# RESOLUTION FOR WEAK MODAL LOGICS

STEPHANE DEMRI

LIFIA-IMAG, 46, Avenue Félix Viallet,

38031 Grenoble Cedex, France

ABSTRACT

We define modal resolution systems for the propositional weak modal logics C, CD, and CT. These systems are based on the Enjalbert and Fariñas Del Cerro's modal resolution systems with minor modifications. Roughly speaking, the axiom $\Sigma(\Diamond\perp, \mathcal{C}) \rightarrow \perp$ is added if each disjunct of the clause $\mathcal{C}$ is a necessity formula ($\perp$ denotes the empty clause) and the simplification rule ($\Diamond\perp \approx \perp$) is deleted. Non clausal resolution systems are also presented for these weak modal logics. They are variants of Abadi and Manna's non clausal resolution systems. The new resolution systems are proved to be sound and refutationnally complete for these logics. The completeness proofs are inspired by the completeness proofs of the systems for normal modal logics with adequate modifications.

## 1 Introduction

Modal logics have shown to be a suitable tool for the formalization of numerous problems in the fields of Artificial Intelligence and Computer Science. In order to mechanize them, two approaches can be distinguished. The direct approach uses specific proof systems for modal logics. For example, resolution methods have modal versions (e.g. [4,1,3] ). The translation approach roughly consists in translating non classical logics to (first-order, second-order ...) classical logic (e.g. [13,12,8]).
Among the non classical logics, weak* modal logics are of special interest since they do not admit in their axiomatic systems the necessitation rule† -see for example in [14] the so-called *regular* logics. If the operator $\Box$ is interpreted as a knowledge operator then the theoremhood of a formula does not entail that this fact is known. So, in the multiagent version of the weak modal logics C, CD, CT, CS4 and CS5 [14,6] the agents are not fully omniscient. That largely explains why these logics can be of special interest for Artificial Intelligence -see for example in [7,5] comprehensive studies about the knowledge operators. From the viewpoint of Automated Deduction, tableaux systems exists for these logics [6]. However there is a cruel lack since no resolution proof system has been defined for them.
The purpose of this paper is to define modal resolution systems for the propositional weak modal logics C, CD, CT (terminology used in [6]). The elegant resolution systems

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*For instance, the term 'weak' has been used in [11].

†If $f$ is a theorem then $\Box f$ is a theorem.

defined in [4] are modified for these logics. We emphasize that the systems in [4] can be naturally adapted to weak modal logics (it is not the least of their merits) and the results presented in [4] are used as much as possible in this work. Non clausal resolution systems are also presented for these logics. They use the systems defined in [1] for normal logics. In order to prove completeness the consistency properties defined in [6] are used. We therefore advice the reader to refer to [6] if needed. In that sense, this note *is not* self-contained. Along this note we shall very often refer to the works [1,4]. The reader is invited to compare the differences in the definitions and main results. Sometimes, the new definitions and the proofs of the propositions are presented to underline the differences with [1,4]. In that sense, the wish of the authors of [4] is partly fulfilled since even for weak modal logics *they have elaborated general techniques for establishing the completeness of decision procedures for modal logics.*

## 2 Weak Modal Logics

The source of the following definitions are [10,11,14]. We recall that the standard language for the propositional modal logics is the language of the propositional calculus with the additional unary operators $\Box$ and $\Diamond$ (by definition $\Diamond A \equiv \neg\Box\neg A$). Unlike the classical Kripke models [9], the models for the regular modal logics distinguish the *normal* worlds and the *queer* worlds.

**Definition 2.1** *(Augmented Model) An augmented model is a 4-uple $\mathcal{M} = (W, Q, R, m)$ where $W$ is a non-empty set of worlds, $Q$ is subset of $W$ (set of queer worlds), $R$ is a binary relation on $W$ called the accessibility relation and $m$ is a function which assigns to each propositional variable $p$ a subset $m(p)$ of $W$.*

Each world in $(W - Q)$ is said to be normal. Given a model $\mathcal{M}$, a world $w \in W$, and a formula $A$, the expression "$w$ satisfies $A$ in $\mathcal{M}$" ($\mathcal{M}, w \models A$) is defined as follows:
- $\mathcal{M}, w \models p$ iff $w \in m(p)$ for $p$ a propositional variable,
- $\mathcal{M}, w \models \neg A$ iff not $\mathcal{M}, w \models A$,
- $\mathcal{M}, w \models A \wedge B$ iff $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$,
- If $w \notin Q$, then $\mathcal{M}, w \models \Box A$ iff $(\forall w' \in W, (wRw') \rightarrow (\mathcal{M}, w' \models A))$,
- If $w \in Q$, then $\mathcal{M}, w \models \Diamond A$.

