Theoremhood Preserving Maps Characterising Cut Elimination for Modal Provability Logics

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Abstract

Propositional modal provability logics like G and Grz have arithmetical interpretations where $\Box\varphi$ can be read as "formula φ is provable in Peano Arithmetic". These logics are decidable but are characterised by classes of Kripke frames which are not first-order definable. By abstracting the aspects common to their characteristic axioms we define the notion of a formula generation map $\mathtt{F}(\mathtt{p})$ in one propositional variable. We then focus our attention on the properly displayable subset of all (first-order definable) Sahlqvist modal logics. For any logic L from this subset, we consider the (provability) logic LF obtained by the addition of an axiom based upon a formula generation map $\mathtt{F}(\mathtt{p})$ so that $\mathtt{LF} = \mathtt{L} + \mathtt{F}(\mathtt{p})$. The class of such logics includes G and Grz. By appropriately modifying the right introduction rules for $\Box$, we give (not necessarily cut-free) display calculi for every such logic. We define the pseudo-displayable subset of these logics as those whose display calculi enjoy cut-elimination for sequents

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of the form $\top \vdash \varphi$ for any formula φ. We then show that for any provability logic LF having a conservative tense extension, there is a map f on formulae such that LF is pseudo-displayable if and only if f maps theorems of LF to theorems of the underlying logic L and vice-versa. By using a standard renaming technique we can guarantee that there is a polynomial-time translation from LF into L. All proofs are purely syntactic and show the versatility of display calculi since similar results using traditional Gentzen calculi are not possible for as broad a range of logics and require further conditions. Our maps generalise previously known maps from G into K4. An application of our results gives an $\mathcal{O}((n.log\ n)^3)$ translation from the ("second order") provability logic Grz into a decidable subset of first-order logic. Since each of our logics L is a Sahlqvist logic, it is first-order definable, and hence each L has a translation into first-order logic. Our results therefore show that all pseudo-displayable logics LF are "essentially first-order" even though their characteristic axiom may not be first-order definable.

Key-words: provability modal logic, display logic, cut elimination, many-one reduction

1 Introduction

Background. Display Logic (**DL**) is a proof-theoretical framework introduced by Belnap [Bel82] that generalises the structural language of Gentzen's sequents in a rather abstract way by using multiple complex structural connectives instead of Gentzen's comma. The term "display" comes from the nice property that any occurrence of a *structure* in a sequent can be displayed either as the entire antecedent or as the entire succedent of some sequent which is *structurally equivalent* to the initial sequent. An important feature of the proof-theoretical framework **DL** is the existence of a very general cut-elimination theorem [Bel82]. Indeed, any display calculus satisfying the conditions (C1)-(C8) [Bel82] enjoys cut-elimination. The generality of **DL** is witnessed by the fact that cut-free display calculi have been defined for substructural logics [Bel82, Bel90, Res98, Gor98], for modal and polymodal logics [Wan94, Kra96, Wan98a, Wan99, DG99a, DG00a], for intuitionistic and subintuitionistic logics [Gor95, Wan97, Gor00] and for relation algebras [Gor97]. Furthermore, numerous enriched versions of sequent calculi can be easily encoded into display calculi; see e.g. [Min97, Wan98b].

Cut-elimination for a display calculus is established by simply checking that it obeys Belnap's conditions (C1)-(C8) [Bel82]. In [Wan98a], such a result is strengthened by proving that any classical modal display calculus admits a strong normalisation theorem. Another important general result about **DL** is Kracht's characterisation of *properly displayable* tense logics [Kra96]. Roughly speaking, every properly displayable tense logic is an extension of the polymodal version of tense logic $\mathtt{K}_t$ obtained from a Hilbert calculus for classical propositional logic by the addition of *primitive axioms*, which are a particular subset of Sahlqvist tense formulae [Sah75]. Conversely, every such extension of $\mathtt{K}_t$ is properly displayable. In [Kra96], it is shown that every properly displayable tense logic admits a display calculus that obeys the conditions (C1)-(C8), and therefore enjoys cut-elimination, and that every primitive axiom can be encoded effectively by purely structural rules. In the paper, we restrict ourselves to monomodal extensions of the modal logic $\mathtt{K}$.

Since every primitive axiom is a Sahlqvist implication, it is first-order definable; that is, the class of modal frames for which the primitive axiom is valid is definable by first-order formulae. Consequently, *a fortiori*, every modal logic characterized by a class of frames that is not first-order definable, is not properly displayable. This includes the well-known provability logics $\mathtt{G}$[1] and $\mathtt{Grz}$ (for Grzegorczyk) which admit important arithmetical interpretations as "logics of provability" [Sol76] (see also [Boo93]). At first glance, this seems to contradict the fact that **DL** generalizes Gentzen-style calculi since the well-known traditional sequent and tableau calculi for these logics [SV80, Lei81, SV82, Val83, Fit83, Avr84, Boo93, Gor99] do enjoy cut-elimination.

Our contribution By abstracting the aspects common to the characteristic axioms for $\mathtt{G}$ and $\mathtt{Grz}$, respectively, we define the notion of a formula generation map $\mathtt{F(p)}$ in one propositional variable. Let ϕ be a modal formula and $\mathtt{F}(\phi)$ be a formula built from $\{\phi\}$ using $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Box$ such that any subformula of the form $\Box\psi$ in $\mathtt{F}(\phi)$ occurs positively (when every $\phi_1 \Rightarrow \phi_2$ is written as $\neg\phi_1 \vee \phi_2$). Let $\mathtt{L}$ be a properly displayable modal logic and $\mathtt{LF}$ be the logic obtained from $\mathtt{L}$ by adding the axiom scheme $\Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi$. Here a *logic* is understood as a set of formulae and therefore is exactly a *(decision) problem* in the usual sense in complexity theory. That is, as a language viewed as a set of strings built upon a given alphabet.

[1] Also called $\mathtt{GL}$ (for Gödel and Löb), $\mathtt{KW}$, $\mathtt{K4W}$, $\mathtt{PrL}$.

For any logic $\mathtt{LF}$, we define a display calculus $\delta\mathtt{LF}$ by slightly modifying the display calculus $\delta\mathtt{L}$ for $\mathtt{L}$ defined in [Kra96]. Indeed, we appropriately modify the right introduction for $\Box$. When $\mathtt{LF}$ has a conservative tense extension, the proof calculus $\delta\mathtt{LF}$ is sound and complete with respect to $\mathtt{LF}$ and this is not very difficult to show using [Wan94, Kra96]. Since $\mathtt{LF}$ is not necessarily properly displayable (e.g. when $\mathtt{LF}$ is $\mathtt{G}$) and since $\delta\mathtt{LF}$ does not necessarily obey Belnap's condition C8, cut-elimination in $\delta\mathtt{LF}$ is not a by-product of [Bel82, Kra96]. We show that (weak) cut-elimination for $\delta\mathtt{LF}$ is equivalent to the theoremhood-preserving nature of certain maps defined in the sequel. Since $\mathtt{LF}$ is not necessarily properly displayable, this provides an alternative way to define display calculi for modal logics: the encoding of the modal Hilbert axioms is done via the logical introduction rules instead of via structural rules. In a sense, we have dropped certain working hypotheses from [Kra96] in order to open new possibilities to define display calculi.

A similar analysis for traditional sequent-style calculi is also given in the paper. Finally, although our initial motivation is the proof-theoretical problem of how to define display calculi for non properly displayable modal logics, we also show that if $\delta\mathtt{LF}$ satisfies (weak) cut elimination, then there is a polynomial-time transformation from $\mathtt{LF}$ into $\mathtt{L}$. See e.g. [Pap94] for a thorough introduction to complexity theory. Since $\mathtt{L}$ itself can be (cleverly) translated into first-order logic in linear-time, this provides an alternative method to mechanise deduction in such an $\mathtt{LF}$ using theorem provers for classical logic. Particular cases of our general results apply to the provability logics $\mathtt{G}$ and $\mathtt{Grz}$. Our results therefore show that all pseudo-displayable logics $\mathtt{LF}$ are "essentially first-order" even though their characteristic axiom may not be first-order definable.

Plan of the paper. In Section 2, we define the class of provability modal logics studied in the paper. In Section 3, we define display calculi for the provability logics and show their soundness and completeness. In Section 4, we give necessary and sufficient conditions to establish that the display calculi admit a (limited) cut-elimination theorem. Section 5 contains a similar analysis for traditional sequent-style calculi. This paper is an extended and corrected version of [DG99b].

2 Provability Logics

Given a set PRP $= \{\mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of *atomic formulae*, the formulae $\phi \in$ FML are inductively defined as follows for $\mathtt{p}_i \in$ PRP:

$$\phi ::= \bot \mid \top \mid \mathtt{p}_i \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi \mid \phi_1 \Rightarrow \phi_2 \mid \Box\phi.$$

Standard abbreviations include $\Leftrightarrow$, $\Diamond$: for instance, $\Diamond\phi \stackrel{\text{def}}{=} \neg\Box\neg\phi$. An occurrence of the subformula ψ in ϕ is *positive* (resp. *negative*) $\stackrel{\text{def}}{\Leftrightarrow}$ it is in the scope of an even [resp. odd] number of negations, where as usual, every occurrence of $\phi_1 \Rightarrow \phi_2$ is treated as an occurrence of $\neg\phi_1 \vee \phi_2$.

The standard Hilbert-style axiomatic calculus $\mathtt{K}$ is composed of the tautologies of the Propositional Calculus (PC), the axiom schema $\Box(\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow (\Box\mathtt{p} \Rightarrow \Box\mathtt{q})$, and the inference rules below:

Modus Ponens: from ϕ and $\phi \Rightarrow \psi$ infer ψ

Necessitation: from ϕ infer $\Box\phi$.

We write $\phi \in \mathtt{K}$ to mean that ϕ is a *theorem*[2] of $\mathtt{K}$. Similarly, when $\mathtt{L}$ is an extension of $\mathtt{K}$, we write $\phi \in \mathtt{L}$ to denote that ϕ is a theorem of $\mathtt{L}$. In the paper, we refer to the following well-known extensions $\mathtt{L}$ of $\mathtt{K}$:

- $\mathtt{T}$ is defined as $\mathtt{K}$ plus the axiom schema $\Box\mathtt{p} \Rightarrow \mathtt{p}$
- $\mathtt{K4}$ is defined as $\mathtt{K}$ plus the axiom schema $\Box\mathtt{p} \Rightarrow \Box\Box\mathtt{p}$
- $\mathtt{S4}$ is defined as $\mathtt{K4}$ plus the axiom schema $\Box\mathtt{p} \Rightarrow \mathtt{p}$
- $\mathtt{G}$ is defined as $\mathtt{K4}$ plus the axiom schema $\Box(\Box\mathtt{p} \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p}$
- $\mathtt{Grz}$ is defined as $\mathtt{S4}$ plus the axiom schema $\Box(\Box(\mathtt{p} \Rightarrow \Box\mathtt{p}) \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p}$.

The logic $\mathtt{Grz}$ can also be axiomatised as $\mathtt{S4}$ plus the axiom schema $\Box(\Box(\mathtt{p} \Rightarrow \Box\mathtt{p}) \Rightarrow \mathtt{p}) \Rightarrow \mathtt{p}$ [GHH97] which came to light in investigations of the connection between intuitionistic and modal logic (see e.g. [Gol78]).

We write $\mathtt{L}^+$ [resp. $\mathtt{L}^+\mathtt{F}$] to denote the extension of $\mathtt{L}$ [resp. $\mathtt{LF}$] obtained by adding the axiom schemata

$$\Box^-(\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow (\Box^-\mathtt{p} \Rightarrow \Box^-\mathtt{q}) \qquad \mathtt{q} \Rightarrow \Box\Diamond^-\mathtt{q} \qquad \mathtt{q} \Rightarrow \Box^-\Diamond\mathtt{q}$$

[2] That is, there is a finite sequence $\langle\phi_1, \ldots, \phi_n\rangle$ such that $\phi_n = \phi$ and for any $i \in \{1, \ldots, n\}$, either ϕ_i is an instance of an axiom schema or ϕ_i is obtained by application of a rule of inference to formulae in $\{\phi_1, \ldots .\phi_{i-1}\}$.

and the necessitation rule: from ϕ infer $\Box^{-}\phi$. The language is extended by adding $\Box^{-}$, and defining $\Diamond^{-}\phi$ as $\neg\Box^{-}\neg\phi$. The traditional Kripke semantics for modal logics interpret the modalities $\Box$ and $\Diamond$ using a binary accessibility relation R, and interpret $\Box^{-}$ and $\Diamond^{-}$ using the converse R^{-1} of R. Thus the modality $\Diamond^{-}$ is the "backward existential" modality.

