# Bisimulation and the reduction of Petri nets

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**Abstract.** We investigate structural equivalences on places of P/T nets that allow reductions compatible with bisimilarity. This comes with a study of two kinds of reductions: fusion of equivalent places, and replacement of some places by other ones. When effectivity issues are considered, we are lead to a variant of place bisimulation that takes into account a set of "relevant" markings.

## 1 Introduction

*Place bisimulation. Place bisimulation* is a formal notion of equivalence between places of P/T nets that combines two distinctive features: it agrees with a notion of *bisimilarity*, and it relates equivalent *places* rather than equivalent markings.

While several early papers adapted Milner's concept of bisimilarity to the transition systems generated by Petri nets [NT84,Pom86], Olderog was the first to propose a notion of bisimulation that comes from lifting a bisimulation between places [Old89,Old91]. The main advantage of this approach is that, as with many structural methods on Petri nets, it considers the elements (places and transitions) of the net rather than the reachability graph (which may be infinite). Using Olderog's method, it was possible to prove that two nets had equivalent behaviours just by exhibiting a relation between their places and checking a finite number of local constraints.

The name "place bisimulation" comes from [ABS91] where Olderog's proposal is improved and where it was proven that place bisimulation could be used to *reduce* P/T nets (rather than just state two nets have equivalent behaviours).

*Reductions for Petri nets. Reductions* are transformations of Petri nets that decrease the size. Their main use is as an optimization technique. For such applications the reduction must return a net behaviorally equivalent to the original one, but smaller in size. This smaller net can be used in place of the original net when it comes to implementing the net, or to analyzing it (e.g., for verification purposes).

Place bisimulations yield reductions that preserve the branching-time behaviour. These reductions are *structural* (one removes places and transitions, and this is what may remove markings) and do not depend on the initial marking (or the set of reachable markings).

The fact that place bisimulations do not depend on the set of reachable markings has its pros and cons. On the positive side, this allows polynomial-time algorithms, and leads to reductions that are correct whatever the initial marking. On the negative side, some reductions that seem obviously correct are not given by place bisimulation simply because there exists an (irrelevant) alternative initial marking for which they are incorrect. This is the drawback we try to alleviate in this paper.

*Our contribution.* In this paper we investigate structural reductions of P/T nets that preserve bisimilarity and that take into account a notion of which markings are relevant. Our starting point is an investigation of what are the correct ways of defining an equivalence on places. This departs from earlier works where behavioural equivalence is the starting point, and reductions only a byproduct.

We first identify two different, equally natural, reductions based on a notion of equivalent places. They are "fusion" and "replacement". There is a strong connection between them but correct fusions only coincide with correct replacements when we impose (rather strong) structural conditions on the set of relevant markings. These theoretical results and the accompanying examples are enlightening and they help understand why it is quite delicate to take into account a set of markings in these structural reduction problems.

Then, decidability issues impose further restrictions. We end up with a variant of place bisimulation that is parameterized by a set of markings, a partial answer to what was felt as the main inconvenience of place bisimulation. We say the answer is only partial because the set of reachable markings has to satisfy some strong closure restrictions.

*Why bisimulation ?* We consider bisimilarity as our basic correctness notion because it has proven to be a fundamental semantic equivalence in the algebraic theory of concurrent systems [Mil89,Mil90,Gla90], because it is more local than more classical language-theoretical notions and then often behaves better algorithmically, because there exists a very successful verification technology for transition systems based on bisimilarity (e.g., [CPS93]), because it has not been much studied in the context of Petri nets.

*Related works.* The work in [ABS91] was continued in [AS92] (studying how true concurrency is preserved, and where an algorithm for place bisimulation is given) and [APS94] (investigating place bisimulations abstracting from silent moves). These works did not start from reduction issues and we find our new approach clearer and more natural (also, these works did not try to take into account a set of relevant markings).

Independently, Voorhoeve studied a concept very similar to place bisimulations [Voo96]. His starting point is also "bisimulation + structural equivalences", but he emphasizes equivalence of behaviours and does not mention applications to reductions of Petri nets.

For optimization purposes there also exists structural reductions that do not preserve bisimilarity (e.g., [NKML95,STMD96]). For more specific purposes, reductions have also been used as a computational device [Ber86]. E.g., telling whether a net is bounded can be done by reducing it to a normal form where boundedness is obvious. Here it is not necessary that the reductions preserve all of the behaviour of the net, only the relevant aspect (boundedness in our example) is enough.