A formula $f$ is said to be *satisfiable* if there is an augmented model $\mathcal{M}$ and a world $w$ in $\mathcal{M}$ such that $\mathcal{M}, w \models f$. A formula $f$ is *valid in an augmented model* $\mathcal{M}$ iff for every world $w \in W$, $(\mathcal{M}, w \models f)$. A formula $f$ is *valid* iff it is valid in every augmented model (noted $\models f$). When the operator '$\Box$' is interpreted as an epistemic operator, the existence of queer worlds naturally entails that there are worlds where nothing is known and everything is believed.
The axiomatic system C[‡] is defined as follows. An axiomatic system for the propositional calculus (resp. for the normal logic K [1]) can be the system below without the axiom scheme (4) and the regularity rule RR (resp. with the addition of the necessitation rule).

[‡]Without undesirable consequence for the sequel, it is assumed that a logic L is identified by an axiomatic system.

*Axiom Schemes*
(1) $A \Rightarrow (B \Rightarrow A)$ ; (2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(3) $(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$ ; (4) $\Box A \Rightarrow (\Box(A \Rightarrow B) \Rightarrow \Box B)$
*Inference rules*
MP (Modus Ponens) : $\frac{A \quad A \Rightarrow B}{B}$ ; RR (Regularity Rule): $\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B}$

We write $\vdash_C A$ if $A$ can be deduced by means of these axioms and inference rules. Observe that if $\vdash_C A \Rightarrow B$ then $\vdash_C \Diamond A \Rightarrow \Diamond B$ [10].

**Fact 2.2** [14] *A formula $f$ is valid iff $\vdash_C f$.*

The systems considered in the present paper are the following:
- CD = C $\cup$ $\{\Box A \Rightarrow \Diamond A\}$, CT = C $\cup$ $\{\Box A \Rightarrow A\}$
- CS4 = CT $\cup$ $\{\Box A \Rightarrow \Box(\Box B \Rightarrow \Box A)\}$, CS5 = CS4 $\cup$ $\{\Box \top \Rightarrow (A \Rightarrow \Box\Diamond A)\}$

All these systems can be given a semantic interpretation in terms of augmented models. As it is common for modal logics (e.g. [15]), each previously presented axiom scheme is characterized by a property on the augmented models. Figure 1 recalls some properties for augmented models. A formula $f$ is said to be *P-valid* iff $f$ is valid in any augmented model satisfying the property $P$.

| Notation | Name | Property |
|---|---|---|
| Refl | Reflexivity on normal worlds | $\forall w \in (W-Q), wRw$ |
| Trans | Transitivity on normal worlds | $\forall x, y \in (W-Q), \forall z \in W (xRy) \wedge (yRz) \rightarrow (xRz)$ |
| Sym | Symmetry on normal worlds | $\forall x, y \in (W-Q), (xRy) \rightarrow (yRx)$ |
| Ideal | Idealization on normal worlds | $\forall x \in (W-Q), \exists y \in W, xRy$ |

Figure 1: Properties of Accessibility Relations

**Fact 2.3** [14] *For any modal formula $F$, $F$ is a theorem of CD (resp. CT, CS4, CS5) iff $F$ is Ideal-valid (resp. Refl, Refl+Trans, Refl+Sym+Trans).*

For any logic L in {C, CT, CD, CS4, CS5}, we use the standard definitions of *L-satisfiability*, *L-validity* and *L-model* (e.g. CS4-valid stands for (Refl+Trans)-valid).

# 3 Clausal Resolution Systems

We use the definition of formulas in conjunctive normal form (CNF) and in disjunctive normal form (DNF) defined in [4]. A modal formula is said to be in disjunctive normal form (DNF) if it is a disjunction of the general form

$$L_1 \vee L_2 \vee \ldots \vee L_n \Box D_1 \vee \Box D_2 \vee \Box D_p \vee \Diamond A_1 \vee \Diamond A_2 \vee \ldots \vee \Diamond A_q$$

where the $L_i$'s are literals, the $D_i$'s are in DNF and the $A_i$'s are in CNF. A formula is in CNF if it is a conjunction of formulas in DNF. The symbol ',' is also used as a conjunction operator. A *clause* is a modal formula in DNF.

