# Prefixed Tableaux Systems for Modal Logics with Enriched Languages*

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## Abstract

We present sound and complete prefixed tableaux systems for various modal logics with enriched languages including the "difference" modal operator $[\neq]$ and the "only if" modal operator $[-R]$. These logics are of special interest in Artificial Intelligence since their expressive power is higher than the standard modal logics and for most of them the satisfiability problem remains decidable. We also include in the paper decision procedures based on these systems. In the conclusion, we relate our work with similar ones from the literature and we propose extensions to other logics.

## 1 Introduction

The definition of logical formalisms that model cognitive and reasoning processes has been always confronted to two issues: how to decrease the expressive power of existing untractable logics in order to obtain tractable fragments and how to increase the expressive power of decidable logics while preserving decidability - this includes for instance the extension of known decidable fragments of the classical logic. These fragments include various modal logics (see e.g. [Hughes and Cresswell, 1984]) if one translates them in the standard way to classical logic. The modal logics have been recognized in the Artificial Intelligence community as serious candidates to capture different aspects of reasoning about knowledge (see e.g. [Fagin *et al.*, 1995]). However the standard modal logics have a restricted expressive power (for instance the class of irreflexive frames is not definable by a modal formula of the logic K).

That is why in the literature various modal logics with enriched languages have been defined. Most of the work done for these logics has been dedicated to study their expressive power (see e.g. [Goranko and Passy, 1992; Rijke, 1993]). In the paper our aim is to analyze various features related to the mechanization of numerous modal logics with enriched languages. To do so, we define prefixed tableaux which are known to be close to the semantics of the logics and they allow a user-friendly presentation of the proofs. Moreover, the use of prefixes (see e.g. [Fitting, 1983; Wallen, 1990; Massacci, 1994; Governatori, 1995]) is known to take advantage of the computational features of the logics. Namely, each prefix occurring at some stage of the proof contains some information about part of the current proof. However we ignore whether a matrix characterization of the logics treated herein exist in order to avoid some redundancies in the tableaux proof search -*notational redundancy*, *irrelevance* and *non-permutability* [Wallen, 1990].

The logics treated in the paper contain various operators that differ from the standard necessity operator $\Box$ (also noted $[R]$):

- the difference operator $[\neq]$ that allows to access to the worlds different from the current world (see e.g. applications of its use in [Segerberg, 1981; Sain, 1988; Koymans, 1992; Rijke, 1993])
- the complement operator $[-R]$ that allows to access to the worlds not accessible from the current world (see e.g. [Humberstone, 1983; Goranko, 1990a; Levesque, 1990; Lakemeyer, 1993])
- and by a side-effect the universal operator $[U]$ that allows to access to any world of the model (see e.g. [Goranko and Passy, 1992]). $[U]\mathtt{A}$ can be defined in various ways: for instance $[U]\mathtt{A} =_{def} \mathtt{A} \wedge [\neq]\mathtt{A}$ or $[U]\mathtt{A} =_{def} [R]\mathtt{A} \wedge [-R]\mathtt{A}$.

Adding these operators to standard modal logics can significantly increase their expressive power. For instance every finite cardinality is definable in a modal logic whose language contains $[\neq]$ [Koymans, 1992]. Most of the logics dealt with in the paper have a decidable satisfiability problem and we shall provide decision procedures based on our systems. However because of the expressive power of the logics our calculi have two original features: a current information $\mathcal{C}$ is associated to each branch of a tableau and a restricted cut rule is included in various calculi that can be viewed as a modal variant of the cut rule in the d'Agostino's calculi [d'Agostino, 1993].

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The rest of the paper is structured as follows. Section 2 presents the logics considered in the paper. The sections 3, 4, 5 and 6 present the calculi for the various logics as well as the decision procedures. Because of lack of space we have omitted part of the proofs as well as the possible extensions where the accessibility relations satisfy standard conditions (reflexivity, symmetry, transitivity, ...). Section 7 compares our calculi with existing ones for other modal logics and concludes the paper by presenting possible extensions.

## 2 Enriched multi-modal logics

### 2.1 Syntax and semantics

A modal language $\mathtt{L}$ is determined by three sets that are supposed to be pairwise disjoint: a set $\mathtt{For}_0 = \{\mathtt{p}, \mathtt{q}, \ldots\}$ of *propositional variables*, a set $\{\neg, \wedge\}$ of propositional operators (the connectives $\vee, \Rightarrow, \Leftrightarrow$ are defined as for the propositional calculus) and a (possibly finite) countable set $OP = \{[i] : i \in I\}$ of *modal operators*. The set of formulae $\mathtt{For}$ of the language $\mathtt{L}$ is defined by the following grammar: $\mathtt{A} ::= \mathtt{p} \mid \neg\mathtt{A} \mid \mathtt{A} \wedge \mathtt{B} \mid \oplus\mathtt{A}$ where $\mathtt{p} \in \mathtt{For}_0$, $\mathtt{A}, \mathtt{B} \in \mathtt{For}$ and $\oplus \in OP$. In the sequel we assume that $OP$ is finite and as usual $\langle i \rangle \mathtt{A} =_{def} \neg[i]\neg\mathtt{A}$. A *frame* is a structure $(W, (R_i)_{i \in I})$ where $W$ is a non-empty set of *worlds* (sometimes also called *knowledge states*) and $(R_i)_{i \in I}$ is a family of binary relations on $W$. A *model* $\mathcal{M}$ is a structure $(W, (R_i)_{i \in I}, V)$ where $(W, (R_i)_{i \in I})$ is a frame and $V$ is mapping $\mathtt{For}_0 \rightarrow \mathcal{P}(W)$, the power set of $W$. For each set $W$, we write $id_W$ (resp. $dif_W$) to denote the binary relation $\{\langle w, w \rangle : w \in W\}$ (resp. $W \times W \setminus id_W$). Let $\mathcal{M} = (W, (R_i)_{i \in I}, V)$ be a model. As usual, we say that a formula $\mathtt{A}$ is *satisfied* by the world $w \in W$ (denoted by $\mathcal{M}, w \models \mathtt{A}$) when the following conditions are satisfied:

- $\mathcal{M}, w \models \mathtt{p}$ iff $w \in V(\mathtt{p})$ for all $\mathtt{p} \in \mathtt{For}_0$,
- $\mathcal{M}, w \models \neg\mathtt{A}$ iff not $\mathcal{M}, w \models \mathtt{A}$,
- $\mathcal{M}, w \models \mathtt{A} \wedge \mathtt{B}$ iff $\mathcal{M}, w \models \mathtt{A}$ and $\mathcal{M}, w \models \mathtt{B}$,
- $\mathcal{M}, w \models [i]\mathtt{A}$ iff for all $w' \in W$ such that $(w, w') \in R_i$, we have $\mathcal{M}, w' \models \mathtt{A}$.

In the sequel by a *logic* $\mathcal{L}$ we understand a pair $\langle \mathtt{For}, \mathcal{S} \rangle$ such that $\mathtt{For}$ is a set of formulae from a given language and $\mathcal{S}$ is a set of models. A formula $\mathtt{A}$ is said to be $\mathcal{L}$*-valid* iff for all models $\mathcal{M} \in \mathcal{S}$ and all $w \in W$, $\mathcal{M}, w \models \mathtt{A}$. A formula $\mathtt{A}$ is said to be $\mathcal{L}$*-satisfiable* iff $\neg\mathtt{A}$ is not $\mathcal{L}$-valid.

### 2.2 Logics in the paper

In the paper we shall consider numerous logics that admit interactions between the modal operators:

1. $K_I = \langle \mathtt{For}, \mathcal{S} \rangle$ is the logic such that $\mathcal{S}$ is the set of all the models. The $K_I$-satisfiability problem is **PSPACE**-complete (see e.g. [Fagin *et al.*, 1995]).
2. $\mathcal{L}([R], [-R]) = \langle \mathtt{For}, \mathcal{S} \rangle$ (see e.g. [Goranko, 1990a]) is the logic such that $I = \{1, 2\}$ and $\mathcal{M} = (W, R_1, R_2, V) \in \mathcal{S}$ iff $R_1 = W \times W \setminus R_2$. The satisfiability problem is decidable and **EXPTIME**-hard [Spaan, 1993]. Similar modal logics are considered in the context of knowledge representation and reasoning (see e.g. [Lakemeyer, 1993]).
3. $\mathcal{L}([\neq]) = \langle \mathtt{For}, \mathcal{S} \rangle$ (see e.g. [Segerberg, 1981]) is the logic such that $I = \{1\}$ and $\mathcal{M} = (W, R_1, V) \in \mathcal{S}$ iff $R_1 = dif_W$. The $\mathcal{L}([\neq])$-satisfiability problem is **NP**-complete when $\mathtt{For}_0$ is infinite and in **P** otherwise (see e.g. [Spaan, 1993; Demri, 1996]).
4. $K_I([\neq]) = \langle \mathtt{For}, \mathcal{S} \rangle$ is the logic such that $1 \in I$ (a distinguished element of $I$), $card(I) \geq 2$ and $\mathcal{M} = (W, (R_i)_{i \in I}, V) \in \mathcal{S}$ iff $R_1 = dif_W$. Axiomatization of $K_I([\neq])$ has been studied in [Rijke, 1993; Balbiani, 1997]. For $I = \{1, 2\}$, the $K_I([\neq])$-satisfiability problem is decidable and **EXPTIME**-complete [Rijke, 1993].

The models for $\mathcal{L}([R], [-R])$ satisfy ($\star$) $R_1 = W \times W \setminus R_2$. If we require ($\star\star$) $R_1 = dif_W$ then $[2]\mathtt{A} \Leftrightarrow \mathtt{A}$ is valid in this new logic. $\mathcal{L}([\neq])$ can be seen as $\mathcal{L}([R], [-R])$ except that the models satisfy ($\star$) and ($\star\star$) and only $[1]$ is in the language. Moreover, $K_I([\neq])$ is obtained from $\mathcal{L}([\neq])$ by adding the operators $\{[i] : i \in I \setminus \{1\}\}$ that behave as in $K_I$. The notion of complementary relations is therefore crucial in the semantics of the logics.

It is not the purpose of this section to recall all the features of the expressive power of the abovementioned logics (see e.g. [Goranko, 1990a; Koymans, 1992; Rijke, 1993]). By way of example we consider the logic $K_I([\neq])$ with $I = \{1, 2\}$. As usual, a class $\mathcal{F}$ of frames $(W, R_1, R_2)$ is said to be $K_I([\neq])$*-definable* iff there exists a $K_I([\neq])$-formula $\mathtt{A}$ such that for all frames $(W, R_1, R_2)$, $(W, R_1, R_2) \in \mathcal{F}$ iff $(W, R_1, R_2) \models \mathtt{A}$ (i.e. for all valuations $V$ and all $w \in W$, $(W, R_1, R_2, V), w \models \mathtt{A}$). A similar notion of definability can be naturally defined for other logics.