We say that $\mathtt{LF}$ has a *conservative tense extension* $\stackrel{\text{def}}{\Leftrightarrow}$ for any formula $\phi \in$ FML which is free of occurrences of $\Box^{-}$, $\phi \in \mathtt{LF}$ iff $\phi \in \mathtt{L^{+}F}$. The logics $\mathtt{K4}$, $\mathtt{S4}$, $\mathtt{G}$ and $\mathtt{Grz}$ are (cofinal subframe) logics that have a conservative tense extension (see e.g. [Wol99, Chapter 4]).

Following [Kra96], a formula is said to be *primitive* (for the monomodal language) $\stackrel{\text{def}}{\Leftrightarrow}$ it is of the form $\phi \Rightarrow \psi$ where both ϕ and ψ are built using members of PRP $\cup \{\top\}$ with the help of $\wedge$, $\vee$, $\Diamond$, and where ϕ contains each atomic proposition at most once. Primitive formulae are a subset of the class of Sahlqvist formulae [Kra96, Sah75]. In the paper, by a **primitive modal logic**, we mean a (mono)modal logic defined from the modal logic $\mathtt{K}$ by adding primitive formulae as axioms.

Example 1 Neither $\Box\mathtt{p} \Rightarrow \mathtt{p}$ and $\Box\mathtt{p} \Rightarrow \Box\Box\mathtt{p}$ are primitive, but their logically equivalent forms $\mathtt{p} \Rightarrow \Diamond\mathtt{p}$ and $\Diamond\Diamond\mathtt{p} \Rightarrow \Diamond\mathtt{p}$ are both primitive.

Let $\delta\mathtt{K}$ be the display calculus for the modal logic $\mathtt{K}$ defined in [Kra96] under the name **DLM** (see Figures 1-4 for its definition). As a consequence of [Kra96, Theorem 16], every primitive modal logic has a sound and complete display calculus $\delta\mathtt{L}$ obtained by adding structural rules to $\delta\mathtt{K}$ that preserve Belnap's properties (C1)-(C8). We therefore refer to primitive modal logics as **properly displayable** modal logics.

By Example 1, the logics $\mathtt{T}$, $\mathtt{K4}$ and $\mathtt{S4}$ are properly displayable. The traditional axioms for many well-known modal logics are not primitive, but most have a primitive equivalent [Kra96]. Every properly displayable modal logic is known to have a display calculus satisfying conditions (C1)-(C8) [Bel82] and therefore enjoying cut-elimination [Kra96]. In what follows, we write $\delta\mathtt{L}$ to denote the display calculus for $\mathtt{L}$ defined in [Kra96].

A *formula generation map* $\mathtt{F}$: FML $\rightarrow$ FML is a function such that

1. there is a formula $\psi_{\mathtt{F}}$ containing only one atomic proposition, say $\mathtt{p}$, and no logical constants, such that for $\phi \in$ FML, $\mathtt{F}(\phi)$ is obtained from $\psi_{\mathtt{F}}$ by replacing every occurrence of $\mathtt{p}$ by ϕ
2. no subformula of the form $\Box\varphi$ occurs negatively in $\psi_{\mathtt{F}}$.

$\mathtt{F}$ is also written $\lambda\mathtt{p}.\psi_{\mathtt{F}}$.

The definition of a formula generation map is actually a restricted form of maps defined in [Avr84]. For instance, no restriction on the polarity of the occurrences of $\Box$ is assumed in [Avr84]. For any properly displayable logic $\mathtt{L}$ and for any formula generation map $\mathtt{F}$, we write $\mathtt{LF}$ to denote the logic obtained from $\mathtt{L}$ by addition of the schema

$$\Box(\mathtt{F}(\mathtt{p}) \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p} \tag{1}$$

Observe that $\Box(\mathtt{F}(\mathtt{q}) \Rightarrow \mathtt{q}) \Rightarrow \Box\mathtt{q}$ is not a Sahlqvist implication (see e.g. [BRV01, Definition 3.51]). Indeed, $\Box(\mathtt{F}(\mathtt{q}) \Rightarrow \mathtt{q}) \Rightarrow \Box\mathtt{q}$ can be rewritten into $\Box(\neg\mathtt{F}(\mathtt{q}) \vee \mathtt{q}) \Rightarrow \Box\mathtt{q}$. Recall that $\phi_1 \Rightarrow \phi_2$ is a Sahlqvist formula iff ϕ_2 is *positive* (all the occurrences of the atomic propositions occur positively) and ϕ_1 is built up from *negative* formulae (all the occurrences of the atomic propositions occur negatively), formulae without occurrences of atomic propositions and formulae of the form $\sigma\mathtt{p}$ with σ a (possibly empty) sequence of $\Box$s, and $\mathtt{p} \in \mathrm{PRP}$ using only $\wedge$, $\vee$ and $\neg\Box\neg$. However, the formula $\Box(\neg\mathtt{F}(\mathtt{q}) \vee \mathtt{q})$ is not negative and $\neg\mathtt{F}(\mathtt{q}) \vee \mathtt{q}$ is not an atomic proposition. So, $\Box(\neg\mathtt{F}(\mathtt{q}) \vee \mathtt{q}) \Rightarrow \Box\mathtt{q}$ is not a Sahlqvist formula. Of course it may be possible to find a Sahlqvist formula logically equivalent (in the basic modal logic $\mathtt{K}$) to it. For instance, this is the case when $\mathtt{F}(\mathtt{q}) = \neg\mathtt{q}$ since then, $\Box(\mathtt{F}(\mathtt{q}) \Rightarrow \mathtt{q})) \Rightarrow \Box\mathtt{q}$ is equivalent $\Box\mathtt{q} \Rightarrow \Box\mathtt{q}$, and this has an equivalent primitive form $\top \Rightarrow \top$. However, in numerous cases $\mathtt{LF}$ is not properly displayable. For instance, let $\mathtt{F}_{\mathtt{G}}$ and $\mathtt{F}_{\mathtt{Grz}}$ be $\lambda\mathtt{p}.\Box\mathtt{p}$ and $\lambda\mathtt{p}.\Box(\mathtt{p} \Rightarrow \Box\mathtt{p})$, respectively. Then, by definition $\mathtt{G} = \mathtt{K4F}_{\mathtt{G}}$ and $\mathtt{Grz} = \mathrm{S4}\mathtt{F}_{\mathtt{Grz}}$. Since $\mathtt{F}_{\mathtt{G}}$ and $\mathtt{F}_{\mathtt{Grz}}$ are modal axioms that correspond to essential second-order conditions on frames (see e.g. [Boo93]), the logics $\mathtt{G}$ and $\mathtt{Grz}$ are not primitive.

3 Display Calculi for Provability Logics

In the rest of this section, $\mathtt{L}$ is a properly displayable modal logic, $\mathtt{F}$ is a formula generation map and $\mathtt{LF}$ is the corresponding extension of $\mathtt{L}$ by the axiom schema (1).

Let us first briefly recall the main features of the modal display calculus $\delta\mathtt{L}$ as defined in [Wan94, Kra96]. On the structural side, we have structural connectives $*$ (unary), $\circ$ (binary), I (nullary) and $\bullet$ (unary). A *structure* $\mathtt{X} \in \mathtt{struc}(\delta\mathtt{L})$ is defined using the BNF grammar below where $\phi \in \mathrm{FML}$:

$$\mathtt{X} ::= \phi \mid *\mathtt{X} \mid \mathtt{X}_1 \circ \mathtt{X}_2 \mid I \mid \bullet\mathtt{X}$$

$$\text{(Id)}\quad \mathtt{p} \vdash \mathtt{p} \qquad\qquad \text{(cut)}\ \frac{\mathtt{X} \vdash \phi \quad \phi \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}$$

Figure 1: Fundamental logical axioms and cut rule

$$\frac{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}{\mathtt{X} \vdash \mathtt{Z} \circ *\mathtt{Y}} \quad \frac{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}{\mathtt{Y} \vdash *\mathtt{X} \circ \mathtt{Z}} \quad \frac{\mathtt{X} \vdash \mathtt{Y} \circ \mathtt{Z}}{\mathtt{X} \circ *\mathtt{Z} \vdash \mathtt{Y}} \quad \frac{\mathtt{X} \vdash \mathtt{Y} \circ \mathtt{Z}}{*\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}$$

$$\frac{*\mathtt{X} \vdash \mathtt{Y}}{*\mathtt{Y} \vdash \mathtt{X}} \quad \frac{\mathtt{X} \vdash *\mathtt{Y}}{\mathtt{Y} \vdash *\mathtt{X}} \quad \frac{* * \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}} \quad \frac{\mathtt{X} \vdash * * \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}} \quad \frac{\mathtt{X} \vdash \bullet\mathtt{Y}}{\bullet\mathtt{X} \vdash \mathtt{Y}}$$

Figure 2: Display postulates

We use *formula variables* like ϕ, ψ, φ to stand for formulae, and use *structure variables* like X, Y and Z to stand for arbitrary structures from struc(δL).

A *sequent* is defined as a pair of structures of the form $\mathtt{X} \vdash \mathtt{Y}$ with X the *antecedent* and Y the *succedent*. The rules of δL are presented in Figures 1-4. Additional structural rules satisfying the conditions (C1)-(C8) are also needed but their presentation is omitted here since they depend on the primitive axioms upon which L is defined (see [Kra96] for details). For instance, the structural rule corresponding to the primitive formula $(\Diamond \mathtt{p} \wedge \Diamond \mathtt{q}) \Rightarrow \Diamond(\mathtt{p} \wedge \Diamond \mathtt{q}) \vee \Diamond(\mathtt{q} \wedge \Diamond \mathtt{p}) \vee \Diamond(\mathtt{p} \wedge \mathtt{q})$ known as .3 is the following [Kra96]:

$$\frac{* \bullet *(\mathtt{X} \circ * \bullet *\mathtt{Y}) \vdash \mathtt{Z} \quad * \bullet * (\mathtt{Y} \circ * \bullet *\mathtt{X}) \vdash \mathtt{Z} \quad * \bullet * (\mathtt{X} \circ \mathtt{Y}) \vdash \mathtt{Z}}{* \bullet *\mathtt{X} \circ * \bullet *\mathtt{Y} \vdash \mathtt{Z}}$$

Observe that a primitive formula can generate more than one structural rule [Kra96]. In all proofs that follow we omit the cases for the structural rules obtained from the primitive axioms of L since they pose no difficulty.

The *display postulates* (reversible rules) in Figure 2 deal with the manipulation of structural connectives.

