p_1, \ldots, p_n$ also agrees on p_0". Let Ψ be an LTL formula initially equivalent to Φ.

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- Therefore $(\mathsf{G}\,\Psi)$ expresses the first property and $|\Psi|$ is in $\Omega(2^n)$!

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The $NLTL$ formula $\mathsf{G}\,\mathsf{N}\,\Phi$ clearly states that "any two future states that agree on $p_1, \ldots, p_n$ also agree on p_0".
Then any equivalent $PLTL$ formula has a size $\Omega(2^n)$.

Verification problems

We are interested in:

- Satisfiability: Given ϕ, is there some π, i such that: $\pi, i \models \phi$?
- Initial satisfiability: Given ϕ, is there some π such that: $\pi, 0 \models \phi$?
- Model checking: Given ϕ and a Kripke structure K, do we have, for any run π of K: $\pi, 0 \models \phi$?
- ...

Deciding satisfiability for linear time TL

- Let Φ be an LTL or $PLTL$ formula. [VW94]
  Build a Büchi automaton $\mathcal{A}_\Phi = \langle Q, \rightarrow, F \rangle$ with $Q = 2^{\mathsf{SubF}(\Phi)}$.
  Φ is satisfiable $\Leftrightarrow$ there exists an accepting run in $\mathcal{A}_\Phi$.

$$\left.\begin{array}{l} |\mathcal{A}_\Phi| \text{ is in } O(2^{|\Phi|}) \\ \text{Emptiness in BA is NLOGSPACE-complete} \end{array}\right\} \Rightarrow \text{Algorithm in PSPACE.}$$

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These algorithms are optimal: LTL satisfiability is PSPACE-hard.

Deciding satisfiability for LTL

$\mathsf{SubF}(\Phi) =$ set of Φ subformulae + negations + $\mathsf{X}(\_\,\mathsf{U}\,\_)$.

States are *Atoms*: coherent subsets of $\mathsf{SubF}(\Phi)$

$a\mathsf{U}b$, b,
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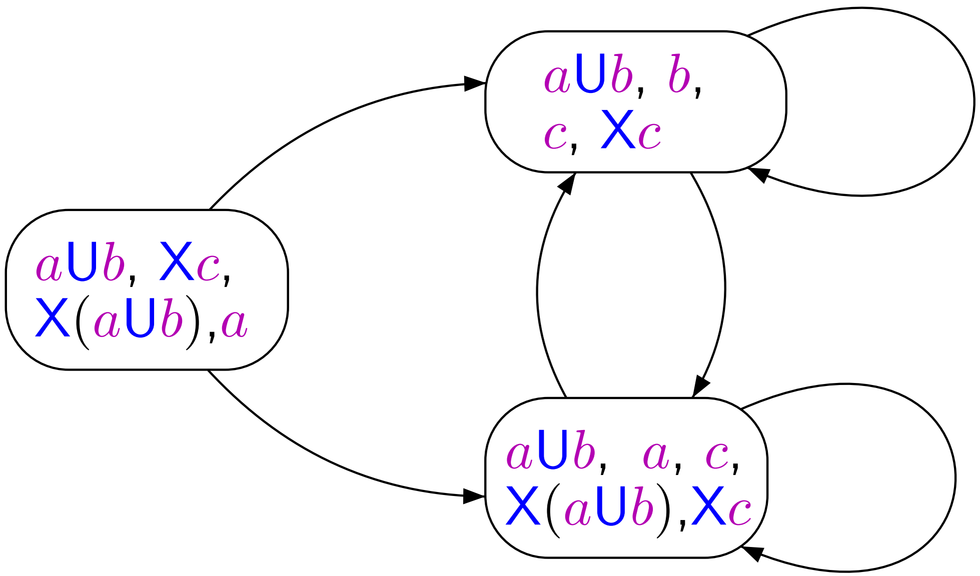

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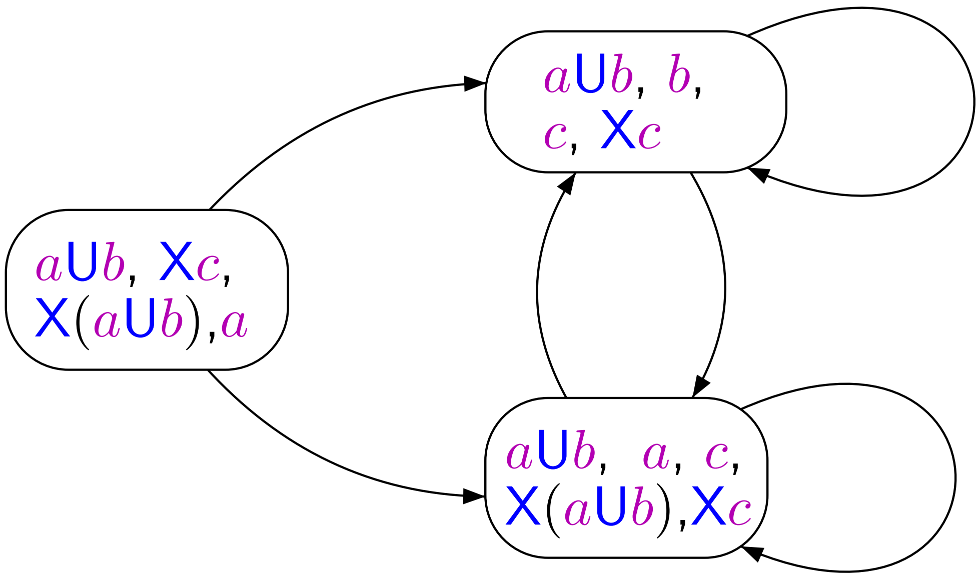

+ fairness conditions for Until formulae.

$\varphi \in \mathsf{SubF}(\Phi)$ is satisfiable **iff** there is an accepting run going through $A \ni \varphi$.

Deciding satisfiability for $PLTL$

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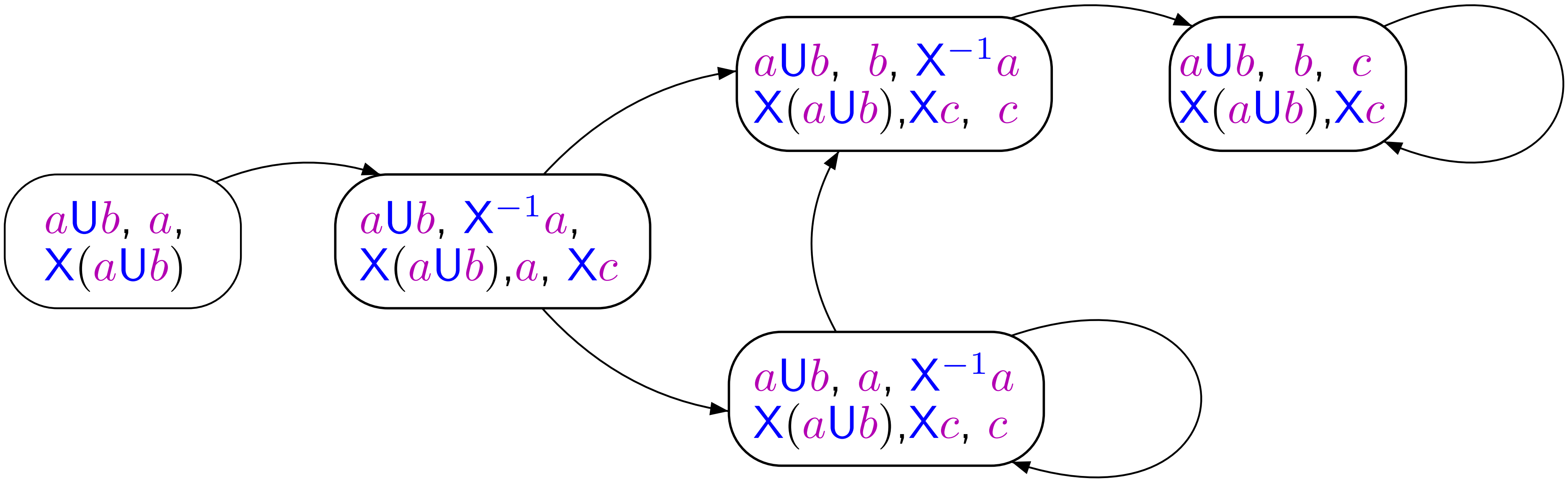