**Definition 3.1** *(Normal set of modal formulas, Necessity clause) A set of modal formulas is said to be normal iff it contains at least one necessity formula (of the form* $\Box f$ *or* $\neg\Diamond f$*). A necessity clause is a clause, different from the empty clause, such that each disjunct is a necessity formula.*

A necessity clause can only be satisfied in a normal world.

**Proposition 3.2** *There is an effective procedure which, given any modal formula* $F$*, constructs a formula* $F'$ *in CNF such that* $\vdash_C F \Leftrightarrow F'$.

The proof of this proposition is similar with the one for Proposition 1.3 in [4].

### *3.1 Resolution System for C*

Like the system $\mathcal{R}K$ defined in [4], the system $\mathcal{R}C$ is composed of rules for computing resolvents, simplification rules and inference rules (by convention, the clausal resolution system for a logic L is noted $\mathcal{R}L$). The system $\mathcal{R}C$ is the system $\mathcal{R}K$ with the new axiom (A3) but without the simplification rule (S1):($\Diamond \perp \approx \perp$). Figure 2 presents the system $\mathcal{R}C$ (except the inference rules) with the notations used in [4]. Two relations are defined on clauses: $C$ is a direct resolvent of $A$ and $B$ (noted $\Sigma(A,B) \to C$) and $C$ is a direct resolvent of $A$ (noted $\Gamma(A) \to C$). As in [4], the relation "$A$ can be simplified in $B$" is noted $A \approx B$. It is the least congruence of the simplification rules (see Figure 2). For every formula $F$, there is a unique $F'$ such that $F \approx F'$ and $F'$ cannot be simplified further ($F'$ normal form of $F$). We define $\Sigma(A,B) \Rightarrow C$[§] (resp. $\Gamma(A) \Rightarrow C$) if there is some $C'$ such that $\Sigma(A,B) \to C'$ holds (resp. $\Gamma(A) \to C'$) and $C$ is the normal form of $C'$. The inference rules of $\mathcal{R}C$ are ($\frac{C}{D}$ if $\Gamma(C) \Rightarrow D$) and ($\frac{C_1,C_2}{D}$ if $\Sigma(C_1,C_2) \Rightarrow D$). We note $\vdash_{\mathcal{R}C}$ the deduction operator in $\mathcal{R}C$.

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**Simplification rules** (S2) $\perp \vee D \approx D$ ; (S3) $\perp, E \approx \perp$ ; (S4) $(A \vee A \vee D) \approx (A \vee D)$

**Axioms** (A1) $\Sigma(p,\neg p) \to \perp$ ; (A2) $\Sigma(\perp, A) \to \perp$ ; (A3) $\Sigma(\Diamond \perp, \mathcal{C}) \to \perp$ with $\mathcal{C}$ a necessity clause

**$\Sigma$-rules** $\vee - rule : \frac{\Sigma(A,B)\to C}{\Sigma(A\vee D_1, B\vee D_2)\to C\vee D_1\vee D_2}$, $\Box\Diamond - rule : \frac{\Sigma(A,B)\to C}{\Sigma(\Box A, \Diamond(B,E))\to\Diamond(B,C,E)}$

$\Box\Box - rule : \frac{\Sigma(A,B)\to C}{\Sigma(\Box A, \Box B)\to\Box C}$

**$\Gamma$-rules** $\Diamond - rule : \frac{\Sigma(A,B)\to C}{\Gamma(\Diamond(A,B,F))\to\Diamond(A,B,C,F)}$ ; $\Diamond - rule2 : \frac{\Gamma(A)\to B}{\Gamma(\Diamond(A,F))\to\Diamond(B,A,F)}$

$\vee - rule2 : \frac{\Gamma(A)\to B}{\Gamma(A\vee C)\to B\vee C}$ ; $\Box - rule : \frac{\Gamma(A)\to B}{\Gamma(\Box A)\to\Box B}$

$A, B, C, D_1, D_2$ denote clauses, $E$, $F$ denote (possibly empty) sets of clauses and $(A, E)$ denotes the result of appending the clause $A$ to the set $E$. $\perp$ denotes the empty clause.

