On Modal Logics Characterized by Models with Relative Accessibility Relations: Part II*

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May 15, 2002

Abstract

This work is divided in two papers (Part I and Part II). In Part I, we introduced the class of Rare-logics for which the set of terms indexing the modal operators are hierarchized in two levels: the set of Boolean terms and the set of terms built upon the set of Boolean terms. By investigating different algebraic properties satisfied by the models of the Rare-logics, reductions for decidability were established by faithfully translating the Rare-logics into more standard modal logics (some of them contain the universal modal operator).

In Part II, we push forward the results from Part I. For Rare-logics with nominals (present at the level of formulae *and* at the level of modal expressions), we show that the constructions from Part I can be extended although it is technically more involved. We also characterize a class of standard modal logics for which the universal modal operator can be eliminated as far as satifiability is concerned. Although the previous results have a semantic flavour, we are also able to define proof systems for Rare-logics from existing proof systems for the corresponding standard modal logics. Last, but not least, decidability results for Rare-logics are established uniformly, in particular for information logics derived from rough set theory.

Since this paper is the continuation of Part I, we do not recall here the definitions of Part I although we refer to them.

Key-words: polymodal logic, relative accessibility relation, decision procedure, translation.

1 Introduction

Background. During the last decade, the information logics derived from Pawlak's *information systems* [Paw81] have been the object of active research (see for example [Orło84a, OP84, Vak91]). In [DG], we have introduced the

*This work has been partly done during a visit of S. Demri at the Imperial College in the group "Logic and Automated Reasoning". The visit has been financed by Action COST #15 "Applications of Many-Valued Logics to Computer Science".

class of Rare-logics that capture many information logics met in the literature. Indeed, the polymodal logics obtained from the information systems are multimodal logics such that the relations in the Kripke-style semantical structures correspond to relations between objects in the underlying information systems. Since this paper is a continuation of [DG], a more detailed introduction can be found there. However, roughly speaking, a Rare-logic $\mathcal{L}$ of type $\mathtt{T}$ is a structure of the form $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, (X_i)_{i \in J}, \mathtt{T} \rangle$ such that

- $\mathtt{L}$ is a *modal language*, possibly admitting n-ary modal operators ($n \geq 1$) and the set of modal expressions is hierarchized in two levels;
- $\mathcal{D}$ is a *dimension map* for $\mathtt{L}$, that is it encodes the dimensions of the relations associated to the modal expressions;
- $\mathcal{I}$ is an *operator interpretation*, that is it encodes how a syntactic modal operator is interpreted (e.g., '$\cap$' is interpreted as set intersection);
- $\mathcal{C}$ is a non-empty class of modal frames depending on $\mathtt{L}$ and $\mathcal{D}$;
- $(X_i)_{i \in J}$ is a finite family of relations that are allowed to occur in the frames of $\mathcal{C}$ (this is a necessary condition);
- $\mathtt{T}$ is a label to name a condition $C_{\mathtt{T}}$. Indeed the frames of $\mathcal{C}$ are precisely the modal frames based on relations from $(X_i)_{i \in J}$ satisfying the condition $C_{\mathtt{T}}$. $C_{\mathtt{T}}$ is a technical trick to capture various kinds of requirements on the class $\mathcal{C}$ of frames, mostly algebraic in nature.

The precise definition can be found in [DG, Section 3].

Our objectives. Our prime objective in the paper is to compare various classes of Rare-logics with their corresponding classes of standard modal logics. The criteria of comparison range from (un)decidability of satisfiability (finite model property, ...) to complete and sound axiomatization. By achieving such a comparison, we provide a framework for studying Rare-logics that allies generality and sheds a new light on the Rare-logics themselves. For instance by defining satisfiability-preserving transformation from Rare-logics into standard modal logics, we shall solve open problems about decidability issues.

Our contribution. The results of this paper are the continuation of [DG]. First, we extend the main proof technique from [DG] in order to deal with Rare-logics admitting nominals in the language. This leads to a generalization of the translations defined in [DG]. By taking advantage of such translations, we define a general construction that allows to define an axiomatization of Rare-logics from calculi for the corresponding standard modal logics. Some other refinements are also performed in order to eliminate the universal modal operator for some cases. Last, but not least, we provide a uniform proof of decidability for various Rare-logics from [Orło84a, Orło93, Orło89, Orło88b, Bal96a, Bal97] for which the issue has been open up to now. Indeed, the decidability of the corresponding standard modal logics is used with possibly some adequate adjustments.

Plan of the paper. The rest of the paper is structured as follows. In Section 2, nominals (names) are added to the logics and we show that the construction in [DG, Section 4] can be adapted to this new case although it is technically more involved. Elimination of the universal modal operator is discussed in Section 3. Section 4 is devoted to show how to build Hilbert-style proof systems for Rare-logics from existing proof systems for the corresponding standard modal logics. In Section 5, we apply the new results to various classes of information logics providing for instance new decidability results. Section 6 contains concluding remarks.

Because of lack of space, we do not recall the definitions from [DG] and we invite the reader to consult [DG] if needed.

2 Nominal Rare-logics

In this section, we consider Rare-logics for which names (also called nominals) are included in the language. First, we consider a countably infinite set $\mathtt{For}_0^N = \{\mathtt{p}^N, \mathtt{q}^N, \ldots\}$ of *world nominals* that behave as propositional variables except that for any interpretation function V, $V(\mathtt{p}^N)$ is a singleton. The addition of world nominals to modal logics has been investigated in the past years with different motivations (see e.g. [Orło84a, PT85, Gar86, Bla90, Bla93a, Gia95, BS95]). A world nominal is usually understood as an atomic proposition that holds true in a unique world of a Kripke model. Most of the time, the addition of names is intended to increase the expressive power of the initial logics. For instance, there is a tense formula with names that characterizes the class of irreflexivity frames [Bla93b]. Another remarkable breakthrough due to the inclusion of names consists in defining the intersection operator (see e.g. [PT91]) although it is known that the intersection is not modally definable in the standard modal language [GT75]. Names have also been introduced in information logics [Orło84a, Kon97b, Kon97a] derived from Pawlak's rough set theory [Paw81] where the motivations concern both definability and axiomatization purposes. We write $\mathcal{L}(\mathtt{For}_0^N)$ to denote a logic whose language is extended with world nominals. Numerous results in [DG, Section 4] still hold true when the logics (the Rare-logics *and* the standard modal logics) admit world nominals. Indeed, most of the time, what is problematic, is the construction of families of relations satisfying given constraints. In all the previous constructions, the interpretation of the atomic formulae are always smoothly preserved.

However, there is another fashion to add nominals to a modal logic. Let $\mathcal{L}$ be a Rare-logic and $\mathtt{P}_0^N$ be a set of *parameter nominals* such that $\mathtt{P}_0^N$ and $\mathtt{P}$ are disjoint, $\{\mathtt{P}_0^{iN} : i \in J\}$ is a partition of $\mathtt{P}_0^N$ and each set $\mathtt{P}_0^{iN} = \{\mathtt{C}^{iN}, \mathtt{D}^{iN}, \ldots\}$ is a countably infinite set of parameter nominals. We write $\mathcal{L}(\mathtt{P}_0^N)$ to denote the Rare-logic $\mathcal{L}$ whose language has been extended with $\mathtt{P}_0^N$. Each set $\mathtt{P}_i$ of parameter expressions is built upon $\mathtt{P}_0^i \cup \mathtt{P}_0^{iN}$ and as usual $\mathtt{P}_i$ is closed under $\cap, \cup$ and $-$. Moreover, for any $\mathtt{C}^{iN}, \mathtt{D}^{iN} \in \mathtt{P}_0^{iN}$, we require $\rho(\mathtt{C}^{iN}) = \mathcal{D}(i) - 1$, $\mathtt{C}^{iN} \neq \mathtt{D}^{iN}$ only if $V(\mathtt{C}^{iN}) \neq V(\mathtt{D}^{iN})$ and for any $\mathtt{P}$-valuation V, $V(\mathtt{C}^{iN})$ is a singleton of PAR_i -in the rest of the section a $\mathtt{P}$-valuation is understood in this way. As a consequence PAR_i is infinite since $\mathtt{P}_0^{iN}$ is assumed to be countably infinite. We write $\mathcal{L}(\mathtt{For}_0^N, \mathtt{P}_0^N)$ to denote the Rare-logic obtained from the Rare-logic $\mathcal{L}$ by adding *world nominals and parameter nominals.* For this kind of extension, the results from [DG, Section 4.2] cannot be so easily applied to the Rare-logics $\mathcal{L}(\texttt{For}_0^N, \texttt{P}_0^N)$. Indeed, the very notion of normalization has to be revised. This section is devoted to define satisfiability-preserving maps between Rare-logics $\mathcal{L}(\texttt{For}_0^N, \texttt{P}_0^N)$ and the corresponding standard modal logics $\mathcal{L}_d^\star(\texttt{For}_0^N)$. In the rest of the section, we assume that the Rare-logics satisfy the hypothesis presented at the beginning of [DG, Section 4.2]. By the way, a proposition similar to [DG, Proposition 3.3] holds true for the Rare-logics $\mathcal{L}(\texttt{For}_0^N, \texttt{P}_0^N)$.

2.1 Elimination of parameter expressions

Let $\texttt{L}$ be a modal language dedicated to some Rare-logic $\mathcal{L}(\texttt{For}_0^N, \texttt{P}_0^N)$ and $\texttt{M}_{0d} = \{\texttt{c}_{\alpha,\beta} : \alpha, \beta \in \omega\}$ be a countable set of modal constants. Using the decidability of the validity problem of the logic CPDL in [PT91], one can prove that the following problems are decidable: (1) Is $\texttt{A} \equiv \perp$ $(\texttt{A} \in \texttt{P}_i)$? (2) Is $\texttt{A} \equiv \top$ $(\texttt{A} \in \texttt{P}_i)$?

Let $\texttt{C}_1, \ldots, \texttt{C}_n \in \texttt{P}_0^i$ and $\texttt{C}_1^{iN}, \ldots, \texttt{C}_l^{iN} \in \texttt{P}_0^{iN}$ for some $i \in J$. We assume $l \neq 0$, otherwise consider the developments from [DG, Section 4.1]. For any integer $k \in \{0, \ldots, 2^n - 1\}$, we write $\texttt{B}_{k,0}^{i*}$ to denote the Boolean expression (also called a *component*)

$$\texttt{B}_{k,0}^{i*} \stackrel{\text{def}}{=} \texttt{A}_k^{i*} \cap -\texttt{C}_1^{iN} \cap \ldots \cap -\texttt{C}_l^{iN}$$

where for $k \in \{0, \ldots, 2^n - 1\}$, we write $\texttt{A}_k^{i*}$ to denote the Boolean expression

$$\texttt{A}_k^{i*} \stackrel{\text{def}}{=} \texttt{A}_1 \cap \ldots \cap \texttt{A}_n$$

where for any $s \in \{1, \ldots, n\}$, $\texttt{A}_s \stackrel{\text{def}}{=} \texttt{C}_s$ if $bit_s(k) = 0$ ($bit_s(k)$ denoting the sth bit in the binary representation of k) otherwise $\texttt{A}_s \stackrel{\text{def}}{=} -\texttt{C}_s$. When $n = 0$, we only consider the Boolean expression $-\texttt{C}_1^{iN} \cap \ldots \cap -\texttt{C}_l^{iN}$ (noted $\texttt{B}_0^{i*}$). For any $\langle k, k' \rangle \in \{0, \ldots, 2^n - 1\} \times \{1, \ldots, l\}$, we write $\texttt{B}_{k,k'}^{i*}$ to denote the Boolean expression $\texttt{B}_{k,k'}^{i*} \stackrel{\text{def}}{=} \texttt{A}_k^{i*} \cap \texttt{C}_{k'}^{iN}$. If $n = 0$, then we only consider the Boolean expression $\texttt{B}_{k'}^{i*} \stackrel{\text{def}}{=} \texttt{C}_{k'}^{iN}$. In the sequel, we omit the case when $n = 0$ since the results can be obtained easily from the non-degenerate case. For any $\texttt{P}$-valuation V (dealing with parameter nominals), the family $\{V(\texttt{B}_{k,k'}^{i*}) : \langle k', k \rangle \in \{0, \ldots, l\} \times \{0, \ldots, 2^n - 1\}\}$ is a partition of PAR_i.

Let $\emptyset \neq X \subseteq \texttt{P}_i$ be such that for any $\texttt{A} \in X$, $\texttt{P}_0^i(\texttt{A}) \subseteq \{\texttt{C}_1, \ldots, \texttt{C}_n\}$ and $\texttt{P}_0^{iN}(\texttt{A}) \subseteq \{\texttt{C}_1^{iN}, \ldots, \texttt{C}_l^{iN}\}$. One can show that for any $\texttt{A} \in X$, either $\texttt{A} \equiv \perp$ or $\texttt{A} \equiv \texttt{B}_{k_1,k_1'}^{i*} \cup \ldots \cup \texttt{B}_{k_u,k_u'}^{i*}$. Such a decomposition has been introduced in [Kon97b] and it generalizes with parameter nominals the canonical disjunctive normal form from the propositional calculus (see e.g. [Lem65]). For any $k' \in \{1, \ldots, l\}$, we write $occ_{k'}^X$ to denote the set

$$occ_{k'}^X \stackrel{\text{def}}{=} \{k \in \{0, \ldots, 2^n - 1\} : \exists \texttt{A} \in X,\ \texttt{A} \equiv \ldots \cup \texttt{B}_{k,k'}^{i*} \cup \ldots\}$$

Informally, $occ_{k'}^X$ is the set of indices $k \in \{0, \ldots, 2^n - 1\}$ such that $\texttt{B}_{k,k'}^{i*}$ occurs in the *normal form* of some element of X. We write $setocc_{k'}^X$ to denote the set

$$setocc_{k'}^X \stackrel{\text{def}}{=} \{Y \subseteq occ_{k'}^X : card(occ_{k'}^X) - 1 \leq card(Y) \leq 2^n - 1\}$$

The definition of $setocc_{k'}^X$ is motivated by the fact that for any $\texttt{P}$-valuation V, there is a *unique* $k \in \{0, \ldots, 2^n - 1\}$ such that $V(\texttt{B}_{k,k'}^{i*}) \neq \emptyset$ (and $V(\texttt{B}_{k,k'}^{i*}) =$

$V(\mathtt{C}^{iN}_{k'})$). For each $Y \in setocc^X_{k'}$ in turn, in the forthcoming constructions we shall enforce $V(\mathtt{B}^{i*}_{k,k'}) = \emptyset$ for any $k \in Y$.

PROPOSITION 2.1. (**Semilattice constructions with atomic parameters**) Let $\mathtt{C}_1, \ldots, \mathtt{C}_n$ be distinct elements of $\mathtt{P}^i_0$ and $\mathtt{C}^{iN}_1, \ldots, \mathtt{C}^{iN}_l$ be distinct elements of $\mathtt{P}^{iN}_0$ for some $i \in J$ ($n > 0$, $l > 0$). Let X be a non-empty subset of $\mathtt{P}_i$ such that $\mathtt{P}^i_0(X) = \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$ and $\mathtt{P}^{iN}_0(X) = \{\mathtt{C}^{iN}_1, \ldots, \mathtt{C}^{iN}_l\}$.

(I) Let $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e\rangle$ be a semilattice with zero element e and V be the restriction to $\mathtt{P}_i$ of a $\mathtt{P}$-valuation such that (H1) $X_\emptyset = e$ and (H2) for any $P, P' \subseteq PAR_i$, $X_{P \cup P'} = X_P \sqcup X_{P'}$.

Then, there is a family $(Y_\mathtt{C})_{\mathtt{C} \in \mathtt{M}_{0d}}$ such that

(a) $\{Y_\mathtt{C} : \mathtt{c} \in \mathtt{M}_{0d}\}$ is a finite subset of $\{X_P : P \subseteq PAR_i\}$;

(b) If $\mathtt{A}$ is a parameter expression built upon $\{\mathtt{C}_1, \ldots, \mathtt{C}_n, \mathtt{C}^{iN}_1, \ldots, \mathtt{C}^{iN}_l\}$ such that $\mathtt{A} \equiv \mathtt{B}^{i*}_{k_1,k'_1} \cup \ldots \cup \mathtt{B}^{i*}_{k_u,k'_u}$, then $X_{V(\mathtt{A})} = Y_{\mathtt{C}_{k_1,k'_1}} \sqcup \ldots \sqcup Y_{\mathtt{C}_{k_u,k'_u}}$;

(c) There is $\langle O_1, \ldots, O_l\rangle \in setocc^X_1 \times \ldots \times setocc^X_l$ such that for any $k' \in \{1, \ldots, l\}$ and any $k \in O_{k'}$, $Y_{\mathtt{C}_{k,k'}} = e$.

(II) Let $\langle Y, \sqcup, e\rangle$ be a semilattice with zero element and $(Y_\mathtt{C})_{\mathtt{C} \in \mathtt{M}_{0d}}$ be a family such that $\{Y_\mathtt{C} : \mathtt{c} \in \mathtt{M}_{od}\} \subseteq Y$. Let $\langle O_1, \ldots, O_l\rangle$ be in $setocc^X_1 \times \ldots \times setocc^X_l$ such that for any $k' \in \{1, \ldots, l\}$ and any $k \in O_{k'}$, $Y_{\mathtt{C}_{k,k'}} = e$. Then, there is a subalgebra $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e\rangle$ of $\langle Y, \sqcup, e\rangle$ satisfying (H1)-(H2) and there is a restriction V to $\mathtt{P}_i$ of a $\mathtt{P}$-valuation such that

(a) (Ib) above and for any $P \subseteq PAR_i$, there exists a (possibly empty) finite subset $\{x_1, \ldots, x_k\}$ of PAR_i such that $X_P = X_{\{x_1\}} \sqcup \ldots \sqcup X_{\{x_k\}} \sqcup X_\emptyset$;

(b) there exists a finite subset Z of PAR_i such that $card(Z) = 2^n + l$ and for any $P \subseteq PAR_i$, $X_P = X_{P \cap Z}$ (Z is the finite relevant part of PAR_i);

(c) for any $k' \in \{1, \ldots, l\}$ and $k \in O_{k'}$, $V(\mathtt{B}^{i*}_{k,k'}) = \emptyset$.