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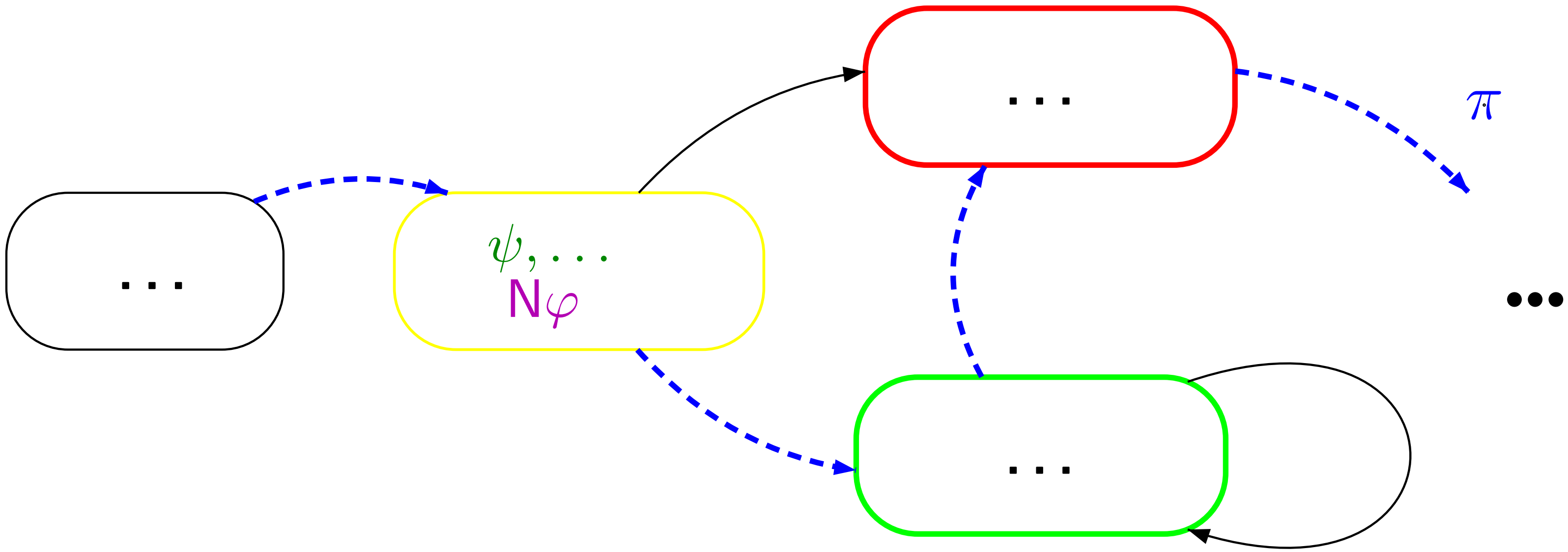
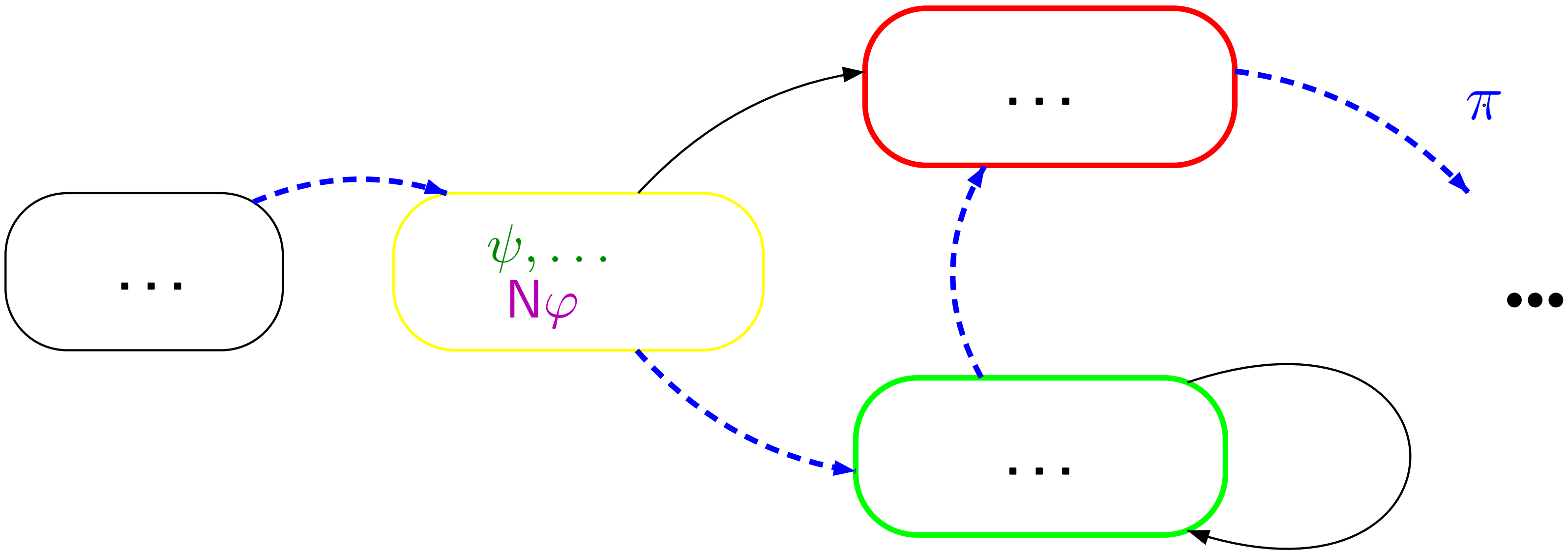


And *NLTL* ?

$$\boxed{\begin{array}{l} \pi, i \models \psi \\ \pi^i, 0 \models \varphi \end{array}}$$



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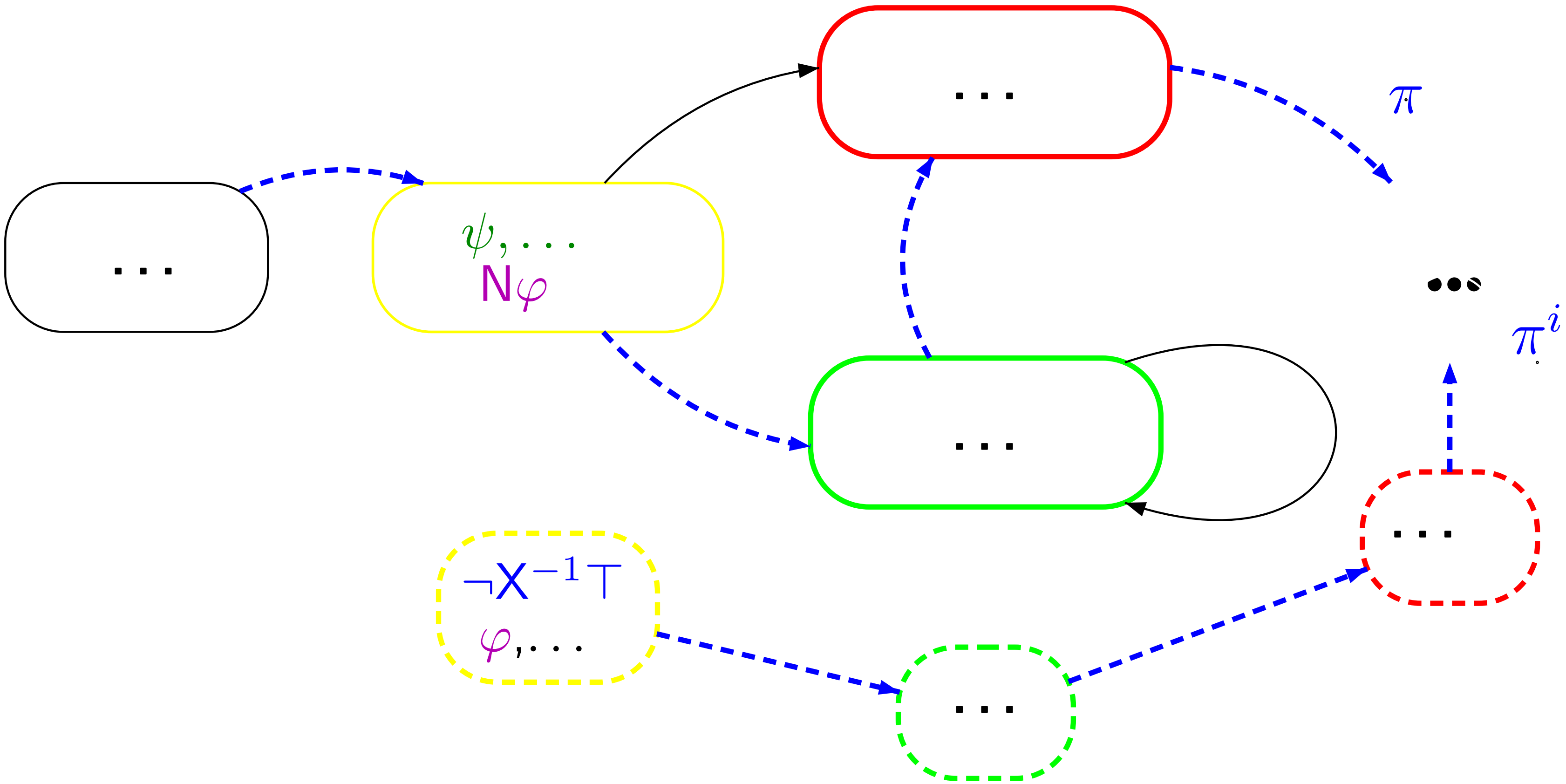