In any structure Z, the structure X occurs *negatively* [resp. *positively*] $\stackrel{\text{def}}{\Leftrightarrow}$ X occurs in the scope of an odd number [resp. an even number] of occurrences of $*$ [Bel82]. In a sequent $\mathtt{X} \vdash \mathtt{Y}$, an occurrence of Z is an

$$\frac{}{I \vdash \top}\ (\vdash \top) \quad \frac{I \vdash \mathtt{X}}{\top \vdash \mathtt{X}}\ (\top \vdash) \quad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash \bot}\ (\vdash \bot) \quad \frac{}{\bot \vdash I}\ (\bot \vdash)$$

$$\frac{\mathtt{X} \circ \phi \vdash \psi}{\mathtt{X} \vdash \phi \Rightarrow \psi}\ (\vdash \Rightarrow) \quad \frac{\mathtt{X} \vdash \phi \quad \psi \vdash \mathtt{Y}}{\phi \Rightarrow \psi \vdash *\mathtt{X} \circ \mathtt{Y}}\ (\Rightarrow \vdash)$$

$$\frac{\mathtt{X} \vdash *\phi}{\mathtt{X} \vdash \neg\phi}\ (\vdash \neg) \quad \frac{*\phi \vdash \mathtt{X}}{\neg\phi \vdash \mathtt{X}}\ (\neg \vdash) \quad \frac{\mathtt{X} \vdash \phi \quad \mathtt{Y} \vdash \psi}{\mathtt{X} \circ \mathtt{Y} \vdash \phi \wedge \psi}\ (\vdash \wedge) \quad \frac{\phi \circ \psi \vdash \mathtt{X}}{\phi \wedge \psi \vdash \mathtt{X}}\ (\wedge \vdash)$$

$$\frac{\mathtt{X} \vdash \phi \circ \psi}{\mathtt{X} \vdash \phi \vee \psi}\ (\vdash \vee) \quad \frac{\phi \vdash \mathtt{X} \quad \psi \vdash \mathtt{Y}}{\phi \vee \psi \vdash \mathtt{X} \circ \mathtt{Y}}\ (\vee \vdash)$$

$$\frac{\phi \vdash \mathtt{X}}{\Box\phi \vdash \bullet\mathtt{X}}\ (\Box \vdash) \quad \frac{\mathtt{X} \vdash \bullet\phi}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_{\mathtt{L}})$$

Figure 3: Operational rules

antecedent part [resp. *succedent part*] $\overset{\text{def}}{\Leftrightarrow}$ it occurs positively in $\mathtt{X}$ [resp. negatively in $\mathtt{Y}$] or it occurs negatively in $\mathtt{Y}$ [resp. positively in $\mathtt{X}$] [Bel82]. Two sequents $\mathtt{X} \vdash \mathtt{Y}$ and $\mathtt{X}' \vdash \mathtt{Y}'$ are said to be *structurally equivalent* $\overset{\text{def}}{\Leftrightarrow}$ there is derivation of the first sequent from the second (and vice-versa) using only the display postulates defined in Figure 2.

Theorem 2 (Display Theorem [Bel82]) For every sequent $\mathtt{X} \vdash \mathtt{Y}$ and every antecedent [resp. succedent] part $\mathtt{Z}$ of $\mathtt{X} \vdash \mathtt{Y}$, there is a structurally equivalent sequent $\mathtt{Z} \vdash \mathtt{Y}'$ [resp. $\mathtt{X}' \vdash \mathtt{Z}$] that has $\mathtt{Z}$ (alone) as its antecedent [resp. succedent]. $\mathtt{Z}$ is said to be *displayed* in $\mathtt{Z} \vdash \mathtt{Y}'$ [resp. $\mathtt{X}' \vdash \mathtt{Z}$].

Theorem 3 (Soundness and Completeness [Kra96]) For all $\phi \in \mathrm{FML}$, $I \vdash \phi$ has a cut-free derivation in $\delta\mathtt{L}$ iff $\phi \in \mathtt{L}$.

We shall now define the calculus $\delta\mathtt{LF}$. Before we give the formal definition of $\delta\mathtt{LF}$, we need to introduce an additional notion. Let m be a map $m : \mathrm{FML} \times \{0,1\} \to \mathtt{struc}(\delta\mathtt{L})$ that transforms certain logical connectives into structural connectives, inductively defined as follows ($i \in \{0,1\}$):

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{I \circ \mathtt{X} \vdash \mathtt{Z}}\ (I_l) \qquad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \vdash I \circ \mathtt{Z}}\ (I_r)$$

$$\frac{I \vdash \mathtt{Y}}{*I \vdash \mathtt{Y}}\ (Q_l) \qquad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash *I}\ (Q_r)$$

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}\ (weak_l) \quad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}\ (weak_r)$$

$$\frac{\mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3) \vdash \mathtt{Z}}{(\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3 \vdash \mathtt{Z}}\ (assoc_l) \qquad \frac{\mathtt{Z} \vdash \mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3)}{\mathtt{Z} \vdash (\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3}\ (assoc_r)$$

$$\frac{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}\ (com_l) \quad \frac{\mathtt{Z} \vdash \mathtt{Y} \circ \mathtt{X}}{\mathtt{Z} \vdash \mathtt{X} \circ \mathtt{Y}}\ (com_r)$$

$$\frac{\mathtt{X} \circ \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (contr_l) \quad \frac{\mathtt{Y} \vdash \mathtt{X} \circ \mathtt{X}}{\mathtt{Y} \vdash \mathtt{X}}\ (contr_r)$$

$$\frac{I \vdash \mathtt{X}}{\bullet I \vdash \mathtt{X}}\ (nec^l) \quad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash \bullet I}\ (nec^r)$$

Figure 4: Other basic structural rules

$$m(\mathtt{p}, i) \stackrel{\text{def}}{=} \mathtt{p} \text{ for any } \mathtt{p} \in \mathrm{PRP}$$

$$\begin{array}{ll}
m(\top, i) \stackrel{\text{def}}{=} \top & m(\bot, i) \stackrel{\text{def}}{=} \bot \\
m(\phi_1 \vee \phi_2, 1) \stackrel{\text{def}}{=} m(\phi_1, 1) \circ m(\phi_2, 1) & m(\phi_1 \vee \phi_2, 0) \stackrel{\text{def}}{=} \phi_1 \vee \phi_2 \\
m(\phi_1 \wedge \phi_2, 1) \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2 & m(\phi_1 \wedge \phi_2, 0) \stackrel{\text{def}}{=} m(\phi_1, 0) \circ m(\phi_2, 0) \\
m(\phi_1 \Rightarrow \phi_2, 1) \stackrel{\text{def}}{=} *m(\phi_1, 0) \circ m(\phi_2, 1) & m(\phi_1 \Rightarrow \phi_2, 0) \stackrel{\text{def}}{=} \phi_1 \Rightarrow \phi_2 \\
m(\Box\phi, i) \stackrel{\text{def}}{=} \Box\phi & m(\neg\phi, i) \stackrel{\text{def}}{=} *m(\phi, 1 - i).
\end{array}$$

The second argument of m merely indicates whether to read the first argument of m as an antecedent part ($i = 0$) or as a succedent part ($i = 1$). The calculus $\delta\mathtt{LF}$ has the same structures as $\delta\mathtt{L}$, so $\mathtt{struc}(\delta\mathtt{LF}) \stackrel{\text{def}}{=} \mathtt{struc}(\delta\mathtt{L})$, and $\delta\mathtt{LF}$ is obtained from $\delta\mathtt{L}$ by replacing the $(\vdash \Box_{\mathtt{L}})$-rule from Figure 3 by the $(\vdash \Box_{\mathtt{LF}})$ rule below:

$$\frac{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}{\mathtt{X} \vdash \Box\phi} \ (\vdash \Box_{\mathtt{LF}})$$

The $(\vdash \Box_{\mathtt{LF}})$-rules for $\delta\mathtt{Grz}$ and $\delta\mathtt{G}$ are respectively:

$$\frac{\mathtt{X} \vdash \bullet(*\Box(\phi \Rightarrow \Box\phi) \circ \phi)}{\mathtt{X} \vdash \Box\phi} \ (\vdash \Box_{\mathtt{Grz}}) \qquad \frac{\mathtt{X} \vdash \bullet(*\Box\phi \circ \phi)}{\mathtt{X} \vdash \Box\phi} \ (\vdash \Box_{\mathtt{G}})$$

Observe that at the present stage, there is no need to define $m(\top, i)$ and $m(\bot, i)$ since in $(\vdash \Box_{\mathtt{LF}})$, $m(., 0)$ is applied to $\mathtt{F}(\phi)$ and $\mathtt{F} = \lambda\mathtt{p}.\psi_{\mathtt{F}}$ is a formula generation map in which neither $\bot$ nor $\top$ occurs. However, $m(\top, i)$ and $m(\bot, i)$ are needed for Lemma 12.

The calculus $\delta\mathtt{LF}$ satisfies conditions (C2)-(C7). In particular, $\delta\mathtt{G}$ satisfies the conditions (C1)-(C7). The $(\vdash \Box_{\mathtt{G}})$-rule in $\delta\mathtt{G}$ is similar to the GLR rule in [SV82] and to the G-rule in [Rau83] (see also [Avr84]). Analogously, the $(\vdash \Box_{\mathtt{Grz}})$-rule in $\delta\mathtt{Grz}$ is similar to the (GRZc) rule in [BG86] and to the $(\Rightarrow \Box)$ rule in [Avr84]. An intuitively obvious way to understand the $(\vdash \Box_{\mathtt{LF}})$-rule is to recall the *double nature* of the $\Box$-formulae in $\mathtt{LF}$ as illustrated by the $\mathtt{LF}$-theorem below:

$$\Box\phi \Leftrightarrow \Box(\mathtt{F}(\phi) \Rightarrow \phi).$$

The rule below would highlight this double nature even more clearly:

$$\frac{\mathtt{X} \vdash \bullet(\mathtt{F}(\phi) \Rightarrow \phi)}{\mathtt{X} \vdash \Box\phi} \ (\vdash \Box')$$

But when $\mathtt{F}(\phi) \notin \{\phi, \Box\phi\}$, the above rule never satisfies Belnap's condition (C1) recalled below, without even mentioning condition (C8):

(C1) Each formula which occurs in the premiss of an inference $\mathcal{I}$ is a subformula of some formula that occurs in the conclusion of $\mathcal{I}$.

Instead of the ($\vdash \Box'$)-rule, we have designed a rule that may satisfy (C1). Actually, the ($\vdash \Box'$)-rule can be shown to be admissible (but not derivable without cut) in $\delta\mathtt{LF}$ thanks to Lemma 4 below:

Lemma 4 The following rules are admissible in $\delta\mathtt{LF}$:

$$\frac{\mathtt{X} \vdash \phi_1 \vee \phi_2}{\mathtt{X} \vdash \phi_1 \circ \phi_2}\ (\vdash \circ) \qquad \frac{\mathtt{X} \vdash \phi_1 \Rightarrow \phi_2}{\mathtt{X} \vdash *\phi_1 \circ \phi_2}\ (adm1) \qquad \frac{\phi_1 \wedge \phi_2 \vdash \mathtt{X}}{\phi_1 \circ \phi_2 \vdash \mathtt{X}}\ (\circ \vdash)$$

$$\frac{\neg\phi \vdash \mathtt{X}}{*\phi \vdash \mathtt{X}}\ (* \vdash) \qquad \frac{\mathtt{X} \vdash \neg\phi}{\mathtt{X} \vdash *\phi}\ (\vdash *) \qquad \frac{\mathtt{F}(\phi) \vdash \mathtt{X}}{m(\mathtt{F}(\phi), 0) \vdash \mathtt{X}}\ (adm2)$$

Moreover, for each of these rules, if the premiss has a cut-free derivation in $\delta\mathtt{LF}$, then the conclusion also has a cut-free derivation in $\delta\mathtt{LF}$.

Proof The proof of admissibility of the rules ($\vdash \circ$), ($adm1$), ($\circ \vdash$), ($* \vdash$) and ($\vdash *$) is similar to [Kra96, Lemma 9][3]. Admissibility of ($adm2$) is a mere consequence of the admissibility of the above rules.

As usual in **DL**, two formula occurrences in an inference $\mathcal{I}$ are *congruent* $\overset{\text{def}}{\Leftrightarrow}$ they occupy similar positions in occurrences of structures assigned to the same structure variable [Bel82]. Observe that in any inference of the ($\vdash \Box_{\mathtt{G}}$)-rule in $\delta\mathtt{G}$, the two instances of the occurrences of $\Box\phi$ are not *congruent* since they are not obtained by instantiating a *structure variable* but a formula variable, namely ϕ.

In what follows, we write

$$\frac{s}{s'}\ (dp)$$

to denote that the sequent s' is obtained from the sequent s by some finite number (possibly zero) of applications of display postulates from Figure 2. The ($\vdash \Box_{\mathtt{L}}$)-rule from $\delta\mathtt{L}$ is derivable in $\delta\mathtt{LF}$ as shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{X} \vdash \bullet\phi}{\bullet\mathtt{X} \vdash \phi}\ (dp)}{m(\mathtt{F}(\phi), 0) \circ \bullet\mathtt{X} \vdash \phi}\ (weak_l)}{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}\ (dp)}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}})$$

[3] Note that these rules are derivable using cut, as shown in [Gor96], but we cannot use this technique here because we do not know if $\delta\mathtt{LF}$ enjoys cut-elimination.

Lemma 5 Sequent $\phi \vdash \phi$ is cut-free derivable in $\delta\mathtt{LF}$ for any formula ϕ.