PROOF: (I) Let $(Y_\mathtt{C})_{\mathtt{C} \in \mathtt{M}_{0d}}$ be the family such that

- for any $\langle k, k'\rangle \in \{0, \ldots, 2^n - 1\} \times \{0, \ldots, l\}$, $Y_{\mathtt{C}_{k,k'}} \stackrel{\text{def}}{=} X_{V(\mathtt{B}^{i*}_{k,k'})}$.
- for any $\mathtt{c} \in \mathtt{M}_{0d} \setminus \{\mathtt{c}_{k,k'} : \langle k', k\rangle \in \{0, \ldots, l\} \times \{0, \ldots, 2^n - 1\}\}$, $Y_\mathtt{C} \stackrel{\text{def}}{=} Y_{\mathtt{C}_{1,1}}$ (arbitrary value).

It is easy to check that $(Y_\mathtt{C})_{\mathtt{C} \in \mathtt{M}_0}$ satisfies (Ia) and (Ib).
(Ic) Let $\langle i_1, \ldots, i_l\rangle \in \{0, \ldots, 2^n - 1\}^l$ be such that for any $k' \in \{1, \ldots, l\}$, $V(\mathtt{B}^{i*}_{i_{k'},k'}) \neq \emptyset$ ($\langle i_1, \ldots, i_l\rangle$ is unique). For any $k' \in \{1, \ldots, l\}$, we take $O_{k'} \stackrel{\text{def}}{=} occ^X_{k'} \setminus \{i_{k'}\}$ ($i_{k'}$ may not belong to $occ^X_{k'}$). By (Ib), for any $k' \in \{1, \ldots, l\}$ and any $k \in O_{k'}$, $Y_{\mathtt{C}_{k,k'}} = X_{V(\mathtt{B}^{i*}_{k,k'})} = X_\emptyset = e$.
(II) Let $(Y_\mathtt{C})_{\mathtt{C} \in \mathtt{M}_0}$ and $\langle O_1, \ldots, O_l\rangle$ satisfy the hypothesis of (II). For each $k' \in \{1, \ldots, l\}$, we *choose* $u_{k'} \in \{0, \ldots, 2^n - 1\} \setminus O_{k'}$ such that if $occ^X_{k'} \neq O_{k'}$, then $\{u_{k'}\} = occ^X_{k'} \setminus O_{k'}$. Let PAR_i be defined as the set ω of natural numbers. Moreover, for any $s \in \{1, \ldots, n\}$,

$$V(\mathtt{C}_s) \stackrel{\text{def}}{=} \{k \in \{0, \ldots, 2^n - 1\} : bit_s(k) = 0\} \cup \{2^n - 1 + k' : k' \in \{1, \ldots, l\},\ bit_s(u_{k'}) = 0\}$$

For the other parameter constants V is not constrained until V is the restriction of a $\mathtt{P}$-valuation. For any $s \in \omega$, $V(\mathtt{C}_s^{iN}) = \{2^n - 1 + s\}$. By construction, for any $k \in \{0, \ldots, 2^n - 1\}$, $\{k\} \subseteq V(\mathtt{A}_k^{i*})$ and for any $\langle k, k' \rangle \in \{0, \ldots, 2^n - 1\} \times \{1, \ldots, l\}$, $2^n - 1 + k' \in V(\mathtt{A}_k^{i*})$ iff $u_{k'} = k$. Let $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e \rangle$ and V be defined such that (H2) holds and

- $X_\emptyset \stackrel{\text{def}}{=} e$; for any $k \in \{0, \ldots, 2^n - 1\}$, $X_{\{k\}} \stackrel{\text{def}}{=} Y_{\mathtt{C}_{k,0}}$;
- for any $k' \in \{1, \ldots, l\}$, $X_{\{2^n - 1 + k'\}} \stackrel{\text{def}}{=} Y_{\mathtt{C}_{u_{k'}, k'}}$;
- for the remaining $P \subseteq PAR_i$, $X_P \stackrel{\text{def}}{=} X_{P \cap \{0, \ldots, 2^n - 1 + l\}}$.

One can check that the definition of $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e \rangle$ and V satisfy the required conditions. Q.E.D.

2.2 The normalization process $\mathtt{N}$

Let $\mathcal{L}$ be a Rare-logic satisfying the hypothesis of [DG, Section 4.2] and $\mathcal{L}(\mathtt{For}_0^N, \mathtt{P}_0^N)$ be its extension with parameter and world nominals. Results about the logic $\mathcal{L}(\mathtt{P}_0^N)$ shall be also stated in this section: they can be easily obtained from the more difficult case with $\mathcal{L}(\mathtt{For}_0^N, \mathtt{P}_0^N)$. Let $\mathtt{F}$ be an $\mathtt{L}$-formula such that for any $i \in J$, $\mathtt{P}_0^i(\mathtt{F}) = \{\mathtt{C}_1^i, \ldots, \mathtt{C}_{n_i}^i\}$ and $\mathtt{P}_0^{iN}(\mathtt{F}) = \{\mathtt{C}_1^{iN}, \ldots, \mathtt{C}_{l_i}^{iN}\}$ (we assume $n_i \neq 0$ and $l_i \neq 0$ since the degenerate cases can be easily derived from the present one). Let $r(\mathtt{A})$ be a basic modal expression occurring in $\mathtt{F}$ such that $\mathtt{A} \in \mathtt{P}_i$ for some $i \in J$ and $\mathtt{A} \equiv \mathtt{B}_{k_1, k_1'}^{i*} \cup \ldots \cup \mathtt{B}_{k_u, k_u'}^{i*}$. The first normal form of $r(\mathtt{A})$, written $\mathtt{N}_1(r(\mathtt{A}))$, is the basic modal expression $r(\mathtt{B}_{k_1, k_1'}^{i*} \cup \ldots \cup \mathtt{B}_{k_u, k_u'}^{i*})$. In the case when $\mathtt{A} \equiv \perp$, $\mathtt{N}_1(r(\mathtt{A})) \stackrel{\text{def}}{=} r(\mathtt{C}_1^i \cap -\mathtt{C}_1^i)$. We write $\mathtt{N}_1(\mathtt{F})$ to denote the formula obtained from $\mathtt{F}$ by substituting each occurrence of $r(\mathtt{A})$ by $\mathtt{N}_1(r(\mathtt{A}))$. The second normal form of $\mathtt{F}$, written $\mathtt{N}_2(\mathtt{F})$ is the formula obtained from $\mathtt{N}_1(\mathtt{F})$ where each occurrence of $r(\mathtt{B}_{k_1, k_1'}^{i*} \cup \ldots \cup \mathtt{B}_{k_u, k_u'}^{i*})$ has been substituted by $r(\mathtt{B}_{k_1, k_1'}^{i*}) \oplus^i \ldots \oplus^i r(\mathtt{B}_{k_u, k_u'}^{i*})$. One can check that $\mathtt{F} \Leftrightarrow \mathtt{N}_2(\mathtt{F})$ is $\mathcal{L}$-valid. We define the mapping $\mathtt{N}$ from the set of $\mathcal{L}(\mathtt{For}_0^N, \mathtt{P}_0^N)$-formulae into the set of $\mathcal{L}_d^{\{\mathtt{U}_{\mathcal{D}(1)}, \ldots, \mathtt{U}_{\mathcal{D}(j)}\}}(\mathtt{For}_0^N)$-formulae in case $i_{\mathcal{L}} = 1$ [resp. $\mathcal{L}_d^{\{\mathtt{O}_{\mathcal{D}(1)}, \ldots, \mathtt{O}_{\mathcal{D}(j)}\}}(\mathtt{For}_0^N)$-formulae in case $i_{\mathcal{L}} = 0$] where $\mathcal{L}_d$ is the standard modal logic from $\mathcal{L}$.

The normal form of $\mathtt{F}$, written $\mathtt{N}(\mathtt{F})$, is the $\mathcal{L}_d^\star(\mathtt{For}_0^N)$-formula defined as follows ($\mathcal{L}_d^\star$ - see [DG, Section 4.2]). Without any loss of generality, we can assume that for any $i \in J$ we have in the language $\mathcal{L}_d^\star(\mathtt{For}_0^N)$ the following stock of distinct modal constants

$$\bigcup_{i \in J} \{\mathtt{c}_{0,0}^i, \ldots, \mathtt{c}_{2^{n_i}-1,0}^i, \ldots, \mathtt{c}_{0,l_i}^i, \ldots, \mathtt{c}_{2^{n_i}-1,l_i}^i\}$$

such that for any $i \in J$ and for any $\langle k, k' \rangle \in \{0, \ldots, 2^{n_i} - 1\} \times \{0, \ldots, l_i\}$, $\rho(\mathtt{c}_{k,k'}^i) = \mathcal{D}(i) - 1$. $\mathtt{N}(\mathtt{F})$ is obtained from $\mathtt{N}_2(\mathtt{F})$ by substituting every occurrence of $r(\mathtt{C}_1^i \cap -\mathtt{C}_1^i)$ by $\mathtt{U}_{\mathcal{D}(i)}$ in the case when $i_{\mathcal{L}} = 1$ [resp. by $\mathtt{O}_{\mathcal{D}(i)}$ in the case when $i_{\mathcal{L}} = 0$] and every occurrence of $r(\mathtt{B}_{k,k'}^{i*})$ by $\mathtt{c}_{k,k'}^i$.

The translation is not quite finished yet. Exactly one of the components $\mathtt{C}_1^{iN} \cap \mathtt{A}_1^i$ or $\mathtt{C}_1^{iN} \cap -\mathtt{A}_1^i$ (when $n_i = 1$) is interpreted by the empty set. However, this fact is not taken into account in $\mathtt{N}$. The following developments provide an answer to this technical problem. Let $\mathtt{G}$ be an $\mathcal{L}_d^\star(\mathtt{For}_0^N)$-formula. For any

$i \in J$, any $k'_i \in \{1, \ldots, l_i\}$ and any $Y \in setocc_{k'_i}^{\mathtt{P}_i(\mathtt{F})}$ we write $\mathtt{G}[k'_i, Y]$ to denote the formula obtained from $\mathtt{G}$ where for any $k \in Y$, $\mathtt{c}^i_{k,k'_i}$ has been substituted by $\mathtt{U}_{\mathcal{D}(i)}$ in the case when $i_{\mathcal{L}} = 1$ [resp. by $\mathtt{0}_{\mathcal{D}(i)}$ in the case when $i_{\mathcal{L}} = 0$].

For any $i \in J$, $\langle O^i_1, \ldots, O^i_{l_i} \rangle \in setocc_1^{\mathtt{P}_i(\mathtt{F})} \times \ldots \times setocc_{l_i}^{\mathtt{P}_i(\mathtt{F})} \stackrel{\text{def}}{=} S_i$ we write $\mathtt{G}[i, \langle O^i_1, \ldots, O^i_{l_i} \rangle]$ to denote the formula

$$\mathtt{G}[i, \langle O^i_1, \ldots, O^i_{l_i} \rangle] \stackrel{\text{def}}{=} \mathtt{G}[1, O^i_1] \ldots [l_i, O^i_{l_i}]$$

Proposition 2.2. (**Faithfulness of** $\mathtt{N}$) Let $\mathtt{F}$ be an $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-formula. The statements below are equivalent:

(I) $\mathtt{F}$ is $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-satisfiable;

(II) $\bigvee\{\mathtt{N}(\mathtt{F})[1, \sigma^1] \ldots [j, \sigma^j] : \sigma^i \in S_i, i \in J\}$ is $\mathcal{L}^\star_d(\mathtt{For}^N_0)$-satisfiable.

Proof: (I) $\rightarrow$ (II) Assume $\mathtt{F}$ is $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-satisfiable. So there exist an $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-model $\langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$. For any $i \in J$, $\langle \{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}} \rangle$ and the restriction of V to $\mathtt{P}_i$ satisfy the hypothesis of Proposition 2.1(I). So for any $i \in J$, there is a family $(Y^i_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}^i_0}$ and $\sigma^i \in S_i$ satisfying (Ia), (Ib) and (Ic) from Proposition 2.1 -we omit the cases $n_i = 0$ or $l_i = 0$.

Let $\mathtt{F}'$ be $\mathtt{N}(\mathtt{F})[1, \sigma^1] \ldots [j, \sigma^j]$ and $\mathcal{M}'$ be the $\mathcal{L}^\star_d(\mathtt{For}^N_0)$-model

$$\langle W, (R_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V' \rangle$$

defined as in the first part of the proof of [DG, Proposition 4.3] where we add: for any $\mathtt{p}^N \in \mathtt{For}^N_0$, $V'(\mathtt{p}^N) \stackrel{\text{def}}{=} V(\mathtt{p}^N)$. One can check that for any $\mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ such that $\mathtt{A} \equiv \mathtt{B}^{i*}_{k_1,k'_1} \cup \ldots \cup \mathtt{B}^{i*}_{k_u,k'_u}$, $V(r(\mathtt{A})) = V'(\mathtt{d}^i_{k_1,k'_1} \oplus^i \ldots \oplus^i \mathtt{d}^i_{k_u,k'_u})$ where for $s \in \{1, \ldots, u\}$,

$$\mathtt{d}^i_{k_s,k'_s} \stackrel{\text{def}}{=} \begin{cases} \mathtt{c}^i_{k_s,k'_s} \text{ if } k_s \notin (\sigma^i)_{k'_s} \text{ - } k'_s\text{th component of } \sigma^i\text{-} \\ \mathtt{U}_{\mathcal{D}(i)} \text{ if } k_s \in (\sigma^i)_{k'_s} \text{ and } i_{\mathcal{L}} = 1 \\ \mathtt{0}_{\mathcal{D}(i)} \text{ if } k_s \in (\sigma^i)_{k'_s} \text{ and } i_{\mathcal{L}} = 0 \end{cases}$$

Such a replacement is allowed thanks to the satisfaction of (Ic). Consider the surjective map $\Phi : \mathtt{M}(\mathtt{F}) \rightarrow \mathtt{M}_d(\mathtt{N}(\mathtt{F}))$ ($\mathtt{M}_d$ denotes the set of modal expressions of $\mathcal{L}^\star_d(\mathtt{For}^N_0)$) such that for any $\mathtt{a} \in \mathtt{M}(\mathtt{F})$, $\Phi(\mathtt{a})$ is obtained from $\mathtt{a}$ by substituting simultaneously

- each $r(\mathtt{A})$ such that $\perp \equiv \mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ by $\mathtt{U}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 1$ [resp. by $\mathtt{0}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 0$];
- each $r(\mathtt{A})$ such that $\mathtt{B}^{i*}_{k_1,k'_1} \cup \ldots \cup \mathtt{B}^{i*}_{k_u,k'_u} \equiv \mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ by $\mathtt{d}^i_{k_1,k'_1} \oplus^i \ldots \oplus^i \mathtt{d}^i_{k_u,k'_u}$.

By Proposition 3.1, $\mathcal{M}, w \models \mathtt{F}$ iff $\mathcal{M}', w \models \mathtt{F}'$. Hence $\mathcal{M}', w \models \mathtt{F}'$.
(II) $\rightarrow$ (I) Assume there exist $\mathtt{F}' = \mathtt{N}(\mathtt{F})[1, \sigma^1][2, \sigma^2] \ldots [j, \sigma^j]$, an $\mathcal{L}^\star_d(\mathtt{For}^N_0)$-model $\mathcal{M} = \langle W, (R_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}'$. Let us define the family $(R'_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}^i_0}$ as follows:

$$R'_{\mathtt{C}} \stackrel{\text{def}}{=} \begin{cases} e^i_{\mathcal{L}} \text{ if } \mathtt{c} = \mathtt{c}_{k,k'} \text{ with } k' \in \{1, \ldots, l_i\} \text{ and } k \in (\sigma^i)_{k'} \\ R_{\mathtt{C}} \text{ otherwise} \end{cases}$$

The semilattice $\langle \mathcal{P}(W^{\mathcal{D}(i)}), \phi^i_2(W), e^i_{\mathcal{L}} \rangle$, $(R'_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}^i_0}$ and σ^i satisfy the hypothesis of Proposition 2.1(II). So for $i \in J$, there exist a structure $\langle \{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}} \rangle$ and V_i such that (take $X = \mathtt{P}_i(\mathtt{F})$ if $\mathtt{P}_i(\mathtt{F}) \neq \emptyset$ otherwise $X = \{\mathtt{C}^i_1, \mathtt{C}^{iN}_1\}$),

- V_i is the restriction of a $\mathtt{P}$-valuation to $\mathtt{P}_i$;
- $\langle \{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}}\rangle$ is a semilattice with zero element $W^{\mathcal{D}(i)}$ [resp. $\emptyset$] if $i_{\mathcal{L}} = 1$ [resp. if $i_{\mathcal{L}} = 0$];
- $R^i_{\emptyset} \stackrel{\text{def}}{=} e^i_{\mathcal{L}}$ and for any $P, P' \subseteq PAR_i$, $R^i_{P\cup P'} \stackrel{\text{def}}{=} \phi^i_2(W)(R^i_P, R^i_{P'})$;
- $PAR_i \stackrel{\text{def}}{=} \omega$ and the *relevant part* of PAR_i is $\{0, \ldots, 2^{n_i} - 1 + l_i\}$;
- If $\mathtt{A} \in \mathtt{P}_i(\mathtt{F})$, $\mathtt{A} \equiv \mathtt{B}^{i*}_{k_1,k'_1} \cup \ldots \cup \mathtt{B}^{i*}_{k_u,k'_u}$, then $R_{V(\mathtt{A})} \stackrel{\text{def}}{=} R'_{\mathtt{C}^i_{k_1,k'_1}} \phi^i_2(W) \ldots \phi^i_2(W) R'_{\mathtt{C}^i_{k_u,k'_u}}$;
- for any $k' \in \{1, \ldots, l_i\}$ and $k \in (\sigma^i)_{k'}$, $V_i(\mathtt{B}^{i*}_{k,k'}) \stackrel{\text{def}}{=} \emptyset$.