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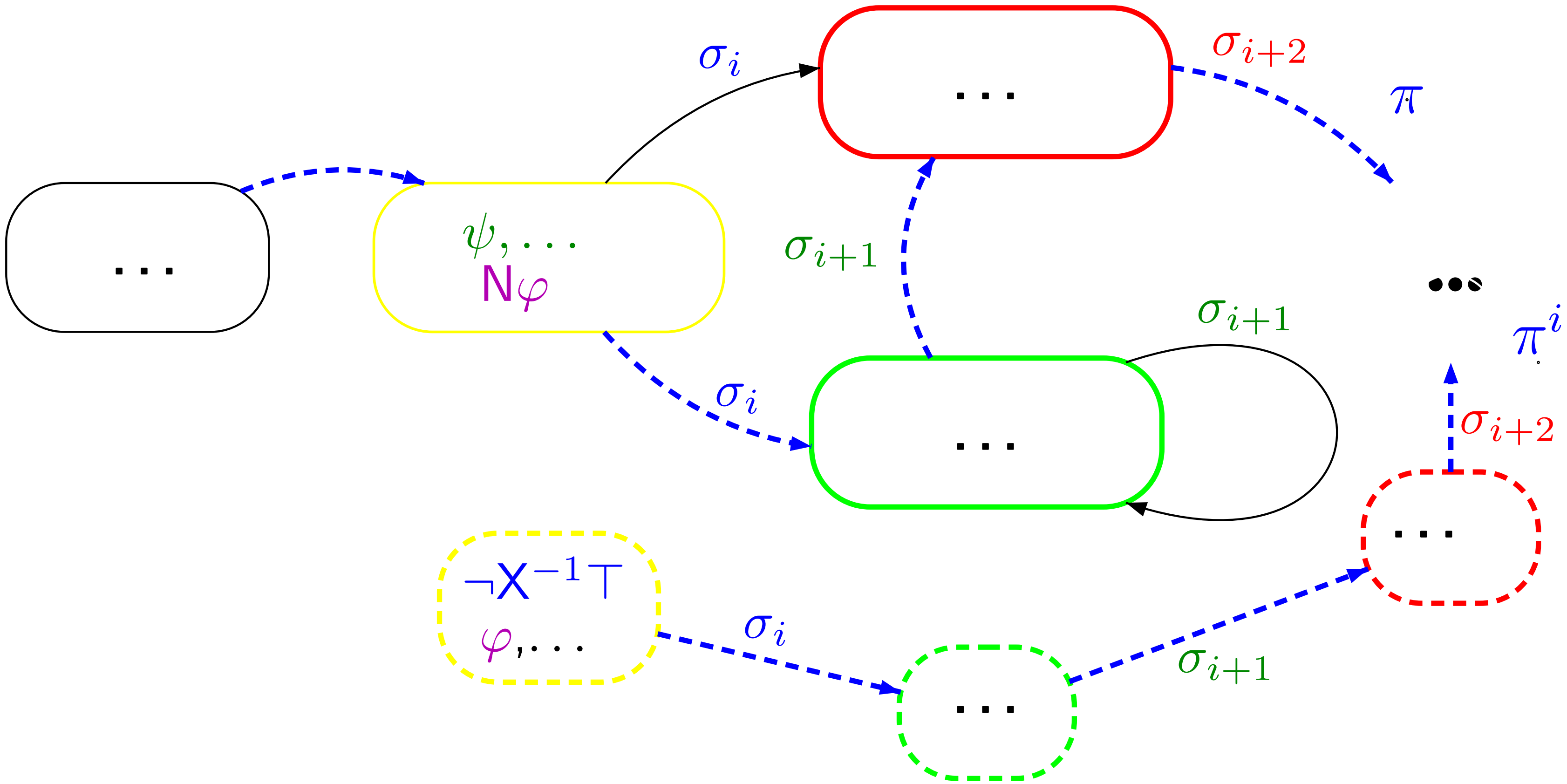
$\sigma_i \in 2^{AP}$: atomic propositions of $\pi(i)$.



Alternating Büchi automata for $NLTL$ formulae

$\mathcal{A}_{\Phi} = \langle \Sigma, S, \rho, S_0, \mathcal{F} \rangle$ is defined as follows:

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$$\mathsf{Now}(A') \stackrel{\text{def}}{=} \{A'' \in S_0 \mid \forall \mathsf{N}\alpha \in \mathsf{SubF}(\Phi), \mathsf{N}\alpha \in A' \Leftrightarrow \mathsf{N}\alpha, \alpha \in A''\}$$

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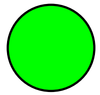
- $\mathcal{F} = \{F_1, \ldots, F_k\}$ with:

$$F_i \stackrel{\text{def}}{=} \{A \in \mathsf{Atom}(\Phi) \mid \neg \mathsf{X}^{-1}\top \in A \text{ or } \psi_i \in A \text{ or } \neg(\varphi_i \mathsf{U} \psi_i) \in A\}$$

An accepting run π

$\xi(\pi(i)) = \sigma_i$



Initial atom

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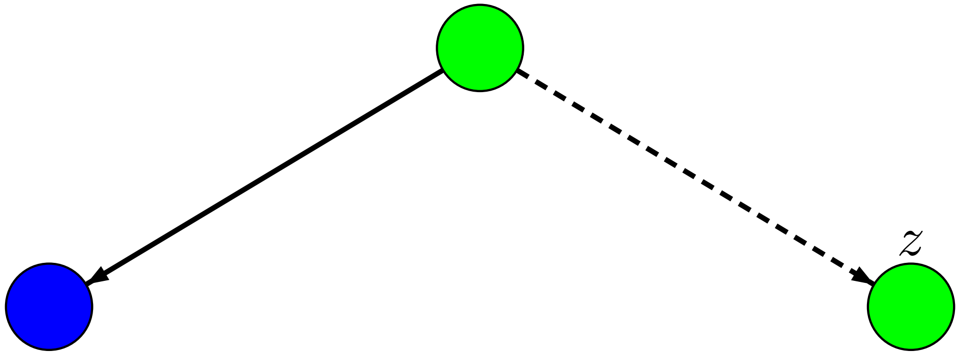

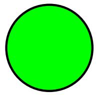

σ_1

An accepting run π

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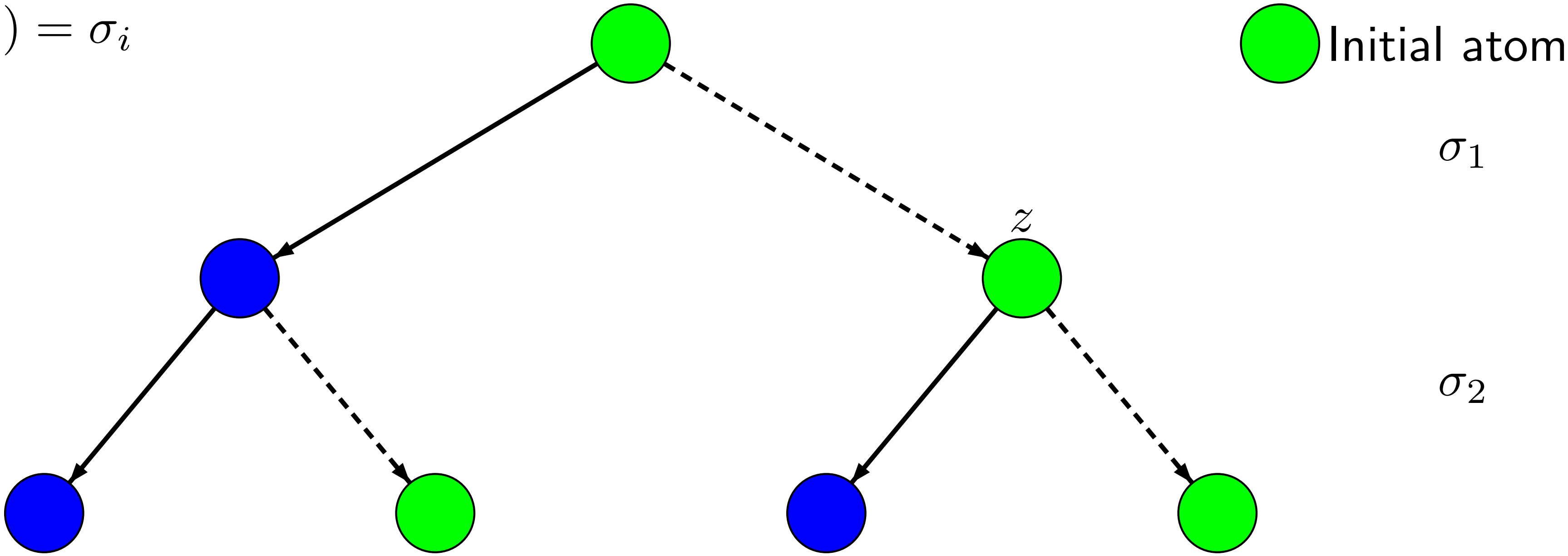

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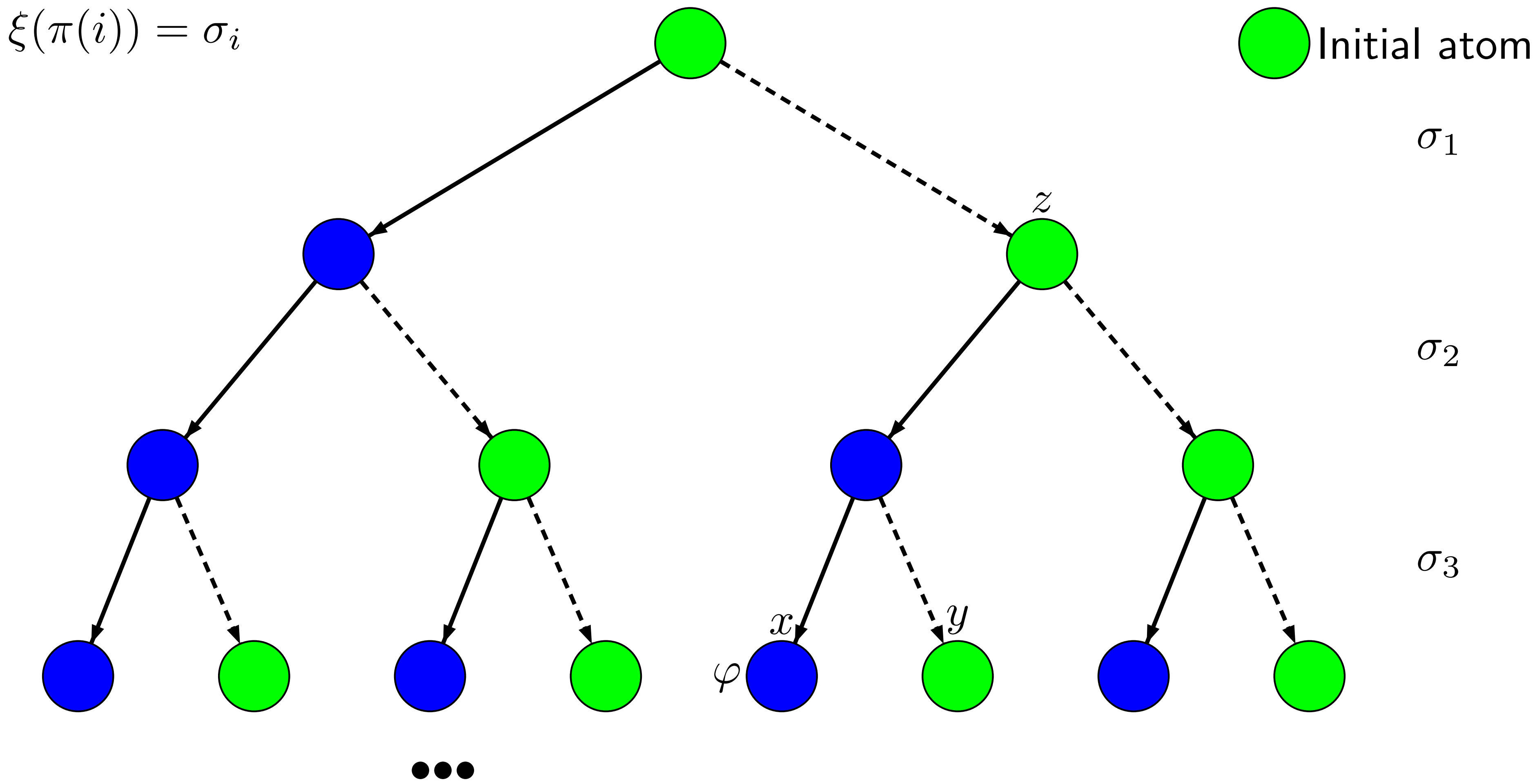