*Plan of the paper.* We first recall basic definitions and notations about P/T nets and bisimilarity (§ 2). Then we define reductions by fusion (§ 3) and by replacement (§ 4). It is then possible to define and study when a correct replacement gives rise to a correct fusion and vice versa (§ 5). We investigate how correct reductions combine (§ 6) and show there exists a largest correct reduction, unfortunately not computable in general (§ 7). Place bisimulation is then introduced as a computable approximation (§ 8) and exemplified (§ 9).

## 2 Basic definitions

$\mathbb{N}$ denotes the set of natural numbers. Let $Act = \{a, b, c, \ldots\}$ be a finite alphabet of *action names.*

### 2.1 Labeled nets and their behaviour

A *labelled Petri net* is a tuple $N = \langle P_N, T_N, F_N, l_N \rangle$, where:

- $P_N$ and $T_N$ are two disjoint non-empty finite sets of *places* and *transitions* respectively;
- $F_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ is a *weight function* (also called *flow relation*);
- $l_N : T_N \to Act$ labels each transition $t \in T_N$ with some action $l_N(t)$ from $Act$.

We drop the $N$ subscript whenever no ambiguity can arise and present nets with the usual boxes and circles graphical presentation.

Markings are configurations of a net. A *marking* $M$ of $N$ can be seen as a multiset over $P$, i.e. a mapping from $P$ into $\mathbb{N}$. This is the viewpoint we adopt here. Then $M(p)$ is the number of appearances of place $p$ in multiset $M$: this represents the number of tokens on $p$ in marking $M$. The set of all possible markings is $\mathbb{N}^P$.

We write $|M|$ for the size of a marking $M$ (its number of tokens) and, given two markings $M, M'$ we let $\Delta(M, M')$ denote the *distance* between $M$ and $M'$, i.e. the number of tokens one has to move to transform $M$ into $M'$.

Given a transition $t \in T$, the *preset* $^\bullet t$ and the *postset* $t^\bullet$ of $t$ are the multisets of places given by $^\bullet t(p) \stackrel{\text{def}}{=} F(p,t)$ and $t^\bullet(p) \stackrel{\text{def}}{=} F(t,p)$ for any $p \in P$.

A transition $t \in T$ is *enabled* in marking $M$ iff $\forall p \in P, M(p) \geq F_N(p,t)$. An enabled transition $t$ may fire, thus performing action $l(t)$. This results in a new marking $M'$ defined by $M' \stackrel{\text{def}}{=} M - {}^\bullet t + t^\bullet$ (that is, $M'(p) \stackrel{\text{def}}{=} M(p) - F(p,t) + F(t,p)$ for any $p \in P$).

We write $N : M \xrightarrow{t} M'$ to denote that $M \xrightarrow{t} M'$ is a step in net $N$. Derived notations are "$M \xrightarrow{t} M'$" when $N$ is implicit, "$M \xrightarrow{a} M'$" when $a$ is $l(t)$, "$M \xrightarrow{a}$" when $M'$ is not relevant, etc.

### 2.2 Bisimulation

Let $N_1 = (P_1, T_1, F_1, l_1)$ and $N_2 = (P_2, T_2, F_2, l_2)$ be two nets and $R \subseteq \mathbb{N}^{P_1} \times \mathbb{N}^{P_2}$ be a relation between their markings.

We say that $R$ has the *transfer property* iff for all $M_1, M_2$ s.t. $M_1 R M_2$, and for all steps $N_1 : M_1 \xrightarrow{t_1} M_1'$, there exists a step $N_2 : M_2 \xrightarrow{t_2} M_2'$ s.t. $l(t_2) = l(t_1)$ and $M_1' R M_2'$.