The proof of Lemma 5 is by induction on the formation of ϕ.

To prove soundness of $\delta\mathtt{LF}$ with respect to $\mathtt{LF}$-theoremhood, we use the mappings $a : \mathtt{struc}(\delta\mathtt{L}) \to \mathrm{FML}$ and $s : \mathtt{struc}(\delta\mathtt{L}) \to \mathrm{FML}$ below:

$$a(\phi) \stackrel{\text{def}}{=} s(\phi) \stackrel{\text{def}}{=} \phi \text{ for any } \phi \in \mathrm{FML}$$

$$\begin{array}{lcllcl}
a(I) & \stackrel{\text{def}}{=} & \top & s(I) & \stackrel{\text{def}}{=} & \bot \\
a(*\mathtt{X}) & \stackrel{\text{def}}{=} & \neg s(\mathtt{X}) & s(*\mathtt{X}) & \stackrel{\text{def}}{=} & \neg a(\mathtt{X}) \\
a(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} & a(\mathtt{X}) \wedge a(\mathtt{Y}) & s(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} & s(\mathtt{X}) \vee s(\mathtt{Y}) \\
a(\bullet\mathtt{X}) & \stackrel{\text{def}}{=} & \Diamond^{-} a(\mathtt{X}) & s(\bullet\mathtt{X}) & \stackrel{\text{def}}{=} & \Box s(\mathtt{X})
\end{array}$$

Theorem 6 If $\mathtt{X} \vdash \mathtt{Y}$ is derivable in $\delta\mathtt{LF}$, then $a(\mathtt{X}) \Rightarrow s(\mathtt{Y}) \in \mathtt{L^{+}F}$.

Proof By induction on the length of the given derivation of $\mathtt{X} \vdash \mathtt{Y}$.

By way of example, assume that the bottom-most rule application in the given derivation is the $(\vdash \Box_{\mathtt{LF}})$ rule. Thus $\mathtt{Y}$ is $\Box\phi$ and the premiss $\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)$ of this rule instance has a derivation in $\delta\mathtt{LF}$. We therefore have to show that $a(\mathtt{X}) \Rightarrow \Box\phi \in \mathtt{L^{+}F}$.

By the induction hypothesis, $a(\mathtt{X}) \Rightarrow s(\bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)) \in \mathtt{L^{+}F}$. That is, $a(\mathtt{X}) \Rightarrow \Box(\neg a(m(\mathtt{F}(\phi), 0)) \vee \phi) \in \mathtt{L^{+}F}$, which is just $a(\mathtt{X}) \Rightarrow \Box(a(m(\mathtt{F}(\phi), 0)) \Rightarrow \phi) \in \mathtt{L^{+}F}$.

By induction on the size of ϕ, one can show that $a(m(\mathtt{F}(\phi), 0)) \Leftrightarrow \mathtt{F}(\phi) \in \mathtt{K}$ and $s(m(\mathtt{F}(\phi), 1)) \Leftrightarrow \mathtt{F}(\phi) \in \mathtt{K}$. So, $a(\mathtt{X}) \Rightarrow \Box(\mathtt{F}(\phi) \Rightarrow \phi) \in \mathtt{L^{+}F}$. Since $\Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi \in \mathtt{L^{+}F}$, then $a(\mathtt{X}) \Rightarrow \Box\phi \in \mathtt{L^{+}F}$.

The maps a and s (for antecedent and succedent) can be found for instance in [Wan94, Kra96] where they are called τ_1 and τ_2. The interest of a and s is not only in the soundness proof but also in the way the structural connectives should be interpreted depending on the polarity of their occurrence (either as antecedent part or as succedent part).

Corollary 7 and Theorem 8 below are the **DL** versions of Theorem 1 in [Avr84] for Gentzen-style calculi.

Corollary 7 If $\mathtt{LF}$ admits a conservative tense extension and $I \vdash \phi$ is derivable in $\delta\mathtt{LF}$, then $\phi \in \mathtt{LF}$.

Theorem 8 If a formula $\phi \in \mathtt{LF}$, then $I \vdash \phi$ is derivable in $\delta\mathtt{LF}$.

Proof The proof is by induction on the length of the derivation in $\mathtt{LF}$. Actually, most of the cases have been already proved in [Wan94, Kra96, Wan98a]. Since $\Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi \in \mathtt{LF}$, it remains to show that

$$I \vdash \Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi$$

has a derivation in $\delta\mathtt{LF}$ which is done below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathtt{F}(\phi) \vdash \mathtt{F}(\phi)}}{m(\mathtt{F}(\phi), 0) \vdash \mathtt{F}(\phi)}\,(adm2) \quad \dfrac{\vdots}{\phi \vdash \phi}}{\mathtt{F}(\phi) \Rightarrow \phi \vdash *m(\mathtt{F}(\phi), 0) \circ \phi}\,(\Rightarrow\vdash)}{\Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}\,(\Box\vdash)}{\Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \Box\phi}\,(\vdash \Box_{\mathtt{LF}})}{I \circ \Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \Box\phi}\,(I_l)}{I \vdash \Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi}\,(\vdash\Rightarrow)$$

By Lemma 5, $\mathtt{F}(\phi) \vdash \mathtt{F}(\phi)$ and $\phi \vdash \phi$ are derivable in $\delta\mathtt{LF}$.

The proof of Theorem 8 *requires* uses of the cut rule to simulate applications of the *modus ponens* rule in $\mathtt{LF}$. A very important feature of the proof-theoretical framework **DL** is the existence of a very general cut-elimination theorem [Bel82]. Indeed, any display calculus satisfying the conditions (C2)-(C8) from [Bel82] admits cut-elimination. In [Wan98a], such a result is strengthened by proving that any classical modal display calculus defined from [Kra96] for a properly displayable classical modal logic admits a strong normalisation theorem; that is, the process of cut-elimination terminates for any sequence of eligible reductions. Unfortunately $\delta\mathtt{LF}$ does not satisfy (C8) recalled below (see e.g. [Wan98a]):

(C8) If there are inferences $\mathcal{I}_1$ and $\mathcal{I}_2$ with respective conclusions $\mathtt{X} \vdash \phi$ and $\phi \vdash \mathtt{Y}$ with ϕ *principal* in both inferences, and if cut is applied to obtain $\mathtt{X} \vdash \mathtt{Y}$, then

- either $\mathtt{X} \vdash \mathtt{Y}$ is identical to one of $\mathtt{X} \vdash \phi$ and $\phi \vdash \mathtt{Y}$;
- or there is a derivation of $\mathtt{X} \vdash \mathtt{Y}$ from the premisses of $\mathcal{I}_1$ and $\mathcal{I}_2$ in which every cut-formula of any application of cut is a proper subformula of ϕ.

Specifically, the cut instance shown below does not obey (C8):

$$\dfrac{\dfrac{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0)\circ\phi)}{\mathtt{X} \vdash \Box\phi}\,(\vdash \Box_{\mathtt{LF}}) \qquad \dfrac{\phi \vdash \mathtt{Y}'}{\Box\phi \vdash \bullet\mathtt{Y}'}\,(\Box\vdash)}{\mathtt{X} \vdash \bullet\mathtt{Y}'}\,(cut)$$

when some formula ψ in $m(\mathtt{F}(\phi),0)$ is not a subformula of ϕ. For instance, such cases are easy to find with the display calculi $\delta\mathtt{G}$ and $\delta\mathtt{Grz}$. Furthermore, to infer $\mathtt{X} \vdash \bullet\mathtt{Y}'$ from $\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0)\circ\phi)$ and $\phi \vdash \mathtt{Y}'$, no cut can be used on ψ if (C8) has to be satisfied. In the display calculus $\delta\mathtt{LF}$, for all the derivations of the sequent $\mathtt{X}'' \vdash \mathtt{Y}''$ from $\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0)\circ\phi)$, if a cut with a cut-formula that is not a subformula of ϕ is forbidden in the derivation, then either $\mathtt{X}'' \vdash \mathtt{Y}''$ contains ψ as the subformula of some formula/structure[4], or $\mathtt{X}'' \vdash \mathtt{Y}''$ contains $\Box\phi$ as the subformula of some formula/structure[5]. So, there is no way to derive $\mathtt{X} \vdash \bullet\mathtt{Y}'$ *in the general case* since neither ψ nor $\Box\phi$ are required to occur in it.

In the sequel, we say that LF is *pseudo displayable* $\overset{\text{def}}{\Leftrightarrow}$ for any $\phi \in$ FML, $I \vdash \phi$ has a derivation in $\delta\mathtt{LF}$ iff $I \vdash \phi$ has a cut-free derivation in $\delta\mathtt{LF}$. "Pseudo" because strong cut-elimination is couched using arbitrary sequents $\mathtt{X} \vdash \mathtt{Y}$ rather than sequents of the form $I \vdash \phi$. For mechanisation, "pseudo" is sufficient for our needs since we want to check whether $\phi \in$ LF. The next sections provide a characterization of a class of pseudo displayable logics and shows that both G and Grz are pseudo displayable.

4 Pseudo Displayable Logics

In this section, L is a properly displayable logic and F is a formula generation map. Let f : FML $\times \{0,1\} \to$ FML be the following map for $i \in \{0,1\}$:

$$\begin{array}{ll}
f(\mathtt{p},0) \overset{\text{def}}{=} \mathtt{p} & f(\mathtt{p},1) \overset{\text{def}}{=} \mathtt{p} \\
f(\top,i) \overset{\text{def}}{=} \top & f(\bot,i) \overset{\text{def}}{=} \bot \\
f(\neg\phi,0) \overset{\text{def}}{=} \neg f(\phi,1) & f(\neg\phi,1) \overset{\text{def}}{=} \neg f(\phi,0) \\
f(\phi_1 \wedge \phi_2,i) \overset{\text{def}}{=} f(\phi_1,i) \wedge f(\phi_2,i) & f(\phi_1 \vee \phi_2,i) \overset{\text{def}}{=} f(\phi_1,i) \vee f(\phi_2,i) \\
f(\phi_1 \Rightarrow \phi_2,1) \overset{\text{def}}{=} f(\phi_1,0) \Rightarrow f(\phi_2,1) & f(\phi_1 \Rightarrow \phi_2,0) \overset{\text{def}}{=} f(\phi_1,1) \Rightarrow f(\phi_2,0) \\
f(\Box\phi,1) \overset{\text{def}}{=} \Box(f(\mathtt{F}(\phi),0) \Rightarrow f(\phi,1)) & f(\Box\phi,0) \overset{\text{def}}{=} \Box f(\phi,0)
\end{array}$$

[4] See the introduction rules different from $(\vdash \Box_{\mathtt{LF}})$.

[5] See the $(\vdash \Box_{\mathtt{LF}})$-rule.

In $f(\phi, i)$, the index i should be seen as information about the *polarity* of ϕ in the translation process as done in [BH94]. The map f is well-defined because no $\Box$ occurs negatively in $\mathtt{F}(\mathtt{p})$. The definition of the map f also generalizes the definition of one of the maps from $\mathtt{G}$ into $\mathtt{K4}$ defined in [BH94]. The forthcoming Lemma 9 and Lemma 10 state that the map f belongs to a class of maps that have interesting properties with respect to $\mathtt{LF}$-theoremhood and $\mathtt{L}$-theoremhood.

Lemma 9 Let $f' : \mathrm{FML} \times \{0,1\} \rightarrow \mathrm{FML}$ be a map defined as f except that the clause for defining $f(\Box\phi, 1)$ is replaced by

$$f'(\Box\phi, 1) \stackrel{\mathrm{def}}{=} \Box(\psi_\phi \Rightarrow f'(\phi, 1))$$

where ψ_ϕ is a formula possibly defined from ϕ such that for any $\phi \in \mathrm{FML}$, $\psi_\phi \Leftrightarrow \mathtt{F}(\phi) \in \mathtt{KF}$. Then, for any $\phi \in \mathrm{FML}$ and any $i \in \{0,1\}$: $\phi \Leftrightarrow f'(\phi, i) \in \mathtt{KF}$.

The proof of Lemma 9 is quite straightforward via the rule of *replacement of equivalents*, which is admissible in $\mathtt{KF}$. Remember that $\mathtt{KF} \subseteq \mathtt{LF}$. A similar proof using simultaneous induction instead allows us to show that for any $\phi \in \mathrm{FML}$ and for any $i \in \{0,1\}$, $\phi \Leftrightarrow f(\phi, i) \in \mathtt{LF}$.