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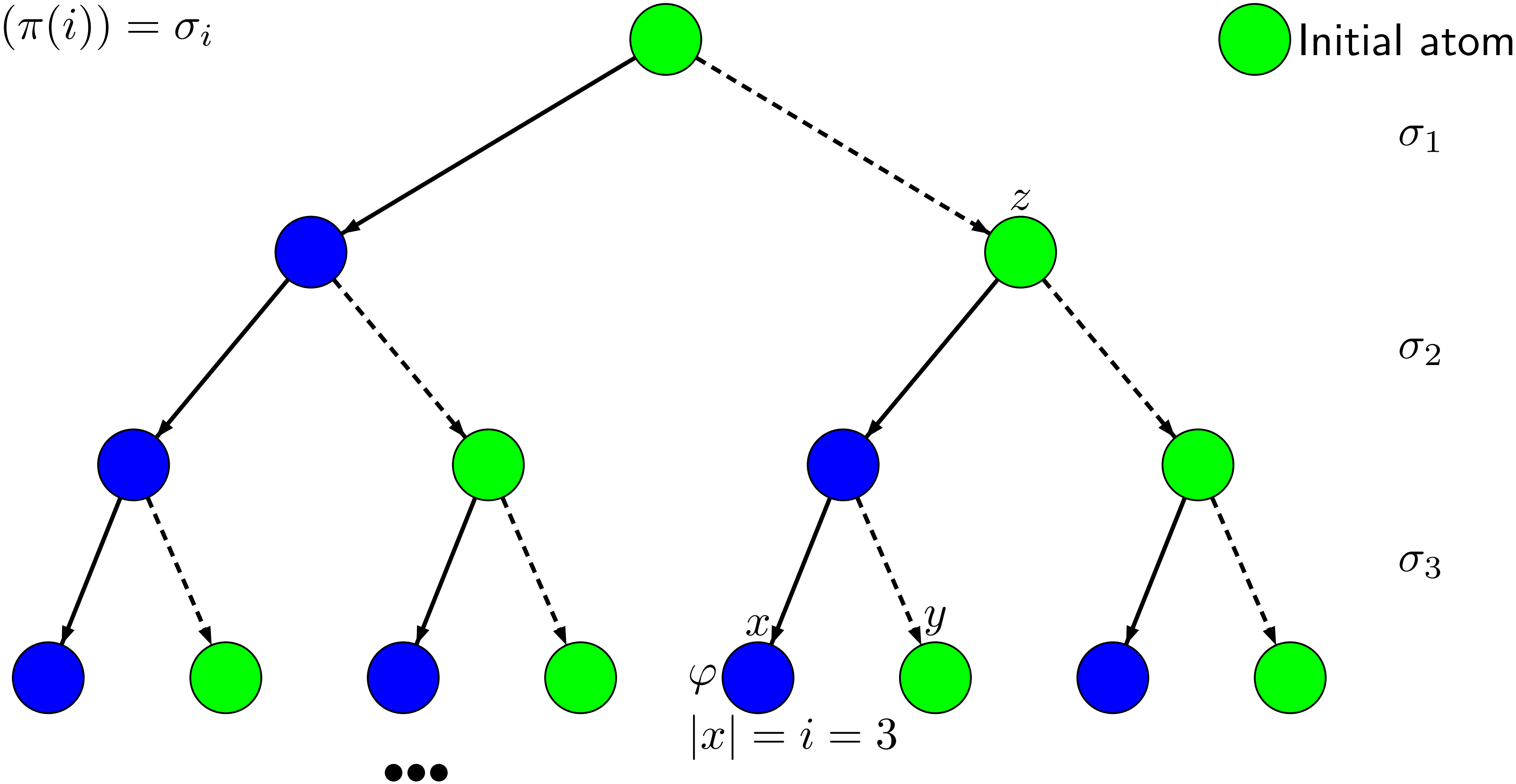

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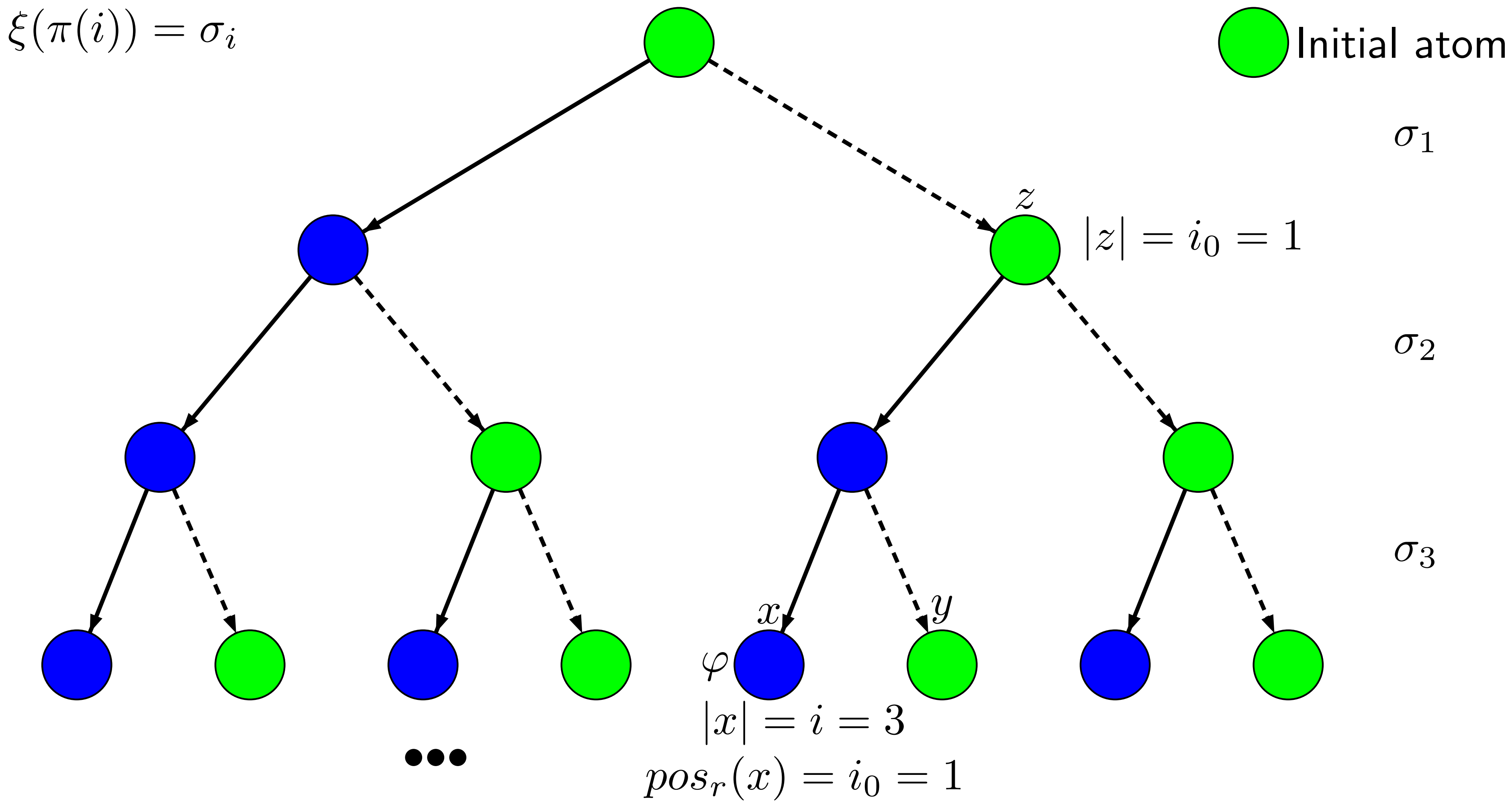