**Fact 2.1**. [Goranko, 1990b; Koymans, 1992]

- All universal first-order conditions on $R, =$ are $K_I([\neq])$-definable.
- Every finite cardinality is $\mathcal{L}([\neq])$-definable.
- Each universal first-order formula on $R$ is $\mathcal{L}([R], [-R])$-definable.

The statements of Fact 2.1 do not hold for the logic $K_I$: for example the class of irreflexive frames is not $K_I$-definable.

## 3 Tableaux for $K_I$

The calculus defined for $K_I$ in this section can be easily obtained from existing ones in the literature (see e.g.

$$\frac{\sigma : \alpha\ [\mathcal{C}]}{\begin{array}{c}\sigma : \alpha_1\ [\mathcal{C}] \\ \sigma : \alpha_2\ [\mathcal{C}]\end{array}}\ \alpha\text{−rule}$$

$$\frac{\sigma : \beta\ [\mathcal{C}]}{\sigma : \beta_1\ [\mathcal{C}] \mid \sigma : \beta_2\ [\mathcal{C}]}\ \beta\text{−rule}$$

$$\frac{\sigma : \pi^i\ [\mathcal{C}]}{\sigma k^i : \pi^i_0\ [\mathcal{C}]}\ \pi^i\text{−rule},\ \text{ new } k \in \omega \text{ on the branch}$$

$$\frac{\sigma : \nu^i\ [\mathcal{C}]}{\sigma' : \nu^i_0\ [\mathcal{C}]}\ \nu^i\text{−rule}$$

if $\sigma'$ is already on the branch and for some $k \in \omega$, $\sigma' = \sigma k^i$.

Figure 1: Tableaux system for $K_I$

[Fitting, 1983]) but it will be the opportunity to introduce various definitions smoothly.

We shall define prefixed tableaux following the methodology described in [Fitting, 1983]. We make substantial use of the uniform notation for modal formulae defined in [Fitting, 1983]. Four types of formulae are usually distinguished: $\nu$ (necessity), $\pi$ (possibility), $\alpha$ (conjunction) and $\beta$ (disjunction). For $i \in I$, we introduce the types $\nu^i$ and $\pi^i$. For instance, $\neg\langle i\rangle\mathtt{A}$ and $[i]\mathtt{A}$ are of type $\nu^i$ ($\nu^i_0$ denotes the formulae $\neg\mathtt{A}$ and $\mathtt{A}$ respectively) and $\neg[i]\mathtt{A}$ and $\langle i\rangle\mathtt{A}$ are of type $\pi^i$ ($\pi^i_0$ denotes the formulae $\neg\mathtt{A}$ and $\mathtt{A}$ respectively).

A prefixed formula is a triple of the form $\sigma : \mathtt{A}\ [\mathcal{C}]$ where $\sigma$ is a *prefix*, i.e. $\sigma$ is a finite sequence of natural numbers possibly superscripted by some $i \in I$, $\mathtt{A}$ is a formula and $\mathcal{C}$ is a couple $\langle \mathcal{C}_1, \mathcal{C}_2\rangle$. Each $\mathcal{C}_i$ is a set of pairs of prefixes. When the context is clear we omit $\sigma$ or $[\mathcal{C}]$. The condition $\mathcal{C}$ is the current information on the branch that is stored during its development. At each step of the development of a branch, $\mathcal{C}$ is identical for all the prefixed formulae on that branch, i.e. $\mathcal{C}$ is an attribute for branches. We refer to a prefixed formula as *atomic* if it is of the form $\sigma : \mathtt{p}\ [\mathcal{C}]$ or $\sigma : \neg\mathtt{p}\ [\mathcal{C}]$ when $\mathtt{p}$ is an atomic formula. Figure 1 presents the prefixed tableau system for the logic $K_I$. Observe that the condition $[\mathcal{C}]$ is of no use in this calculus.

In the sequel we omit the *presentation* of the $\alpha$-rule (decomposition of conjunctions) and the $\beta$-rule (decomposition of disjunctions) but these rules are included in any forthcoming calculus. A branch is *closed* if it contains contradictory prefixed formulae (for any formula $\mathtt{A}$, $\sigma : \mathtt{A}$ and $\sigma : \neg\mathtt{A}$ are contradictory). A tableau is *closed* if every branch is closed. A formula $\mathtt{A}$ is said *to have a closed tableau* iff there is a closed tableau which root is $0 : \neg\mathtt{A}\ [\langle\emptyset,\emptyset\rangle]$. Termination occurs when no operation is possible. A branch is *open* if it is not closed and a tableau is *open* if at least one branch is such.