Lemma 10 Let $f' : \mathrm{FML} \times \{0,1\} \rightarrow \mathrm{FML}$ be a map defined as f except that the clause for defining $f(\Box\phi, 1)$ is replaced by

$$f'(\Box\phi, 1) \stackrel{\mathrm{def}}{=} \Box(\psi_\phi \Rightarrow f'(\phi, 1))$$

where ψ_ϕ is a formula possibly defined from ϕ. Then, for any $\phi \in \mathrm{FML}$,

1. $\phi \Rightarrow f'(\phi, 1) \in \mathtt{K}$.
2. $f'(\phi, 0) \Rightarrow \phi \in \mathtt{K}$.

Proof The proof is by simultaneous induction on the size of ϕ. The base case when $\phi \in \mathrm{PRP}$ is immediate. By way of example, let us treat the cases below in the induction step:

(i) $\phi_1 \wedge \phi_2 \Rightarrow f'(\phi_1 \wedge \phi_2, 1) \in \mathtt{K}$ (ii) $\Box\phi_1 \Rightarrow f'(\Box\phi_1, 1) \in \mathtt{K}$

(iii) $f'(\neg\phi_1, 0) \Rightarrow \neg\phi_1 \in \mathtt{K}$ (iv) $f'(\Box\phi_1, 0) \Rightarrow \Box\phi_1 \in \mathtt{K}$

(i) By induction hypothesis, $\phi_1 \Rightarrow f'(\phi_1, 1) \in \mathtt{K}$ and $\phi_2 \Rightarrow f'(\phi_2, 1) \in \mathtt{K}$. By easy manipulations in propositional logic, $\phi_1 \wedge \phi_2 \Rightarrow f'(\phi_1, 1) \wedge f'(\phi_2, 1) \in \mathtt{K}$. By definition of f', $\phi_1 \wedge \phi_2 \Rightarrow f'(\phi_1 \wedge \phi_2, 1) \in \mathtt{K}$.

(ii) By induction hypothesis, $\phi_1 \Rightarrow f'(\phi_1, 1) \in \mathtt{K}$. By easy manipulation at the propositional level, $\phi_1 \Rightarrow (\psi_{\phi_1} \Rightarrow f'(\phi_1, 1)) \in \mathtt{K}$. It is known that the *regular rule*, from $\psi_1 \Rightarrow \psi_2$ infer $\Box\psi_1 \Rightarrow \Box\psi_2$, is admissible in K. So, $\Box\phi_1 \Rightarrow \Box(\psi_{\phi_1} \Rightarrow f'(\phi_1, 1)) \in \mathtt{K}$. By definition of f', $\Box\phi_1 \Rightarrow f'(\Box\phi_1, 1) \in \mathtt{K}$.

(iii) By induction hypothesis, $\phi_1 \Rightarrow f'(\phi_1, 1) \in \mathtt{K}$. By easy manipulation at the propositional level, $\neg f'(\phi_1, 1) \Rightarrow \neg\phi_1 \in \mathtt{K}$. By definition of f', $f'(\neg\phi_1, 0) \Rightarrow \neg\phi_1 \in \mathtt{K}$.

(iv) By induction hypothesis, $f'(\phi_1, 0) \Rightarrow \phi_1 \in \mathtt{K}$. Since the regular rule is admissible in K, $\Box f'(\phi_1, 0) \Rightarrow \Box\phi_1 \in \mathtt{K}$. By definition of f', $f'(\Box\phi_1, 0) \Rightarrow \Box\phi_1 \in \mathtt{K}$.

As a corollary of Lemma 10, for any $\phi \in \mathrm{FML}$:

$$f(\phi, 0) \Rightarrow \phi \in \mathtt{L} \qquad \phi \Rightarrow f(\phi, 1) \in \mathtt{L} \qquad f(\phi, 0) \Rightarrow f(\phi, 1) \in \mathtt{L}$$

All the maps from G into K4 defined in [BH94] satisfy the hypothesis of Lemma 9 (and *a fortiori* also the hypothesis of Lemma 10).

Lemma 11 Every positive [resp. negative] occurrence of

1. $\Box\psi$ in $f(\phi, 1)$ is of the form $\Box(f(\mathtt{F}(\varphi), 0) \Rightarrow f(\varphi, 1))$ [resp. $\Box f(\varphi, 0)$] for some subformula φ of ϕ;
2. $\neg\psi$ in $f(\phi, 1)$ is of the form $\neg f(\varphi, 0)$ [resp. $\neg f(\varphi, 1)$] for some subformula φ of ϕ;
3. $\psi_1 \Rightarrow \psi_2$ in $f(\phi, 1)$ is of the form $f(\varphi_1, 0) \Rightarrow f(\varphi_2, 1)$ [resp. $f(\varphi_1, 1) \Rightarrow f(\varphi_2, 0)$] for some subformulae φ_1, φ_2 of ϕ;
4. $\psi_1 \oplus \psi_2$ ($\oplus \in \{\wedge, \vee\}$) in $f(\phi, 1)$ is of the form $f(\varphi_1, 1) \oplus f(\varphi_2, 1)$ [resp. $f(\varphi_1, 0) \oplus f(\varphi_2, 0)$] for some subformulae φ_1, φ_2 of ϕ;

The proof of Lemma 11 is by an easy verification. We extend the map f to structures in the following way ($i \in \{0, 1\}$):

$$f(I, i) \stackrel{\text{def}}{=} I \qquad\qquad f(*\mathtt{X}, i) \stackrel{\text{def}}{=} *f(\mathtt{X}, 1 - i)$$

$$f(\mathtt{X} \circ \mathtt{Y}, i) \stackrel{\text{def}}{=} f(\mathtt{X}, i) \circ f(\mathtt{Y}, i) \qquad\qquad f(\bullet\mathtt{X}, i) \stackrel{\text{def}}{=} \bullet f(\mathtt{X}, i)$$

Lemma 12 The following rules are admissible in δ`L`:

$$\frac{\mathtt{X} \vdash f(m(\phi,1),1)}{\mathtt{X} \vdash f(\phi,1)}\ (adm3) \qquad\qquad \frac{f(m(\phi,0),0) \vdash \mathtt{X}}{f(\phi,0) \vdash \mathtt{X}}\ (adm4)$$

Proof By induction on the size of ϕ. The base case when $\phi \in \mathrm{PRP}$ is obvious. By way of example, we treat here the cases $\phi = \phi_1 \wedge \phi_2$. Since $m(\phi,1) = \phi$, the proof of this case for $(adm3)$ is trivial. For $(adm4)$ the derivation below suffices, remember $f(\phi_1,0) \wedge f(\phi_2,0) = f(\phi_1 \wedge \phi_2, 0)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ f(m(\phi_1,0),0) \circ f(m(\phi_2,0),0) \vdash \mathtt{X}\end{array}}{f(m(\phi_2,0),0) \vdash * f(m(\phi_1,0),0) \circ \mathtt{X}}\ (dp)}{f(\phi_2,0) \vdash * f(m(\phi_1,0),0) \circ \mathtt{X}}\ (adm4), IH}{f(m(\phi_1,0),0) \vdash \mathtt{X} \circ * f(\phi_2,0)}\ (dp)}{f(\phi_1,0) \vdash \mathtt{X} \circ * f(\phi_2,0)}\ (adm4), IH}{f(\phi_1,0) \circ f(\phi_2,0) \vdash \mathtt{X}}\ (dp)}{f(\phi_1,0) \wedge f(\phi_2,0) \vdash \mathtt{X}}\ (\wedge \vdash)$$

Let us also treat the case $\phi = \Box\phi_1$. Since $m(\Box\phi_1, i) = \Box\phi_1$ for $i \in \{0,1\}$, the proof is trivial.

In order to prove the next theorem, we define a *partial* function f^{-1} : `struc`(δ`L`) $\times \{0,1\} \rightarrow$ `struc`(δ`L`) in Figure 5.

Lemma 13 For any $\phi \in \mathrm{FML}$,

1. $f^{-1}(f(\phi,1),1) = \phi$ and $f^{-1}(f(\phi,0),0) = \phi$;
2. for any positive [resp. negative] occurrence of ψ in $f(\phi,1)$, $f^{-1}(\psi,1)$ is defined [resp. $f^{-1}(\psi,0)$ is defined].

Proof By induction on the size of the formula ϕ.

Theorem 14 below is maybe the most surprising result of the paper. Indeed, (weak) cut-elimination of δ`LF` is equivalent to the theoremhood preserving nature of f from `LF` into `L`. Its proof is purely syntactic and therefore does not depend on the class of modal frames that possibly characterises `LF`.

$f^{-1}(I, i) \stackrel{\text{def}}{=} I$ $\qquad f^{-1}(\top, i) \stackrel{\text{def}}{=} \top$ $\qquad f^{-1}(\bot, i) \stackrel{\text{def}}{=} \bot$

$f^{-1}(\mathtt{X} \circ \mathtt{Y}, i) \stackrel{\text{def}}{=} f^{-1}(\mathtt{X}, i) \circ f^{-1}(\mathtt{Y}, i)$ or undefined

$f^{-1}(\bullet\mathtt{X}, i) \stackrel{\text{def}}{=} \bullet f^{-1}(\mathtt{X}, i)$ or undefined

$f^{-1}(*\mathtt{X}, 1-i) \stackrel{\text{def}}{=} *f^{-1}(\mathtt{X}, i)$ or undefined

$f^{-1}(\mathtt{p}, i) \stackrel{\text{def}}{=} \mathtt{p}$ for any $\mathtt{p} \in$ PRP

for $\oplus \in \{\wedge, \vee\}$, $f^{-1}(\phi \oplus \psi, i) \stackrel{\text{def}}{=} f^{-1}(\phi, i) \oplus f^{-1}(\psi, i)$ or undefined

$f^{-1}(\phi \Rightarrow \psi, 1) \stackrel{\text{def}}{=} f^{-1}(\phi, 0) \Rightarrow f^{-1}(\psi, 1)$ or undefined

$f^{-1}(\phi \Rightarrow \psi, 0) \stackrel{\text{def}}{=} f^{-1}(\phi, 1) \Rightarrow f^{-1}(\psi, 0)$ or undefined

$f^{-1}(\neg\phi, 1-i) \stackrel{\text{def}}{=} \neg f^{-1}(\phi, i)$ or undefined

$f^{-1}(\Box\phi, 0) \stackrel{\text{def}}{=} \Box f^{-1}(\phi, 0)$ or undefined

$$f^{-1}(\Box\phi, 1) \stackrel{\text{def}}{=} \begin{cases} \Box f^{-1}(\phi_2, 1) \text{ if } \phi = (\phi_1 \Rightarrow \phi_2) \text{ and } f^{-1}(\phi_2, 1) \text{ is defined} \\ \text{undefined otherwise} \end{cases}$$

where "$x \stackrel{\text{def}}{=} y$ or undefined" means:

$$x \stackrel{\text{def}}{=} \begin{cases} y \text{ if all components of } y \text{ are defined} \\ \text{undefined otherwise} \end{cases}$$

Figure 5: Definition of $f^{-1}(.)$ for $i \in \{0, 1\}$.

Theorem 14 Let L be a primitive modal logic (properly displayable), F be a formula generation map such that LF admits a conservative tense extension. Then the statements below are equivalent:

1. For all $\phi \in$ FML, $\phi \in$ LF iff $f(\phi, 1) \in$ L.
2. LF is pseudo displayable.

Proof (2) implies (1): Since LF is an extension of L, if $f(\phi, 1) \in$ L, then *a fortiori* $f(\phi, 1) \in$ LF. By Lemma 9, $\phi \Leftrightarrow f(\phi, 1) \in$ LF and therefore $\phi \in$ LF.

Now assume $\phi \in$ LF. So $I \vdash \phi$ has a derivation in δLF. Since LF is pseudo displayable, $I \vdash \phi$ has a cut-free derivation in δLF. We show that:

($\star$) in the given cut-free derivation of $I \vdash \phi$, for every sequent $\mathtt{X} \vdash \mathtt{Y}$ with cut-free derivation Π, the sequent $f(\mathtt{X}, 0) \vdash f(\mathtt{Y}, 1)$ admits a cut-free derivation, say $f(\Pi)$, in δL.