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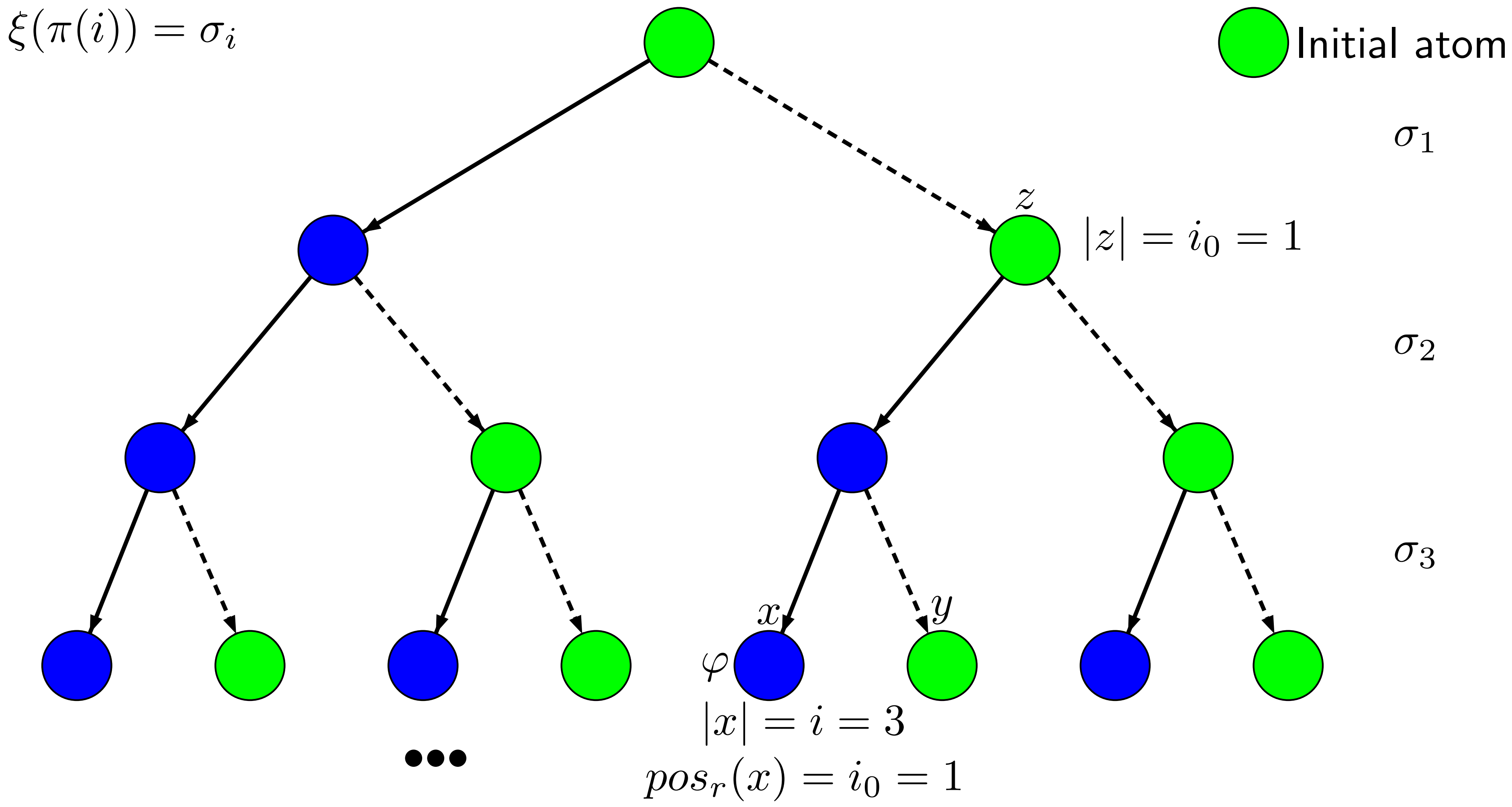

Lemma: $\pi^{i_0}, i - i_0 \models \varphi \quad \Leftrightarrow \quad \exists\, x \; s.t. \left(|x| = i, \quad pos_r(x) = i_0, \quad \varphi \in x \right)$

Alternating Büchi automata for $NLTL$ formulae

Proposition: Let Φ be an $NLTL$ formula, then Φ is satisfiable **iff** there exists an accepting run in $\mathcal{A}_{\mathsf{F}\Phi}$ starting from a node containing $\mathsf{F}\Phi$.

Proposition: Non-emptiness problem of alternating Büchi automaton can be solved in space polynomial in the size of the automaton. [VW94]

Theorem: Satisfiability for $NLTL$ formulae can be decided in EXPSPACE.

Theorem: Model checking Kripke structures for $NLTL$ formulae can be decided in EXPSPACE.

EXPSPACE-hardness of $NLTL$ model checking

Proposition: Satisfiability and model checking for $NLTL$ are $\mathrm{EXPSPACE}$-hard.

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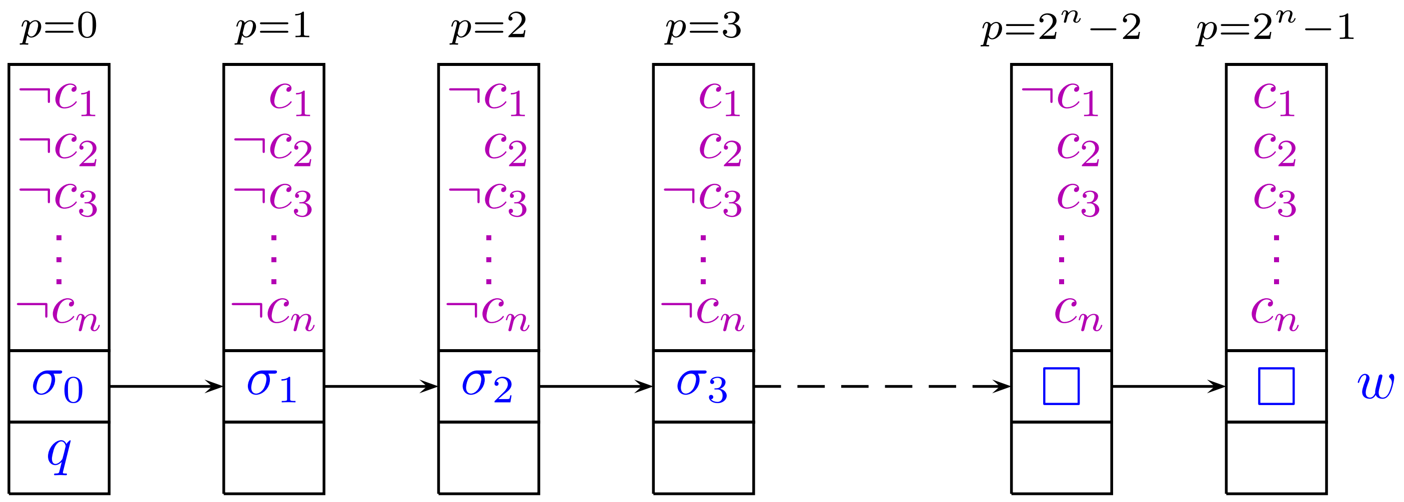
Let $\mathcal{M} = \langle \Sigma, Q_{\mathcal{M}}, q_0, q_F, \mathcal{T} \rangle$ be a Turing machine that operates in exponential space. The run of $\mathcal{M}$ on some input word w of length n can be described by an $NLTL$-formula Φ:

- The set AP contains Σ, $Q_{\mathcal{M}}$, and $\{c_1, \ ..., c_n\}$.
- A configuration of $\mathcal{M}$ is a sequence of 2^n states:



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- the 2^n cells of the tape are all present, in increasing order:

$$\mathsf{G}\Big((c_1 \Leftrightarrow \mathsf{X}\neg c_1) \wedge \bigwedge_{i=2}^{n} \Big((\neg(c_i \Leftrightarrow \mathsf{X}c_i)) \Leftrightarrow (c_{i-1} \wedge \mathsf{X}\neg c_{i-1})\Big)\Big)$$

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$$\mathsf{GN}\Big(\bigwedge_{(a,b,c)\rightarrow b'} (a \wedge \mathsf{X}b \wedge \mathsf{X}^2 c) \Rightarrow (\neg \phi_{p=0} \mathsf{U}(\phi_{p=0} \wedge \mathsf{X}(\neg \phi_{p=0} \mathsf{U}(\phi_{sv} \wedge \mathsf{X}b')))\Big)$$

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This is to check whether we can apply the transition.

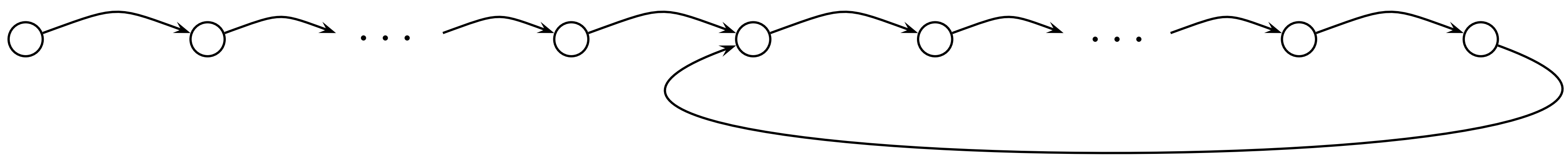
This ensures that we go to the very next configuration.

Here we control that the transition has been correctly applied.

Model checking a path

Theorem: Model checking an $NLTL$-formula ϕ along an ultimately-periodic path can be done in polynomial-time.

For LTL, we can directly apply CTL algorithm since each state has exactly one successor.