Figure 2: Resolution rules for $\mathcal{R}C$

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**Proposition 3.3** *(Soundness) (1) If* $\Sigma(A,B) \to C$ *then* $\vdash_C A \wedge B \Rightarrow C$*; (2)If* $\Gamma(A) \to C$ *then* $\vdash_C A \Rightarrow C$ *; (3) If* $S \vdash_{\mathcal{R}C} C$ *then* $\vdash_C S \Rightarrow C$

[§]From the context, '$\Rightarrow$' has to be understood either as a logical connective (implication) or as a derivation operator.

**Proof:** The proof is similar to the proof of Theorem 1 in [4] except that the following points must be established:
(i) If $\vdash_C A \Rightarrow B$ then $\vdash_C \Box A \Rightarrow \Box B$ (see the Regularity Rule) and if $\vdash_C A \Rightarrow B$ then $\vdash_C \Diamond A \Rightarrow \Diamond B$ (proof by an easy verification).
(ii) If $\Sigma(\Diamond \perp, \mathcal{C}) \rightarrow C$ with $\mathcal{C}$ a necessity clause then $\vdash_C \Diamond \perp \wedge \mathcal{C} \Rightarrow \perp$. The formula $\Diamond \perp \wedge \mathcal{C}$ is C-unsatisfiable since $\Diamond \perp$ cannot be satisfied in any normal world and $\mathcal{C}$ cannot be satisfied in any queer world. Therefore $\Diamond \perp \wedge \mathcal{C} \Rightarrow \perp$ is C-valid. Hence, from Fact 2.2, $\vdash_C \Diamond \perp \wedge \mathcal{C} \Rightarrow \perp$. Q.E.D.

**C-trees and Completeness** We use the definition of trees defined in [4]. A tree is noted $(\mathcal{A}, \mathcal{T}, r)$ where $\mathcal{A}$ is the set of nodes, $\mathcal{T}$ is the binary relation on $\mathcal{A}$ and $r$ is the root. In order to prove the completeness of $\mathcal{R}C$ we have to slightly modify the notion of K-trees [4].

**Definition 3.4** *(C-tree) Let $S$ be a set of clauses. A C-tree for $S$ is a tree $u$ whose nodes are sets of clauses such that (1) the root of $u$ is $S$ itself and (2) $u$ is constructed by performing the operations 1 and 2 alternately until Operation 2 is inapplicable.*
Operation 1*: Repeat the following steps as long as possible:*
- *choose a leaf $n$ of $u$ and a clause $C$ in $n$ of the form $C_1 \vee C_2$,*
- *append two children to $n$, $(n - \{C\}) \cup \{C_1\}$ and $(n - \{C\}) \cup \{C_2\}$.*

Operation 2*: for each leave $n$ of $u$*
- *if some propositional variable $p$, both $p$ and $\neg p$ are in $n$, do nothing;*
- *otherwise we can write $n = \{L_1, \ldots, L_m, \Box A_1, \ldots, \Box A_k, \Diamond P_1, \ldots, \Diamond P_q\}$ -the $L_i$'s are literals.* If $n$ is normal[¶] $(k \geq 1)$ then form the sets $n_i = \{P_i, A_1, \ldots, A_k\}$ *for $i = 1, \ldots, q$ and append them as children of $n$. The $n_i$'s are called the C-projections of $n$.*

The construction of a C-tree for a set of clauses always terminates. Moreover, a C-tree is said to be closed iff it satisfies the conditions for a K-tree to be closed [4]. Every node to which operation 1 has been applied is said to be of *type 1.* The others are of *type 2.*

**Lemma 3.5** *If the set of clauses $S$ has a nonclosed C-tree then $S$ has a C-model.*

**Proof:** The proof is strongly inspired by the proof of Lemma 2.7 in [4]. However, a distinction must be operated in order to define the normal worlds and the queer worlds. A detailed proof is presented in order to convince the reader that the changes are very local. Let $u = (\mathcal{A}, \mathcal{T}, r)$ be a nonclosed C-tree for $S$. The trick of the proof is to build an augmented model from $u$. It can be easily seen by induction on the depth of $u$ that there exists a subtree $u'$ of $u$ (in the sense of Definition 2.3 in [4]) such that (1) every node of $u'$ is nonclosed, (2) every node of type 1 has exactly one descendant and (3) if $w$ is of type 2, the children of $w$ in $u'$ are exactly the children of $w$ in $u$.
Let $\rho$ be the smallest equivalence relation containing the couples $(w, w')$ of $u'$ such that $w \mathcal{T} w'$ and $w$ is of type 1. The equivalence class of $w$ for this relation will be noted $|w|$. We define an augmented model $\mathcal{M} = (W, Q, R, m)$ such that $(W, R)$ has a tree