Let $\mathcal{M}' = \langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V'\rangle$ be the $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-model defined as in the second part of the proof of [DG, Proposition 4.3] except that we add: for any $\mathtt{p}^N \in \mathtt{For}^N_0$, $V'(\mathtt{p}^N) = V(\mathtt{p}^N)$. We claim $\mathcal{M}'$ is an $\mathcal{L}$-model. The verification of the properties is straightforward.

By applying [DG, Proposition 3.1] with Φ described in the proof of (I) $\rightarrow$ (II), we conclude that $\mathcal{M}, w \models \mathtt{F}'$ iff $\mathcal{M}', w \models \mathtt{F}$. Q.E.D.

The construction in the proof of Proposition 2.2 can be seen as a generalization of the proof of [DG, Proposition 4.3].

Proposition 4.4 in [DG] can be also naturally extended:

Proposition 2.3. (**Reducing $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$ to $\mathcal{L}(\mathtt{For}^N_0)$**) There exists a polynomial-time transformation from $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$-satisfiability into $\mathcal{L}(\mathtt{For}^N_0)$-satisfiability.

The proof of Proposition 2.3 is analogous to the proof of [DG, Proposition 4.4]. Hence,

Corollary 2.4. (**Decidability and finite model property correspondences**) Let $\mathcal{L}$ be a Rare-logic of the type described at the beginning of [DG, Section 4.2] and $\mathcal{L}(\mathtt{For}^N_0)$, $\mathcal{L}(\mathtt{P}^N_0)$, $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ be extensions with (world and/or parameter) nominals.

(I) $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ [resp. $\mathcal{L}(\mathtt{P}^N_0)$, $\mathcal{L}(\mathtt{For}^N_0)$] is decidable iff $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$ [resp. $\mathcal{L}^{\star}_d$, $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$] is decidable;

(II) $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ [resp. $\mathcal{L}(\mathtt{P}^N_0)$, $\mathcal{L}(\mathtt{For}^N_0)$] has the fmp iff $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$ [resp. $\mathcal{L}^{\star}_d$, $\mathcal{L}^{\star}_d(\mathtt{For}^N_0)$] has the fmp;

(III) If for any $i \in J$, $\mathcal{L}^{\star}_d$ is $\mathtt{U}_{\mathcal{D}(i)}$-simplifiable or $\mathtt{0}_{\mathcal{D}(i)}$-simplifiable (according to $i_{\mathcal{L}}$) then $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ [resp. $\mathcal{L}(\mathtt{P}^N_0)$, $\mathcal{L}(\mathtt{For}^N_0)$] is decidable iff $\mathcal{L}^{\star -}_d(\mathtt{For}^N_0)$ [resp. $\mathcal{L}^{\star -}_d$, $\mathcal{L}^{\star -}_d(\mathtt{For}^N_0)$] is decidable.

Example 2.1. Let $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 4\rangle$ such that $j = 1$, $\mathcal{D}(1) = 2$, $X_1 = Fr^2$ and $OP = \{\cup, \circ, ^*\}$ (interpreted in the standard way). For any $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$-model $\mathcal{M} = \langle W, PAR, (R_P)_{P\subseteq PAR}, V\rangle$ and for any $P, P' \subseteq PAR$, $R_{P\cap P'} = R_P \cup R_{P'}$ and $R_{PAR} = \emptyset$. Here is an example of formula: $[r(\mathtt{C}^N \cup -\mathtt{D})^* \circ r(\mathtt{D})](\mathtt{p} \Rightarrow [r(\mathtt{C}^N \cap \mathtt{D})]\mathtt{q}^N)$. Then, $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ has a decidable validity problem. Indeed, let $\mathcal{L}'(\mathtt{For}^N_0, \mathtt{P}^N_0)$ be the logic of type 2 similar to $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$. By Corollary 2.4(III), $\mathcal{L}'(\mathtt{For}^N_0, \mathtt{P}^N_0)$ is decidable iff $\mathcal{L}'^{\mathtt{0}_2 -}_d(\mathtt{For}^N_0)$ is decidable (see [DG, Proposition 4.6(III)]). However, $\mathcal{L}'^{\mathtt{0}_2 -}_d(\mathtt{For}^N_0)$ is a fragment of the Combinatory Dynamic Logic CPDL defined in [PT91]. Since CPDL has been proved decidable (see e.g. [PT91]), $\mathcal{L}'(\mathtt{For}^N_0, \mathtt{P}^N_0)$ has a decidable validity problem. Hence $\mathcal{L}(\mathtt{For}^N_0, \mathtt{P}^N_0)$ is decidable by using [DG, Proposition 3.3] for names.

3 Elimination of the universal modal operator

In this section we concentrate on the elimination of $[\mathtt{U}_2]$ (noted $[\mathtt{U}]$) as far as decidability of satisfiability for standard modal logics is concerned. Instead of stating, $\mathcal{L}$ is decidable iff $\mathcal{L}_d^\star$ is decidable, we identify cases when $\mathcal{L}$ is decidable iff $\mathcal{L}_d$ is decidable. As a consequence, we take advantage of the (un)decidability of standard modal logics (see e.g. [Har84, PT91]) to deduce (un)decidability of Rare-logics. Adding the universal operator to modal logics increases the expressive power (see e.g. [GP92]) and it may modify the decidability status of the logic (see e.g. [GP92, Spa93, Mar97]). The main result on the present section (Proposition 3.1) can be viewed as a consequence of results from [GP92]. However, when in [GP92] the focus is on proof systems, we provide a semantical proof whose consequences have been mostly ignored in the literature.

PROPOSITION 3.1. (**Reduction of logical consequence to validity**) Let $\mathcal{L}_d$ be a standard modal logic closed under disjoint unions and isomorphic copies (see [DG, Section 2]) such that for any $\mathtt{a} \in \mathtt{M}$, $\mathcal{D}(\mathtt{a}) = 2$ (the $\mathcal{L}_d$-models contain only binary relations). The statements below are equivalent:

(I) the logical $\mathcal{L}_d$-consequence problem is decidable;

(II) the $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-validity problem is decidable.

PROOF: (II) $\rightarrow$ (I) Let $\mathtt{F}, \mathtt{G}$ be $\mathcal{L}_d$-formulas. It is easy to see that $\mathtt{F} \models_{\mathcal{L}_d} \mathtt{G}$ iff $[\mathtt{U}]\mathtt{F} \Rightarrow [\mathtt{U}]\mathtt{G}$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid. Hence, the $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-validity problem is decidable only if the logical $\mathcal{L}_d$-consequence problem is decidable.
(I) $\rightarrow$ (II) Let us recall the notion of *elementary disjunction* from [GP92]. An *elementary disjunction* $\mathtt{ED}$ is an $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-formula of the form

$$\mathtt{F}_{-1} \vee \langle \mathtt{U} \rangle \mathtt{F}_0 \vee [\mathtt{U}]\mathtt{F}_1 \vee \ldots \vee [\mathtt{U}]\mathtt{F}_n$$

where $\mathtt{F}_{-1}, \mathtt{F}_0, \mathtt{F}_1, \ldots, \mathtt{F}_n$ are $\mathtt{U}$-free formulae. So, $\mathtt{F}_{-1}, \mathtt{F}_0, \mathtt{F}_1, \ldots, \mathtt{F}_n$ are $\mathcal{L}_d$-formulae. Using arguments from the proof of [GP92, Theorem 3.7], for any $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-formula $\mathtt{F}$ there exists a finite set $\{\mathtt{ED}_1, \ldots, \mathtt{ED}_N\}$ of elementary disjunctions such that $\mathtt{F} \Leftrightarrow (\mathtt{ED}_1 \wedge \ldots \wedge \mathtt{ED}_N)$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid. $\mathtt{ED}_1 \wedge \ldots \wedge \mathtt{ED}_N$ is called here a *conjunctive form* of $\mathtt{F}$. Actually, there exists an effective procedure to compute such a set. Let $\mathtt{a} \in \mathtt{M}$ and $\mathtt{H}, \mathtt{H}'$ be $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-formulas such that $\mathtt{H}'$ is a Boolean combination of formulae prefixed by $[\mathtt{U}]$ or $\langle \mathtt{U} \rangle$. The $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-formulas below are $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid:

(i) $[\mathtt{a}](\mathtt{H} \vee \mathtt{H}') \Leftrightarrow [\mathtt{a}]\mathtt{H} \vee \mathtt{H}'$

(ii) $[\mathtt{U}](\mathtt{H} \vee \mathtt{H}') \Leftrightarrow [\mathtt{U}]\mathtt{H} \vee \mathtt{H}'$

By induction on the structure of $\mathtt{F}$, one can show that $\mathtt{F}$ is equivalent to a conjunction of elementary disjunctions. The base case ($\mathtt{F}$ is a propositional variable) and the cases in the induction step when the outermost connective of $\mathtt{F}$ is Boolean are standard and they are omitted here. Let $\mathtt{F} = [\mathtt{a}]\mathtt{F}_1$. By induction hypothesis, there is a finite set $\{\mathtt{F}_1^1, \ldots, \mathtt{F}_m^1\}$ of elementary disjunctions such that $\mathtt{F}_1^1 \wedge \ldots \wedge \mathtt{F}_m^1$ is a conjunctive form of $\mathtt{F}_1$. So, $[\mathtt{a}]\mathtt{F}_1 \Leftrightarrow [\mathtt{a}]\mathtt{F}_1^1 \wedge \ldots \wedge [\mathtt{a}]\mathtt{F}_m^1$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid since $\langle \mathtt{a} \rangle$ is a *normal* modal operator. (i) guarantees that each

$[\mathtt{a}]\mathtt{F}^1_j$, $1 \leq j \leq m$, has a conjunctive form. So $\mathtt{F}$ has an equivalent conjunctive form. Let $\mathtt{F} = [\mathtt{U}]\mathtt{F}_1$. The proof is similar to the proof of the previous case except that (ii) is used instead of (i).

Now let $\mathtt{ED} = \mathtt{F}_{-1} \vee \langle\mathtt{U}\rangle\mathtt{F}_0 \vee [\mathtt{U}]\mathtt{F}_1 \vee \ldots \vee [\mathtt{U}]\mathtt{F}_n$ be an elementary disjunction. For any $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-model $\mathcal{M}$, $\mathcal{M} \models [\mathtt{U}]\mathtt{ED}$ iff $\mathcal{M} \models \mathtt{ED}$. Using (ii), $\mathcal{M} \models [\mathtt{U}]\mathtt{ED}$ iff $\mathcal{M} \models [\mathtt{U}]\mathtt{F}_{-1} \vee \langle\mathtt{U}\rangle\mathtt{F}_0 \vee [\mathtt{U}]\mathtt{F}_1 \vee \ldots \vee [\mathtt{U}]\mathtt{F}_n$. By easy manipulation at the propositional level, $\mathcal{M} \models \mathtt{ED}$ iff (iii) $\mathcal{M} \models [\mathtt{U}]\neg\mathtt{F}_0 \Rightarrow ([\mathtt{U}]\mathtt{F}_{-1} \vee [\mathtt{U}]\mathtt{F}_1 \vee \ldots \vee [\mathtt{U}]\mathtt{F}_n)$. Moreover (iii) holds iff for some $i \in \{-1, 1, \ldots, n\}$, $\mathcal{M} \models [\mathtt{U}]\neg\mathtt{F}_0 \Rightarrow [\mathtt{U}]\mathtt{F}_i$. So, $\mathtt{ED}$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid iff (iv) for any $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-model $\mathcal{M}$, there is $i \in \{-1, 1, \ldots, n\}$, $\mathcal{M} \models [\mathtt{U}]\neg\mathtt{F}_0 \Rightarrow [\mathtt{U}]\mathtt{F}_i$. Let us show that (iv) iff (v) there is $i \in \{-1, 1, \ldots, n\}$ such that for any $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-model $\mathcal{M}$, $\mathcal{M} \models [\mathtt{U}]\neg\mathtt{F}_0 \Rightarrow [\mathtt{U}]\mathtt{F}_i$ (or equivalently, $\neg\mathtt{F}_0 \models_{\mathcal{L}_d} \mathtt{F}_i$). The equivalence between (iv) and (v) corresponds to the permutation of two quantifiers.

(v) implies (iv) is obvious. Now assume (iv) and suppose (v) does not hold. By definition of $\mathcal{L}_d^{\{\mathtt{U}\}^-}$, (v) holds iff there is $i \in \{-1, 1, \ldots, n\}$ such that for any $\mathcal{L}_d$-model $\mathcal{M}$, $\mathcal{M} \models \neg\mathtt{F}_0$ implies $\mathcal{M} \models \mathtt{F}_i$. It is worth mentioning that $\mathcal{M}$ is an $\mathcal{L}_d$-model (not an $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-model) which is however correct since $\mathtt{F}_{-1}, \mathtt{F}_0, \mathtt{F}_1, \ldots, \mathtt{F}_n$ are $\mathcal{L}_d$-formulas. Since (v) is supposed not to hold, for $i \in \{-1, 1, \ldots, n\}$, there exist an $\mathcal{L}_d$-model $\mathcal{M}^i = \langle W^i, (R^i_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V^i\rangle$ $\mathcal{M}^i \models \mathtt{F}_0$ and $y_i \in W^i$ such that $\mathcal{M}^i, y_i \not\models \mathtt{F}_i$. Let $\mathcal{M}'^{-1} = \langle W'^{-1}, (R'^{-1}_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V'^{-1}\rangle$, $\ldots$, $\mathcal{M}'^n = \langle W'^n, (R'^n_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V'^n\rangle$ be isomorphic copies of $\mathcal{M}^{-1}, \ldots, \mathcal{M}^n$ respectively such that for $i, i' \in \{-1, 1, \ldots, n\}$, $i \neq i'$ implies $W'^i \cap W'^{i'} = \emptyset$. By assumption, $\mathcal{L}_d$ is closed under isomorphic copies. So $\mathcal{M}'^{-1}$, $\ldots$, $\mathcal{M}'^n$ are $\mathcal{L}_d$-models. Let $\mathcal{M}' = \langle W', (R'_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V'\rangle$ be the structure such that:

- $W' \stackrel{\text{def}}{=} W'^{-1} \cup W'^1 \cup \ldots \cup W'^n$;
- for $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) = V'^{-1}(\mathtt{p}) \cup V'^1(\mathtt{p}) \cup \ldots \cup V'^n(\mathtt{p})$;
- for $\mathtt{c} \in \mathtt{M}_0$, $R'_{\mathtt{C}} \stackrel{\text{def}}{=} R'^{-1}_{\mathtt{C}} \cup R'^1_{\mathtt{C}} \cup \ldots \cup R'^n_{\mathtt{C}}$;
- for $\mathtt{a} \in \mathtt{M}$, $V'(\mathtt{a}) = V'^{-1}(\mathtt{a}) \cup V'^1(\mathtt{a}) \cup \ldots \cup V'^n(\mathtt{a})$.

By assumption, $\mathcal{L}_d$ is closed under disjoint unions, so $\mathcal{M}'$ is an $\mathcal{L}_d$-model. By induction on the structure of the $\mathcal{L}_d$-formula $\mathtt{G}$, for any $i \in \{-1, 1, \ldots, n\}$ and for $x \in W'^i$, $\mathcal{M}'^i, x \models \mathtt{G}$ iff $\mathcal{M}', x \models \mathtt{G}$. Consequently, $\mathcal{M}' \models \neg\mathtt{F}_0$ and for $i \in \{-1, 1, \ldots, n\}$, $\mathcal{M}' \not\models \mathtt{F}_i$. By definition of $\mathcal{L}_d^{\{\mathtt{U}\}^-}$, (iv) holds iff for any $\mathcal{L}_d$-model $\mathcal{M}$, there is $i \in \{-1, 1, \ldots, n\}$, $\mathcal{M} \models \neg\mathtt{F}_0$ implies $\mathcal{M} \models \mathtt{F}_i$, a contradiction.

It is now possible to conclude the proof. Let $\mathtt{F}$ be a $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-formula such that $\mathtt{ED}_1 \wedge \ldots \wedge \mathtt{ED}_N$ is a conjunctive form of $\mathtt{F}$. Let us say that for $i \in \{1, \ldots, N\}$, $\mathtt{ED}_i$ is of the form

$$\mathtt{F}^i_{-1} \vee \langle\mathtt{U}\rangle\mathtt{F}^i_0 \vee [\mathtt{U}]\mathtt{F}^i_1 \vee \ldots \vee [\mathtt{U}]\mathtt{F}^i_{n_i}$$

(vi) $\mathtt{F}$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid iff for all $i \in \{1, \ldots, N\}$, $\mathtt{ED}_i$ is $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-valid. That is, (vi) holds iff there is $\langle j_1, \ldots, j_N\rangle \in \{-1, 1, \ldots, n_1\} \times \ldots \times \{-1, 1, \ldots, n_N\}$ such that for any $l \in \{1, \ldots, N\}$,$\neg\mathtt{F}^l_0 \models_{\mathcal{L}_d} \mathtt{F}^l_{j_l}$. By assumption, the $\mathcal{L}_d$-consequence problem is decidable and there is an effective procedure to build the conjunctive forms, so the $\mathcal{L}_d^{\{\mathtt{U}\}^-}$-validity problem is decidable.

Q.E.D.

Although Proposition 3.1 is interesting for its own sake, it has also natural consequences for some classes of Rare-logics.

Corollary 3.2. Let $\mathcal{L}$ be a Rare-logic of the type described at the beginning of [DG, Section 4.2] such that for any $i \in J$, $\mathcal{D}(i) = 2$, $i_{\mathcal{L}} = 1$ and $\mathcal{L}_d$ is closed under disjoint unions and isomorphic copies. If $\mathcal{L}_d^{\star}$ is $\mathtt{U}_2$-simplifiable, then the statements below are equivalent:

(I) $\mathcal{L}$ is decidable;

(II) the logical $\mathcal{L}_d$-consequence problem is decidable.

Proposition 3.3 below describes a set OP of operators that allows us to simplify Corollary 3.2(II).