From ($\star$), we can conclude that $I \vdash f(\phi, 1)$ is derivable in δL and therefore $f(\phi, 1) \in$ L (see Theorem 3).

The proof of ($\star$) is by induction on the structure of the given derivation. All the structural rules (those involving only structure variables) pose no difficulties because f is homomorphic for the structural connectives. By way of example, the derivation step in $\delta\mathtt{LF}$

$$\frac{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\psi),0) \circ \psi)}{\mathtt{X} \vdash \Box\psi}\ (\vdash \Box_{\mathtt{LF}})$$

is transformed into the derivation steps in $\delta\mathtt{L}$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{f(\mathtt{X},0) \vdash \bullet(*f(m(\mathtt{F}(\psi),0),0) \circ f(\psi,1))}{\bullet f(\mathtt{X},0) \vdash *f(m(\mathtt{F}(\psi),0),0) \circ f(\psi,1)}\ (dp)}{\bullet f(\mathtt{X},0) \vdash f(\psi,1) \circ *f(m(\mathtt{F}(\psi),0),0)}\ (com_r)}{f(m(\mathtt{F}(\psi),0),0) \vdash * \bullet f(\mathtt{X},0) \circ f(\psi,1)}\ (dp)}{f(\mathtt{F}(\psi),0) \vdash * \bullet f(\mathtt{X},0) \circ f(\psi,1)}\ (adm4)}{\bullet f(\mathtt{X},0) \circ f(\mathtt{F}(\psi),0) \vdash f(\psi,1)}\ (dp)}{\bullet f(\mathtt{X},0) \vdash f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1)}\ (\vdash\Rightarrow)}{f(\mathtt{X},0) \vdash \bullet(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\ (dp)}{f(\mathtt{X},0) \vdash \Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\ (\vdash \Box_{\mathtt{L}})$$

The other cases are quite straightforward. For instance, the derivation step in $\delta\mathtt{LF}$ shown below left is transformed into the derivation step in $\delta\mathtt{L}$ shown below right:

$$\frac{\phi_1 \circ \phi_2 \vdash \mathtt{X}}{\phi_1 \wedge \phi_2 \vdash \mathtt{X}}\ (\wedge \vdash) \qquad\qquad \frac{f(\phi_1,0) \circ f(\phi_2,0) \vdash f(\mathtt{X},1)}{f(\phi_1,0) \wedge f(\phi_2,0) \vdash f(\mathtt{X},1)}\ (\wedge \vdash)$$

since $f(\phi_1 \wedge \phi_2, 0) = f(\phi_1,0) \wedge f(\phi_2,0)$ and $f(\phi_1 \circ \phi_2, 0) = f(\phi_1,0) \circ f(\phi_2,0)$.

(1) implies (2): Assume that $I \vdash \phi$ has a derivation in $\delta\mathtt{LF}$. Since $\mathtt{LF}$ has a conservative tense extension, by Corollary 7, $\phi \in \mathtt{LF}$ and by assumption (1), $f(\phi,1) \in \mathtt{L}$. Thus $I \vdash f(\phi,1)$ has a cut-free derivation in $\delta\mathtt{L}$. Let us show that $I \vdash \phi$ has a cut-free derivation in $\delta\mathtt{LF}$. We show that

($\star\star$) in the given cut-free derivation of $I \vdash f(\phi,1)$ in $\delta\mathtt{L}$, for every sequent $\mathtt{X} \vdash \mathtt{Y}$ with cut-free derivation Π, the sequent $f^{-1}(\mathtt{X},0) \vdash f^{-1}(\mathtt{Y},1)$ admits a cut-free derivation, say $f^{-1}(\Pi)$, in $\delta\mathtt{LF}$.

It is worth observing that thanks to Lemma 13, and to the fact that $\delta\mathtt{L}$ satisfies (C1)-(C8), $f^{-1}(\mathtt{X},0)$ and $f^{-1}(\mathtt{Y},1)$ are always defined. If ($\star\star$)

holds, then $I \vdash f^{-1}(f(\phi,1),1)$ has a cut-free derivation in $\delta\mathtt{LF}$: that is $I \vdash \phi$ has a cut-free derivation in $\delta\mathtt{LF}$.

As expected, the proof of ($\star\star$) is by induction on the structure of the given $\delta\mathtt{L}$ derivation. The base cases when $\mathtt{X} \vdash \mathtt{Y}$ is of the form $\mathtt{p} \vdash \mathtt{p}$, $I \vdash \top$ or $\bot \vdash I$ are immediate. For every rule of $\delta\mathtt{L}$ except (cut) with premisses $\mathtt{X}_1 \vdash \mathtt{Y}_1, \ldots, \mathtt{X}_k \vdash \mathtt{Y}_k$ and conclusion $\mathtt{X} \vdash \mathtt{Y}$, we must now show that: if each of the sequents $f^{-1}(\mathtt{X}_1,0) \vdash f^{-1}(\mathtt{Y}_1,1), \ldots, f^{-1}(\mathtt{X}_k,0) \vdash f^{-1}(\mathtt{Y}_k,1)$ is cut-free provable in $\delta\mathtt{LF}$, then $f^{-1}(\mathtt{X},0) \vdash f^{-1}(\mathtt{Y},1)$ is also cut-free provable in $\delta\mathtt{LF}$.

This poses no difficulty when the rule is a basic structural rule (from Figure 2 and Figure 4) or a structural rule obtained from the axioms for $\mathtt{L}$ or an operational rule introducing a Boolean connective of the form $\wedge, \vee$ because f^{-1} is homomorphic for these connectives and because each rule satisfies Belnap's condition (C4), recalled shortly.

By Lemma 11, the $\Box$-formulae occurring as succedent parts in the $\delta\mathtt{L}$-derivation of $I \vdash f(\phi,1)$ have a particular form. Therefore, all instances of the rule $(\vdash \Box_{\mathtt{L}})$ in the given $\delta\mathtt{L}$-derivation must be of the form shown below left, and these are replaced by the $\delta\mathtt{LF}$ rule applications shown below right:

$$\dfrac{\mathtt{X} \vdash \bullet(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}{\mathtt{X} \vdash \Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\,(\vdash \Box_{\mathtt{L}})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f^{-1}(\mathtt{X},0) \vdash \bullet(\mathtt{F}(\psi) \Rightarrow \psi)}{\bullet f^{-1}(\mathtt{X},0) \vdash \mathtt{F}(\psi) \Rightarrow \psi}\,(dp)}{\bullet f^{-1}(\mathtt{X},0) \vdash (*\mathtt{F}(\psi) \circ \psi)}\,(adm1)}{\mathtt{F}(\psi) \vdash \psi \circ * \bullet f^{-1}(\mathtt{X},0)}\,(dp)}{m(\mathtt{F}(\psi),0) \vdash \psi \circ * \bullet f^{-1}(\mathtt{X},0)}\,(adm2)}{f^{-1}(\mathtt{X},0) \vdash \bullet(*m(\mathtt{F}(\psi),0) \circ \psi)}\,(dp)}{f^{-1}(\mathtt{X},0) \vdash \Box\psi}\,(\vdash \Box_{\mathtt{LF}})$$

since

$$f^{-1}(f(\psi,1),1) = \psi$$
$$f^{-1}(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1),1) = \mathtt{F}(\psi) \Rightarrow \psi$$

and

$$f^{-1}(\Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1)),1) = \Box\psi.$$

If the $\delta\mathtt{L}$ rule application is as shown below left, then it is transformed into the $\delta\mathtt{LF}$ rule application shown below right:

$$\frac{f(\psi,0) \vdash \mathtt{X}}{\Box f(\psi,0) \vdash \bullet\mathtt{X}}\ (\Box\vdash) \qquad\qquad \frac{\psi \vdash f^{-1}(\mathtt{X},1)}{\Box\psi \vdash \bullet f^{-1}(\mathtt{X},1)}\ (\Box\vdash)$$

since $f^{-1}(f(\psi,0),0) = \psi$.

Similarly, the derivation step in $\delta\mathtt{L}$ shown below left is transformed into the derivation step in $\delta\mathtt{LF}$ shown below right

$$\frac{\mathtt{X} \vdash *f(\psi,0)}{\mathtt{X} \vdash \neg f(\psi,0)}\ (\vdash\neg) \qquad\qquad \frac{f^{-1}(\mathtt{X},0) \vdash *\psi}{f^{-1}(\mathtt{X},0) \vdash \neg\psi}\ (\vdash\neg)$$

since $f^{-1}(\neg f(\psi,0),1) = \neg\psi$ and $f^{-1}(*f(\psi,0),1) = *\psi$.

The other cases for the Boolean connectives are left to the reader. Lemma 11 is of course used.

Remark 15 In both subproofs, it is crucial that $\delta\mathtt{L}$ and $\delta\mathtt{LF}$ satisfy Belnap's condition (C4), and that the given initial derivation of $I \vdash \phi$ in $\delta\mathtt{LF}$ and $I \vdash f(\phi,1)$ in $\delta\mathtt{L}$ is cut-free. The condition (C4) is recalled below:

(C4) Congruent parameters are either all antecedent or all succedent parts of their respective sequent.

The *parameters* are substructures of some structure obtained by instantiating some structural variable [Bel82].

For instance, in the induction step, the $\delta\mathtt{LF}$ derivation step shown below left does not guarantee that the corresponding $\delta\mathtt{L}$ derivation step below right is a correct application of (cut)

$$\frac{\mathtt{X} \vdash \psi \quad \psi \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (cut) \qquad\qquad \frac{f(\mathtt{X},0) \vdash f(\psi,1) \quad f(\psi,0) \vdash f(\mathtt{Y},1)}{f(\mathtt{X},0) \vdash f(\mathtt{Y},1)}\ (?)$$

since, in general, $f(\psi,1) \neq f(\psi,0)$.

Similarly, if $\delta\mathtt{LF}$ and $\delta\mathtt{L}$ contained the rule (r) shown below left, which breaks (C4), then the step below right is not generally a correct application of (r) in $\delta\mathtt{L}$:

$$\frac{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (r) \qquad\qquad \frac{f(\mathtt{X},0) \circ f(\mathtt{Y},0) \vdash f(\mathtt{Y},1)}{f(\mathtt{X},0) \vdash f(\mathtt{Y},1)}\ (?)$$

The proof of Theorem 14 shows once more that **DL** is particularly well-designed to reason about polarity, succedent and antecedent parts. One of the translations from $\mathtt{G}$ into $\mathtt{K4}$ defined in [BH94] is exactly the map f when $\mathtt{L}$ is $\mathtt{K4}$ and $\mathtt{F}$ is $\mathtt{F}_{\mathtt{G}}$. Consequently, by Theorem 14 we obtain

Corollary 16 $\mathtt{G}$ is pseudo displayable.

Under reasonable hypotheses, there exist polynomial-time reductions from $\mathtt{LF}$ into $\mathtt{L}$.

Theorem 17 Let $\mathtt{L}$ be a primitive modal logic, $\mathtt{F}$ be a formula generation map such that $\mathtt{LF}$ is a pseudo displayable logic that has a conservative tense extension. There exists a polynomial-time transformation[6] g such that for any $\phi \in \mathrm{FML}$, $\phi \in \mathtt{LF}$ iff $g(\phi) \in \mathtt{L}$.

Proof The right-hand side of the definition of $f(\Box\psi, 1)$ may require several calls to $f(\psi, 0)$ and $f(\psi, 1)$, so f is not necessarily computable in polynomial-time (unless $\mathtt{F}$ contains a unique occurrence of a propositional variable). However, we can use a variant of f using a standard renaming technique (see e.g. [Min88]). Let $md(\phi)$ denote the *modal depth* of ϕ, let $\Box^0\varphi \stackrel{\text{def}}{=} \varphi$ and $\Box^{k+1}\varphi \stackrel{\text{def}}{=} \Box\Box^k\varphi$ for $k \in \mathbb{N}$. Then for any extension $\mathtt{L}$ of $\mathtt{K}$,

$$\phi \in \mathtt{L} \text{ iff } (\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{new} \Leftrightarrow \psi)) \Rightarrow \phi' \in \mathtt{L}$$

where ϕ' is obtained by replacing every occurrence of ψ in ϕ by $\mathtt{p}_{new}$, a new propositional variable not occurring in ϕ.