**Theorem 3.1**. A formula $\mathtt{A}$ is $K_I$-valid iff $\mathtt{A}$ has a closed tableau built with the rules presented in Figure 1.

$$\frac{\sigma : \nu^i\ [\mathcal{C}]}{\sigma' : \nu^i_0\ [\mathcal{C}]}\ \nu^i\text{−rule},\ i \in \{1,2\}$$

if $\mathtt{C}_i(\langle\sigma,\sigma'\rangle,\mathcal{C})$ holds and $\sigma'$ already occurs on the branch.

$$\frac{\sigma : \pi^i\ [\mathcal{C}]}{\sigma k^i : \pi^i_0\ [\mathcal{C}]}\ \pi^i\text{−rule},\ \text{ new } k \in \omega \text{ on the branch}$$

if there is no $\sigma'$ such that $\sigma' : \pi^i_0$ on the branch and either $\mathtt{C}_i(\langle\sigma,\sigma'\rangle,\mathcal{C})$ or (for all $\sigma : \nu^i$ on the branch, $\sigma' : \nu^i_0$ is on the branch).

$$\frac{\sigma'' : \mathtt{A}\ [\mathcal{C}]}{\sigma'' : \mathtt{A}\ [\mathcal{C}'] \mid \sigma'' : \mathtt{A}\ [\mathcal{C}''] \mid \sigma'' : \mathtt{A}\ [\mathcal{C}''']}$$

$\sigma, \sigma'$ not already applied with this rule

Figure 2: Tableaux system for $K^-_{1,2}$

The proof of Theorem 3.1 can be easily obtained from existing ones from the literature [Fitting, 1983].

## 4 Tableaux for $\mathcal{L}([R],[-R])$

Instead of defining a sound and complete calculus for the logic $\mathcal{L}([R],[-R])$ we define a sound and complete calculus for the logic $K^-_{1,2}$ ($I = \{1,2\}$) characterized by the models $(W, R_1, R_2, V)$ where $R_1 \cup R_2 = W \times W$ (we do not require $R_1 \cap R_2 = \emptyset$). It is known that $\mathcal{L}([R],[-R])$ and $K^-_{1,2}$ have the same class of valid formulae [Goranko, 1990a] and we shall provide a decision procedure for the set of $K^-_{1,2}$-valid formulae based on our tableaux approach. Actually from the calculus for $K^-_{1,2}$ the careful reader will observe that a calculus for $\mathcal{L}([R],[-R])$ can be easily defined. However the calculus for $K^-_{1,2}$ is more adequate to define a decision procedure. The rules for the logic $K^-_{1,2}$ are those in Figure 2 where

- $\mathcal{C}' = \langle \mathcal{C}_1 \cup \{\langle\sigma,\sigma'\rangle\}, \mathcal{C}_2\rangle$, $\mathcal{C}'' = \langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle\sigma,\sigma'\rangle\}\rangle$,
- $\mathcal{C}''' = \langle \mathcal{C}_1 \cup \{\langle\sigma,\sigma'\rangle\}, \mathcal{C}_2 \cup \{\langle\sigma,\sigma'\rangle\}\rangle$.

For the logic $K^-_{1,2}$, $\mathtt{C}_i(\langle\sigma,\sigma'\rangle,\mathcal{C})$ holds ($i \in \{1,2\}$) iff either $\langle\sigma,\sigma'\rangle \in \mathcal{C}_i$ or $\sigma' = \sigma k^i$ for some $k \in \omega$. Intuitively, $\mathtt{C}_i$ encodes the accessibility relation $R_i$. The condition $\mathcal{C}$ could be deleted in the definition of the calculus since it only stores some information about the way the rules have been applied on the branch. However, if one wishes to implement our calculi, the actual presentation is well-suited for this purpose. For instance the $\nu^i$-rule can be read as follows. If the formula $\sigma : \nu^i$ occurs on the branch and if the current information on the branch is $\mathcal{C}$ then add $\sigma' : \nu^i_0$ on the branch and $\mathcal{C}$ remains unchanged. It is worth observing that the cut rule cannot be deleted unless completeness is lost. This property is also shared by the cut rule in the calculi defined in [d'Agostino, 1993]. It is also worth noting that the condition of the restricted cut rule in Figure 2 is equivalent to: either not $\mathtt{C}_1(\langle\sigma,\sigma'\rangle,\mathcal{C})$ or not $\mathtt{C}_2(\langle\sigma,\sigma'\rangle,\mathcal{C})$. Moreover, by applying the restricted cut rule, the current information $\mathcal{C}$ on the branch is updated.

### 4.1 Soundness

Let $X$ be a set of prefixed formulae having the same condition $\mathcal{C}$ (what happens at a current stage of the development of a given branch). Let $\mathcal{M} = (W, R_1, R_2, V)$ be a $K^-_{1,2}$-model. By an *interpretation of* $X$ *in* $\mathcal{M}$ we mean a mapping $\mathcal{I} : \{\sigma : \sigma : \mathtt{A} \in X\} \to W$ such that if $\sigma, \sigma'$ occur in $X$, then $\mathtt{C}_i(\langle\sigma,\sigma'\rangle,\mathcal{C})$ implies $\langle\mathcal{I}(\sigma),\mathcal{I}(\sigma')\rangle \in R_i$ $(i = 1, 2)$. We say that $X$ is $K^-_{1,2}$*-satisfiable under the interpretation* $\mathcal{I}$ if for each $\sigma : \mathtt{A} \in X$, $\mathcal{M}, \mathcal{I}(\sigma) \models \mathtt{A}$. We say that $X$ is $K^-_{1,2}$-satisfiable if $X$ is $K^-_{1,2}$-satisfiable under some interpretation. We say that a branch of a tableau is $K^-_{1,2}$*-satisfiable* if the set of prefixed formulae on it is $K^-_{1,2}$-satisfiable. A tableau is $K^-_{1,2}$*-satisfiable* if some branch is.