¶See also the notion of normal branch defined in [6] p. 270.

structure:
• $W$ is the set of equivalence classes of $u'$ for $\rho$
• If $w' \in |w|$ and $w'$ contains only literals or possibility formulas then $|w| \in Q$.
• for $|w|$ and $|w'|$ in $W$, $|w|R|w'|$ iff $|w| \neq |w'|$ and there are some $w_1 \in |w|$ and $w_1' \in |w'|$ such that $w_1 \mathcal{T} w_1'$ .
• $|w| \in m(p)$ iff $p \in w_1$ for some $w_1 \in |w|$

It can be shown by induction of the length of $A$ that for every node $w$ of $u'$ and every $A$ of $w$, $(\mathcal{M}, |w| \models A)$ holds. Only the case $A = \Diamond A_1$ is detailed here since the other cases are very similar with the proof of Fact 2.8 in [4] (for the case $A = \Box A_1$ the normality of $|w|$ has to be used).

There is a node $w'$ of type 2 in $|w|$ such that $A \in w'$. Assume $w'$ has a child. $|w|$ is normal and according to Operation 2, there is a child $w''$ of $w'$ such that $A_1 \in w''$. By induction hypothesis, $\mathcal{M}, |w''| \models A_1$ and therefore $\mathcal{M}, |w'| \models \Diamond A_1$. Now assume $w'$ has no child. $|w|$ is a queer world and therefore $\mathcal{M}, |w| \models \Diamond A_1$. As a consequence if $r$ is the root of the tree, we have $\mathcal{M}, |r| \models S$ and $\mathcal{M}$ is a model for $S$. Q.E.D.

**Lemma 3.6** *(Upward Lemma)*
*(1) If* $A_1, \ldots, A_n \vdash_{\mathcal{RC}} B$ *then* $\Box A_1, \ldots, \Box A_n \vdash_{\mathcal{RC}} \Box B$
*(2) If* $A_1, \ldots, A_n, Q_1, \ldots, Q_r \vdash_{\mathcal{RC}} B$ $(r \geq 1,\ n \geq 1)$ *and the proof uses at least one of the* $Q_i$*'s then if* $B \neq \perp$ *then* $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \Diamond(B, Q_1, \ldots, Q_r, E)$ *for some set* $E$ *of clauses, otherwise* $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \perp$

**Proof:** The proof of (1) is identical with the proof of (i) of Lemma 2.9 in [4]. The first case of the proof of (2) - $B \neq \perp$ - is identical with the first case of the proof of (ii) of Lemma 2.9. If $B = \perp$, using the argument of (ii) we obtain that there exists a set $E$ of clauses such that $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \Diamond(\perp \wedge E)$. By using (A2), we get $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \Diamond \perp$. By using (A3) we get $\Box A_1, \Diamond \perp \vdash_{\mathcal{RC}} \perp$. Q.E.D.

**Corollary 3.7** *(Refutation)*
*(1) If* $A_1, \ldots, A_n, Q_1, \ldots, Q_r \vdash_{\mathcal{RC}} \perp$ $(n \geq 1)$ *then* $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \perp$.
*(2) Let* $S$ *be a set of clauses. If* $S$ *has a refutable C-projection then* $S$ *is refutable.*

**Proof:** (1) Two cases have to be distinguished.
If $A_1, \ldots, A_n \vdash_{\mathcal{RC}} \perp$ then $\Box A_1, \ldots, \Box A_n \vdash_{\mathcal{RC}} \Box \perp$. By using the axiom (A2) and the $\Box\Diamond$-rule, $\Sigma(\Box \perp, \Diamond(Q_1, \ldots, Q_r)) \rightarrow \Diamond \perp$ holds. $\Sigma(\Box \perp, \Diamond \perp) \rightarrow \perp$ holds by using the axiom (A3). Hence $\Box A_1, \ldots, \Box A_n, \Diamond(Q_1, \ldots, Q_r) \vdash_{\mathcal{RC}} \perp$. Otherwise, if the proof of $\perp$ depends upon $Q_1, \ldots, Q_r$ then we apply Lemma 3.6(2).
(2) is an immediate consequence of (1). Q.E.D.