The key point to define g is to observe that there is a map $\mathtt{F}'$: $\mathrm{FML} \times \mathrm{FML} \rightarrow \mathrm{FML}$ and a formula $\psi_{\mathtt{F}'}$ containing at most *two* atomic propositions, say $\mathtt{p}$ and $\mathtt{q}$, such that

- $\mathtt{F}'(\varphi_1, \varphi_2)$ is obtained from $\psi_{\mathtt{F}'}$ by simultaneously replacing every occurrence of $\mathtt{p}$ [resp. $\mathtt{q}$] by φ_1 [resp. φ_2];
- for any $\varphi \in \mathrm{FML}$, $f(\mathtt{F}(\varphi), 0) = \mathtt{F}'(f(\varphi, 0), f(\varphi, 1))$.

For instance, if $\mathtt{F} = \lambda\mathtt{p}.\mathtt{p} \wedge \neg\mathtt{p}$ then $\mathtt{F}' = \lambda\mathtt{pq}.\mathtt{p} \wedge \neg\mathtt{q}$.

Let ϕ be a modal formula we wish to translate from $\mathtt{LF}$ into $\mathtt{L}$. Let $\phi_1, \ldots, \phi_m$ be an enumeration (without repetition) of all the subformulae of ϕ in increasing order with respect to the size. We shall build a formula

[6] Also called a "many-one reduction", see e.g. [Pap94].

Form of ϕ_i	ψ_i
$\top$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow \top) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow \top))$
$\bot$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow \bot) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow \bot))$
$\mathtt{p}$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,0} \Leftrightarrow \mathtt{p}_{i,1}))$
$\neg\phi_j$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow \neg\mathtt{p}_{j,0}) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow \neg\mathtt{p}_{j,1}))$
$\phi_{i_1} \wedge \phi_{i_2}$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow (\mathtt{p}_{i_1,1} \wedge \mathtt{p}_{i_2,1})) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow (\mathtt{p}_{i_1,0} \wedge \mathtt{p}_{i_2,0})))$
$\phi_{i_1} \vee \phi_{i_2}$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow (\mathtt{p}_{i_1,1} \vee \mathtt{p}_{i_2,1})) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow (\mathtt{p}_{i_1,0} \vee \mathtt{p}_{i_2,0})))$
$\phi_{i_1} \Rightarrow \phi_{i_2}$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow (\mathtt{p}_{i_1,0} \Rightarrow \mathtt{p}_{i_2,1})) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow (\mathtt{p}_{i_1,1} \Rightarrow \mathtt{p}_{i_2,0})))$
$\Box\phi_j$	$(\bigwedge_{k=0}^{md(\phi)} \Box^k(\mathtt{p}_{i,1} \Leftrightarrow \Box(\mathtt{F}'(\mathtt{p}_{j,0}, \mathtt{p}_{j,1}) \Rightarrow \mathtt{p}_{j,1})) \wedge \Box^k(\mathtt{p}_{i,0} \Leftrightarrow \Box\mathtt{p}_{j,0}))$

Figure 6: Definition of ψ_i

$g(\phi)$ using the set $\{\mathtt{p}_{i,j} : 1 \leq i \leq m,\ j \in \{0,1\}\}$ of atomic propositions[7] such that $g(\phi) \in \mathtt{L}$ iff $f(\phi, 1) \in \mathtt{L}$.

Moreover, $g(\phi)$ can be computed in time $\mathcal{O}(|\phi|^3.log\ |\phi|)$. For $i \in \{1,\ldots,m\}$, we associate a formula ψ_i as shown in Figure 6 and let

$$g(\phi) \stackrel{\text{def}}{=} (\bigwedge_{i=1}^{m} \psi_i) \Rightarrow \mathtt{p}_{m,1}.$$

Each $|\psi_i|$ is in $\mathcal{O}(|\phi|^2 \times log\ |\phi|)$ since $\Sigma_{i=0}^{md(\phi)} i$ is in $\mathcal{O}(|\phi|^2)$. So $|g(\phi)|$ is in $\mathcal{O}(|\phi| \times (|\phi|^2 \times log\ |\phi|))$. As usual in complexity theory, the extra $log\ |\phi|$ factor in the size of ϕ is because we need an index of size $\mathcal{O}(log\ |\phi|)$ for these different atomic propositions. That is, these indices are represented in binary writing.

Similarly, one can show that if LF is pseudo-displayable and LF has a conservative tense extension, then there is a polynomial-time transformation g' such that for any $\phi \in$ FML, $\neg\phi \not\in \mathtt{LF}$ iff $\neg g'(\phi) \not\in \mathtt{L}$. So, if both L and LF are characterized by classes of modal frames, LF is pseudo displayable and LF has a conservative tense extension, then there is a polynomial-time map from LF-satisfiability into L-satisfiability. But note

[7]We could also just consider the set $\{\mathtt{p}_i : i \in \mathbb{N}\}$ of atomic propositions and use a 1-1 mapping from $\mathbb{N}^2 \rightarrow \mathbb{N}$ but for simplicity, the present option is the most convenient.

$$\text{(initial sequents)} \qquad \psi \vdash \psi \qquad \bot \vdash \qquad \vdash \top$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \Delta, \neg\phi}\ (\vdash \neg) \qquad \frac{\Gamma \vdash \Delta, \phi}{\Gamma, \neg\phi \vdash \Delta}\ (\neg \vdash)$$

$$\frac{\Gamma \vdash \Delta, \phi_1 \quad \Gamma \vdash \Delta, \phi_2}{\Gamma \vdash \Delta, \phi_1 \wedge \phi_2}\ (\vdash \wedge) \qquad \frac{\Gamma, \phi_1, \phi_2 \vdash \Delta}{\Gamma, \phi_1 \wedge \phi_2 \vdash \Delta}\ (\wedge \vdash)$$

$$\frac{\Gamma, \phi_1 \vdash \Delta, \phi_2}{\Gamma \vdash \Delta, \phi_1 \Rightarrow \phi_2}\ (\vdash \Rightarrow) \qquad \frac{\Gamma \vdash \Delta, \phi_1 \quad \Gamma, \phi_2 \vdash \Delta}{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \Delta}\ (\Rightarrow \vdash)$$

Figure 7: Standard rules

once again that none of the proofs in this paper hinge on semantical notions since all proofs are syntactic.

Remark 18 One of the maps from G into K4 from [BH94] is in polynomial time and does not use the renaming technique (which allows us to treat the general case). We are currently investigating if their map can be generalised by considering the map $f' : \mathrm{FML} \times \{0,1\} \to \mathrm{FML}$ inductively defined as f except that $f'(\Box\phi, 1) \stackrel{\mathrm{def}}{=} \Box(\mathtt{F}(\phi) \Rightarrow f'(\phi, 1))$. Another map in polynomial time from G into K4 is given in [Fit83, Chapter 5]. Kracht [Kra99] notes that such maps exist for nearly all "classical" logics.

5 Pseudo Gentzenisable Logics

Theorem 14 admits a natural counterpart when LF has a traditional Gentzen-style calculus, see forthcoming Theorem 23. However, the **DL** framework appears to be much more flexible. Typically, δLF is sound and complete as soon as LF admits a conservative tense extension. The assumptions in Definition 19 below illustrate that the sequent-style formulation of Theorem 14 require more restrictions. That is why the present section is designed to allow a comparison with **DL** but our main technical contribution is in Section 4.

In the rest of this section, L is a properly displayable modal logic and F is a formula generation map.

Definition 19 Let $\mathtt{L}$ be a primitive modal logic such that $\mathtt{L}$ admits a traditional Gentzen-style calculus $\mathtt{GenL}$ in which:

1. $\mathtt{GenL}$ is an extension of a standard Gentzen calculus with contraction, weakening, exchange and cut (see the standard introduction rules in Figure 7), with $\mathtt{GenL}$ satisfying the cut-elimination theorem;
2. Any sequent $\Gamma \vdash \Delta$ is derivable in $\mathtt{GenL}$ iff $(\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow (\bigvee_{\phi \in \Delta} \phi) \in \mathtt{L}$;
3. The $(\Box \vdash)$-rule (if any) and the $(\vdash \Box)$-rule (possibly augmented with a side condition) have the form

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \Box\phi \vdash \Delta}\ (\Box \vdash) \qquad\qquad \frac{\Sigma' \vdash \phi}{\Sigma \vdash \Box\phi}\ (\vdash \Box)$$

where there is a map $h : \mathrm{FML} \to \mathrm{FML}$ such that

$$h(\Sigma) = \Sigma' \qquad\qquad h(f(\Sigma, 0)) = f(\Sigma', 0)$$

for any sequence Σ to which the $(\vdash \Box)$-rule is applicable. Both h and f (defined via $\mathtt{F}$) are naturally extended to sequences of formulae. Moreover, $f(\Sigma, 0)$ satisfies the side conditions on the $(\vdash \Box)$-rule iff Σ does too.
4. Sequents are understood as pairs of sequences of formulae and "," is the concatenation operator.

Then, $\mathtt{LF}$ is *pseudo Gentzenisable* $\overset{\text{def}}{\Leftrightarrow}$ the Gentzen-style calculus $\mathtt{GenLF}$ obtained from $\mathtt{GenL}$ by replacing the $(\vdash \Box)$-rule by the one below enjoys cut-elimination

$$\frac{\mathtt{F}(\phi), \Sigma' \vdash \phi}{\Sigma \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}})$$

For instance, when h is the identity function, the condition 3. in Definition 19 is satisfied. Although it is interesting in its own sake that $\mathtt{LF}$ is pseudo Gentzenisable, we shall use this property to establish that $\mathtt{LF}$ is pseudo displayable.

In [Avr84], it is shown that when $\mathtt{L}$ is among $\mathtt{K}$, $\mathtt{K4}$, $\mathtt{S4}$, $\mathtt{GenLF}$ is sound and complete for $\mathtt{LF}$.

Condition 3 in Definition 19 ensures in some sense that Belnap's condition (C4) is satisfied when the *structures* are simply sequences of formulae.

Example 20 Two particular Gentzen calculi which obey Definition 19 are the traditional calculi GenK4 and GenS4 containing the modal rules:

$$(\Box_{S4}\vdash)\ \frac{\phi,\Gamma\vdash\Delta}{\Box\phi,\Gamma\vdash\Delta}\qquad(\vdash\Box_{S4})\ \frac{\Box\Gamma\vdash\phi}{\Box\Gamma\vdash\Box\phi}\qquad(\vdash\Box_{K4})\ \frac{\Box\Gamma,\Gamma\vdash\phi}{\Box\Gamma\vdash\Box\phi}$$

These are known to enjoy cut-elimination; see [Gor99].

Here, f is extended to sets of formulae in the natural way: for $i\in\{0,1\}$, $f(\phi_1,\ldots,\phi_m,i)\stackrel{\text{def}}{=}f(\phi_1,i),\ldots,f(\phi_m,i)$. Lemma 21 below is used in the forthcoming Theorem 23.

Lemma 21 Let $\Gamma\vdash\Delta$ be a sequent and Π be a derivation of $f(\Gamma,0)\vdash f(\Delta,1)$ in GenL. Then, for any sequent $\Gamma'\vdash\Delta'$ in Π, Γ' [resp. Δ'] is of the form $f(\phi_1,0),\ldots,f(\phi_m,0)$ [resp. of the form $f(\phi_1,1),\ldots,f(\phi_m,1)$].

Proof For any sequent $\Gamma'\vdash\Delta'$ in Π, we write $d(\Gamma'\vdash\Delta')$ to denote the length of the path between the root sequent $f(\Gamma,0)\vdash f(\Delta,1)$ and $\Gamma'\vdash\Delta'$. The proof is by induction on $d(\Gamma'\vdash\Delta')$. Lemma 11 is used together with the fact that $f(\Box\psi,1)=\Box(f(\mathtt{F}(\psi)\Rightarrow\psi,1))$.