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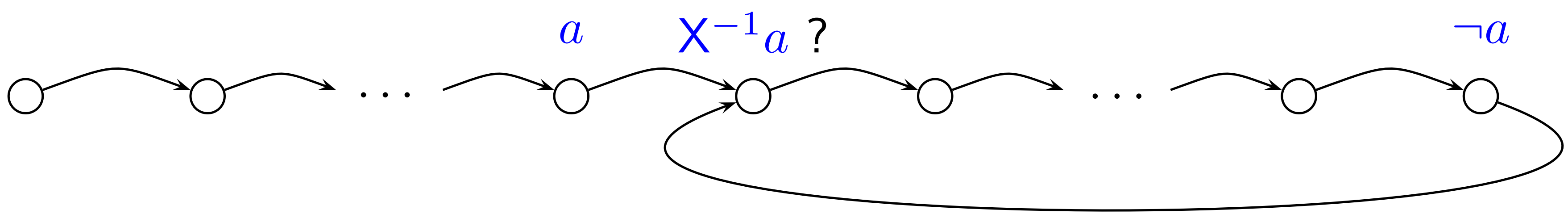

But this does not apply for $PLTL$ because some states don't have exactly one predecessor.

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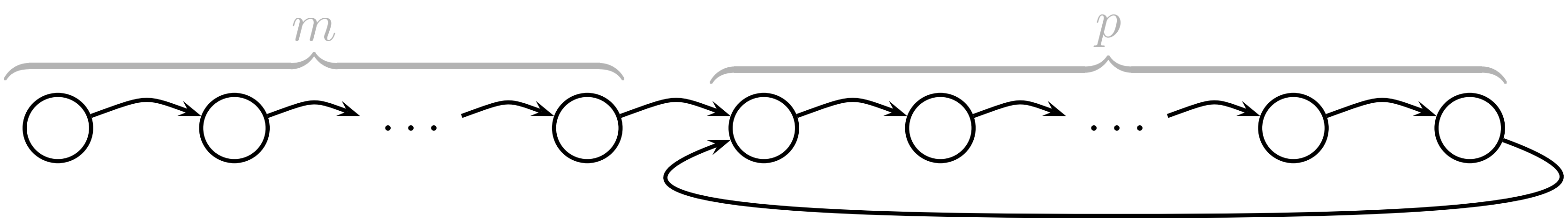
A *loop of type* (m, p) is an ultimately periodic KS where the initial part has length m and the periodic part has length p.



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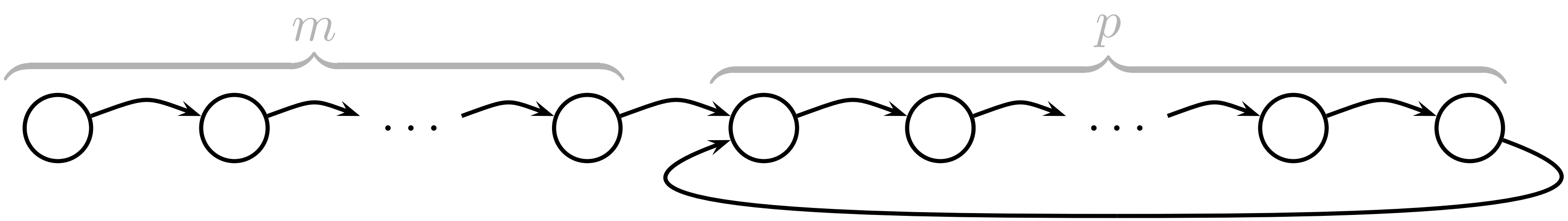

Lemma: For any loop L of type (m, p), for any $NLTL$ formula ϕ with at most $h(\phi)$ nested past-time modalities, and for any $k \geq m + p \cdot h(\phi)$, we have

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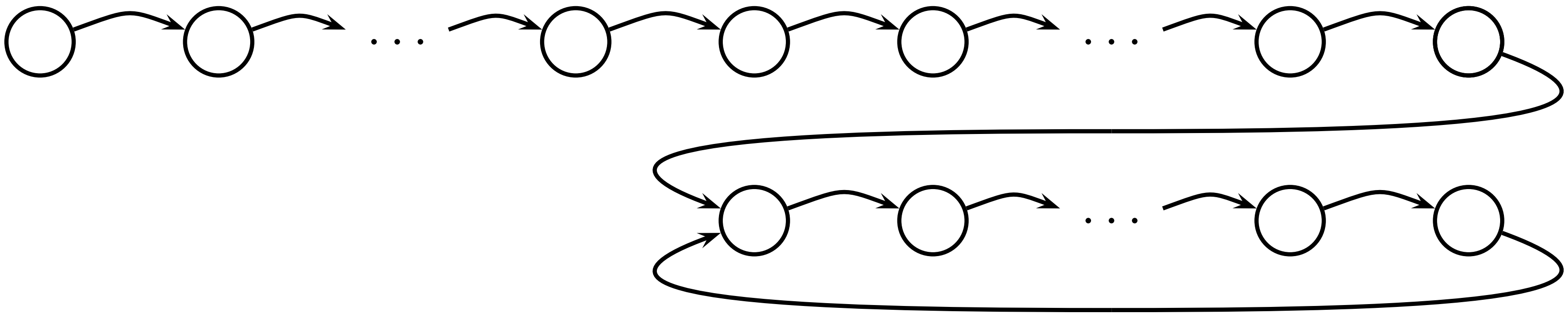

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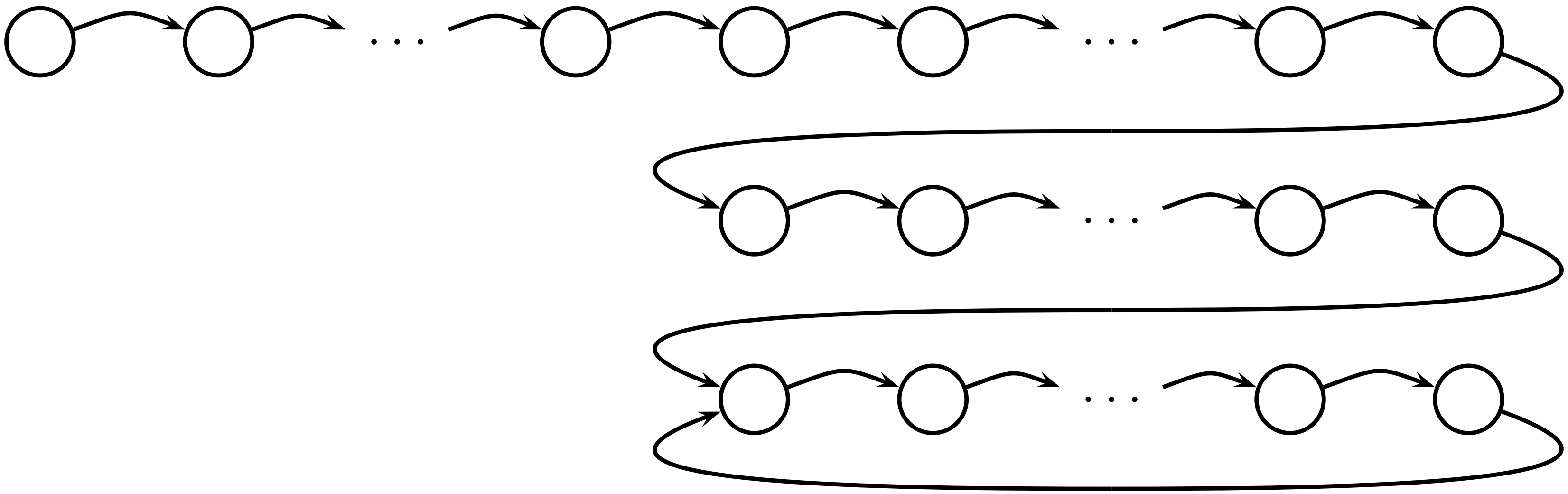

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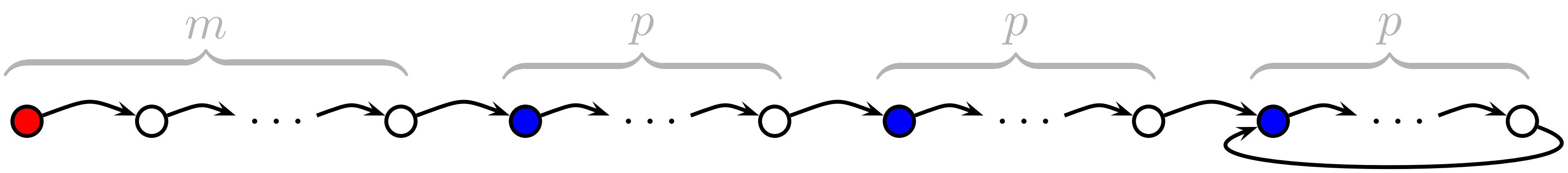

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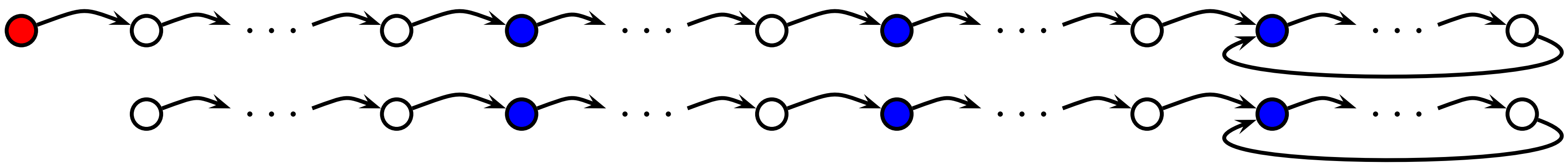