In Lemma 3.6 and Corollary 3.7 it is required that $n \geq 1$ whereas in the original results for $\mathcal{RK}$ in [4] $n$ can be equal to zero (see Operation 2 in Definition 3.4).

**Lemma 3.8** *In a C-tree, every closed node is* $\mathcal{RC}$*-refutable.*

The proof is identical to the proof of Lemma 2.11 in [4].

**Proposition 3.9** *Any C-unsatisfiable set of clauses is $\mathcal{R}C$-refutable.*

**Proof:** Let $S$ be an C-unsatisfiable set of clauses. By Lemma 3.5 its C-tree must be closed. By Lemma 3.8, its root $S$ (closed) is $\mathcal{R}C$-refutable. Q.E.D.

### *3.2 Resolution for CD, CT*

Figure 3 presents the additional rules[‖] for the systems $\mathcal{R}CD = \mathcal{R}C \cup \{S_Q\}$, $\mathcal{R}T = \mathcal{R}CD \cup \{T-rule\}$ and $\mathcal{R}CS4 = (\mathcal{R}CT \cup \{S4\Box\Box-rule, S4\Box\Diamond-rule\})$. The rules $S_Q$,

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$S_Q : (\Box \perp) \approx \perp$ ; $T - rule : \frac{\Sigma(A,B) \to C}{\Sigma(\Box A,B) \to C}$ ;

S4$\Box\Diamond$-rule: $\frac{\Sigma(\Box A,B) \to C}{\Sigma(\Box A, \Diamond(B,E)) \to \Diamond(B,C,E)}$ $B, E$ contains a necessity clause

S4$\Box\Box$ $- rule$: $\frac{\Sigma(\Box A,B) \to C}{\Sigma(\Box A,\Box B) \to \Box C}$ $B$ is a necessity clause

Figure 3: Additional resolution rules for $\mathcal{R}CD$, $\mathcal{R}CT$, $\mathcal{R}CS4$

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T-rule, S4$\Box\Diamond$-rule and S4$\Box\Box$rule have been defined in [4] but without the conditions on the necessity clauses.

**Proposition 3.10** *Let $L$ be a logic in $\{CD, CT\}$.*
*(1) If $\Sigma(A, B) \to C$ then $\vdash_L A \wedge B \Rightarrow C$ ; (2) If $\Gamma(A) \to C$ then $\vdash_L A \Rightarrow C$*
*($\Sigma$ and $\Gamma$ are defined according to the rules for each logic $L$)*
*(3) If $S \vdash_{\mathcal{R}L} C$ then $\vdash_L S \Rightarrow C$*
*(4) Any L-unsatisfiable set of clauses is $\mathcal{R}L$-refutable.*

To prove this proposition, it is enough to combine the modifications of the previous section for the logic C with the modifications for Q, T in [4]. A preliminary version of this work stated the completeness and soundness of $\mathcal{R}CS4$. Until now, only the soundness has been proved. We conjecture the completeness of the system $\mathcal{R}CS4$ for CS4.
A resolution system for CS5 cannot be defined from a system for CT (as it has been done for RS5 with RT in [4]) since every CS5-formula is not equivalent to a formula without nesting of modal operators (for example $\not\vdash_{CS5} \Box\Box A \Leftrightarrow \Box A$).

## 4 Non Clausal Resolution Systems

In [1], non clausal systems have been defined for first-order modal logics. One claim for non clausal resolution is that *the formulas do not need to be rephrased in unnatural and sometimes long clausal forms* [1]. In this section, we outline the definition of non clausal resolution systems for propositional logics C, CT, CD. We assume the reader familiar with [1] (for the non clausal resolution system for K) and with [6] (for the definition of *consistency property*). Herein, for brevity, the non clausal resolution system is detailed

[‖]The new systems have the same inference rules than $\mathcal{R}C$ but they have different rules for computing resolvents and for simplification.

only for C. We note $\mathcal{NR}K$ the nonclausal resolution system defined in [1] for propositional logic K. We note $\mathcal{NR}C = (\mathcal{NR}K \setminus \{(\Diamond false \Rightarrow false)\} \cup \{(\mathcal{C}, \Diamond false \Rightarrow false)\}$ with $\mathcal{C}$ a necessity formula. Figure 4 presents the system $\mathcal{NR}C$. We recall that the