Here is another (standard) property about GenL and GenLF we need :

Lemma 22 Let $\Gamma\vdash\Delta$ be a sequent and ϕ,ψ be formulae. Then, $\Gamma,\phi\vdash\psi,\Delta$ has a cut-free derivation in GenL [resp. in GenLF] iff $\Gamma\vdash\phi\Rightarrow\psi,\Delta$ has a cut-free derivation in GenL [resp. in GenLF]

Proof The proof for the direction from left to right is obvious by using the $(\vdash\Rightarrow)$-rule. To prove the other direction, from a cut-free derivation of $\Gamma\vdash\phi\Rightarrow\psi,\Delta$ one can build a cut-free derivation of $\Gamma,\phi\vdash\psi,\Delta$ by analysing how $\phi\Rightarrow\psi$ was introduced (by an initial sequent, by weakening or by the $(\vdash\Rightarrow)$-rule) and appropriately blocking such introductions.

Here is the sequent counterpart of Theorem 14.

Theorem 23 Let L be a primitive modal logic and F a formula generation map. If L satisfies the assumptions 1-4 from Definition 19 and GenLF is sound and complete for LF, then the statements below are equivalent:

1. For all $\phi\in$ FML, $\phi\in$ LF iff $f(\phi,1)\in$ L.
2. LF is pseudo Gentzenisable.

Proof (2) implies (1): By Lemma 9, $\phi \Leftrightarrow f(\phi,1) \in \mathtt{LF}$. So, if $f(\phi,1) \in \mathtt{L}$, then *a fortiori* $f(\phi,1) \in \mathtt{LF}$ and therefore $\phi \in \mathtt{LF}$.

Now assume $\phi \in \mathtt{LF}$, hence the sequent $\vdash \phi$ has a cut-free derivation in GenLF. We can show that in the given cut-free derivation of $\vdash \phi$, for every sequent $\Gamma \vdash \Delta$ with cut-free derivation Π', the sequent $f(\Gamma,0) \vdash f(\Delta,1)$ admits a cut-free derivation in GenL. So, we shall conclude that $\vdash f(\phi,1)$ is derivable in GenL and therefore $f(\phi,1) \in \mathtt{L}$. The proof is by induction on the structure of the derivations.

Base case: When $\Gamma \vdash \Delta$ is an initial sequent $\psi \vdash \psi$, it is immediate that $f(\psi,0) \vdash f(\psi,1)$, has a cut-free derivation in GenL since $f(\psi,0) \Rightarrow f(\psi,1) \in \mathtt{L}$.

Induction step: The structural rules pose no difficulties because by definition f is homomorphic with respect to the comma. By way of example, the derivation step in GenLF shown below left is transformed into the derivation steps in GenL shown below right

$$\dfrac{\begin{array}{c}\vdots\\ \mathtt{F}(\psi), h(\Sigma) \vdash \psi\end{array}}{\Sigma \vdash \Box\psi}\,(\vdash \Box)_{\mathtt{LF}} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ h(f(\Sigma,0)), f(\mathtt{F}(\psi),0) \vdash f(\psi,1)\end{array}}{h(f(\Sigma,0)) \vdash f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1)}\,(\vdash\Rightarrow)}{f(\Sigma,0) \vdash \Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\,(\vdash \Box)_{\mathtt{L}}$$

By the assumption about GenL,

$$h(f(\Sigma,0)) = f(h(\Sigma),0)$$

and

$$f(\Sigma,0) \vdash \Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))$$

satisfy the condition to apply the $(\vdash \Box)_{\mathtt{L}}$-rule. The derivation in GenLF shown below left is transformed into the derivation in GenL shown below right

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, \psi \vdash \Delta\end{array}}{\Gamma, \Box\psi \vdash \Delta}\,(\Box \vdash) \qquad \dfrac{\begin{array}{c}\vdots\\ f(\Gamma,0), f(\psi,0) \vdash f(\Delta,1)\end{array}}{f(\Gamma,0), \Box f(\psi,0) \vdash f(\Delta,1)}\,(\Box \vdash)$$

Indeed, $f(\Box\psi,0) = \Box f(\psi,0)$. The other cases are not difficult to obtain and they are omitted here.

(1) implies (2): We define a map $f^{-1} : \mathtt{Seq} \times \{0,1\} \rightarrow \mathtt{Seq}$ as in Figure 5 where Seq denotes the set of sequents (pair of sequences of formulae) except that $f^{-1}(\phi_1, \ldots, \phi_m, i) = f^{-1}(\phi_1, i), \ldots, f^{-1}(\phi_m, i)$. Here f^{-1} is just a simplified form of the reverse map defined in Figure 5.

Assume that $\vdash \phi$ has a derivation in `GenLF`. By soundness of `GenLF` with respect to `LF`, $\phi \in$ `LF` and by assumption (1), $f(\phi, 1) \in$ `L`. Thus $\vdash f(\phi, 1)$ has a cut-free derivation in `GenL`. Let us show that $\vdash \phi$ has a cut-free derivation in `GenLF`. We show that

($\star\star\star$) in the given cut-free derivation of $\vdash f(\phi, 1)$ in `GenL`, for every sequent $\Gamma \vdash \Delta$ with cut-free derivation Π, the sequent $f^{-1}(\Gamma, 0) \vdash f^{-1}(\Delta, 1)$ admits a cut-free derivation, say $f^{-1}(\Pi)$, in `GenLF`.

It is worth observing that thanks to Lemma 11, and finally thanks to Lemma 21, $f^{-1}(\Gamma, 0)$ and $f^{-1}(\Delta, 1)$ are always defined. If ($\star\star\star$) holds, then $\vdash f^{-1}(f(\phi, 1), 1)$ has a cut-free derivation in `GenLF`: that is $\vdash \phi$ has a cut-free derivation in `GenLF`.

The proof of ($\star\star\star$) is by induction on the structure of the given derivation. The base case when $\Gamma \vdash \Delta$ is of the form $\psi \vdash \psi$ are immediate. For every rule of `GenL` except (cut) with premisses $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_k \vdash \Delta_k$ and conclusion $\Gamma \vdash \Delta$, we must now show that: if each of the sequents $f^{-1}(\Gamma_1, 0) \vdash f^{-1}(\Delta_1, 1), \ldots, f^{-1}(\Gamma_k, 0) \vdash f^{-1}(\Delta_k, 1)$ is cut-free derivable in `GenLF`, then $f^{-1}(\Gamma, 0) \vdash f^{-1}(\Delta, 1)$ is also cut-free derivable in `GenLF`.

This poses no difficulty when the rule is a structural rule (contraction, weakening and exchange) or an operational rule introducing a Boolean connective.

By Lemma 11, the $\Box$-formulae occurring as succedent parts in the `GenL`-derivation of $\vdash f(\phi, 1)$ have a particular form. Therefore, all instances of the rule ($\vdash \Box_{\mathtt{L}}$) in the given `GenL`-derivation must be of the form shown below:

$$\frac{h(f(\psi_0, 0), \ldots, f(\psi_m, 0)) \vdash f(\mathtt{F}(\psi), 0) \Rightarrow f(\psi, 1)}{f(\psi_0, 0), \ldots, f(\psi_m, 0) \vdash \Box(f(\mathtt{F}(\psi), 0) \Rightarrow f(\psi, 1))} \ (\vdash \Box_{\mathtt{L}})$$

These are replaced by the `GenLF` rule applications shown below:

$$\frac{h(\psi_0, \ldots, \psi_m), \mathtt{F}(\psi) \vdash \psi}{\psi_0, \ldots, \psi_m \vdash \Box\psi} \ (\vdash \Box_{\mathtt{LF}})$$

where

- $f^{-1}(\Box(f(\mathtt{F}(\psi), 0) \Rightarrow f(\psi, 1)), 1) = \Box\psi$;
- $f^{-1}(f(\psi_0, 0), \ldots, f(\psi_m, 0), 0) = \psi_0, \ldots, \psi_m$;

- $f^{-1}(h(f(\psi_0, 0), \ldots, f(\psi_m, 0)), 0) = f^{-1}(f(h(\psi_0), 0), \ldots, f(h(\psi_m), 0)) = h(\psi_0), \ldots, h(\psi_m)$.

Of course the conditions on the rule $(\vdash \Box_{\mathtt{L}})$ are used to permute h and f and to allow the application of the $(\vdash \Box_{\mathtt{LF}})$ inference. By the induction hypothesis, $f^{-1}(h(f(\psi_0, 0), \ldots, f(\psi_m, 0)), 0) \vdash f^{-1}(f(\mathtt{F}(\psi), 0) \Rightarrow f(\psi, 1), 1)$ has a cut-free derivation in GenLF.

So, by Lemma 22, $h(\psi_0, \ldots, \psi_m), \mathtt{F}(\psi) \vdash \psi$ has a cut-free derivation in GenLF. So, $\psi_0, \ldots, \psi_m \vdash \Box\psi$ has a cut-free derivation in GenLF

If the GenL rule application is as shown below left, then it is transformed into the GenLF rule application shown below right:

$$\frac{\Gamma, f(\psi, 0) \vdash \Delta}{\Gamma, \Box f(\psi, 0) \vdash \Delta} (\Box \vdash) \qquad\qquad \frac{f^{-1}(\Gamma, 0), \psi \vdash f^{-1}(\Delta, 1)}{f^{-1}(\Gamma, 0), \Box\psi \vdash f^{-1}(\Delta, 1)} (\Box \vdash)$$

since $f^{-1}(f(\psi, 0), 0) = \psi$.

Finally, we can partially summarize the situation as follows.

Corollary 24 Let L be a primitive modal logic satisfying the assumptions 1-4. from Definition 19. Let F be a formula generation map such that GenLF is sound and complete for LF and LF admits a conservative tense extension. Then,

1. The statements I-III below are equivalent:

   (I) LF is pseudo Gentzenisable.

   (II) LF is pseudo displayable.

   (III) For all $\phi \in \mathrm{FML}$, $\phi \in \mathtt{LF}$ iff $f(\phi, 1) \in \mathtt{L}$ (f is defined via F).

2. When (I) above holds, there is a polynomial time transformation from LF into L (f is not necessarily such a transformation).

By [Avr84, Corollary 3.1], Grz is pseudo Gentzenisable and therefore

Corollary 25 Grz is pseudo displayable and for any $\phi \in \mathrm{FML}$, $\phi \in \mathtt{Grz}$ iff $f(\phi, 1) \in \mathtt{S4}$ where f is defined with $\mathtt{F} = \mathtt{F}_{\mathtt{Grz}}$.

An alternative proof of Corollary 16 can be given from Theorem 23. Indeed, from Example 20, it is easy to see that G is pseudo Gentzenisable (see e.g. [Val83, Avr84]). Hence for all $\phi \in \mathrm{FML}$, $\phi \in \mathtt{G}$ iff $f(\phi, 1) \in \mathtt{K4}$ where f is defined via $\mathtt{F} = \mathtt{F}_{\mathtt{G}}$. By Theorem 14, G is pseudo displayable.

6 Concluding Remarks

For any logic LF, we have defined a display calculus δLF by slightly modifying the display calculus δL for L defined in [Kra96]. When LF has a conservative tense extension, the calculus δLF is shown to be sound and complete with respect to LF. Although LF is not necessarily properly displayable, we have shown that (weak) cut-elimination for δLF is equivalent to the theoremhood-preserving nature of the map f. Using this fact, we have defined cut-free display calculi for the provability logics G and Grz satisfying Belnap's conditions (C2)-(C7). The calculus for G also satisfies (C1). As a side-effect, we have defined theoremhood-preserving mappings from Grz into S4, and these in turn can be used to translate Grz into a decidable fragment of first-order logic; see [DG00b]. We have also provided a proof for mapping G into K4 which is different from the one in [BH94], although the translation itself is identical. Although none of the calculi δLF satisfies the conditions (C1)-(C8), we have characterized the classes of such calculi for which cut-elimination holds (condition on translations).

Because our base logic L can be any properly displayable modal logic with a conservative tense extension, our results provide a means to show that the logics G.3 and Grz.3 (and some others) *could* be pseudo-displayable, using the methodology provided by Corollary 24. A natural question is: What extensions are required to handle other "second-order" logics like S4.3.1 [Gor99] ? But Wolter has shown that the tense extension of S4.3.1 is *not* conservative, so our display methodology is unlikely to answer this question.

There is one caveat: our display calculi enjoy cut-elimination only for sequents of the form $I \vdash \phi$, not for sequents of the general form X $\vdash$ Y.

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