**Lemma 4.1**. Suppose $\mathtt{T}$ is a prefixed tableau that is $K^-_{1,2}$-satisfiable. Let $\mathtt{T}'$ be the tableau that results from a single tableau rule being applied to $\mathtt{T}$. Then $\mathtt{T}'$ is also $K^-_{1,2}$-satisfiable.

**Proof:** By an easy verification. **Q.E.D.**

**Proposition 4.2**. (soundness) If $\mathtt{A}$ has a closed tableau built with the rules in Figure 2 then $\mathtt{A}$ is $K^-_{1,2}$-valid.

**Proof:** Similar to the proof of Theorem 3.2 in [Fitting, 1983] (p.400). **Q.E.D.**

### 4.2 Completeness

Let $\mathtt{A}$ be a formula. As done in [Fitting, 1983], we define a systematic attempt to produce a proof of $\mathtt{A}$. The procedure is in stages and the stage 1 consists in placing $0 : \neg\mathtt{A}\ [\langle\emptyset,\emptyset\rangle]$ at the root. Now suppose $n$ stages of the construction have been done. If the tableau is closed then we stop. Similarly if every occurrence of a prefixed formula is *finished* (see the definition of 'finished' below) then we stop. Otherwise we go on. If $n + 1$ is even, $\sigma, \sigma'$ satisfies the condition of the cut rule on some open branch $\mathtt{BR}$ (chosen in some *fair* way) and $\langle\sigma,\sigma'\rangle$ is the smallest pair (for some encoding in the set of natural numbers $\omega$) satisfying this property then split the end of branch $\mathtt{BR}$ in three sub-branches by applying the restricted cut rule with $\langle\sigma,\sigma'\rangle$. Otherwise $(n + 1$ odd) any stage $n + 1$ consists in choosing an occurrence of a prefixed formula $\sigma : \mathtt{B}\ [\mathcal{C}]$ as high up in the tree as possible (as close to the origin as possible) that has not been finished. If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is atomic then the occurrence is declared finished. This ends the stage $n + 1$ otherwise we extend the tableau as follows. For each open branch $\mathtt{BR}$ through the occurrence of $\sigma : \mathtt{B}\ [\mathcal{C}]$ (*under the proviso the conditions to apply the rules hold*):

P1 If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is of the form $\sigma : \alpha\ [\mathcal{C}]$ add $\sigma : \alpha_1\ [\mathcal{C}]$ and $\sigma : \alpha_2\ [\mathcal{C}]$ to the end of $\mathtt{BR}$.

P2 If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is of the form $\sigma : \beta\ [\mathcal{C}]$ split the end of $\mathtt{BR}$ and add $\sigma : \beta_1\ [\mathcal{C}]$ to the end of one *sub-branch* and $\sigma : \beta_2\ [\mathcal{C}]$ to the end of the other one.

P3 If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is of the form $\sigma : \nu^i\ [\mathcal{C}]$ then for all $\sigma'$ satisfying the condition of the $\nu^i$-rule add $\sigma' : \nu^i_0\ [\mathcal{C}]$ to the end of $\mathtt{BR}$, after which add a fresh occurrence of $\sigma : \nu^i\ [\mathcal{C}]$ to the end of $\mathtt{BR}$.

P4 If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is of the form $\sigma : \pi^i\ [\mathcal{C}]$ then add $\sigma k^i : \pi^i_0\ [\mathcal{C}]$ to the end of $\mathtt{BR}$. Moreover for $\sigma : \nu^i\ [\mathcal{C}]$ on the branch add $\sigma k^i : \nu^i_0\ [\mathcal{C}]$ to the end of $\mathtt{BR}$ (applications of the $\nu^i$-rule)

Having done this for each branch $\mathtt{BR}$ through the particular occurrence of $\sigma : \mathtt{B}\ [\mathcal{C}]$ being considered, declare that occurrence of $\sigma : \mathtt{B}\ [\mathcal{C}]$ finished. This ends stage $n + 1$.

**Definition 4.1**. Let $X$ be a set of prefixed formulae and $\mathcal{C}$ be a condition. We say $X$ is *downward-saturated with respect to* $\mathcal{C}$ iff:

C1 For all $\sigma, \sigma' \in X$, (C1.1) either $\mathtt{C}_1(\langle\sigma,\sigma'\rangle,\mathcal{C})$ or $\mathtt{C}_2(\langle\sigma,\sigma'\rangle,\mathcal{C})$ and, (C1.2) for all $\mathtt{p} \in \mathtt{For}_0$, $\{\sigma : \mathtt{p}, \sigma' : \neg\mathtt{p}\} \subseteq X$ implies $\sigma \neq \sigma'$.

C2 if $\sigma : \alpha \in X$ then $\{\sigma : \alpha_1, \sigma : \alpha_2\} \subseteq X$.

C3 if $\sigma : \beta \in X$ then either $\sigma : \beta_1 \in X$ or $\sigma : \beta_2 \in X$.

C4 if $\sigma : \nu^i \in X$ then for all $\sigma'$ in $X$ satisfying the condition of the $\nu^i$-rule, we have $\sigma' : \nu^i_0 \in X$.

C5 if $\sigma : \pi^i \in X$ then there is $\sigma'$ such that $\sigma' : \pi^i_0 \in X$ and, either $\mathtt{C}_1(\langle\sigma,\sigma'\rangle,\mathcal{C})$ or (for all $\sigma : \nu^i \in X$, $\sigma' : \nu^i_0 \in X$).