**Simplification rules**

$false \vee u \Rightarrow u$ ; $false, u \Rightarrow false$ ; $\mathcal{C}, \Diamond false \Rightarrow false$ with $\mathcal{C}$ a necessity formula (true-false simplification rules)

$\neg\Box u \Rightarrow \Diamond\neg u$ ; $\neg\Diamond u \Rightarrow \neg u$ ; $\neg(u \wedge v) \Rightarrow (\neg u \vee \neg v)$ ; $\neg(u \vee v) \Rightarrow (\neg u \wedge \neg v)$ ; $\neg\neg u \Rightarrow u$ (Negation rules)

$u, v \Rightarrow u$ (Weakening rule); $u, v_1 \vee \ldots \vee v_k \Rightarrow (u \wedge v_1) \vee \ldots \vee (u \wedge v_k)$ (distribution rule)

**Inference rules**

$A < u, \ldots, u >, B < u, \ldots, u > \rightarrow A < true > \vee B < false >$ (nonclausal resolution rule). The occurrences of u in A or B that are replaced by true or false, respectively, are not in the scope of any $\Box$ or $\Diamond$ in A or B.

$\Box u, \Diamond v \rightarrow \Diamond(u \wedge v)$

Figure 4: Resolution rules for $\mathcal{NR}C$

simplification rules replace formulas (use of the derivation symbol '$\Rightarrow$') whereas the inference rules add formulas (use of the derivation symbol '$\rightarrow$').

**Proposition 4.1** *$\mathcal{NR}C$ is sound for the weak modal logic C.*

In the system $\mathcal{NR}K$ only the simplification rule $(\Diamond false \Rightarrow false)$ is not sound for the weak modal logic C. Proof of Proposition 3.3 is sufficient to prove that $(\mathcal{C}, \Diamond false \Rightarrow false)$ with $\mathcal{C}$ a necessity formula, is sound for C.

**Proposition 4.2** *$\mathcal{NR}C$ is complete for the weak modal logic C.*

**Proof:** Only differences with the proof of the completeness of $\mathcal{NR}K$ [1] are presented here. *Admissible* sets of sentences are defined as in [1] except that the system $\mathcal{NR}C$ is considered. More precisely, a set $S$ of sentences is *admissible* (for C) if no finite conjunction of members of $S$ can be refuted in $\mathcal{NR}C$. In order to show that admissibility is a *consistency property* (see [6] p. 282) if $S$ is admissible then we have to check that if $\Diamond A \in S$ *and* $S$ *is normal* then $S^\sharp \cup \{A\}$ is admissible. We recall that for the logic C, $S^\sharp = \{B \mid \Box B \in S\} \cup \{\neg B \mid \neg\Diamond B \in S\}$.
Assume $S^\sharp \cup \{A\}$ is not admissible. From the definition of *admissibility*, there is a finite sequence $\mathcal{S}$ of distinct elements of $S^\sharp \cup \{A\}$, namely, $B_1, \ldots B_k$, such that *false* can be derived from $\mathcal{S}$. If $A \notin \{B_1, \ldots B_k\}$ then after $k$ applications of the rule $(\Box u, \Diamond v \rightarrow \Diamond(u \wedge v))$, the modal formula $\Diamond(B_1 \wedge \ldots \wedge B_k \wedge A)$ can be derived from $\Box B_1, \ldots, \Box B_k, \Diamond A$. From the hypothesis and with applications of the simplification rule $(u, false \Rightarrow false)$, $\Diamond false$ can be derived from $\Diamond(B_1 \wedge \ldots \wedge B_k \wedge A)$. The new simplification rule in $\mathcal{NR}C$ allows us to derive $false$ ($S$ is normal). So there is a finite sequence of distinct elements of $S$ that can be refuted. Hence, $S$ is not admissible. If $A \in \{B_1, \ldots B_k\}$ then a reasoning similar to the previous one can be used. Admissibility is a consistency property -other (non presented here) conditions also have to be checked- which entails that $\mathcal{NR}C$ is complete (see the full argument of this entailment in [1,6]). Q.E.D.

We define the nonclausal resolution system for CD (resp. CT) from the nonclausal resolution system for D (resp. T) [1] by deleting the simplification rule ($\Diamond false \Rightarrow false$) and by adding the simplification rule ($\mathcal{C}, \Diamond false \Rightarrow false$) with $\mathcal{C}$ a necessity formula. Soundness and completeness can be proved for these systems as it has been done for $\mathcal{NRC}$.