Proposition 3.3. (**Particular cases of elimination of** $[\mathtt{U}]$) Let $\mathcal{L}_d = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C} \rangle$ be a standard modal logic closed under restrictions such that, for any $\mathtt{a} \in \mathtt{M}$, $\mathcal{D}(\mathtt{a}) = 2$ and either $\cup^* \in OP$ or $\{\cup, ^*\} \subseteq OP$ with $OP \subseteq \{\cap, \cup, \cup^*, ^*, ^{-1}, \circ\}$ (interpreted in the standard way). Then, the statements below are equivalent:

(I) the logical $\mathcal{L}_d$-consequence problem is decidable;

(II) the $\mathcal{L}_d$-validity problem is decidable.

A result similar to Proposition 3.3 has been proved in [KT90] for particular dynamic logics.

Proof:(I) $\rightarrow$ (II) The proof is immediate.

(II) $\rightarrow$ (I) We can prove using standard techniques for PDL that (i) $\mathtt{F} \models_{\mathcal{L}_d} \mathtt{G}$ iff (ii) $[\mathtt{a}]\mathtt{F} \Rightarrow \mathtt{G}$ is $\mathcal{L}_d$-valid where (the $\mathtt{c}_i$'s are the modal constants occurring in $\{\mathtt{F}, \mathtt{G}\}$):

$$\mathtt{a} \stackrel{\text{def}}{=} \begin{cases} (\mathtt{c}_1 \cup \ldots \cup \mathtt{c}_n)^* \text{ if } ^{-1} \not\in OP \text{ and } \{\cup, ^*\} \subseteq OP \\ \mathtt{c}_1 \cup^* \ldots \cup^* \mathtt{c}_n \text{ if } ^{-1} \not\in OP,\ \cup^* \in OP \text{ and } \{\cup, ^*\} \not\subseteq OP \\ (\mathtt{c}_1 \cup \ldots \cup \mathtt{c}_n \cup \mathtt{c}_1^{-1} \cup \ldots \cup \mathtt{c}_n^{-1})^* \text{ if } ^{-1} \in OP \text{ and } \{\cup, ^*\} \subseteq OP \\ (\mathtt{c}_1 \cup^* \ldots \cup^* \mathtt{c}_n \cup^* \mathtt{c}_1^{-1} \cup^* \ldots \cup^* \mathtt{c}_n^{-1}) \text{ if } ^{-1} \in OP,\ \cup^* \in OP \text{ and } \{\cup, ^*\} \not\subseteq OP \end{cases}$$

(ii) $\rightarrow$ (i) Assume $[\mathtt{a}]\mathtt{F} \Rightarrow \mathtt{G}$ is $\mathcal{L}_d$-valid and $\mathcal{M} \models \mathtt{F}$ for some $\mathcal{L}_d$-model $\mathcal{M}$. So $\mathcal{M} \models [\mathtt{a}]\mathtt{F}$ and $\mathcal{M} \models \mathtt{G}$.

(i) $\rightarrow$ (ii) Assume $\mathtt{F} \models_{\mathcal{L}} \mathtt{G}$. Let $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$ be an $\mathcal{L}_d$-model and $x \in W$ be such that $\mathcal{M}, x \models [\mathtt{a}]\mathtt{F}$. Let us show that $\mathcal{M}, x \models \mathtt{G}$. It is easy to show that

$$(\bigcup_{\mathtt{b} \in \mathtt{M}([\mathtt{a}]\mathtt{F} \Rightarrow \mathtt{G})} V(\mathtt{b}))^*(x) \subseteq V(\mathtt{a})(x)$$

and (iii) for any $y \in V(\mathtt{a})(x)$ and $\mathtt{F}' \in sub([\mathtt{a}]\mathtt{F} \Rightarrow \mathtt{G})$ (set of subformulae of $[\mathtt{a}]\mathtt{F} \Rightarrow \mathtt{G}$), $\mathcal{M}_{|V(\mathtt{a})(x)}, y \models \mathtt{F}'$ iff $\mathcal{M}, y \models \mathtt{F}'$. So $\mathcal{M}_{|V(\mathtt{a})(x)}, x \models [\mathtt{a}]\mathtt{F}$ and $\mathcal{M}_{|V(\mathtt{a})(x)} \models \mathtt{F}$. Since $\mathcal{M}_{|V(\mathtt{a})(x)}$ is an $\mathcal{L}_d$-model ($\mathcal{L}_d$ is closed under restrictions), $\mathcal{M}_{|V(\mathtt{a})(x)} \models \mathtt{G}$. In particular, $\mathcal{M}_{|V(\mathtt{a})(x)}, x \models \mathtt{G}$ and by (iii) $\mathcal{M}, x \models \mathtt{G}$ (note that $x \in V(\mathtt{a})(x)$ since $V(\mathtt{a})$ is reflexive).

Q.E.D.

Proposition 3.1 above is a crucial result since it allows to refine various statements in [DG, Corollary 4.5].

EXAMPLE 3.1. Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 1\rangle$ be the Rare-logic such that $j = 1$, $\mathcal{D}(1) = 2$, $OP = \{\cup^*, \cap\}$ (binary operators interpreted by transitive and reflexive closure of union and set intersection, respectively) and $X_1 = Fr^2$. $\mathcal{L}$ is decidable iff $\mathcal{L}_d^{\{\mathtt{U}\}-}$ is decidable by [DG, Corollary 4.5(II)]. Since $\mathcal{L}_d$ is closed under isomorphic copies, disjoint unions and restrictions, by Proposition 3.1 and Proposition 3.3, $\mathcal{L}_d$ is decidable iff $\mathcal{L}_d^{\{\mathtt{U}\}-}$ is decidable. Since $\mathcal{L}_d$ is a fragment of PDL+Intersection that is known to be decidable [Dan84], $\mathcal{L}$ is therefore decidable.

4 Proof system translations

This section is devoted to show how to build Hilbert-style proof systems for Rare-logics from existing proof systems for the corresponding standard modal logics. We propose a generic translation between calculi using the results of the previous sections. The idea of the translation consists in providing an analogous treatment to the basic modal expressions (more precisely to the components) in the Rare-logics and to the modal constants in the corresponding standard modal logics.

In what follows, $\mathcal{L}$ denotes a Rare-logic of the type described at the beginning of [DG, Section 4.2], including for instance the Rare-logics of type 1,2 and 7. For Rare-logics of type 0, 3 or 4, similar results can be easily obtained. They are omitted here to avoid the boredom of repetitive arguments. $\mathcal{L}_d$ denotes the corresponding standard modal logic and $\mathcal{L}'$ the logic $\mathcal{L}_d^\star$. We subscript the sets of syntactic objects related to $\mathcal{L}'$ by the symbol 'd'.

4.1 Proof systems

The rules of the proof systems do not involve formulae but *formula schemes*. That is why, we use extensions of the languages of the logics in order to define calculi. The propositional variables, modal constants and parameter constants should be read as variables in formula schemes. However, we shall enrich the set $\mathtt{M}_0$ by adding *modal constants* since it is not true that for any $\mathtt{a} \in \mathtt{M}$, $\mathcal{D}(\mathtt{a}) \in \{\mathcal{D}(i) : i \in J\}$. Let J' be the finite set of integers $J' = \{1, \ldots, j + card(\{i_{n+1} : \oplus \in OP,\ \mathcal{D}(\oplus) = \langle i_1, \ldots, i_{n+1}\rangle\})\}$ (remember OP is finite). We extend $\mathcal{D}$ such that $\{\mathcal{D}(i) : i \in J' \setminus J\} = \{\rho^2_{OP}(\oplus) : \oplus \in OP\}$. Let $\mathtt{M}_0^{ax}$ be the infinite countable set of constants defined as follows

1. $\{\mathtt{M}_0^{i,ax} : i \in J'\}$ is a partition of $\mathtt{M}_0^{ax}$ and each $\mathtt{M}_0^{i,ax}$ is countably infinite;
2. for any $i \in J$, $\mathtt{M}_0^{i,ax} \stackrel{\text{def}}{=} \mathtt{M}_{0d}^i$ (and therefore $\mathtt{M}_{0d} \subseteq \mathtt{M}_0^{ax}$);
3. for any $i \in J' \setminus J$ and any $\mathtt{c} \in \mathtt{M}_0^{i,ax}$, we extend ρ with $\rho(\mathtt{c}) \stackrel{\text{def}}{=} \mathcal{D}(i) - 1$ and $\mathcal{D}(\mathtt{c}) \stackrel{\text{def}}{=} \rho(\mathtt{c}) + 1$.

First, we define various syntactic notions related for the Rare-logic $\mathcal{L}$. Let $\mathtt{M}^{ax}$ be the set of well-formed *modal expression schemes* and $\mathtt{For}^{ax}$ be the set of *formula schemes* that are obtained by substituting in the definitions from [DG, Section 2 and Section 3], $\mathtt{M}_0$ by $\mathtt{M}_0 \cup \mathtt{M}_0^{ax}$. Consequently, $\mathtt{For} \subseteq \mathtt{For}^{ax}$. An $\mathcal{L}$-*substitution* σ is a mapping $\sigma : \mathtt{For}_0 \cup \mathtt{M}_0^{ax} \cup \mathtt{P}_0 \to \mathtt{For} \cup \mathtt{M} \cup \mathtt{P}$ such that

1. for any $\mathtt{p} \in \mathtt{For}_0$, $\sigma(\mathtt{p}) \in \mathtt{For}$;

2. for any $i \in J'$, for any $\mathtt{c} \in \mathtt{M}_0^{i,ax}$, $\sigma(\mathtt{c}) \in \mathtt{M}$ and $\mathcal{D}(\sigma(\mathtt{c})) = \mathcal{D}(\mathtt{c})$;
3. for any $i \in J$, for any $\mathtt{C} \in \mathtt{P}_0^i$, $\sigma(\mathtt{C}) \in \mathtt{P}_i$.

Let $\mathtt{F}$ be in $\mathtt{For}^{ax}$. The formula $\mathtt{F}\sigma \in \mathtt{For}$ (also written $\sigma(\mathtt{F})$) is obtained from $\mathtt{F}$ by simultaneously replacing every occurrence of the syntactic object $\mathtt{O} \in \mathtt{For}_0 \cup \mathtt{M}_0^{ax} \cup \mathtt{P}_0$ by $\sigma(\mathtt{O})$. From a formula scheme $\mathtt{F} \in \mathtt{For}^{ax}$, the set of $\mathcal{L}$-substitutions generates a countable set of $\mathcal{L}$-formulae of the form $\mathtt{F}\sigma$. A *rule* $\mathtt{Ru}$ is a pair $\mathtt{Ru} = \langle\langle \mathtt{F}_1, \ldots, \mathtt{F}_n\rangle, \lambda \mathtt{x}_1 \ldots \mathtt{x}_m C\rangle$ such that

- $n \geq 1$; for any $i \in \{1, \ldots, n\}$, $\mathtt{F}_i \in \mathtt{For}^{ax}$;
- $\{\mathtt{x}_1, \ldots, \mathtt{x}_m\} \subseteq (\mathtt{For}_0 \cup \mathtt{M}_0^{ax} \cup \mathtt{P}_0)(\{\mathtt{F}_1, \ldots, \mathtt{F}_n\})$;
- C is a *condition* of the meta-language.

Another way to represent $\langle \mathtt{F}_1, \ldots, \mathtt{F}_n\rangle$ is the standard $\frac{\mathtt{F}_1, \ldots, \mathtt{F}_{n-1}}{\mathtt{F}_n}$ whereas $\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C$ is a condition for its application. As usual, when $n = 1$, $\mathtt{Ru}$ is also called an axiom scheme.

Example 4.1. Some examples of rules:

- $\langle\langle \mathtt{p}, \mathtt{p} \Rightarrow \mathtt{q}, \mathtt{q}\rangle, \lambda \mathtt{p}.\mathrm{true}\rangle$ (modus ponens);
- $\langle\langle \mathtt{p}, [\mathtt{c}](\mathtt{p}, \mathtt{q})\rangle, \lambda \mathtt{p}.\mathrm{true}\rangle$ (necessitation rule);
- $\langle\langle [r(\mathtt{C} \cap -\mathtt{C})]\mathtt{p} \Rightarrow [\mathtt{c}]\mathtt{p}\rangle, \lambda \mathtt{p}.\mathrm{true}\rangle$; $\langle\langle [r(\mathtt{C})]\mathtt{p} \Rightarrow [\mathtt{c}]\mathtt{p}\rangle, \lambda \mathtt{C}.\mathtt{C} \equiv \perp\rangle$.

An *axiom system*[1] $\mathtt{Ax}$ is defined as a countable set of rules. The present notion of axiom system is strongly related to Hilbert-style calculi. However, various technical results in this section can be adapted to other kinds of calculi (see e.g. [DG98]). The set of $\mathtt{Ax}$-*theorems*, denoted $\vdash_{\mathtt{Ax}}$, is the smallest set of formulae satisfying the following condition. For any rule $\mathtt{Ru} = \langle\langle \mathtt{F}_1, \ldots, \mathtt{F}_n\rangle, \lambda \mathtt{x}_1 \ldots \mathtt{x}_m C\rangle \in \mathtt{Ax}$, for any substitution σ, if for any $i \in \{1, \ldots, n-1\}$, $\mathtt{F}_i\sigma \in \vdash_{\mathtt{Ax}}$, $\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C \sigma(\mathtt{x}_1) \ldots \sigma(\mathtt{x}_m)$ holds true, and $\mathtt{F}_n\sigma \in \mathtt{For}$ then $\mathtt{F}_n\sigma \in \vdash_{\mathtt{Ax}}$. Here, we abusively write $\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C \sigma(\mathtt{x}_1) \ldots \sigma(\mathtt{x}_m)$ to denote that C' holds true where C' is obtained from C by replacing simultaneously $\mathtt{x}_1 \ldots \mathtt{x}_m$ by $\sigma(\mathtt{x}_1), \ldots, \sigma(\mathtt{x}_m)$, respectively. We write indifferently $\mathtt{F} \in \vdash_{\mathtt{Ax}}$ or $\vdash_{\mathtt{Ax}} \mathtt{F}$. An $\mathtt{Ax}$-*derivation* $\mathtt{Der}$ is a finite sequence $\mathtt{Der} = \langle \mathtt{G}_1, \ldots, \mathtt{G}_n\rangle$ of $\mathcal{L}$-formulae such that for any $i \in \{1, \ldots, n\}$, there exist a rule $\mathtt{Ru} = \langle\langle \mathtt{F}_1, \ldots, \mathtt{F}_N\rangle, \lambda \mathtt{x}_1 \ldots \mathtt{x}_M C\rangle \in \mathtt{Ax}$ and an $\mathcal{L}$-substitution σ such that $\{\mathtt{F}_1\sigma, \ldots, \mathtt{F}_{N-1}\sigma\} \subseteq \{\mathtt{G}_1, \ldots, \mathtt{G}_{i-1}\}$ $\lambda \mathtt{x}_1 \ldots \mathtt{x}_M C \sigma(\mathtt{x}_1) \ldots \sigma(\mathtt{x}_M)$ holds true and $\mathtt{G}_i = \mathtt{F}_N\sigma \in \mathtt{For}$.

Until now, the defined notions have been introduced for the Rare-logic $\mathcal{L}$. In order to define the set $\mathtt{M}_d^{ax}$ for the standard modal logic $\mathcal{L}'$, take the definition of $\mathtt{M}_d$ and replace the set $\mathtt{M}_{0d}$ by $\mathtt{M}_0^{ax}$. The set $\mathtt{For}_d^{ax}$ for $\mathcal{L}'$ is built accordingly. An $\mathcal{L}'$-*substitution* σ is a mapping $\sigma : \mathtt{For}_0 \cup \mathtt{M}_0^{ax} \rightarrow \mathtt{For}_d \cup \mathtt{M}_d$ such that σ satisfying the conditions (1)-(2) in the definition of $\mathcal{L}$-substitution. Recall that $\mathtt{For}_0 = \mathtt{For}_{0d}$ and $\mathtt{M}_0^{i,ax} = \mathtt{M}_{0d}^{i,ax}$ for $i \in J$. The notions of rules, axiom systems, $\mathtt{Ax}$-theorems and $\mathtt{Ax}$-derivations for the logic $\mathcal{L}'$ are defined as previously for $\mathcal{L}$.

[1] This definition of axiom systems does not deviate from what is usually considered in the literature. It incorporates features from the notion of *schematic axiomatic system* in [Acz94] and from the notion of *Hilbert-style inference system* in [NA94]. However, our motivation merely consists in expressing translations between proof systems for Rare-logics and standard modal logics.