$\nabla$

**Lemma 4.3**. If $X$ is downward-saturated with respect to $\mathcal{C}$ then $X$ is $K^-_{1,2}$-satisfiable.

**Proof:** Assume $X$ is downward-saturated wrt $\mathcal{C}$. Let $\mathcal{M} = (W, R_1, R_2, V)$ be the structure such that $W = \{\sigma : \sigma : \mathtt{B} \in X\}$, for all $\mathtt{p} \in \mathtt{For}_0$ $V(\mathtt{p}) = \{\sigma : \sigma : \mathtt{p} \in X\}$ and for all $\sigma, \sigma'$ in $X$ and $i \in \{1, 2\}$ $\sigma R_i \sigma'$ iff either $\mathtt{C}_i(\langle\sigma,\sigma'\rangle,\mathcal{C})$ or $\{\nu^i_0 : \sigma : \nu^i \in X\} \subseteq \{\mathtt{B} : \sigma' : \mathtt{B} \in X\}$. One can easily check that the definition of $\mathcal{M}$ is correct, i.e. $\mathcal{M}$ is a $K^-_{1,2}$-model. It can be shown by induction on the structure of the formulae that for every formula $\mathtt{B}$ and every prefix $\sigma$, if $\sigma : \mathtt{B} \in X$ then $\mathcal{M}, \sigma \models \mathtt{B}$ (and therefore $X$ is $K^-_{1,2}$-satisfiable). **Q.E.D.**

**Proposition 4.4**. (completeness) If $\mathtt{A}$ is $K^-_{1,2}$-valid then $\mathtt{A}$ has a closed tableau built with the rules presented in Figure 2.

**Proof:** Suppose $\mathtt{A}$ has no closed prefixed tableau. So the systematic procedure does not generate a closed tableau.

We build a tableau with this procedure by considering $0 : \neg \mathtt{A}\ [\langle \emptyset, \emptyset \rangle]$ at the root. If the procedure terminates then the tableau contains a non-closed branch. If the procedure does not terminate, by König's Lemma, there is an infinite non-closed branch. The systematic procedure guarantees that the non-closed branch BR is downward-saturated wrt some $\mathcal{C}$. By Lemma 4.3, BR is $K^{-}_{1,2}$-satisfiable. Since $0 : \neg \mathtt{A} \in \mathtt{BR}$, there is a $K^{-}_{1,2}$-model $\mathcal{M}$ and a world $w$ such that $\mathcal{M}, w \models \neg \mathtt{A}$, which leads to a contradiction. **Q.E.D.**

In the systematic procedure, we require that if $\sigma : \mathtt{B}$ is a conclusion of some inference of the $\nu^i$-rule and if an occurrence of $\sigma : \mathtt{B}$ has already been introduced on the branch then no new occurrence is added on the branch. The systematic procedure still guarantees completeness but it terminates since the $\pi^i$-rule can be applied only a finite number of times. Actually, each $\pi^i$-rule is applied at most $mw^i(\mathtt{A}) \times 2^{card(\{\mathtt{B},\neg\mathtt{B}:\mathtt{B} \text{ subformula of } \mathtt{A}\})}$ times on a branch where $mw^i(\mathtt{A})$ is the number modal operators of the form $[i]$ or $\langle i \rangle$ occurring in A. The other rules do not introduce new prefixes which guarantees termination since their applications are restricted (while insuring completeness). The systematic procedure above is therefore a decision procedure for the $\mathcal{L}([R], [-R])$-validity problem.

## 5 Tableaux for $\mathcal{L}([\neq])$

For any finite set $X$ of pairs we write $X(a, b)$ to denote that $\langle a, b \rangle$ belongs to the smallest equivalence relation containing $X$. The rules for $\mathcal{L}([\neq])$ are those for $K^{-}_{1,2}$ except that the $\pi^1$-rule becomes

$$\frac{\sigma : \pi^1\ [\mathcal{C}]}{\sigma k^1 : \pi^1_0\ [\langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle \sigma, \sigma k^1 \rangle\}\rangle]} \text{ new } k \in \omega \text{ on the branch}$$

and the restricted cut rule is replaced by:

$$\frac{\sigma'' : \mathtt{A}\ [\mathcal{C}]}{\sigma'' : \mathtt{A}\ [\langle \mathcal{C}_1 \cup \{\langle \sigma, \sigma' \rangle\}, \mathcal{C}_2 \rangle] \mid \sigma'' : \mathtt{A}\ [\langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle \sigma, \sigma' \rangle\}\rangle]}$$

$\sigma, \sigma'$ occur on the branch and neither $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ nor $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$ holds.

The definitions of $\mathtt{C}_1$ and $\mathtt{C}_2$ are modified as follows: $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ holds iff either $\mathcal{C}_1(\sigma, \sigma')$ holds or $\sigma = \sigma'$ and $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$ holds iff there exist $\sigma_1$ and $\sigma'_1$ such that $\{\langle \sigma_1, \sigma'_1 \rangle, \langle \sigma'_1, \sigma_1 \rangle\} \cap \mathcal{C}_2 \neq \emptyset$, $\mathtt{C}_1(\langle \sigma, \sigma_1 \rangle, \mathcal{C})$ and $\mathtt{C}_1(\langle \sigma', \sigma'_1 \rangle, \mathcal{C})$. For instance $\mathtt{C}_1(\langle \sigma, \sigma_1 \rangle, \mathcal{C})$ can be interpreted by "$\sigma$ and $\sigma_1$ are equal modulo $\mathcal{C}$". A branch is *closed* if there exist prefixed formulae $\sigma : \mathtt{A}$ and $\sigma' : \neg \mathtt{A}$ on that branch such that $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ holds. This calculus for $\mathcal{L}([\neq])$ strongly differs from the one in [Demri, 1996] due to the machinery associated to $\mathcal{C}$ and to the restricted cut rule.