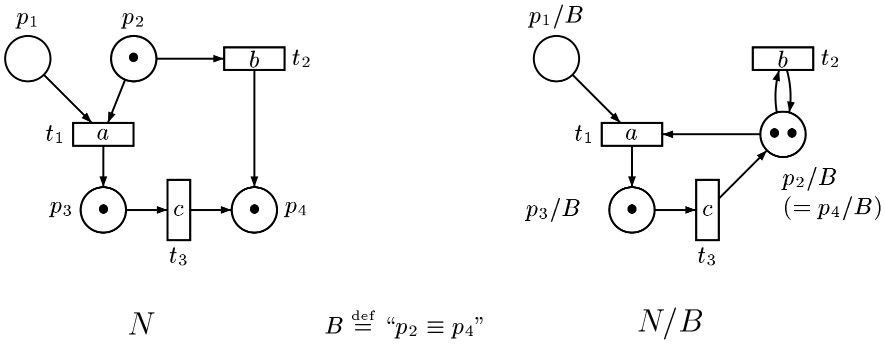
If $R$ and $R^{-1}$ have the transfer property (we sometimes say that $R$ has the transfer property in both directions) then $R$ is a *bisimulation* between $N_1$ and $N_2$ [Par81,Mil89]. When $M_1 R M_2$ for a bisimulation $R$ we say that $(N_1, M_1)$ and $(N_2, M_2)$ are bisimilar, written $N_1, M_1 \sim N_2, M_2$. When $N_1$ and $N_2$ are the same net, we simply write $M_1 \sim M_2$.

It is well-known that, given any two nets, there exists a largest bisimulation between them, and we use $\sim_{[N_1,N_2]}$ to denote it. When $N_1 = N_2$, the largest bisimulation is an equivalence. Our proofs that relations are bisimulation often use basic properties (like $\sim_{[N_2,N_3]} \circ \sim_{[N_1,N_2]} \subseteq \sim_{[N_1,N_3]}$) and Milner's "up to" technique [Mil89]: we say $R$ has the transfer property *up to bisimulation* if $M_1 R M_2$ and $M_1 \xrightarrow{t_1} M_1'$ imply that there exists a $M_2 \xrightarrow{t_2} M_2'$ with $M_1 R' M_2$ for $R' \stackrel{\text{def}}{=} \sim_{[N_2,N_2]} \circ R \circ \sim_{[N_1,N_1]}$ (and $l(t_2) = l(t_1)$). Then $R'$ has the transfer property.

## 3 Reduction by fusion

*Fusion* is a natural way of reducing nets, based on the general idea of quotients of structures.

Consider a net $N$ and assume $B \subseteq P \times P$ is an equivalence relation between places. Then $N$ can be reduced by fusing $B$-equivalent places into a single place. This yields a new net denoted $N/B$. We write $p/B$ for the equivalence class of $p$ (and then for the fused place in $N/B$). The transitions are untouched but the arcs between places and transitions follow the fusion process: formally, if $^\bullet t$ (resp. $t^\bullet$) is $\{p_1, \ldots, p_k\}$ in $N$, then in $N/B$, $^\bullet t$ (resp. $t^\bullet$) is $\{p_1/B, \ldots, p_k/B\}$. A marking $M$ in $N$ is fused into a marking $m \stackrel{\text{def}}{=} M/B$ in $N/B$. $m$ and $M$ have the same number of tokens. Fig. 1 gives an example where $B$ relates $p_2$ and $p_4$.




$B \stackrel{\text{def}}{=}$ "$p_2 \equiv p_4$"

**Fig. 1.** Fusion of places as a reduction method

Fusing places modifies the behaviour, but only in one direction: it adds new behaviours. This is captured as

**Fact 3.1.** *If $M \xrightarrow{t} M'$ in $N$, then $M/B \xrightarrow{t} M'/B$ in $N/B$.*

This implies that $\_/B$, defined as $\{(M, M/B) \mid M \in \mathbb{N}^P\}$, has the transfer property from $N$ to $N/B$.

A fundamental property is the following weak converse:

**Proposition 3.2.** *If $m \xrightarrow{t} m'$ in $N/B$, then there exist two markings $M$ and $M'$ in $N$ s.t. $m = M/B$, $m' = M'/B$ and $M \xrightarrow{t} M'$.*

*Proof.* Assume $m \xrightarrow{t} m'$. Write $M_1$ for ${}^\bullet t$ in $N$ and pick any $M_2$ s.t. $M_2/B$ is $m - (M_1/B)$. Then, in $N$, $M \xrightarrow{t} M'$ for $M = M_1 + M_2$ and $M' = t^\bullet + M_2$. Clearly, $M'/B = m'$. □

An equivalence $B$ between places can be lifted to an equivalence, written $\overline{B}$, between markings. Formally, if $p_1 B p'_1, \ldots, p_n B p'_n$ then $\{p_1, \ldots, p_n\} \overline{B} \{p'_1, \ldots, p'_n\}$. Note that $M \overline{B} M'$ iff $M/B = M'/B$, i.e.

$$\overline{B} = (\_/B)^{-1} \circ (\_/B). \tag{1}$$

Now, a direct reading of Prop. 3.2 gives the following

**Fact 3.3.** *When $\overline{B}$ relates bisimilar markings only, i.e. $\overline{B} \subseteq \sim_{[N,N]}$, then $(\_/B)^{-1}$ has the transfer property up to bisimulation.*

## 4 Reduction by replacement

*Replacement* is a second natural way of reducing nets, this time based on the general idea of homomorphic images of structures.