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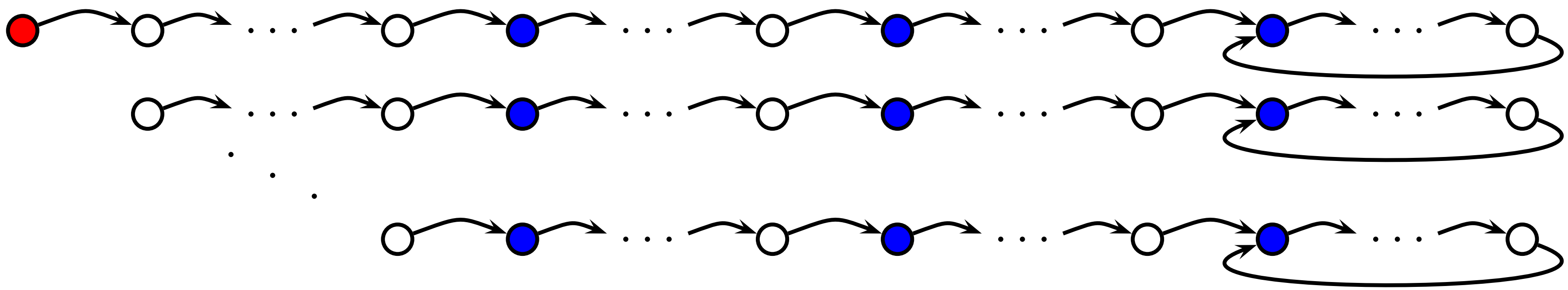

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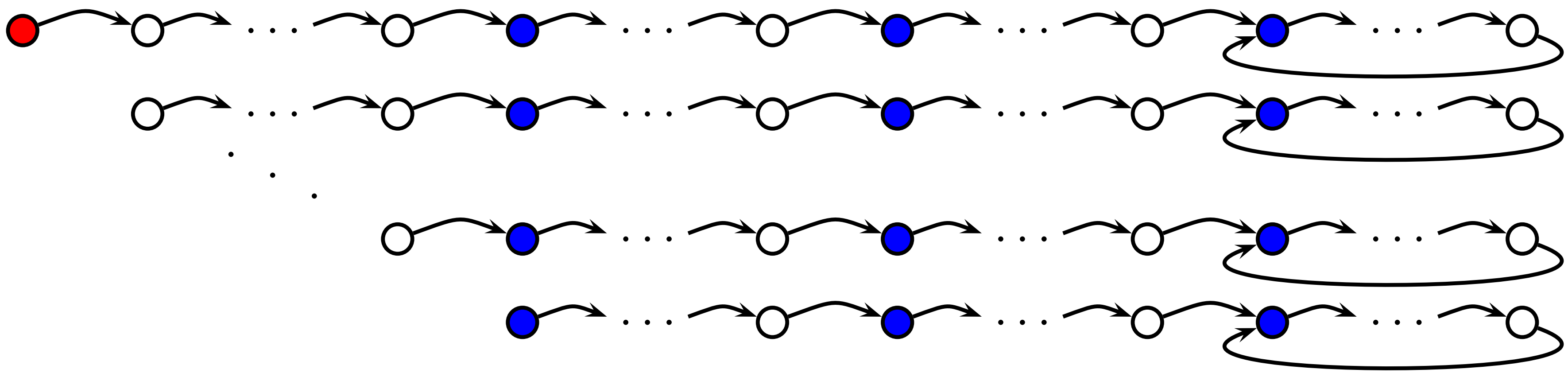

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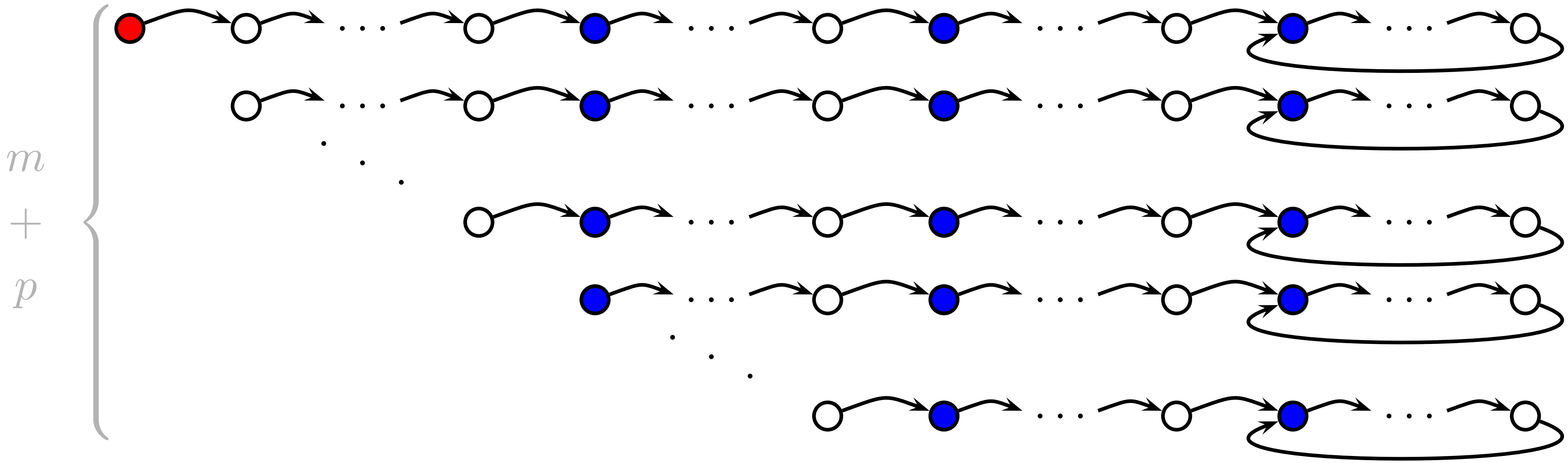

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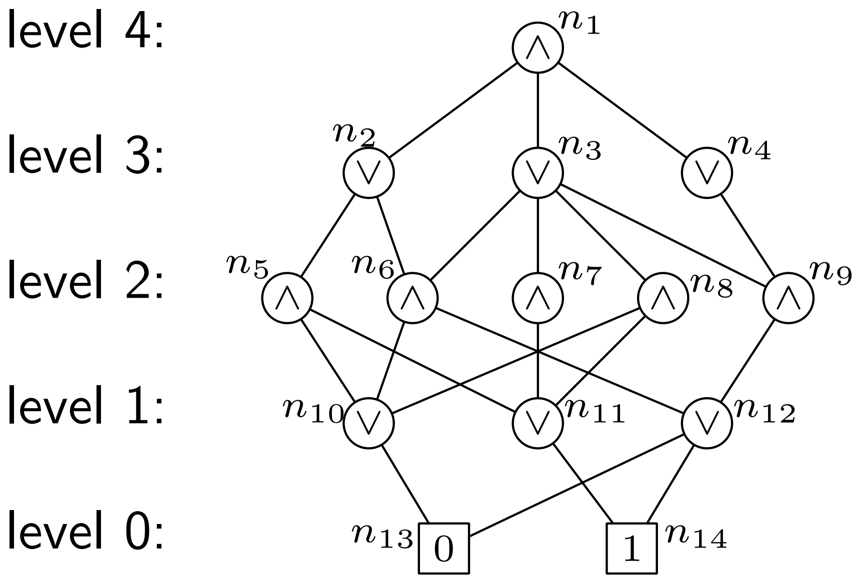
Proposition Model checking *NLTL* along one path is PTIME-hard.

Proof: Reduction from CIRCUIT-VALUE.



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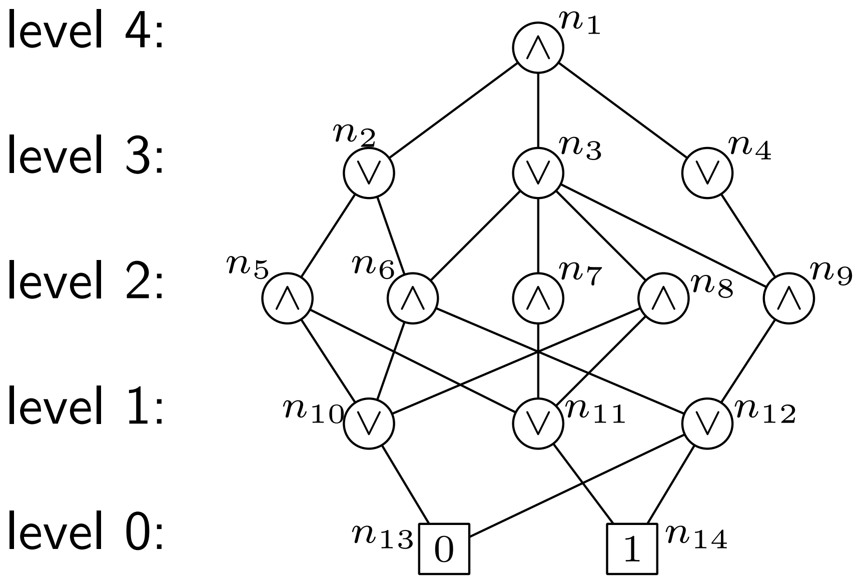