**Theorem 5.1**. (soundness and completeness) A formula A is $\mathcal{L}([\neq])$-valid iff A has a closed tableau built with the rules for $\mathcal{L}([\neq])$.

In order to provide a decision procedure for $\mathcal{L}([\neq])$ it is sufficient to consider the decision procedure in Section 4 adequately modified for $\mathcal{L}([\neq])$ except that the following conditions are required to apply the $\pi^1$-rule:

- $\rho 1$ it is not possible to apply the restricted cut rule (that is the restricted cut rule is *saturated* before applying the $\pi^1$-rule),
- $\rho 2$ there is no $\sigma' : \pi^1_0$ on the branch such that $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$,
- $\rho 3$ there are no $\sigma_1 : \pi^1_0$ and $\sigma_2 : \pi^1_0$ on the branch such that $\mathtt{C}_2(\langle \sigma_1, \sigma_2 \rangle, \mathcal{C})$.

It is possible to show that the calculus is sound and complete and the systematic procedure defined above always terminates (each formula $\pi^1$ occurring in $\neg$A can be used at most twice as a premise of a $\pi^1$-rule inference on a given branch). Actually, at most $1 + 2 \times mw(\mathtt{A})$ different prefixes can occur on a given branch where $mw(\mathtt{A})$ is the so-called *modal weight* of A, i.e. the number of modal operators occurring in A. Hence the above systematic procedure constructs a polynomial-size $\mathcal{L}([\neq])$-model for $\neg$A (with respect to the *size* of A) if A is not $\mathcal{L}([\neq])$-valid.

## 6 Tableaux for $K_I([\neq])$

The conditions $\mathtt{C}_1$ and $\mathtt{C}_2$ are defined as in Section 5 as well as the closure conditions. The tableaux rules for $K_I([\neq])$ are given in Figure 3. Let $X$ be a set of prefixed formulae having the same condition $\mathcal{C}$ and $\mathcal{M} = (W, (R_i)_{i \in I}, V)$ be a $K_I([\neq])$-model. By an *interpretation of $X$ in $\mathcal{M}$* we mean a mapping $\mathcal{I} : \{\sigma : \sigma : \mathtt{A} \in X\} \rightarrow W$ such that if $\sigma, \sigma'$ occur in $X$, then

- $\sigma' = \sigma k^i$ for some $k^i$ implies $\langle \mathcal{I}(\sigma), \mathcal{I}(\sigma') \rangle \in R_i$,
- $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ implies $\mathcal{I}(\sigma) = \mathcal{I}(\sigma')$ and $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$ implies $\mathcal{I}(\sigma) \neq \mathcal{I}(\sigma')$.

Lemma 4.1 can be shown to hold for $K_I([\neq])$ associated with the calculus presented in Figure 3: if A has a closed tableau built with the rules in Figure 3 then A is $K_I([\neq])$-valid. We also use the systematic procedure defined in Section 4.2 (with the binary restricted cut rule) except that (P4) is replaced by:

P4′ If $\sigma : \mathtt{B}\ [\mathcal{C}]$ is of the form $\sigma : \pi^i\ [\mathcal{C}]$ with $i \neq 1$ (resp. $\sigma : \pi^1\ [\mathcal{C}]$) then add $\sigma k^i : \pi^i_0\ [\mathcal{C}]$ (resp. $\sigma k^1 : \pi^1_0\ [\langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle \sigma, \sigma k^1 \rangle\}\rangle]$) to the end of BR.

Similarly, we say $X$ is *downward-saturated wrt $\mathcal{C}$* iff:

C1′ For all $\sigma, \sigma' \in X$, (C1′.1) $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ iff not $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$ (note the difference with C1.1 in Section 4) and, (C1′.2) for all $\mathtt{p} \in \mathtt{For}_0$, $\{\sigma : \mathtt{p}, \sigma' : \neg \mathtt{p}\} \subseteq X$ implies $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$.

- Conditions C2,C3 from Section 4.2 and C4 for $i \neq 1$

$$\frac{\sigma : \pi^i \ [\mathcal{C}]}{\sigma k^i : \pi^i_0 \ [\mathcal{C}]} \ \pi^i\text{-rule}, \ \text{new } k \in \omega, i \in I \setminus \{1\}$$

$$\frac{\sigma : \pi^1 \ [\mathcal{C}]}{\sigma k^1 : \pi^1_0 \ [\langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle \sigma, \sigma k^1 \rangle\}\rangle]} \ \pi^1\text{-rule}, \ \text{new } k \in \omega$$

$$\frac{\sigma : \nu^i \ [\mathcal{C}]}{\sigma' : \nu^i_0 \ [\mathcal{C}]} \ \nu^i\text{-rule}, \ i \in I \setminus \{1\}$$

if there exist $\sigma_1, \sigma_1 k^i$ on the branch such that $\mathtt{C}_1(\langle \sigma, \sigma_1 \rangle, \mathcal{C})$ and $\mathtt{C}_1(\langle \sigma', \sigma_1 k^i \rangle, \mathcal{C})$.

$$\frac{\sigma : \nu^1 \ [\mathcal{C}]}{\sigma' : \nu^1_0 \ [\mathcal{C}]} \ \nu^1\text{-rule}, \ \text{if } \mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$$

$$\frac{\sigma'' : \mathtt{A} \ [\mathcal{C}]}{\sigma'' : \mathtt{A} \ [\langle \mathcal{C}_1 \cup \{\langle \sigma, \sigma' \rangle\}, \mathcal{C}_2 \rangle] \mid \sigma'' : \mathtt{A} \ [\langle \mathcal{C}_1, \mathcal{C}_2 \cup \{\langle \sigma, \sigma' \rangle\}\rangle]}$$

$\sigma, \sigma'$ on the branch and neither $\mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})$ nor $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$ holds.