Consider a net $N$ and assume $h : P \to P$ is a projection, i.e. for all $p \in P$ we have $h(h(p)) = h(p)$. In $N$ a place $p$ can be replaced by $h(p)$ whenever $h(p) \neq p$. This yields a new net, denoted $h(N)$.

More precisely, $h(N)$ is obtained by redirecting all output edges of transitions by their projections: if $t^\bullet = M = \{p_1, \ldots, p_k\}$ in $N$, it becomes $h(M) \stackrel{\text{def}}{=} \{h(p_1), \ldots, h(p_k)\}$ in $h(N)$. All places not in $h(P)$ are removed, all transitions that have some input place removed are removed as well. A marking $M$ in $N$ yields a marking $m = h(M)$ in $h(N)$. $m$ and $M$ have the same number of tokens. Fig. 2 gives an example where $p_3$, and then $t_3$, are removed.

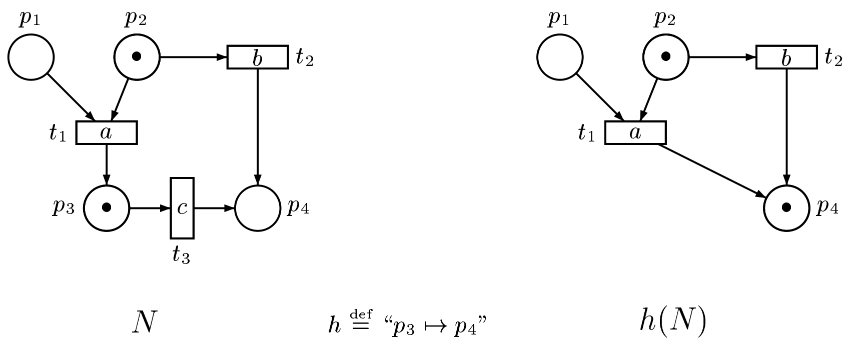



**Fig. 2.** Replacement of places as a reduction method

Replacing places modify the behaviour in the following way:

**Fact 4.1.** *If $m \xrightarrow{t} m'$ in $h(N)$, then $m \xrightarrow{t} M'$ in $N$ for some $M'$ s.t. $h(M') = m'$.*

There is a weak converse:

**Fact 4.2.** *If $M \xrightarrow{t} M'$ in $N$ and $h(M) = M$, then $M \xrightarrow{t} h(M')$ in $h(N)$.*

## 5 Correct reductions

We are only interested in *correct reductions.* Informally, a correct reduction is a reduction such that the reduced net $N'$ is a correct variant of the original net $N$.

In this paper, we investigate formal notions of correctness

1. grounded into bisimilarity, and
2. only applying to a given set of markings (called *relevant markings*).

### 5.1 Relevant markings

In all the following, we assume $\mathcal{A} \subseteq \mathbb{N}^P$ is a set of relevant markings. Typical examples are "the set of all markings reachable from some initial $M_0$", or "all markings with three tokens".

Compared to earlier works on place bisimulation, we remove the basic (implicit) assumption that all and every markings of the nets under study are meaningful. It was a very strong assumption, resulting into a very conservative view of when two places are bisimilar.

This is why our new assumption is that we only aim at correction relative to a given set of relevant markings. In a sense, this work can be seen as extending the earlier place bisimulation theory to a framework with relevant markings. But, in addition, the paper also brings a new viewpoint: here we focus on reductions while the earlier works focused on semantical equivalences. We think the new viewpoint is clearer and more natural.

### 5.2 Correctness for $N$ with $\mathcal{A}$

Assume $N$ comes with a set $\mathcal{A}$ of relevant markings. A reduction is correct if all relevant markings are reduced into bisimilar markings in the reduced net. The other (= irrelevant) markings are not considered for the correctness issues.