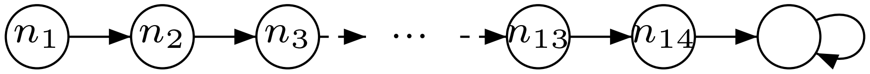

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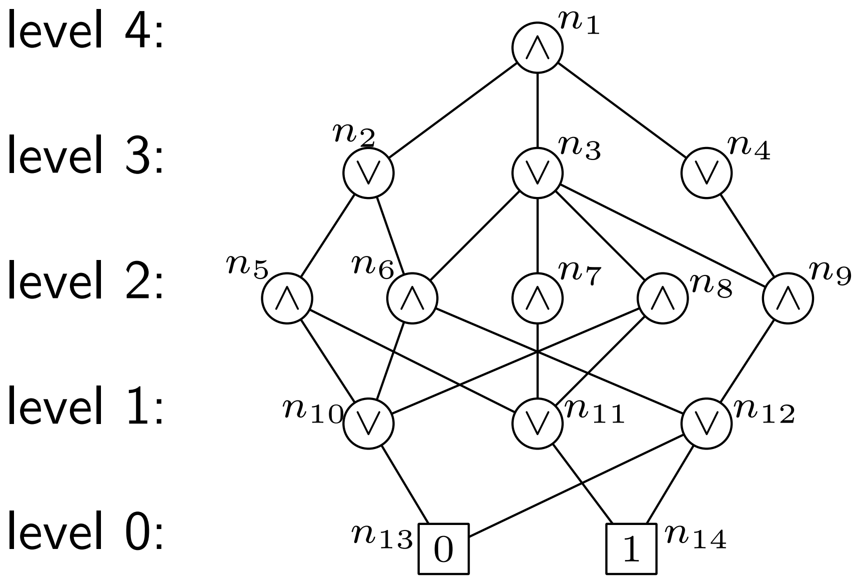

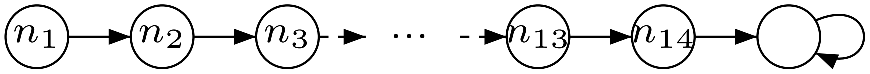

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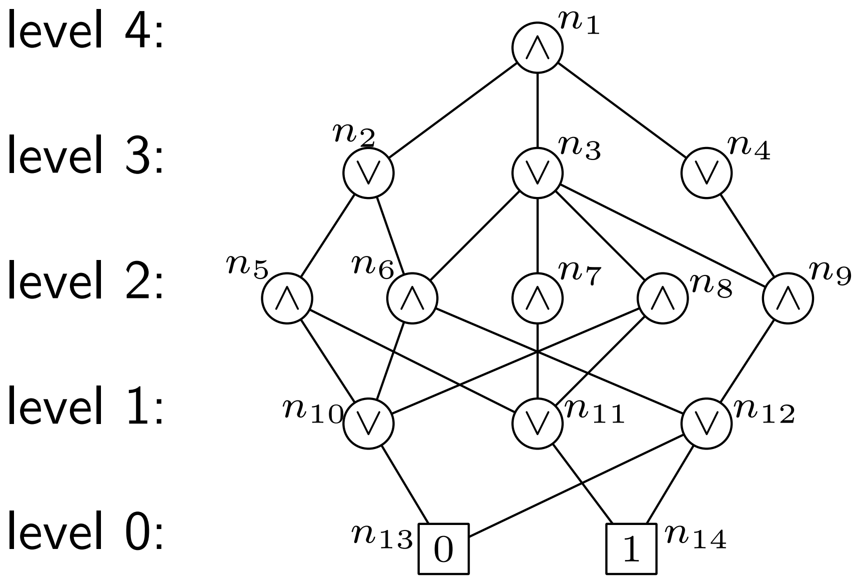

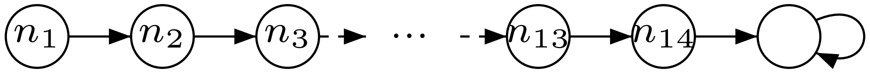

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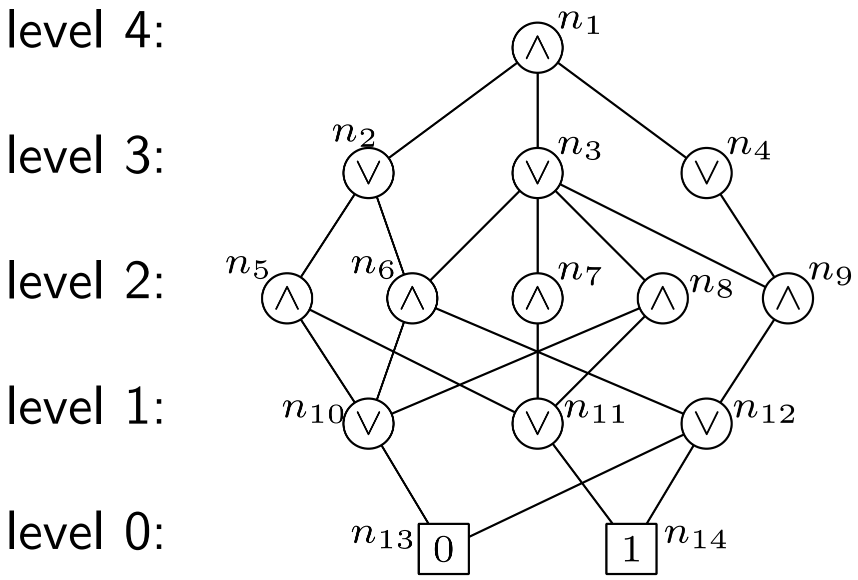

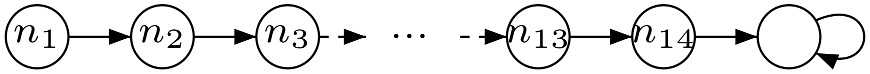

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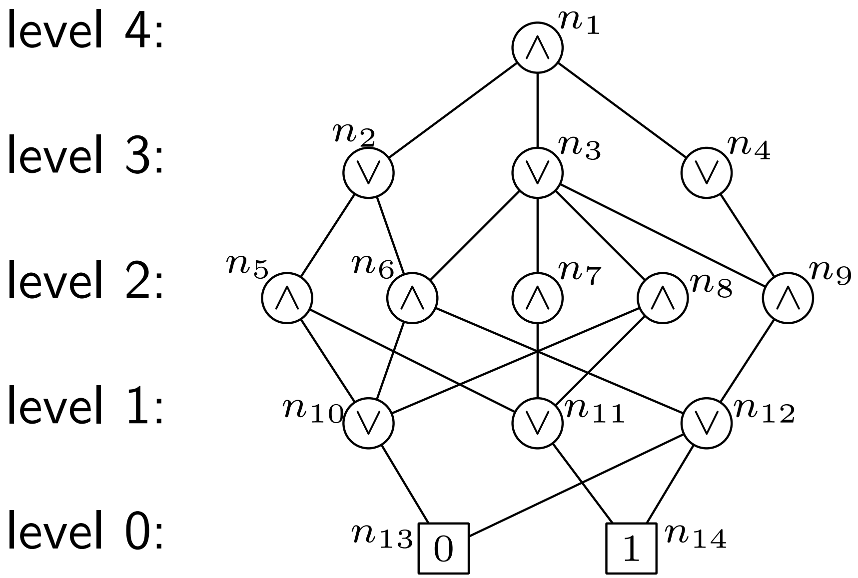

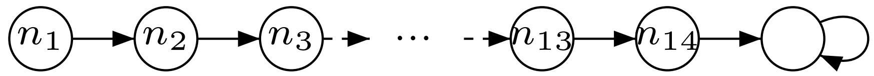

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where $\phi_{next} \stackrel{\text{def}}{=} \bigvee_{(n,n')\in E} (n' \wedge \mathsf{F}^{-1}(n \wedge \neg \mathsf{X}^{-1}\top))$

Conclusion

- $NLTL$ can make specifications easier and more natural.
- That $NLTL$ offers more expressive power can be stated formally as a succinctness gap. The same holds between $PLTL$ and LTL.
- Satisfiability and model checking are EXPSPACE-complete for $NLTL$.

  There is a price for N !
- Model checking a path is PTIME-complete.

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Examples of separation

$$\mathsf{G}(\,\mathsf{alarm} \;\Rightarrow\; \mathsf{F}^{-1}\,\mathsf{problem}\,)$$

$$\equiv$$

$$\mathsf{F}^{-1}\mathsf{problem} \;\vee\; \neg\Big(\,(\neg\,\mathsf{problem})\;\mathsf{U}\;(\mathsf{alarm} \;\wedge\; \neg\,\mathsf{problem}\,)\,\Big)$$

$$\equiv_i$$

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And:

$$\mathsf{G}\,\Big(\,\mathsf{reset} \;\Rightarrow\; \mathsf{N}\;\mathsf{G}\,(\,\mathsf{alarm} \;\Rightarrow\; \mathsf{F}^{-1}\,\mathsf{problem}\,)\Big)$$

$$\equiv_i$$

$$\mathsf{G}\Big[\mathsf{reset} \;\Rightarrow\; \neg\Big(\,(\neg\,\mathsf{problem})\;\mathsf{U}\;(\mathsf{alarm} \;\wedge\; \neg\,\mathsf{problem}\,)\,\Big)$$

Proof of the result of [EVW97]

Claim: Let ϕ_n be a $PLTL$ formula expressing the following property:

"any two future positions that agree on $p_1, \ldots, p_n$ also agree on p_0."

We denote $L_n = \{u \in \{p_0, \ldots, p_n\} \mid u \models \phi_n\}$. From [VW86], we know that L_n is recognized by a Generalized Büchi Automaton $\mathcal{B}$ with $2^{O(|\phi_n|)}$ states.

Let $\{a_0, \ldots, a_{2^n-1}\}$ be a sequence containing all subsets of $\{p_1, \ldots, p_n\}$.

For $K \subseteq \{0, \ldots, 2^n - 1\}$, $w_K = b_0 \cdots b_{2^n-1}$ with $\begin{cases} b_i = a_i & \text{if } i \in K \\ b_i = a_i \cup \{p_0\} & \text{otherwise} \end{cases}$

There are 2^{2^n} such words.

Assume $K \neq K'$. Then $w_K^\omega \models \phi_n$ and $w_{K'} w_K^\omega \not\models \phi_n$. The executions of $\mathcal{B}$ on w_K and $w_{K'}$ cannot lead to the same state. The automaton thus needs at least 2^{2^n} states...