Figure 3: Tableaux system for $K_I([\neq])$

C5′ if $\sigma : \pi^i \in X$ with $i \neq 1$ then there exist $\sigma' : \pi^i_0 \in X$ and $\sigma k^i$ in $X$ such that $\mathtt{C}_1(\langle \sigma', \sigma k^i \rangle, \mathcal{C})$.

C6 if $\sigma : \nu^1 \in X$ then for all $\sigma'$ in $X$ such that $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$, we have $\sigma' : \nu^1_0 \in X$

C7 if $\sigma : \pi^1 \in X$ then there is $\sigma'$ such that $\sigma' : \pi^1_0 \in X$ and $\mathtt{C}_2(\langle \sigma, \sigma' \rangle, \mathcal{C})$.

**Lemma 6.1**. If $X$ is downward-saturated wrt $\mathcal{C}$ then $X$ is $K_I([\neq])$-satisfiable.

**Proof:** Assume $X$ is downward-saturated wrt $\mathcal{C}$. Let $\mathcal{M} = (W, (R_i)_{i \in I}, V)$ be the structure such that,

- $W = \{|\sigma| : \sigma : \mathtt{B} \in X\}$ where $|\sigma| = \{\sigma' : \sigma : \mathtt{B} \in X, \mathtt{C}_1(\langle \sigma, \sigma' \rangle, \mathcal{C})\}$.
- for all $\mathtt{p} \in \mathtt{For}_0$ $V(\mathtt{p}) = \{|\sigma| : \sigma : \mathtt{p} \in X\}$.
- $R_1 = dif_W$ and for all $\sigma, \sigma'$ in $X$, $|\sigma| R_i |\sigma'|$ $(i \neq 1)$ iff $\exists \sigma_1, \sigma_1 k^i$ in $X$, $\mathtt{C}_1(\langle \sigma, \sigma_1 \rangle, \mathcal{C})$ and $\mathtt{C}_1(\langle \sigma', \sigma_1 k^i \rangle, \mathcal{C})$.

$\mathcal{M}$ is a $K_I([\neq])$-model. It can be shown (by induction on $\mathtt{B}$) that if $\sigma : \mathtt{B} \in X$ then $\mathcal{M}, |\sigma| \models \mathtt{B}$. **Q.E.D.**

**Proposition 6.2**. (completeness) If $\mathtt{A}$ is $K_I([\neq])$-valid then $\mathtt{A}$ has a closed prefixed tableau built with the rules presented in Figure 3.

In order to obtain a decision procedure, take the systematic procedure, incorporate the restrictions $\rho 1$, $\rho 2$ and $\rho 3$ from Section 5 and for $i \neq 1$, add the following restriction to the $\pi^i$-rule: there is no $\sigma' : \pi^i_0$ on the branch such that $\mathtt{C}_1(\langle \sigma k^i, \sigma' \rangle, \mathcal{C})$ holds for some $k \in \omega$.

## 7 Concluding remarks

The use of *prefixes* for tableaux systems dedicated to modal logics has been thoroughly developed in [Fitting, 1983] whereas our treatment of the condition $\mathcal{C}$ (see e.g. Sections 4, 5, 6) can be viewed as a means to parametrize our calculi by the *theory* of the accessibility relations. Hence, the idea of *theory resolution* [Stickel, 1985] in which a theory is separately dealt with from the rest of the calculus is present in our calculi. This idea is not new in the realm of the mechanization of modal logics (see e.g. [Frisch and Scherl, 1990; Gent, 1993]) but the originality of our work is related to the conditions satisfied by the accessibility relations of the models.

The second important feature of our calculi is the use of a restricted cut rule. Recently, various works have *tamed* the cut rule for calculi dedicated to modal logics (see e.g. [d'Agostino, 1993; Governatori, 1995]). However our calculi do not have a cut rule with a branching for formulae. In that sense, the cut rule in our calculi is even more restricted than the one in [Governatori, 1995].

We have defined sound and complete prefixed tableaux calculi for the logics $\mathcal{L}([R], [-R])$, and $K_I([\neq])$ (also for $K_I$ and $\mathcal{L}([\neq])$) and decision procedures have been designed from these systems. It is worth noting that the expressive power of the modal logics with enriched languages is attractive in the Artificial Intelligence community since for instance the operator $[\neq]$ has already been shown to be useful to reason about time [Sain, 1988; Koymans, 1992] or space [Balbiani *et al.*, 1997].

Future work could be oriented towards the incorporation of our calculi into existing tableaux-based theorem provers for modal logics and towards the definition of other prefixed tableaux for modal logics with enriched languages including for instance, the logics in the paper where standard conditions for the accessibility relations are required -reflexivity, symmetry, ....

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