4.2 Syntactic correspondences

First, we describe a generalization of the normalization process $\mathtt{N}$ that makes explicit the correspondences between basic modal expressions in $\mathcal{L}$ and modal constants in $\mathcal{L}'$. For $i \in J$, let $\{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ be a non-empty finite subset of $\mathtt{P}^i_0$. Let $\mathcal{X} = (\mathcal{X}_i)_{i \in J}$ be a (finite) *family* of 1-1 mappings $\mathcal{X}_i : \{\mathtt{A}^{i*}_k : k \in \{0, \ldots, 2^{n_i} - 1\}\} \rightarrow subset(\mathtt{M}^i_{0d})$ such that $\{\mathtt{A}^{i*}_k : k \in \{0, \ldots, 2^{n_i} - 1\}\}$ is the set of components computed from $\{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ and $subset(\mathtt{M}^i_{0d})$ is a (finite) subset of $\mathtt{M}^i_{0d}$. We write $\mathtt{For}_{\mathcal{X}}$ [resp. $\mathtt{For}_{d\mathcal{X}}$] to denote the set of $\mathcal{L}$-formulas $\mathtt{F}$ [resp. $\mathcal{L}'$-formulas $\mathtt{F}$] such that for $i \in J$, $\mathtt{P}^i_0(\mathtt{F}) \subseteq \{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ [resp. $\mathtt{M}^i_0(\mathtt{F}) \subseteq subset(\mathtt{M}^i_{0d})$]. Similarly, we write $\mathtt{M}_{\mathcal{X}}$ [resp. $\mathtt{M}_{d\mathcal{X}}$] to denote the set of modal expressions from $\mathtt{M}$ [resp. $\mathtt{M}_d$] such that for $i \in J$, $\mathtt{P}^i_0(\mathtt{a}) \subseteq \{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ [resp. $\mathtt{M}^i_0(\mathtt{a}) \subseteq subset(\mathtt{M}^i_{0d})$]. Let us define the map $\mathtt{N}_{\mathcal{X}} : \mathtt{For}_{\mathcal{X}} \cup \mathtt{M}_{\mathcal{X}} \rightarrow \mathtt{For}_{d\mathcal{X}} \cup \mathtt{M}_{d\mathcal{X}}$ that is indeed a natural extension of $\mathcal{X}$ to formulas. Roughly speaking, the basic modal expressions in $\mathcal{L}$ are replaced by modal expressions in $\mathcal{L}'$, consistently with $\mathcal{X}$. $\mathtt{N}_{\mathcal{X}}$ is inductively defined as follows:

- for $\mathtt{p} \in \mathtt{For}_0$, $\mathtt{N}_{\mathcal{X}}(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{p}$;
- $\mathtt{N}_{\mathcal{X}}$ is homomorphic with respect to the Boolean connectives;
- $\mathtt{N}_{\mathcal{X}}([\mathtt{a}](\mathtt{F}_1, \ldots, \mathtt{F}_{\rho(\mathtt{a})})) \stackrel{\text{def}}{=} [\mathtt{N}_{\mathcal{X}}(\mathtt{a})](\mathtt{N}_{\mathcal{X}}(\mathtt{F}_1), \ldots, \mathtt{N}_{\mathcal{X}}(\mathtt{F}_{\rho(\mathtt{a})}))$;
- for $i \in J$ and for $\mathtt{A} \in \mathtt{P}_i$ such that $\mathtt{P}^i_0(\mathtt{A}) \subseteq \{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ and $\mathtt{A} \equiv \mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l}$, $\mathtt{N}_{\mathcal{X}}(r(\mathtt{A})) \stackrel{\text{def}}{=} \mathcal{X}_i(\mathtt{A}^{i*}_{i_1}) \oplus^i \ldots \oplus^i \mathcal{X}_i(\mathtt{A}^{i*}_{i_l})$;
- for $i \in J$ and $\mathtt{A} \in \mathtt{P}_i$ such that $\mathtt{P}^i_0(\mathtt{A}) \subseteq \{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$ and $\mathtt{A} \equiv \perp$,
$$\mathtt{N}_{\mathcal{X}}(r(\mathtt{A})) \stackrel{\text{def}}{=} \begin{cases} \mathtt{U}_{\mathcal{D}(i)} & \text{if } i_{\mathcal{L}} = 1 \\ \mathtt{0}_{\mathcal{D}(i)} & \text{if } i_{\mathcal{L}} = 0 \end{cases}$$
- for $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n) \in \mathtt{M}_{\mathcal{X}}$, $\mathtt{N}_{\mathcal{X}}(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) \stackrel{\text{def}}{=} \oplus(\mathtt{N}_{\mathcal{X}}(\mathtt{a}_1), \ldots, \mathtt{N}_{\mathcal{X}}(\mathtt{a}_n))$.

For any $\mathcal{L}$-formula $\mathtt{F}$, there is a family $\mathcal{X}$ such that $\mathtt{N}(\mathtt{F}) = \mathtt{N}_{\mathcal{X}}(\mathtt{F})$ ($\mathtt{N}$ is the normalization map defined in [DG, Section 4.2]). Analogously, we inductively define the "reverse" map $\mathtt{N}^{-1}_{\mathcal{X}} : \mathtt{For}_{d\mathcal{X}} \cup \mathtt{M}_{d\mathcal{X}} \rightarrow \mathtt{For}_{\mathcal{X}} \cup \mathtt{M}_{\mathcal{X}}$:

- for $\mathtt{p} \in \mathtt{For}_0$, $\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{p}$;
- $\mathtt{N}^{-1}_{\mathcal{X}}$ is homomorphic with respect to the Boolean connectives;
- $\mathtt{N}^{-1}_{\mathcal{X}}([\mathtt{a}](\mathtt{F}_1, \ldots, \mathtt{F}_{\rho(\mathtt{a})})) \stackrel{\text{def}}{=} [\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{a})](\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{F}_1), \ldots, \mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{F}_{\rho(\mathtt{a})}))$;
- for $i \in J$ and $\mathtt{c} \in subset(\mathtt{M}^i_0)$, $\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{c}) \stackrel{\text{def}}{=} r(\mathcal{X}^{-1}_i(\mathtt{c}))$;
- for $i \in J$, if $i_{\mathcal{L}} = 1$, then $\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{U}_{\mathcal{D}(i)}) \stackrel{\text{def}}{=} r(\mathtt{C}^i_1 \cap -\mathtt{C}^i_1)$ otherwise $\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{0}_{\mathcal{D}(i)}) \stackrel{\text{def}}{=} r(\mathtt{C}^i_1 \cap -\mathtt{C}^i_1)$;
- for $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n) \in \mathtt{M}_{d\mathcal{X}}$, $\mathtt{N}^{-1}_{\mathcal{X}}(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) \stackrel{\text{def}}{=} \oplus(\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{a}_1), \ldots, \mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{a}_n))$.

Let $\mathcal{X}$ be a family. The formula $\mathtt{F} \in \mathtt{For}_{\mathcal{X}}$ is said to be $\mathcal{X}$-*normalized* $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{N}_{\mathcal{X}}(\mathtt{F})) = \mathtt{F}$. So for any $\mathcal{L}$-formula $\mathtt{F}$, there is a family $\mathcal{X}$ such that $\mathtt{N}_2(\mathtt{F})$ is $\mathcal{X}$-normalized. Moreover, for any $\mathcal{L}'$-formula $\mathtt{F}$, there is a family $\mathcal{X}$ such that $\mathtt{F}$ and $\mathtt{N}(\mathtt{N}^{-1}_{\mathcal{X}}(\mathtt{F}))$ are identical modulo the renaming of modal constants. The proof of [DG, Proposition 4.4] uses this property.

Now, let $\mathcal{X}$ be a family and σ [resp. σ'] be an $\mathcal{L}$-substitution [resp. $\mathcal{L}'$-substitution] such that the codomain of σ [resp. σ'] allows us to write the expressions below. We write $\mathcal{X}(\sigma)$ to denote the following $\mathcal{L}'$-substitution:

- for $\mathtt{p} \in \mathtt{For}_{0d} = \mathtt{For}_0$, $\mathcal{X}(\sigma)(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{N}_{\mathcal{X}}(\sigma(\mathtt{p}))$;
- for any $i \in J'$ and any $\mathtt{c} \in \mathtt{M}_0^{i,ax}$, $\mathcal{X}(\sigma)(\mathtt{c}) \stackrel{\text{def}}{=} \mathtt{N}_{\mathcal{X}}(\sigma(\mathtt{c}))$;
- the rest of $\mathcal{X}(\sigma)$ is irrelevant.

We write $\mathcal{X}^{-1}(\sigma')$ to denote the following $\mathcal{L}$-substitution:

- for any $\mathtt{p} \in \mathtt{For}_0$, $\mathcal{X}^{-1}(\sigma')(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{N}_{\mathcal{X}}^{-1}(\sigma'(\mathtt{p}))$;
- for any $i \in J'$ and all $\mathtt{c} \in \mathtt{M}_0^{i,ax}$, $\mathcal{X}^{-1}(\sigma')(\mathtt{c}) \stackrel{\text{def}}{=} \mathtt{N}_{\mathcal{X}}^{-1}(\sigma'(\mathtt{c}))$;
- for any $i \in J$ and for $\mathtt{C} \in \mathtt{P}_0^i$, $\mathcal{X}^{-1}(\sigma')(\mathtt{C}) \stackrel{\text{def}}{=} \mathtt{C}$;
- the rest of $\mathcal{X}^{-1}(\sigma')$ is irrelevant.

Definition 4.1. Let $\mathtt{Ru} = \langle\langle \mathtt{F}_1, \ldots, \mathtt{F}_n\rangle, \lambda \mathtt{x}_1 \ldots \mathtt{x}_m.C\rangle$ be a rule of a system $\mathtt{Ax}_d$. $\mathtt{Ru}$ is said to be *$\mathcal{L}$-transformable* $\stackrel{\text{def}}{\Leftrightarrow}$

1. If $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C]\sigma(\mathtt{x}_1) \ldots \sigma(\mathtt{x}_m)$ holds for some $\mathcal{L}'$-substitution σ, then $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C]\sigma'(\sigma(\mathtt{x}_1)) \ldots \sigma'(\sigma(\mathtt{x}_m))$ holds for any other $\mathcal{L}'$-substitution σ';
2. there exists a condition C' such that for any family $\mathcal{X}$ and for any $\mathcal{L}$-substitution σ [resp. $\mathcal{L}'$-substitution σ'] that make the expressions below meaningful, we have:

   2.1. $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C']\mathcal{X}^{-1}(\sigma')(\mathtt{x}_1) \ldots \mathcal{X}^{-1}(\sigma')(\mathtt{x}_m)$ iff $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C]\sigma'(\mathtt{x}_1) \ldots \sigma'(\mathtt{x}_m)$;
   2.2. $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C']\sigma(\mathtt{x}_1) \ldots \sigma(\mathtt{x}_m)$ iff $[\lambda \mathtt{x}_1 \ldots \mathtt{x}_m C]\mathcal{X}(\sigma)(\mathtt{x}_1) \ldots \mathcal{X}(\sigma)(\mathtt{x}_m)$.

∇

Condition 1. above not only allows us to rename derived formulae but it implies also that no atomic proposition and no modal constants play a "special rôle" in the axiomatization. Observe that when $C = \mathit{true}$, $\mathtt{Ru}$ is always $\mathcal{L}$-transformable: take $C' = \mathit{true}$.

4.3 The translations

In this section from an axiom system $\mathtt{Ax}_d$ for $\mathcal{L}'$ we build an axiom system $\mathtt{Ax}$ for $\mathcal{L}$ in a systematic way. Let $\mathtt{Ax}_d$ be an axiom system for the logic $\mathcal{L}'$ such that,

- for any $\mathtt{F} \in \mathtt{For}_d$, $\vdash_{\mathtt{Ax}_d} \mathtt{F}$ iff $\mathtt{F}$ is $\mathcal{L}'$-valid (soundness and completeness);
- the following rules belong to $\mathtt{Ax}_d$:
  - $\langle\langle \mathtt{p}, \mathtt{p} \Rightarrow \mathtt{q}, \mathtt{q}\rangle, \mathit{true}\rangle \in \mathtt{Ax}_d$ (modus ponens);
  - for any tautology $\mathtt{F}$ of the propositional calculus, $\langle \mathtt{F}, \mathit{true}\rangle \in \mathtt{Ax}_d$;
  - for any $i \in J'$, for any $i' \in \{1, \ldots, \mathcal{D}(i) - 1\}$ ($\mathtt{c}_i \in \mathtt{M}_{0d}^{i,ax}$),

    $$\langle\langle \mathtt{p}, [\mathtt{c}_i](\mathtt{p}_1, \ldots, \mathtt{p}_{i'-1}, \mathtt{p}, \mathtt{p}_{i'+1}, \ldots, \mathtt{p}_{\rho(\mathtt{c}_i)})\rangle, \mathit{true}\rangle \in \mathtt{Ax}_d$$

    (necessitation rules);
  - for any $i \in J'$, for any $i' \in \{1, \ldots, \rho(\mathtt{c}_i)\}$ ($\mathtt{c}_i \in \mathtt{M}_{0d}^{i,ax}$), $\langle\langle \mathtt{F}_1 \Rightarrow (\mathtt{F}_2 \Rightarrow \mathtt{F}_3)\rangle, \mathit{true}\rangle \in \mathtt{Ax}_d$ where the $\mathtt{F}_i$'s are defined as follows

$$\mathtt{F}_1 \stackrel{\text{def}}{=} [\mathtt{c}_i](\mathtt{p}_1, \ldots, \mathtt{p}_{i'-1}, \mathtt{p}, \mathtt{p}_{i'+1}, \ldots, \mathtt{p}_{\rho(\mathtt{c}_i)})$$
$$\mathtt{F}_2 \stackrel{\text{def}}{=} [\mathtt{c}_i](\mathtt{p}_1, \ldots, \mathtt{p}_{i'-1}, \mathtt{p} \Rightarrow \mathtt{q}, \mathtt{p}_{i'+1}, \ldots, \mathtt{p}_{\rho(\mathtt{c}_i)})$$
$$\mathtt{F}_3 \stackrel{\text{def}}{=} [\mathtt{c}_i](\mathtt{p}_1, \ldots, \mathtt{p}_{i'-1}, \mathtt{q}, \mathtt{p}_{i'+1}, \ldots, \mathtt{p}_{\rho(\mathtt{c}_i)})$$

$\mathtt{Ax}_d$ contains the classical tautologies, the modus ponens and the many-dimensional version of the necessitation rule and the normality axiom (see e.g. [Ven91]). The constraints on the rules in $\mathtt{Ax}_d$ are rather weak since for all axiom systems $\mathtt{Ax}_d$ for $\mathcal{L}'$ such that for any $\mathtt{F} \in \mathtt{For}_d$, $\vdash_{\mathtt{Ax}_d} \mathtt{F}$ iff $\mathtt{F}$ is $\mathcal{L}'$-valid, all these rules are $\mathtt{Ax}_d$-admissible. A rule $\mathtt{Ru}$ is $\mathtt{Ax}_d$-*admissible* $\stackrel{\text{def}}{\Leftrightarrow}$ $\vdash_{\mathtt{Ax}_d} = \vdash_{\mathtt{Ax}_d \cup \{\mathtt{Ru}\}}$.

Now let us define the corresponding system $\mathtt{Ax}$ for $\mathcal{L}$. Let $\Phi : \mathtt{For}_d^{ax} \rightarrow \mathtt{For}^{ax}$ be the mapping where $\Phi(\mathtt{F})$ is computed from $\mathtt{F}$ by simultaneously substituting every occurrence of $\mathtt{U}_{\mathcal{D}(i)}$ [resp. $\mathtt{0}_{\mathcal{D}(i)}$] by $r(\mathtt{C}^i \cap -\mathtt{C}^i)$ where $\mathtt{C}^i \in \mathtt{P}_0^i$ is some *representative element* of $\mathtt{P}_0^i$. In the sequel, for any formula $\mathtt{F} \in \mathtt{For}$ such that $\mathtt{P}_0^i(\mathtt{F}) \neq \emptyset$ without any loss of generality we assume that $\mathtt{C}^i$ occurs in $\mathtt{F}$ ($\mathcal{L}$-satisfiability and $\mathcal{L}$-validity are not sensitive to the renaming of the constants) and $\mathtt{C}^i$ is the first parameter constants of $\mathtt{P}_0^i$ in any family $\mathcal{X}$.

Definition 4.2. Let $\equiv$ [resp. $\equiv_{\cup\phi_2^i}$] be the smallest transitive relation on $\mathtt{M}$ such that for any $\mathtt{a}, \mathtt{b} \in \mathtt{M}$, if the condition below holds, then $\mathtt{a} \equiv \mathtt{b}$ [resp. $\mathtt{a} \equiv_{\cup\phi_2^i} \mathtt{b}$]: $\mathtt{b}$ is obtained from $\mathtt{a}$ by substituting one occurrence of some $\mathtt{A}$ by $\mathtt{B}$ such that $\mathtt{A} \equiv \mathtt{B}$ [resp. of some $r(\mathtt{A} \cup \mathtt{B})$ by $r(\mathtt{A}) \oplus^i r(\mathtt{B})$] (here $\mathtt{A}, \mathtt{B} \in \mathtt{P}_i$ for some $i \in J$). ∇

Lemma 4.1. The relations $\equiv$, $\equiv_{\cup\phi_2^i}$ are decidable and $(\equiv \cup \equiv_{\cup\phi_2^i})^* \subseteq \equiv_{\mathcal{L}}$.

Definition 4.3. Let $\mathtt{Ax}_d$ be an axiom system for $\mathcal{L}'$ of the kind above such that all the rules are $\mathcal{L}$-transformable. The axiom system $\mathtt{Ax}$ for $\mathcal{L}$ is composed of the following rules:

1. for each $\langle\langle \mathtt{F}_1, \ldots, \mathtt{F}_n\rangle, C\rangle \in \mathtt{Ax}_d$, $\langle\langle \Phi(\mathtt{F}_1), \ldots, \Phi(\mathtt{F}_n)\rangle, C'\rangle \in \mathtt{Ax}$ (C' is from Definition 4.1);
2. for any $i \in J'$ ($\mathtt{c}, \mathtt{c}' \in \mathtt{M}_0^{i,ax}$),

$$\langle\langle [\mathtt{c}](\mathtt{p}_1, \ldots, \mathtt{p}_{\rho(\mathtt{c})}) \Leftrightarrow [\mathtt{c}'](\mathtt{p}_1, \ldots, \mathtt{p}_{\rho(\mathtt{c})})\rangle, \lambda \mathtt{c}\mathtt{c}'.\text{ either } \mathtt{c} \equiv \mathtt{c}' \text{ or } \mathtt{c} \equiv_{\cup\phi_2^i} \mathtt{c}'\rangle \in \mathtt{Ax}$$

∇

Alternatively, in Definition 4.3(2.) one can also replace the relations $\equiv$ and $\equiv_{\cup\phi_2^i}$ by more primitive substitutions obtained from Definition 4.2. Indeed, $\Leftrightarrow$ is logically transitive.

Proposition 4.2. (**Soundness and completeness**) Let $\mathtt{Ax}_d$ be an axiom system for $\mathcal{L}'$ such that all the rules are $\mathcal{L}$-transformable and $\mathtt{Ax}$ be the axiom system defined from $\mathtt{Ax}_d$ as done in Definition 4.3. For any $\mathcal{L}$-formula $\mathtt{F}$, the statements below are equivalent:

(I) $\vdash_{\mathtt{Ax}} \mathtt{F}$;

(II) $\mathtt{F}$ is $\mathcal{L}$-valid.

PROOF: (Sketch) The proof is rather long and tedious and it is based on the following simple principle.