With this in mind, the following definitions are natural:

**Definition 5.1.** *An equivalence $B \subseteq P \times P$ is a* correct fusion *for net $N$ with relevant markings $\mathcal{A}$ iff $N, M \sim N/B, M/B$ for all $M \in \mathcal{A}$.*

**Definition 5.2.** *A projection $h : P \to P$ is a* correct projection *for net $N$ with relevant markings $\mathcal{A}$ iff $N, M \sim h(N), h(M)$ for all $M \in \mathcal{A}$.*

We are interested into the connections between these two definitions. Indeed, any projection $h : P_N \to P_N$ naturally induces an equivalence $B_h$ given by $pB_hp' \overset{\text{def}}{\Leftrightarrow} h(p) = h(p')$. Can we say that $B_h$ is a correct fusion iff $h$ is a correct projection ? It turns out this is not the case in general.

**Fact 5.3.** *There exist situations where $h$ is a correct projection for $N, \mathcal{A}$ and $B_h$ is not a correct fusion.*

*Example 5.4.* Figure 3 gives an example where $h \overset{\text{def}}{=} \{p_3 \mapsto p_1, p_4 \mapsto p_1\}$ is correct (for $\mathcal{A}$ the set of reachable markings) but $B_h$, "$p_1 \equiv p_3 \equiv p_4$", is not. □

*Example 5.5.* Another example is provided in Figure 4 where $h \overset{\text{def}}{=} \{p_3 \mapsto p_2, p_4 \mapsto p_2\}$ and $\mathcal{A} \overset{\text{def}}{=} \{\{p_1\}\}$. □

**Fact 5.6.** *There exist situations where $B$ is a correct fusion while there is no correct projection $h$ s.t. $B = B_h$.*

*Example 5.7.* Figure 5 gives an example where $B$, "$p_1 \equiv p_2$", is a correct fusion (for $\mathcal{A}$ the set of reachable markings) but where no $h$ is correct (the figure illustrates $h \overset{\text{def}}{=} \{p_1 \mapsto p_2\}$). □

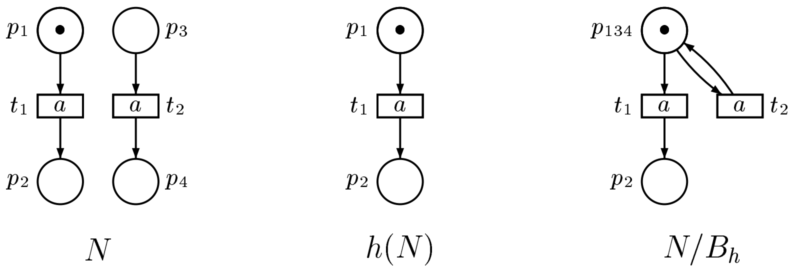



**Fig. 3.** $h = \{p_3 \mapsto p_1, p_4 \mapsto p_1\}$ is correct and $B_h$, "$p_1 \equiv p_3 \equiv p_4$", is not

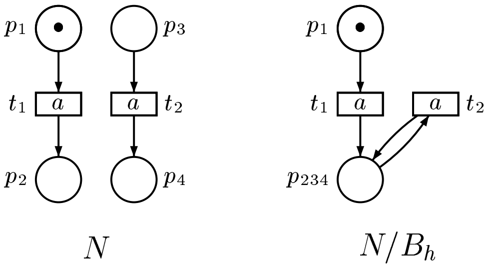



**Fig. 4.** $h = \{p_3 \mapsto p_2, p_4 \mapsto p_2\}$ is correct and $B_h$, "$p_2 \equiv p_3 \equiv p_4$", is not

*Example 5.8.* Another example is provided in Figure 6 with $B$ given by "$p_2 \equiv p_3$" and $h \stackrel{\text{def}}{=} \{p_2 \mapsto p_3\}$. Here $\mathcal{A} \stackrel{\text{def}}{=} \{\{p_1\}\}$. □

It turns out that all these difficulties come from the sets $\mathcal{A}$ we picked in these examples. They disappear when we impose restrictions on possible choices for $\mathcal{A}$.

Given $N$ and a set of markings $\mathcal{A}$, we say that $\mathcal{A}$ is *succ-closed*, if $M \in \mathcal{A}$ and $M \to M'$ entail $M' \in \mathcal{A}$. The sets $\mathcal{A}$ in examples 5.4 and 5.7 are succ-closed by construction.

Secondly, given an equivalence $B$, we say that $\mathcal{A}$ is $B$-closed if $M \in \mathcal{A}$ and $M\overline{B}M'$ entail $M' \in \mathcal{A}$. The sets $\mathcal{A}$ in examples 5.5 and 5.8 are $B$-closed.