## 5 Conclusion

Clausal resolution systems for propositional weak modal logics have been defined from those defined in [4] for propositional normal modal logics. Minor but relevant modifications are provided to the original systems. Soundness and completeness have been proved. The techniques used in [4] have been used quite often, which confirms, once more, that they are general enough to handle a large class of propositional modal logics. The new proof systems can be easily implemented from an implementation of the systems for normal logics. Non clausal resolution systems have also been presented for these logics. They are variants of the systems defined in [1]. Completeness and soundness have also been proved. It should be noted that the clausal systems have more structural constraints on the applications of rules. However these constraints become an obvious asset from the viewpoint of Automated Deduction. We also conjecture that this work can be extended to the propositional logics CS4 and CS5 (see the system $\mathcal{R}CS4$ in Section 3.2).
Though only propositional logics have been considered in this work, we believe that the works in [12,8,3,1] can be extended to first-order weak modal logics. Furthermore, numerous complete strategies defined in [2] could be reasonably adapted to the modal resolution systems for weak modal logics. In other respects, every set of modal clauses containing clauses such that at least one disjunct is a possibility formula, is satisfiable in any queer world of any augmented model. Set of support strategies could therefore be adequate strategies for resolution systems for weak modal logics. However the completeness of the set of support strategy[**] remains an open question even for the resolution systems defined in [4]. These possibilities are presently under investigation.

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## References

1. M. Abadi and Z. Manna. Modal theorem proving. In J. H. Siekmann, editor, *CADE-8*, pages 172–189. Springer Verlag, LNCS 230, July 1986.

2. Y. Auffray, P. Enjalbert, and J-J. Herbrard. Strategies for modal resolution: results and problems. *Journal of Automated Reasoning*, **6**:1–38, 1990.

[**] The proof for the completeness of the set of support strategy for the first-order classical logic presented in [16] cannot be straightforwardly adapted to the modal resolution systems. Similarly, though the subsumption strategy defined in [2] seems intuitively complete, the techniques used to prove the completeness of the subsumption for classical resolution cannot be used for the modal case. As noticed in [2], *the development of more powerful logical techniques* seems necessary.

3. M. Cialdea. Resolution for some first-order modal systems. *Theoretical Computer Science*, **85**:213–229, 1991.

4. P. Enjalbert and L. Fariñas del Cerro. Modal resolution in clausal form. *Theoretical Computer Science*, **65**:1–33, 1989.

5. R. Fagin, J. Halpern, and M. Vardi. A model-theoretic analysis of knowledge. *Journal of the Association for Computing Machinery*, **38**(2):382–428, April 1991.

6. M. C. Fitting. *Proof methods for modal and intuitionistic logics*. D. Reidel Publishing Co., 1983.

7. J. Y. Halpern and Y. Moses. A guide to the modal logics of knowledge and belief: preliminary draft. In *IJCAI-9*, pages 480–490, 1985.

8. A. Herzig. *Raisonnement automatique en logique modale et algorithmes d'unification*. PhD thesis, Université P. Sabatier, Toulouse, 1989.

9. S. Kripke. Semantical considerations on modal logics. *Modal and Many-valued logics, Acta Philosophica Fennica*, 1963.

10. S. Kripke. Semantical analysis of modal logic II: non-normal modal propositional calculi. In *Symposium on the Theory of Models, Amsterdam*, pages 206–220. North-Holland Publ. Co., 1965.

11. E. J. Lemmon. Algebraic semantics for modal logics I and II. *Journal of Symbolic Logic*, **31**(1-2):46–65, 191–208, June 1966.

12. H. Ohlbach. Optimized translation of multi modal logic into predicate logic. In A. Voronkov, editor, *LPAR'93*, pages 253–264. Springer-Verlag, LNAI 698, 1993.

13. E. Orlowska. Resolution systems and their applications I. *Fundamenta Informaticae*, **3**:253–268, 1979.

14. K. Segerberg. An essay in classical modal logic (three vols.). Technical Report Filofiska Studier nr 13, Uppsala Universitet, 1971.

15. J. van Benthem. Correspondence Theory. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, pages 167–247. D. Reidel Publishing Company, 1984.

16. L. Wos, G. Robinson, and D. Carson. Efficiency and completeness of the set of support strategy in theorem proving. *Journal of the Association for Computing Machinery*, **12**(4):536–541, 1965.