We know that for any $\mathcal{L}$-formula $\mathtt{F}$, $\mathtt{F}$ is $\mathcal{L}$-valid iff $\mathtt{N}_2(\mathtt{F})$ is $\mathcal{L}$-valid iff $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}'$-valid. We shall show that we can define mappings between derivations of $\mathtt{N}(\mathtt{F})$ in $\mathtt{Ax}_d$ and derivations of $\mathtt{F}$ in $\mathtt{Ax}$. Since $\mathtt{Ax}$ and $\mathtt{Ax}_d$ essentially share numerous inference rules, we define mappings between different kinds of syntactic objects (modal expressions, formulas, ...) in order to effectively transform $\mathtt{Ax}_d$-derivations of $\mathtt{N}(\mathtt{F})$ [resp. $\mathtt{Ax}$-derivations of $\mathtt{F}$] into $\mathtt{Ax}$-derivations of $\mathtt{F}$ [resp. $\mathtt{Ax}_d$-derivations of $\mathtt{N}(\mathtt{F})$]. In that sense, although the proof is rather technical, it does not say anything deeper than what is informally described above. However, the technique is general enough to capture a wide range of Rare-logics.

The proof is based on the following lemmas:

(i) Let $\mathcal{X}$ be a family and $\mathtt{F} \in \mathtt{For}_{\mathcal{X}}$. Then, $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{N}_{\mathcal{X}}(\mathtt{F})) \Leftrightarrow \mathtt{F}$;

(ii) Let $\mathcal{X}$ be a family and $\mathtt{F} \in \mathtt{For}_{d\mathcal{X}}$. Then, $\mathtt{N}_{\mathcal{X}}(\mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{F})) = \mathtt{F}$;

(iii) Let $\mathcal{X}$ be a family, $\mathtt{F} \in \mathtt{For}_d^{ax}$ and σ [resp. σ'] be an $\mathcal{L}$-substitution [resp. $\mathcal{L}'$-substitution]. Then,

  (iiia) If $\mathtt{F}\sigma' \in \mathtt{For}_{d\mathcal{X}}$, then $\mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{F}\sigma') = \Phi(\mathtt{F})\mathcal{X}^{-1}(\sigma')$;
  (iiib) If $\Phi(\mathtt{F})\sigma \in \mathtt{For}_{\mathcal{X}}$, then $\mathtt{N}_{\mathcal{X}}(\Phi(\mathtt{F})\sigma) = \mathtt{F}\mathcal{X}(\sigma)$;

(iv) Let $\mathtt{Der} = \langle \mathtt{G}_1, \dots, \mathtt{G}_N \rangle$ be an $\mathtt{Ax}_d$-derivation and $\mathcal{X}$ be a family such that for $i \in J$, $\mathtt{M}_{0d}^i(\{\mathtt{G}_1, \dots, \mathtt{G}_N\}) \subseteq subset(\mathtt{M}_{0d}^i)$. Then, $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{G}_N)$.

(v) Let $\mathtt{Der} = \langle \mathtt{G}_1, \dots, \mathtt{G}_N \rangle$ be an $\mathtt{Ax}$-derivation and $\mathcal{X}$ be a family such that for $i \in J$, $\mathtt{P}_0^i(\{\mathtt{G}_1, \dots, \mathtt{G}_N\}) \subseteq \{\mathtt{C}_1^i, \dots, \mathtt{C}_{n_i}^i\}$ (set from which are built the components involved in $\mathcal{X}_i$). Then, $\vdash_{\mathtt{Ax}_d} \mathtt{N}_{\mathcal{X}}(\mathtt{G}_N)$.

The proof of (i) is by an easy verification. It is based on the principle of replacement of equivalents (that can be shown to hold in $\mathtt{Ax}$) and on the rule 2. from Definition 4.3. The proof of (ii) is straightforward. The proof of (iii) is also not very difficult. The proofs of (iv) and (v) are the difficult parts.

(iv) The proof is by induction on N. The base case is omitted here.
Induction step: Let $\mathtt{Der} = \langle \mathtt{G}_1, \dots, \mathtt{G}_{N+1} \rangle$ be an $\mathtt{Ax}_d$-derivation and $\mathcal{X}$ be a family such that for $i \in J$, $\mathtt{M}_{0d}^i(\{\mathtt{G}_1, \dots, \mathtt{G}_{N+1}\}) \subseteq subset(\mathtt{M}_{0d}^i)$. There exist a rule $\mathtt{Ru} = \langle \langle \mathtt{H}_1, \dots, \mathtt{H}_n \rangle, \lambda \mathtt{x}_1 \dots \mathtt{x}_m.C \rangle \in \mathtt{Ax}_d$ and σ an $\mathcal{L}'$-substitution such that $\mathtt{G}_{N+1} = \mathtt{H}_n\sigma$, $\{\mathtt{H}_1\sigma, \dots, \mathtt{H}_{n-1}\sigma\} \subseteq \{\mathtt{G}_1, \dots, \mathtt{G}_N\}$ and $[\lambda \mathtt{x}_1 \dots \mathtt{x}_m.C]\sigma(\mathtt{x}_1)\dots\sigma(\mathtt{x}_m)$ holds true. By induction hypothesis, for $k \in \{1, \dots, N\}$, $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{G}_k)$. In particular, for $k \in \{1, \dots, n-1\}$, $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{H}_k\sigma)$. By (iiia), for $k \in \{1, \dots, n-1\}$, $\vdash_{\mathtt{Ax}} \Phi(\mathtt{H}_k)\mathcal{X}^{-1}(\sigma)$. Since $\mathtt{Ru}$ is $\mathcal{L}$-transformable,

$$\mathtt{Ru}' = \langle \langle \Phi(\mathtt{H}_1), \dots, \Phi(\mathtt{H}_n) \rangle, \lambda \mathtt{x}_1 \dots \mathtt{x}_m.C' \rangle \in \mathtt{Ax}$$

and $[\lambda \mathtt{x}_1 \dots \mathtt{x}_m.C']\mathcal{X}^{-1}(\sigma)(\mathtt{x}_1)\dots\mathcal{X}^{-1}(\sigma)(\mathtt{x}_m)$ holds true by Definition 4.1(2.1). Moreover, by (iiia) $\mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{G}_{N+1}) = \Phi(\mathtt{H}_n)\mathcal{X}^{-1}(\sigma)$. So, $\mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{G}_{N+1})$ can be derived in $\mathtt{Ax}$ by applying the rule $\mathtt{Ru}'$ with the substitution $\mathcal{X}^{-1}(\sigma)$.
(v) The proof is by induction on N. By way of example, we treat only one base case below.
Base case 1: $N = 1$ and there exists an $\mathcal{L}$-substitution σ such that $\mathtt{G}_1 = ([\mathtt{c}](\mathtt{p}_1, \dots, \mathtt{p}_{\rho(\mathtt{c})}) \Leftrightarrow [\mathtt{c}'](\mathtt{p}_1, \dots, \mathtt{p}_{\rho(\mathtt{c})}))\sigma$ and either $\sigma(\mathtt{c}) \equiv \sigma(\mathtt{c}')$ or $\sigma(\mathtt{c}) \equiv_{\cup \phi_2^i} \sigma(\mathtt{c}')$ for some $i \in J$.

Base case 1.1: $\sigma(\mathtt{c}) \equiv \sigma(\mathtt{c}')$
It is not difficult to see that $\mathtt{N}_{\mathcal{X}}([\mathtt{c}](\mathtt{p}_1,\ldots,\mathtt{p}_{\rho(\mathtt{c})})\sigma) = \mathtt{N}_{\mathcal{X}}([\mathtt{c}'](\mathtt{p}_1,\ldots,\mathtt{p}_{\rho(\mathtt{c})})\sigma)$. So, $\vdash_{\mathtt{Ax}_d} \mathtt{N}_{\mathcal{X}}(\mathtt{G}_1)$.
Base case 1.2: $\sigma(\mathtt{c}) \equiv_{\cup\phi_2^i} \sigma(\mathtt{c}')$ for some $i \in J$.
So, $\sigma(\mathtt{c})$ is obtained from $\sigma(\mathtt{c}')$ by replacing (a finite amount of times) occurrences of $r(\mathtt{A}\cup\mathtt{B})$ by $r(\mathtt{A}) \oplus^i r(\mathtt{B})$. Since for any $\mathtt{a},\mathtt{a}',\mathtt{a}'' \in \mathtt{M}_d$,

- $\mathtt{a} \oplus^i \mathtt{a}' \equiv_{\mathcal{L}'} \mathtt{a}' \oplus^i \mathtt{a}$;
- $\mathtt{a} \oplus^i \mathtt{a} \equiv_{\mathcal{L}'} \mathtt{a}$;
- $(\mathtt{a} \oplus^i \mathtt{a}') \oplus^i \mathtt{a}'' \equiv_{\mathcal{L}'} \mathtt{a} \oplus^i (\mathtt{a}' \oplus \mathtt{a}'')$;
- $\mathtt{U}_{\mathcal{D}(i)} \oplus^i \mathtt{a} \equiv_{\mathcal{L}'} \mathtt{a}$ [resp. $\mathtt{0}_{\mathcal{D}(i)} \oplus^i \mathtt{a} \equiv_{\mathcal{L}'} \mathtt{a}$];

then $\mathtt{N}_{\mathcal{X}}(\sigma(\mathtt{c})) \equiv_{\mathcal{L}'} \mathtt{N}_{\mathcal{X}}(\sigma(\mathtt{c}'))$. Note that the $\phi_2^i(W)$s are semilattice operations in $\mathcal{L}$-models and $\mathcal{L}'$-models (see the beginning of [DG, Section 4.2]). Since $\mathtt{Ax}_d$ is complete with respect to $\mathcal{L}'$-validity, $\vdash_{\mathtt{Ax}_d} \mathtt{N}_{\mathcal{X}}(\mathtt{G}_1)$.
Induction step: Let $\mathtt{Der} = \langle \mathtt{G}_1,\ldots,\mathtt{G}_{N+1}\rangle$ be an $\mathtt{Ax}$-derivation and $\mathcal{X}$ be a family such that for $i \in J$, $\mathtt{P}_0^i(\{\mathtt{G}_1,\ldots,\mathtt{G}_{N+1}\}) \subseteq \{\mathtt{C}_1^i,\ldots,\mathtt{C}_{n_i}^i\}$.
Case 1: There exist a rule $\mathtt{Ru} = \langle\langle \mathtt{H}_1,\ldots,\mathtt{H}_n\rangle, \lambda\mathtt{x}_1\ldots\mathtt{x}_m.C\rangle \in \mathtt{Ax}_d$ and σ an $\mathcal{L}$-substitution such that $\mathtt{G}_{N+1} = \Phi(\mathtt{H}_n)\sigma$, $\{\Phi(\mathtt{H}_1)\sigma,\ldots,\Phi(\mathtt{H}_{n-1})\sigma\} \subseteq \{\mathtt{G}_1,\ldots,\mathtt{G}_N\}$ and $[\lambda\mathtt{x}_1\ldots\mathtt{x}_m.C']\sigma(\mathtt{x}_1)\ldots\sigma(\mathtt{x}_m)$ holds true. Since $\mathtt{Ru}$ is $\mathcal{L}$-transformable, we indeed apply a rule $\mathtt{Ru}'$ obtained from $\mathtt{Ru}$ as done in Definition 4.3(1). By induction hypothesis, for $k \in \{1,\ldots,N\}$, $\vdash_{\mathtt{Ax}_d} \mathtt{N}_{\mathcal{X}}(\mathtt{G}_k)$. In particular, for $k \in \{1,\ldots,n-1\}$, $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}(\Phi(\mathtt{H}_k)\sigma)$. By (iiib), for $k \in \{1,\ldots,n-1\}$, $\vdash_{\mathtt{Ax}_d} \mathtt{H}_k\mathcal{X}(\sigma)$. $[\lambda\mathtt{x}_1\ldots\mathtt{x}_m.C]\mathcal{X}(\sigma)(\mathtt{x}_1)\ldots\mathcal{X}(\sigma)(\mathtt{x}_m)$ holds true by Definition 4.1(2.2). Moreover, by (iiib) $\mathtt{N}_{\mathcal{X}}(\mathtt{G}_{N+1}) = \mathtt{H}_n\mathcal{X}(\sigma)$. So, $\mathtt{N}_{\mathcal{X}}(\mathtt{G}_{N+1})$ can be derived in $\mathtt{Ax}_d$ by applying the rule $\mathtt{Ru}$ with the substitution $\mathcal{X}(\sigma)$.
Case 2: There exists an $\mathcal{L}$-substitution σ such that $\mathtt{G}_{N+1} = ([\mathtt{c}](\mathtt{p}_1,\ldots,\mathtt{p}_{\rho(\mathtt{c})}) \Leftrightarrow [\mathtt{c}'](\mathtt{p}_1,\ldots,\mathtt{p}_{\rho(c)}))\sigma$ and either $\sigma(\mathtt{c}) \equiv \sigma(\mathtt{c}')$ or $\sigma(\mathtt{c}) \equiv_{\cup\phi_2^i} \sigma(\mathtt{c}')$ for some $i \in J$. The proof is similar to the Base case 1 above and it is omitted here.
Now we are in position to conclude the proof.
(I) $\rightarrow$ (II) Let $\vdash_{\mathtt{Ax}} \mathtt{F}$ and suppose that $\mathtt{F}$ is not $\mathcal{L}$-valid. Let $\mathtt{Der} = \langle \mathtt{F}_1,\ldots,\mathtt{F}_N\rangle$ be an $\mathtt{Ax}$-derivation of $\mathtt{F}$ ($\mathtt{F}_N = \mathtt{F}$) and $\mathcal{X}$ be a family such that for $i \in J$, $\mathtt{P}_0^i(\{\mathtt{F}_1,\ldots,\mathtt{F}_N\}) \subseteq \{\mathtt{C}_1^i,\ldots,\mathtt{C}_{n_i}^i\}$. Without loss of generality, we can assume that $\mathtt{P}_0^i(\{\mathtt{F}\}) = \{\mathtt{C}_1^i,\ldots,\mathtt{C}_{n_i}^i\}$ (if it is not the case, then $\mathtt{F}$ can be easily replaced by an equivalent formula with this property). By (v), $\vdash_{\mathtt{Ax}_d} \mathtt{N}_{\mathcal{X}}(\mathtt{F})$. Since $\vdash_{\mathtt{Ax}_d}$ is sound with respect to $\mathcal{L}'$-validity, $\mathtt{N}_{\mathcal{X}}(\mathtt{F}) = \mathtt{N}(\mathtt{F})$ is $\mathcal{L}'$-valid, which is equivalent to $\mathtt{F}$ $\mathcal{L}$-valid, a contradiction.
(II) $\rightarrow$ (I) Assume $\mathtt{F}$ is $\mathcal{L}$-valid and suppose $\not\vdash_{\mathtt{Ax}} \mathtt{F}$. So there is a family $\mathcal{X}$ such that $\mathtt{N}_{\mathcal{X}}(\mathtt{F}) = \mathtt{N}(\mathtt{F})$. We know also that $\mathtt{F}$ is $\mathcal{L}$-valid iff $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}'$-valid. Since $\vdash_{\mathtt{Ax}_d}$ is complete with respect to $\mathcal{L}'$-validity, $\vdash_{\mathtt{Ax}_d} \mathtt{N}(\mathtt{F})$. Let $\mathtt{Der} = \langle \mathtt{G}_1,\ldots,\mathtt{G}_N\rangle$ be an $\mathtt{Ax}_d$-derivation of $\mathtt{N}(\mathtt{F})$. Without any loss of generality, we can assume that for $i \in J$, $\mathtt{M}_{0d}^i(\{\mathtt{G}_1,\ldots,\mathtt{G}_N\}) \subseteq subset(\mathtt{M}_{0d}^i)$ (if it is not the case there is a simple way to be in this situation). By (iv), $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{N}_{\mathcal{X}}(\mathtt{F}))$. By (i), $\vdash_{\mathtt{Ax}} \mathtt{N}_{\mathcal{X}}^{-1}(\mathtt{N}_{\mathcal{X}}(\mathtt{F})) \Leftrightarrow \mathtt{F}$. Since the principle of replacement of equivalents holds in $\mathtt{Ax}$, $\vdash_{\mathtt{Ax}} \mathtt{F}$, a contradiction. Q.E.D.

4.4 Example

Let RIL be an extension of the Rare-logic defined in [Orło84b] by adding the intersection operator. RIL is defined as the structure $\langle \mathtt{L},\mathcal{D},\mathcal{I},\mathcal{C},X_1,1\rangle$

such that $j = 1$, $\mathcal{D}(1) = 2$, $OP = \{\cap\}$ (interpreted as intersection) and $X_1 = \{\langle W, R\rangle \in Fr^2 : R$ is an equivalence relation$\}$. The logic $RIL_d^\star$ is actually a standard modal logic whose modal operators are either of the form $[\mathtt{U}]$ or of the form $[\mathtt{c}_1 \cap \ldots \cap \mathtt{c}_n]$ where each $\mathtt{c}_i$ is a constant interpreted as an equivalence relation and as usual $\cap$ is interpreted as intersection. A sound and complete axiom system for $RIL_d^\star$ is defined as the set of rules below (see e.g. [HM92, Bal97]):

1. $\langle\langle \mathtt{p}, \mathtt{p} \Rightarrow \mathtt{q}, \mathtt{q}\rangle, true\rangle$ (modus ponens);
2. $\langle\langle \mathtt{p}, [\mathtt{c}]\mathtt{p}\rangle, true\rangle$ (necessitation rule);
3. for any tautology $\mathtt{F}$ of the propositional calculus, $\langle\langle \mathtt{F}\rangle, true\rangle$;
4. $\langle\langle [\mathtt{c}](\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow ([\mathtt{c}]\mathtt{p} \Rightarrow [\mathtt{c}]\mathtt{q})\rangle, true\rangle$;
5. $\langle\langle [\mathtt{c}]\mathtt{p} \Rightarrow \mathtt{p}\rangle, true\rangle$;
6. $\langle\langle \neg[\mathtt{c}]\mathtt{p} \Rightarrow [\mathtt{c}]\neg[\mathtt{c}]\mathtt{p}\rangle, true\rangle$;
7. $\langle\langle [\mathtt{c}']\mathtt{p} \Rightarrow [\mathtt{c}]\mathtt{p}\rangle, \lambda \mathtt{c}\mathtt{c}'.\mathtt{M}_{0d}(\mathtt{c}') \subseteq \mathtt{M}_{0d}(\mathtt{c})\rangle$.