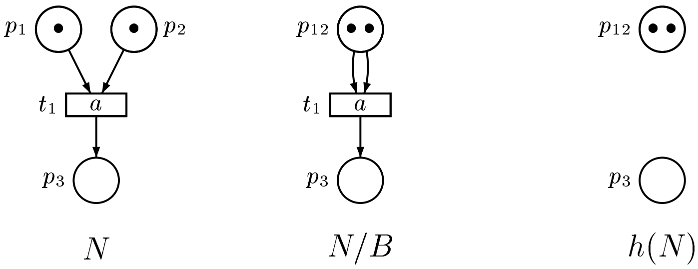



**Fig. 5.** $B$, "$p_1 \equiv p_2$", is correct and $h = \{p_1 \mapsto p_2\}$ is not

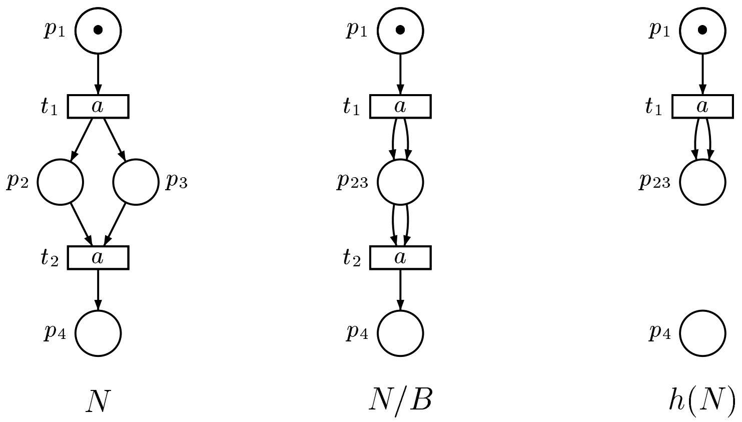



**Fig. 6.** $B$, "$p_2 \equiv p_3$", is correct and $h = \{p_2 \mapsto p_3\}$ is not

We can now state the following crucial property, explaining how when $\mathcal{A}$ is succ-closed and $B$-closed, checking whether $B$ is correct can be done by inspecting $N$ only. This lemma will be of great help because, in general, it is difficult to tell whether a given $B$ or $h$ are correct: indeed, for this we have to compare behaviours in $N$ and in $N/B$, or $h(N)$.

**Lemma 5.9.** *Assume $\mathcal{A}$ is succ-closed and $B$-closed. Then $B$ is a correct fusion iff $\overline{B} \cap (\mathcal{A} \times \mathcal{A})$ is included in $\sim$, the largest bisimulation between markings of $N$.*

*Proof.* ($\Rightarrow$) This is obvious when $\mathcal{A} = \mathbb{N}^{P_N}$, using (1) and the assumption that $\_/B$ is a bisimulation. In the general case, $B$-closure of $\mathcal{A}$ gives that $\overline{B} \cap (\mathcal{A} \times \mathcal{A}) = R^{-1} \circ R$ where $R \stackrel{\text{def}}{=} \{(M, M/B) \mid M \in \mathcal{A}\}$ is included into $\sim_{[N,N/B]}$ (by assumption).

($\Leftarrow$) Now assume $\overline{B} \cap (\mathcal{A} \times \mathcal{A})$ is included in $\sim$, and define $R \stackrel{\text{def}}{=} \{(M, M/B) \mid M \in \mathcal{A}\}$. Fact 3.1 and succ-closure of $\mathcal{A}$ implies that $R$ has the transfer property.

For the other direction, the reasoning underlying Fact 3.3 applies even when $\mathcal{A}$ is taken into account. As a consequence, $R$ is included into $\sim_{[N,N/B]}$, so that $B$ is a correct fusion. $\square$

**Theorem 5.10.** *Assume $h$ is a projection and $\mathcal{A}$ is $B_h$-closed and succ-closed. Then $h$ is a correct projection for $N, \mathcal{A}$ iff $B_h$ is a correct fusion for $N, \mathcal{A}$.*

*Proof.* ($\Rightarrow$) Assume $h$ is a correct projection and consider $M_1 \in \mathcal{A}$. Let $M_2$ s.t. $h(M_1) = h(M_2)$. Correctness of $h$ implies $N, M_1 \sim h(N), h(M_1) \sim N, M_2$ (since $M_2 \in \mathcal{A}$), hence $M_1 \sim M_2$. Thus $\overline{B_h} \subseteq \sim_{[N,N]} \cap (\mathcal{A} \times \mathcal{A})$ and Lemma 5.9 concludes.