$\mathtt{M}_{0d}$ denotes the set of modal constants of $RIL_d^\star$. Now, let us define an axiom system for RIL using Proposition 4.2. First, for any modal expression $\mathtt{a}, \mathtt{b} \in \mathtt{M}$ (in RIL) $\mathtt{a} \sqsubseteq \mathtt{b} \stackrel{\text{def}}{\Leftrightarrow}$ for any RIL-model $\langle W, PAR, (R_P)_{P \subseteq PAR}, V\rangle$, $V(\mathtt{a}) \subseteq V(\mathtt{b})$. $\sqsubseteq$ can be shown to be decidable by taking advantage of the normalization processes $\mathtt{N}_1$ and $\mathtt{N}_2$. All the rules in the system for $RIL_d^\star$ are $\mathcal{L}$-transformable. Indeed, all the rules except the last one have a condition always true. For the rule 7., one can show that the condition $C' = \mathtt{c} \sqsubseteq \mathtt{c}'$ allows us to show that even this rule is $\mathcal{L}$-transformable. So, by applying Proposition 4.2, a sound and complete axiom system for RIL contains the rules 1.-6. above plus the following ones:

7.′ $\langle\langle [\mathtt{c}']\mathtt{p} \Rightarrow [\mathtt{c}]\mathtt{p}\rangle, \lambda \mathtt{c}\mathtt{c}'.\mathtt{c} \sqsubseteq \mathtt{c}'\rangle$.

8. $\langle\langle [\mathtt{c}']\mathtt{p} \Leftrightarrow [\mathtt{c}]\mathtt{p}\rangle, \lambda \mathtt{c}\mathtt{c}'.$ either $\mathtt{c} \equiv \mathtt{c}'$ or $\mathtt{c}' \equiv_{\cup\cap} \mathtt{c}\rangle$.

The axiom schemes 7.′ and 8. are obtained by Definition 4.3 but in the present case 8. is superfluous.

5 Applications to information logics

In this section we show how the results of the previous sections allow us to state new decidability results (and equivalence between decidability problems) for *information logics*. Actually, this was the original motivation to develop most of the results in the previous sections.

5.1 A logic of indiscernibility relation

The logic of indiscernibility relation LIR has been introduced by Orłowska in [Orło93] (see also [Orło84b]). LIR is the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 1\rangle$ such that $j = 1$, $\mathcal{D}(1) = 2$, $OP = \{\cap, \cup^*\}$ are binary operators interpreted respectively as intersection and the transitive closure of the union and $X_1 = \{\langle W, R\rangle \in Fr^2 : R$ is an equivalence relation $\}$. This logic is an extension of RIL defined in Section 4.4. The logic LIR$_d$, the standard modal logic corresponding to LIR is precisely the logic DAL defined in [FdCO85].

Proposition 5.1. (**Reduction for decidability of LIR**) The LIR-satisfiability problem is decidable iff the DAL-satisfiability problem is decidable.

Proof: Since $\mathrm{LIR}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable (by [DG, Proposition 4.6]), then by [DG, Corollary 4.5(II)], LIR is decidable iff $\mathrm{LIR}_d^{\{\mathtt{U}_2\}-}$ is decidable. By Proposition 3.1, $\mathrm{LIR}_d^{\{\mathtt{U}_2\}-}$ is decidable iff the LIR_d-consequence problem is decidable. Observe that LIR_d is closed under isomorphic copies, disjoint unions and restrictions. By Proposition 3.3, the LIR_d-consequence problem is decidable iff the LIR_d-validity problem is decidable. Hence the LIR-validity problem is decidable iff the LIR_d-validity problem is decidable. Moreover, LIR_d is the logic DAL defined in [FdCO85]. Q.E.D.

Decidability of DAL is open although various attempts[2] to prove such a result can be found in the literature (see e.g. [AT89]). This fact is rather surprising considering that after all, DAL is similar to various other polymodal logics, among them the Propositional Dynamic Logics. It is not difficult to show that if PDL with converse and intersection is decidable (which is commonly conjectured in the literature) then DAL is also decidable.

Corollary 5.2. $\mathrm{LIR}(\mathtt{P}_0^N)$ is decidable iff DAL is decidable and LIR has the fpsp.

5.2 A logic for reasoning about concepts

The logic for reasoning about concepts LRC is not *stricto sensu* a Rare-logic in the sense of [DG, Section 3]. It shares the same language with LIR but the semantics differs. This logic has been introduced in [Orło88a]. An *LRC-model* is a triple $\langle OB, AT, V\rangle$ such that $\langle OB, AT\rangle$ is an information system (in the sense of [DG, Section 1]) and V is a mapping $(\mathtt{For}_0 \cup \mathtt{P} \cup \mathtt{M}) \rightarrow (\mathcal{P}(OB) \cup \mathcal{P}(AT) \cup \mathcal{P}(OB \times OB))$ The only condition that differs with the definition in [DG, Section 3] is the following:

$$V(r(\mathtt{A})) \stackrel{\mathrm{def}}{=} \{\langle x, y\rangle \in OB \times OB : \forall at \in V(\mathtt{A}),\ at(x) = at(y)\} = ind(V(\mathtt{A}))$$

The satisfiability relation $\models$ for LRC is defined in the standard way as well as the notions of LRC-validity, LRC-satisfiability, etc .... The LRC-models are very closely related to the notion of information system. Indeed, numerous so-called information logics that are Rare-logics have been designed to reason about information systems. Roughly speaking, in the LIR-model $\langle OB, AT, V\rangle$, OB plays the rôle of W, AT plays the rôle of PAR using the terminology for the Rare-logics.

Proposition 5.3. (**LRC and LIR are identical**) Let $\mathtt{F}$ be an LRC-formula. The statements below are equivalent:

(I) $\mathtt{F}$ is LRC-satisfiable;

(II) $\mathtt{F}$ is LIR-satisfiable.

Proof:The proof is a consequence of the following propositions:

[2] A preliminary version of this article used the decidability of DAL stated in [AT89]. Unfortunately, the logic named DAL in [AT89] is quite different from the logic DAL considered in this work. As far as we know, decidability of DAL is still an open problem.

(i) for any LRC-model $\langle OB, AT, V\rangle$ there exists an LIR-model

$$\langle OB, PAR, (R_P)_{P\subseteq PAR}, V'\rangle$$

such that V and V' agree on propositional variables, parameter expressions and basic modal expressions.

(ii) for any LIR-model $\langle OB, PAR, (R_P)_{P\subseteq PAR}, V'\rangle$ such that $\langle OB, PAR, (R_P)_{P\subseteq PAR}\rangle$ satisfies the condition C_1 and PAR is finite (cf. [DG, Proposition 4.7]) there exists an LRC-model $\langle OB, AT, V\rangle$ such that V and V' agree on propositional variables and basic modal expressions (not necessarily on parameter expressions).

(i) Consider $PAR \stackrel{\text{def}}{=} AT$ and $V'(r(\mathtt{A})) = R_{V'(\mathtt{A})} \stackrel{\text{def}}{=} V(r(\mathtt{A}))$ for any basic modal expression $r(\mathtt{A})$.
(ii) Take an LIR-model $\langle OB, PAR, (R_P)_{P\subseteq PAR}, V'\rangle$ such that $\langle OB, PAR, (R_P)_{P\subseteq PAR}\rangle$ satisfies C_1 and PAR is finite.

- $AT \stackrel{\text{def}}{=} \{at_x : x \in PAR\}$ where for any $o \in OB$ and any $x \in PAR$, $at_x(o) \stackrel{\text{def}}{=} R_{\{x\}}(o)$
- for any parameter expression $\mathtt{A}$, $V(\mathtt{A}) \stackrel{\text{def}}{=} \{at_x : x \in V'(\mathtt{A})\}$;
- for any basic modal expression $r(\mathtt{A})$, $V(r(\mathtt{A})) \stackrel{\text{def}}{=} \{\langle x, y\rangle \in OB \times OB : \forall at_z \in V(\mathtt{A})\ at_z(x) = at_z(y)\}$.

We show below that $V(r(\mathtt{A})) = V'(r(\mathtt{A}))$.

$V(r(\mathtt{A})) = \{\langle x, y\rangle \in OB \times OB : \forall at_z \in V(\mathtt{A})\ at_z(x) = at_z(y)\}$
(by definition of $V(r(\mathtt{A}))$)
$= \{\langle x, y\rangle \in OB \times OB : \forall at_z \in \{at_{z'} : z' \in V'(\mathtt{A})\}\ at_z(x) = at_z(y)\}$
(by definition of $V(\mathtt{A})$)
$= \{\langle x, y\rangle \in OB \times OB : \forall z \in V'(\mathtt{A})\ R_{\{z\}}(x) = R_{\{z\}}(y)\}$
(by definition of the at_z's)
$= \{\langle x, y\rangle \in OB \times OB : \langle x, y\rangle \in R_{V'(\mathtt{A})}\}$
(by hypothesis $\bigcap_{z\in V'(\mathtt{A})} R_{\{z\}} = R_{V'(\mathtt{A})}$)
$= R_{V'(\mathtt{A})}$
$= V'(r(\mathtt{A}))$

Q.E.D.

Corollary 5.4. (I) LRC-satisfiability is decidable iff DAL-satisfiability is decidable;
(II) LRC has the fpsp.

5.3 Logics for reasoning about similarity

Let $SIM = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 1\rangle$ be the Rare-logic such that $j = 1$, $\mathcal{D}(1) = 2$, $OP = \emptyset$ and X_1 is the set of frames having a reflexive and symmetric relation. The extension $SIM(\mathtt{For}_0^N, \mathtt{P}_0^N)$ with world and parameter nominals has been defined in [Kon97b] except that we do not assume here that the set of parameters is fixed. We write $SIM^{\cap}$ to denote the extension of SIM obtained by adding the binary modal operator $\cap$ interpreted as the set intersection.

Proposition 5.5. $SIM(\mathtt{For}_0^N, \mathtt{P}_0^N)$-satisfiability is decidable.

Proof: By Corollary 2.4, $SIM^{\cap}(\mathtt{For}_0^N, \mathtt{P}_0^N)$ is decidable iff $SIM_d^{\cap\{\mathtt{U}_2\}^-}(\mathtt{For}_0^N)$ is decidable. However, there exists a satisfiability-preserving mapping ST' from $SIM_d^{\cap\{\mathtt{U}_2\}^-}(\mathtt{For}_0^N)$ into the two-variable logic $\mathrm{FO}^2[=]$ (without function symbols and with equality). The map ST' is an extension of the Standard Translation ST from modal logic into classical logic [Ben83] using additional arguments from [GG93]. Since $\mathrm{FO}^2[=]$-satisfiability is decidable [Mor75] and in **NEXPTIME** [GKV97], $SIM(\mathtt{For}_0^N, \mathtt{P}_0^N)$-satisfiability is decidable. Q.E.D.

A similar proof is described in [DK98] where a faithful translation is *directly* defined from $SIM(\mathtt{For}_0^N, \mathtt{P}_0^N)$ into $\mathrm{FO}^2[=]$. Furthermore, in [DK98], it is shown that decidability of the logics defined in [Kon97b] is a consequence of the decidability of $SIM(\mathtt{For}_0^N, \mathtt{P}_0^N)$.

5.4 Another logic of indiscernibility relations

Let RIL be the Rare-logic defined in Section 4.4 that is an extension of a Rare-logic defined in [Orło84b]. RIL is a fragment of LIR and RIL is actually a Rare-logic of type 1 satisfying the hypotheses at the beginning of Section 4.2.

Proposition 5.6. RIL-satisfiability is decidable.

Proof: Since $\mathrm{RIL}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable then by [DG, Corollary 4.5(II)], RIL is decidable iff $\mathrm{RIL}_d^{\{\mathtt{U}_2\}-}$ is decidable. By Proposition 3.1, $\mathrm{RIL}_d^{\{\mathtt{U}_2\}-}$ is decidable iff the RIL_d-consequence problem is decidable. Observe that RIL_d is closed under isomorphic copies, disjoint unions and restrictions. By Proposition 3.3, the RIL_d-consequence problem is decidable iff RIL_d is decidable. Hence, RIL is decidable iff RIL_d is decidable. Moreover, RIL_d is known to be decidable (see e.g. [HM92, Dem99]). Q.E.D.

5.5 A logic with knowledge operators

The Logic with Knowledge Operators LKO has been introduced in [Orło89]. Given a set $\mathtt{For}_0 = \{\mathtt{p}_0, \mathtt{p}_1, \ldots\}$ of propositional variables, the formulae $\mathtt{F}$ are defined as follows:

$$\mathtt{F} ::= \mathtt{p}_i \mid \mathtt{F}_1 \wedge \mathtt{F}_2 \mid \neg\mathtt{F} \mid K(\mathtt{A})\mathtt{F}$$

for $\mathtt{p}_i \in \mathtt{For}_0$ and $\mathtt{A} \in \mathtt{P}$ (set of parameter expressions). Here is an example of formula: $K(\mathtt{A} \cup \mathtt{B})\mathtt{F} \Leftrightarrow K(-(-\mathtt{A} \cup -\mathtt{B}))\mathtt{G}$. An *LKO-model* $\mathcal{M} = \langle W, PAR, (R_P)_{P \subseteq PAR}, V\rangle$ is a structure such that:

- W and PAR are non-empty sets;
- for any $P \subseteq PAR$, R_P is an equivalence relation and $\langle OB, PAR, (R_P)_{P \subseteq PAR}\rangle$ satisfies the condition C_1 (with $j = 1$);
- V is a mapping $\mathtt{For}_0 \cup \mathtt{P} \to \mathcal{P}(W) \cup \mathcal{P}(PAR)$ such that $V(\mathtt{p}) \subseteq W$ for any $\mathtt{p} \in \mathtt{For}_0$ and V restricted to $\mathtt{P}$ is a $\mathtt{P}$-valuation.

The satisfiability relation $\models$ is defined as usual, except for the following condition: $\mathcal{M}, w \models K(\mathtt{A})\mathtt{F} \stackrel{\text{def}}{\Leftrightarrow}$ either for any $w' \in R_{V(\mathtt{A})}(w)$, $\mathcal{M}, w' \models \mathtt{F}$ or for any $w' \in R_{V(\mathtt{A})}(w)$, $\mathcal{M}, w' \models \neg\mathtt{F}$. $K(\mathtt{A})\mathtt{F}$ can be interpreted by: the set $\mathtt{A}$ of agents knows whether $\mathtt{F}$ holds (see e.g. [Hin62, Orło89]). It also corresponds to the modal operators in logics of non-contingency (see e.g. [MR66, Hum95]). The notion of LKO-validity, LKO-satisfiability, ..., are defined in the standard way. Let f be the mapping from the set of LKO-formulae into the set of RIL-formulae:

- $f(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{p}$ for any $\mathtt{p} \in \mathtt{For}_0$;
- f is homomorphic with respect to the Boolean connectives;
- $f(K(\mathtt{A})\mathtt{F}) \stackrel{\text{def}}{=} [r(\mathtt{A})]f(\mathtt{F}) \vee [r(\mathtt{A})]f(\neg\mathtt{F})$.

It is easy to show:

Proposition 5.7. Any LKO-formula $\mathtt{F}$ has an LKO-model of the form $\langle W, PAR, (R_P)_{P \subseteq PAR}, V\rangle$ iff $f(\mathtt{F})$ has an RIL-model of the form $\langle W, PAR, (R'_P)_{P \subseteq PAR}, V'\rangle$.

Corollary 5.8. (**Decidability of LKO**) The LKO-satisfiability problem is decidable and LKO has the finite parameter set property.

Using ideas similar to those developed in Section 4, an adequate axiomatization of LKO has been defined in [Dem99].

5.6 Parameter logics

5.6.1 Definition

The language $\mathtt{L}$ of the *parameter logics* contains a *fixed* set $\mathtt{PAR}$ of parameters. The modal expressions of the language are the subsets of $\mathtt{PAR}$. All the modal connectives are unary and they are indexed by sets of parameters. Such logics have been considered in [Bal96a, Bal96b, Bal97]. Before going any further, we wish to observe that it is rather unusual to build syntactic expressions (formulae for instance) from sets that are mathematical structures. This is certainly one of the peculiarities of the work [Bal97]. In this section, we wish to be on a safer track and we shall not deal with sets but rather with a *finite representation* of certain sets. This representational aspect shall be emphasize when needed but this is really necessary since we wish to establish decidability results. For example, we want to be able to decide whether two sets that occur in a formula are equal, which seems to be a reasonable requirement. This means for instance that we shall be able to work at most with a countably infinite subset of $\mathcal{P}(\mathtt{PAR})$. So, the mode of representation of the subsets of $\mathtt{PAR}$ plays an important rôle to establish decidability results. A *mode of representation* mr can be seen as a partial function $mr : \Sigma^* \to \mathcal{P}(\mathtt{PAR})$ such that Σ is an alphabet (finite set of symbols). So, in the sequel to a set of parameters $\mathtt{PAR}$ we attach a mode of representation and all the logics defined from $\mathtt{PAR}$ can only used subsets P of $\mathtt{PAR}$ that can be represented in formulae, that is there is $\sigma \in \Sigma^*$ such that $mr(\sigma) = P$.

By an $\mathtt{L}$-*frame* we understand a pair $\langle W, (R_P)_{P \subseteq \mathtt{PAR}}\rangle$ such that W is a non-empty set and for any $P \subseteq \mathtt{PAR}$, R_P is a binary relation on W. By an $\mathtt{L}$-model

$\mathcal{M}$, we understand a triple $\langle W, (R_P)_{P \subseteq \mathtt{PAR}}, V \rangle$ such that $\mathcal{F} = \langle W, (R_P)_{P \subseteq \mathtt{PAR}} \rangle$ is an $\mathtt{L}$-frame and V is a mapping $\mathtt{For}_0 \rightarrow \mathcal{P}(W)$. The satisfiability relation is defined in the usual way. In the sequel, by a *parameter logic* $\mathcal{L}_p$ of type $\mathtt{T}$, we understand a quadruple $\langle \mathtt{L}, X, Mod, \mathtt{T} \rangle$ such that

- $\mathtt{L}$ is a language for parameter logics;
- $\emptyset \neq X \subseteq Fr^2$;
- Mod is the set of $\mathtt{L}$-models such that for any $\mathtt{L}$-models $\mathcal{M} = \langle W, (R_P)_{P \subseteq \mathtt{PAR}}, V \rangle$, $\mathcal{M} \in Mod$ iff for any $P \subseteq \mathtt{PAR}$, $\langle W, R_P \rangle \in X$ and $\langle W, \mathtt{PAR}, (R_P)_{P \subseteq \mathtt{PAR}} \rangle$ satisfies the condition $C_{\mathtt{T}}$ (by taking $J = \{1\}$).