($\Leftarrow$) Now assume that $B_h$ is a correct fusion. Lemma 5.9 and the fact that $M \overline{B_h} h(M)$ give that $N, M \sim h(N), h(M)$. $\square$

Our earlier examples show that succ-closure or $B$-closure alone is not sufficient for Theorem 5.10.

When $\mathcal{A}$ is the set of all markings, closure is guaranteed and fusions or replacements coincide. In the following, whenever $\mathcal{A}$ is succ-closed and $B$-closed, we speak of correct equivalences without more precision on whether we mean fusions or replacements.

## 6 Combining correct reductions

Correct reductions can be composed, but the natural ways for composing projections and for composing fusions are not the same.

**Lemma 6.1.** *If $h_1$ is a correct projection for $N, \mathcal{A}$ and $h_2$ is a correct projection for $h_1(N), h_1(\mathcal{A})$, where $h_1(\mathcal{A}) \stackrel{def}{=} \{h_1(M) \mid M \in \mathcal{A}\})$, then $h_2 \circ h_1$ is a correct projection for $N, \mathcal{A}$.*

*Proof.* Direct from Def. 5.2 and transitivity of bisimilarity. □

Furthermore, if $\mathcal{A}$ is succ-closed and $B_{h_1}$-closed, and $h_1(\mathcal{A})$ is $B_{h_2}$-closed, then $\mathcal{A}$ is succ-closed and $B_{h_2 \circ h_1}$-closed.

Now let $B_1$ and $B_2$ be two equivalences and assume $\mathcal{A}$ is $B_1$-closed, $B_2$-closed and succ-closed. Then

**Lemma 6.2.** *If $B_1$ and $B_2$ are correct fusions for $N, \mathcal{A}$, then $B \stackrel{def}{=} (B_1 \cup B_2)^*$, the smallest equivalence larger than $B_1$ and $B_2$, is a correct fusion for $N, \mathcal{A}$.*

*Proof.* Once we notice $B$-closure of $\mathcal{A}$ and $\overline{B} = (\overline{B_1} \cup \overline{B_2})^*$, Lemma 5.9 does all the work. □

A related question is whether, assuming $B$ is a correct fusion, all more conservative reductions are correct too? Here, the intuition underlying the question is that an equivalence relation can only be incorrect by making too many identifications, not by making too few.

Let $B$ be a correct fusion for some $N, \mathcal{A}$ (assuming $\mathcal{A}$ is $B$-closed and succ-closed), then

**Lemma 6.3.** *If $B'$ is an equivalence s.t. $B' \subseteq B$ then $B'$ is a correct fusion.*

*Proof.* Use Lemma 5.9 and $\overline{B'} \subseteq \overline{B}$. □

This question can also be investigated in terms of projections: let $h$ be a correct projection and $h = h_2 \circ h_1$. Is $h_1$ correct?

**Lemma 6.4.** *If $h = h_2 \circ h_1$ is a correct projection for $N, \mathcal{A}$, where $\mathcal{A}$ is $B_h$-closed and succ-closed, then $h_1$ is a correct projection for $N, \mathcal{A}$.*

*Proof.* First, note that $B_{h_1} \subseteq B_h$ and that $\mathcal{A}$ is $B_{h_1}$-closed. Now, if $h$ is a correct projection then (Theorem 5.10) $B_h$ is a correct fusion, hence (Lemma 6.3) $B_{h_1}$ too, hence (Theorem 5.10) $h_1$ is a correct projection. □

## 7 The largest correct fusion

An immediate corollary of Lemma 6.2 is the following:

**Theorem 7.1.** *If $\mathcal{A}$ is succ-closed, there exists a largest correct fusion for $N, \mathcal{A}$ (only equivalences under which $\mathcal{A}$ is closed are considered).*

We write $R(N, \mathcal{A})$ for this largest correct fusion. All the equivalences that are subsets of $R(N, \mathcal{A})$ are correct fusions (Lemma 6.3).