The notion of $\mathcal{L}_p$-satisfiability, $\mathcal{L}_p$-validity, logical $\mathcal{L}_p$-consequence etc, ... are defined in the usual way. Let $\mathcal{L}_p$ be a parameter logic $\langle \mathtt{L}, X, Mod, \mathtt{T} \rangle$. The Rare-logic $\mathcal{L}_r$ defined by $\langle \mathtt{L}_r, \mathcal{D}_r, \mathcal{I}_r, \mathcal{C}_r, X_r, \mathtt{T} \rangle$ (see below) is called the *Rare-logic from* $\mathcal{L}_p$:

- $\mathtt{L}_r$ is a language for Rare-logic such that $j \stackrel{\text{def}}{=} 1$, $\mathcal{D}(1) \stackrel{\text{def}}{=} 2$,
  - $OP \stackrel{\text{def}}{=} \{\cap\}$ (interpreted as intersection), $\rho^1_{OP}(\cap) = 2$, $\rho^2_{OP}(\cap) = 1$ if $\mathtt{T} \in \{1, 3\}$;
  - $OP = \{\cup\}$ (interpreted as union) $\rho^1_{OP}(\cup) = 2$, $\rho^2_{OP}(\cup) = 1$ if $\mathtt{T} \in \{2, 4\}$;
  - otherwise $OP = \emptyset$.
- $X_r \stackrel{\text{def}}{=} X$;
- $\mathcal{C}_r$ is the unique set of frames making $\mathcal{L}_r$ a Rare-logic of type $\mathtt{T}$.

In the sequel, we study parameter logics of type 1 or 2. Indeed,

PROPOSITION 5.9. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be similar parameter logics of type 1 [resp. 2] and 3 [resp. 4], respectively. They share the same set of parameters (and the same mode of representation) and we assume that if P can be represented, then so can $\mathtt{PAR} \setminus P$. For any formula $\mathtt{F}$, we write $f_{dual}(\mathtt{F})$ to denote the formula obtained from $\mathtt{F}$ by replacing a parameter set P by $\mathtt{PAR} \setminus P$. Then,

(I) $\mathtt{F}$ is $\mathcal{L}_1$-satisfiable [resp. $\mathcal{L}_2$-satisfiable] iff $f_{dual}(\mathtt{F})$ is $\mathcal{L}_2$-satisfiable [resp. $\mathcal{L}_1$-satisfiable];

(II) $\mathcal{L}_1$ has the fmp iff $\mathcal{L}_2$ has the fmp.

When we are interested in decidability issues, we may add assumptions so that f_{dual} is indeed computable.

5.6.2 Translation

Let $\mathcal{L}_p$ be a parameter logic of type 1 [resp. of type 2] and $\mathcal{L}_r$ be the corresponding Rare-logic. Let $\mathtt{F}$ be an $\mathcal{L}_p$-formula such that the only *subsets* of $\mathtt{PAR}$ occurring in $\mathtt{F}$ are $X_1, \ldots, X_n$. For any integer $k \in \{0, \ldots, 2^n - 1\}$, as usual we write X^*_k to denote the set $X^*_k \stackrel{\text{def}}{=} Y_1 \cap \ldots \cap Y_n$ where for any $i \in \{1, \ldots, n\}$, $Y_i \stackrel{\text{def}}{=} X_i$ if $bit_i(k) = 0$ otherwise $Y_i \stackrel{\text{def}}{=} \mathtt{PAR} \setminus X_i$. Hence X^*_k is a *concrete* set (not a Boolean expression as done in [DG, Section 4.1]). The family $\{X^*_k : k \in \{0, \ldots, 2^n - 1\}, X^*_k \neq \emptyset\}$ is a partition of $\mathtt{PAR}$. For each set

X_i we associate a parameter constant $\mathtt{C}_i$ (from the language of $\mathcal{L}_r$). For any integer $k \in \{0, \ldots, 2^n - 1\}$, we write $\mathtt{A}^*_k$ to denote the Boolean expression

$$\mathtt{A}^*_k \stackrel{\text{def}}{=} \mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$$

where for any $i \in \{1, \ldots, n\}$, $\mathtt{A}_i \stackrel{\text{def}}{=} \mathtt{C}_i$ if $bit_i(k) = 0$ otherwise $\mathtt{A}_i \stackrel{\text{def}}{=} -\mathtt{C}_i$. We write $\mathtt{N}(\mathtt{F})$ to denote the $\mathcal{L}_r$-formula obtained from $\mathtt{F}$ by substituting $X_i \neq \emptyset$ by $\mathtt{N}(X_i) \stackrel{\text{def}}{=} r(\mathtt{N}'(X_i))$

$$\mathtt{N}'(X_i) \stackrel{\text{def}}{=} \bigcup \{\mathtt{A}^*_k : k \in \{0, \ldots, 2^n - 1\}, X^*_k \neq \emptyset, bit_i(k) = 0\}$$

In the case when $X_i = \emptyset$, $\emptyset$ is substituted by $r(\mathtt{C}_1 \cap -\mathtt{C}_1)$.

Proposition 5.10. (**Faithfulness of** $\mathtt{N}$) Let $\mathtt{F}$ be an $\mathcal{L}_p$-formula. The statements below are equivalent:

(I) $\mathtt{F}$ is $\mathcal{L}_p$-satisfiable;

(II) $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}_r$-satisfiable.

Proof: (I) $\rightarrow$ (II) Let $\mathcal{M} = \langle W, (R_P)_{P \subseteq \mathtt{PAR}}, V\rangle$ be an $\mathcal{L}_p$-model and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$. We assume that the only subsets of $\mathtt{PAR}$ occurring in $\mathtt{F}$ are $X_1, \ldots, X_n$. The case when no modal connective occurs in $\mathtt{F}$ is omitted here but its proof can be easily obtained from the proof below for $n \geq 1$. Consider the $\mathcal{L}_r$-model $\mathcal{M}' = \langle W, \mathtt{PAR}, (R_P)_{P \subseteq \mathtt{PAR}}, V'\rangle$ such that

- the restriction of V' to the set of propositional variables is V;
- for any $k \in \{1, \ldots, n\}$, $V'(\mathtt{C}_k) \stackrel{\text{def}}{=} X_k$.

Hence, for any $k \in \{1, \ldots, n\}$, $V'(\mathtt{N}'(X_k)) = X_k$. It is a routine task to check that $\mathcal{M}', w \models \mathtt{N}(\mathtt{F})$ and $\mathcal{M}'$ is an $\mathcal{L}_r$-model.
(II) $\rightarrow$ (I) Let $\mathcal{M}' = \langle W, PAR, (R'_P)_{P \subseteq PAR}, V'\rangle$ be an $\mathcal{L}_r$-model and $w \in W$ such that $\mathcal{M}', w \models \mathtt{N}(\mathtt{F})$. The set PAR *does not have to be equal* to $\mathtt{PAR}$. Consider the $\mathcal{L}_p$-model $\mathcal{M} = \langle W, (R_P)_{P \subseteq \mathtt{PAR}}, V\rangle$ such that

- V is the restriction of V' to the set of propositional variables;
- $R_\emptyset \stackrel{\text{def}}{=} W \times W$ if $\mathcal{L}$ is of type 1 [resp. $R_\emptyset \stackrel{\text{def}}{=} \emptyset$ if $\mathcal{L}$ is of type 2];
- for any $\emptyset \neq P \subseteq \mathtt{PAR}$, we write $\{X^*_{i_1}, \ldots, X^*_{i_l}\}$ to denote the smallest set of non-empty sets (with respect to set inclusion) such that $P \subseteq X^*_{i_1} \cup \ldots \cup X^*_{i_l}$. The set $\{X^*_{i_1}, \ldots, X^*_{i_l}\}$ always exists and it is unique because of the condition of minimality and $\{X^*_k : k \in \{0, \ldots, 2^n - 1\}, X^*_k \neq \emptyset\}$ is a partition of $\mathtt{PAR}$. We define R_P as follows $R_P \stackrel{\text{def}}{=} \bigcap Y$ if $\mathcal{L}$ is of type 1 [resp. $R_P = \bigcup Y$] with $Y \stackrel{\text{def}}{=} \{R'_{V'(\mathtt{A}^*_{i_k})} : k \in \{1, \ldots, l\}\}$.

By an easy manipulation, we can show that for any $P, P' \subseteq \mathtt{PAR}$ $R_{P \cup P'} = R_P \cap R_{P'}$ if $\mathcal{L}$ is of type 1 [resp. if $\mathcal{L}$ is of type 2 $R_{P \cup P'} = R_P \cup R_{P'}$]. It remains to prove that $R_{X_{k'}} = R'_{V'(\mathtt{N}'(X_{k'}))}$ for $k' \in \{1, \ldots, n\}$. First notice that for any $k \in \{0, \ldots, 2^n - 1\}$, if $X^*_k \neq \emptyset$, then $R_{X^*_k} = R'_{V'(\mathtt{A}^*_k)}$. The sequence of equality below is valid if $\mathcal{L}$ is of type 1 but a similar one can be obtained if $\mathcal{L}$ is of type 2 (replace the finite generalized intersections by finite generalized unions).

$$
\begin{aligned}
R_{X_{k'}} &= R_{\bigcup\{X_k^*:k\in\{0,\ldots,2^n-1\},bit_{k'}(k)=0,X_k^*\neq\emptyset\}} \\
&\text{(by definition of the } X_k^*\text{'s)} \\
&= \bigcap\{R_{X_k^*} : k \in \{0,\ldots,2^n-1\}, bit_{k'}(k) = 0, X_k^* \neq \emptyset\} \\
&\text{(since } \mathcal{L}_p \text{ is of type 1)} \\
&= \bigcap\{R'_{V'(\mathtt{A}_k^*)} : k \in \{0,\ldots,2^n-1\}, bit_{k'}(k) = 0, X_k^* \neq \emptyset\} \\
&\text{(by definition)} \\
&= R'_{V'(\cup\{\mathtt{A}_k^*:k\in\{0,\ldots,2^n-1\},bit_{k'}(k)=0,X_k^*\neq\emptyset\})} \\
&(\mathcal{M}' \text{ is an } \mathcal{L}_r\text{-model)} \\
&= R'_{V'(\mathtt{N}'(X_{k'}))}
\end{aligned}
$$

Q.E.D.

5.6.3 Some decidable parameter logics

In this section we establish various decidability results based on the consequences of Section 5.6.2.

Proposition 5.11. (**Decidable fragments of $\mathcal{L}_p$**) Let $\mathcal{L}_p$ be a parameter logic of type $\mathtt{T} \in \{0,\ldots,4\}$. Let Z be a class of $\mathcal{L}_p$-formulae such that for any $\mathtt{F} \in Z$ it is decidable whether (see the notations above)

(I) X_i is empty ($X_1,\ldots,X_n$ are the sets occurring in $\mathtt{F}$);

(II) X_k^* is empty ($0 \leq k \leq 2^n - 1$).

Then, if the corresponding Rare-logic $\mathcal{L}_r$ has a decidable satisfiability problem, then $\mathcal{L}_p$-satisfiability restricted to Z is decidable.

Proof: For any $\mathtt{T} \in \{1,\ldots,4\}$, the satisfaction of (I) and (II) implies that the mapping $\mathtt{N}$ is an *effective* procedure. By Proposition 5.10 we obtain the decidability of the $\mathcal{L}_p$-satisfiability problem restricted to Z.

Now assume $\mathtt{T} = 0$. For any Y, Y' occurring in a formula $\mathtt{F} \in Z$, it is decidable whether $Y = Y'$ since $Y = Y'$ iff $Y \cap (\mathtt{PAR} \setminus Y') = \emptyset$ and $(\mathtt{PAR} \setminus Y) \cap Y' = \emptyset$ (particular instances of (II)). Let $\mathtt{N}$ be the mapping from Z into the set of $\mathcal{L}_r$-formulae that consists in substituting each set Y by a basic modal expression $r(\mathtt{C}_Y)$ - $\mathtt{C}_Y$ is a fixed parameter constant for each Y. By assumption, $\mathtt{N}$ is an effective procedure. One can show that for any $\mathtt{F} \in Z$, $\mathtt{F}$ is $\mathcal{L}_p$-satisfiable iff $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}_r$-satisfiable (one can use arguments similar to the ones in the proof of Proposition 5.10) which terminates the proof. Q.E.D.

The condition (I) in Proposition 5.11 means that for any representation $\sigma \in \Sigma^*$ of the set $P = mr(\sigma)$ of parameters, one can decide whether $P = \emptyset$.

Let $\mathtt{PAR} = \{p_1, p_2, \ldots\}$ be a countable set of parameters (not necessarily finite). A natural representation of the finite subset $\{p_1,\ldots,p_k\}$ of $\mathtt{PAR}$ is $\{^f p_1,\ldots,p_k\}^f$ where '$\{^f$', '$\}^f$' and ',' are symbols of the language. Each cofinite subset $\mathtt{PAR} \setminus \{p_1,\ldots,p_k\}$ can also be represented by $\{^c p_1,\ldots,p_k\}^c$ where '$\{^c$' and '$\}^c$' are symbols of the language. The empty set is represented by $\{^f\}^f$ whereas $\mathtt{PAR}$ is represented by $\{^c\}^c$. Let Z^{fc} be the set of $\mathcal{L}_p$-formulae such that only finite or cofinite sets of parameters occur and the representation above is used. Then Z^{fc} satisfies the hypothesis of Proposition 5.11. Moreover, if the representation of the set Y occurs in a formula, then the representation of the set $\mathtt{PAR} \setminus Y$ can also occur in a formula of Z^{fc} (see also

Proposition 5.9). This class of formulae has been considered for instance in [Bal96a, Bal97]. Proposition 5.11 above can be applied to Z^{fc}.

PROPOSITION 5.12. Let $\mathcal{L}_p = \langle \mathtt{L}, X, Mod, \mathtt{T}\rangle$ be parameter logic such that $\mathtt{T} \in \{0, \ldots, 4\}$. $\mathcal{L}_p$-satisfiability restricted to some Z satisfying (I) and (II) from Proposition 5.11 is decidable in the following cases:

(1) X is the set of all frames or the set of reflexive frames;

(2) X is either the set of serial frames or the set of symmetrical frames or the set of reflexive and symmetrical frames or the set of equivalence relations and $\mathtt{T} \in \{0, 2, 4\}$;

(3) If X is the set of transitive frames or the set of reflexive and transitive frames and $\mathtt{T} \in \{0, 1, 3\}$.

PROOF: By way of example, let us consider in (2) the case $\mathcal{L}_p$ of type 2 and X is the set of equivalence relations ($OP = \{\cup\}$ and X is *not* closed under union). By [DG, Corollary 4.5(II)] and [DG, Proposition 4.6(III)], $\mathcal{L}_r$ is decidable iff $(\mathcal{L}_r)_d$ is decidable (one can get rid of the occurrences of $[\mathtt{0}_2]$ since $[\mathtt{0}_2]\mathtt{G} \Leftrightarrow \top$ is $(\mathcal{L}_r)^*_d$-valid). However $(\mathcal{L}_r)_d$ is precisely a fragment of the Data Analysis Logic with Local Agreement [Gar86] that is known to be decidable [Dem98]. Hence the $\mathcal{L}_p$-satisfiability problem restricted to Z is decidable. We invite the reader to analyse the consequences on Mod of the fact that X is not closed under union. Q.E.D.

Although it is known that PDL with intersection [Dan84] and PDL with converse [Seg82] have both a decidable validity problem, as far as we know, it is an open problem whether PDL with intersection and converse is decidable.

PROPOSITION 5.13. Let $\mathcal{L}_p = \langle \mathtt{L}, X, Mod, \mathtt{T}\rangle$ be a parameter logic of type $\mathtt{T} \in \{1, 3\}$ such that X is the set of symmetric frames [resp. the set of reflexive and symmetric frames, the set of equivalence relations]. If the validity problem for PDL with intersection and converse is decidable, then the $\mathcal{L}_p$-satisfiability problem restricted to some Z satisfying (I) and (II) from Proposition 5.11 is decidable.

6 Concluding remarks

In [DG] we have introduced a class of polymodal logics with relative accessibility relations, the Rare-logics. Particular instances are the information logics from [Orło84a, Orło88a, Orło95, Kon97a, Bal97]. We have shown how to translate Rare-logics into more standard modal logics and the other way around. Various kinds of algebraic properties for the families of relations in the models have been taken into account. The translations are interesting for their own sake, for instance they help understanding what is brought by adding a Boolean dimension to a logic.

In the present paper, we are able to prove new decidability results about some Rare-logics in a unifying framework. The flexibility of the translations allows an extension when nominals are included in the language for atomic propositions and above all for atomic parameters (it is technically more involved). Some refinements to eliminate the universal operator are also presented. Most of our results have a semantical flavour except that we have defined translations of calculi between Rare-logics and corresponding standard modal logics. This is all the more significant because the transformation is general enough to be used for numerous logics and types of calculi. So when Rare-logics can be translated into well-known modal logics, we obtain straightforward results about the Rare-logics (decidability, complexity upper bounds, proof systems, ...). This is not always the case, especially when names are allowed in the language. For instance, the resolution of the following open problems in the realm of (combinatory) dynamic logics will have straightforward consequences for numerous classes of Rare-logics (for instance for the logics defined in [Orło84a]): (un)decidability of CPDL+$\{\cap\}$, CPDL+$\{\cap, ^{-1}\}$ (see e.g. [PT91]).

Acknowledgements: The authors warmly thank the anonymous referee for so many valuable suggestions and corrections about preliminary versions of this work improving very significantly the quality of the paper. In particular, we are thankful to the referee for finding a major flaw in the proof of previous Proposition 3.1.

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