Now let $N$ be a net with a succ-closed set of relevant markings $\mathcal{A}$. We would like to compute $R(N, \mathcal{A})$ and use it for reducing $N$ in the most efficient way (w.r.t. $\mathcal{A}$). Unfortunately this is impossible because of:

**Theorem 7.2 (Undecidability).** *The largest correct fusion $R(N, \mathcal{A})$ is not computable in general, even when $\mathcal{A} = \mathbb{N}^P$.*

The proof is long, technical and is omitted. Basically, it shows that the problem of telling whether two places $p$ and $q$ are such that $Id \cup \{(p, q), (q, p)\}$ is a correct fusion for $N, \mathbb{N}^P$ is undecidable. This reuses ideas from [Qui95], extending Jančar's technique [Jan95] to a setting where all possible markings may be considered.

## 8 Place bisimulation

Since $R(N, \mathcal{A})$ cannot be computed in general, we settle for a computable approximation: *place bisimulation.*

In the following we assume a net $N$ and a succ-closed set $\mathcal{A}$ of relevant markings are fixed. The following definition extends earlier proposals from [AS92,APS94]:

**Definition 8.1.** *A reflexive symmetric relation $B \subseteq P \times P$ is a* place bisimulation *over $N, \mathcal{A}$ if $\mathcal{A}$ is $B$-closed and for all $M_1, M_2 \in \mathcal{A}$ s.t. $M_1 \overline{B} M_2$, for all steps $M_1 \xrightarrow{t} M_1'$ there is a step $M_2 \xrightarrow{u} M_2'$ with $M_1' \overline{B} M_2'$ and $l(t) = l(u)$.*

Hence a place bisimulation is a $B$ s.t. $\overline{B} \cap (\mathcal{A} \times \mathcal{A})$ has the transfer property.

**Fact 8.2.** *Any transitive place bisimulation over $N, \mathcal{A}$ is a correct fusion.*

**Proposition 8.3.** *There exists a largest place bisimulation over $N, \mathcal{A}$, denoted $B(N, \mathcal{A})$.*

*Proof (Idea).* Similar to the proof of Theo. 7.1. □

Clearly $B(N, \mathcal{A}) \subseteq R(N, \mathcal{A})$.

The definition we gave is not very effective, and it does not give a direct way to see whether $B(N, \mathcal{A})$ is computable. Indeed, given a potential $B$, checking whether $\overline{B} \cap (\mathcal{A} \times \mathcal{A})$ has the transfer property involves checking an infinite number of situations (if $\mathcal{A}$ is infinite).

We now explain how to reduce this to a finite number of checks.

**Definition 8.4.** *The set of all* minimal covers *for a transition $t$, denoted $C(t)$, is the set of all minimal markings $M \in \mathcal{A}$ s.t. ${}^\bullet t \subseteq M$.*

Here "$M$ minimal" means that $M' \in \mathcal{A}$ and ${}^\bullet t \subseteq M' \subseteq M$ imply $M = M'$.

Note that Dickson's lemma implies the finiteness of $C(t)$ for any $t$. $C(t)$ can be empty if there is no way to enlarge ${}^\bullet t$ into some element of $\mathcal{A}$.

**Definition 8.5.** *We say that relation $B \subseteq P \times P$ has the* weak transfer property *if $\mathcal{A}$ is $B$-closed, and for all $t \in T$, for all $p \in {}^\bullet t$, for all $q \in P$ s.t. $pBq$, for all $M_1 \in C(t)$ there is a $u \in T$ s.t. $l(u) = l(t)$ and, writing $M_2$ for $M_1 - p + q$, we have $M_2 \xrightarrow{u} M_2'$ with $M_1' \overline{B} M_2'$ (assuming $M_1 \xrightarrow{t} M_1'$).*

Hence the weak transfer property is the transfer property restricted to pairs $M_1, M_2$ and steps $M_1 \xrightarrow{t} M_1'$ where $M_1$ is in the minimal cover $C(t)$ and where $M_2$ differs from $M_1$ by only one token. Since this is a weaker property, we have

**Fact 8.6.** *$B(N, \mathcal{A})$ has the weak transfer property.*

**Lemma 8.7.** *Let $B$ be an equivalence s.t. $\mathcal{A}$ is $B$-closed. If $B$ has the weak transfer property, then it is a place bisimulation.*

*Proof.* We show that $\overline{B} \cap (\mathcal{A} \times \mathcal{A})$ has the transfer property, i.e. "for all $M_1 \overline{B} M_2$, for all $M_1 \xrightarrow{t_1} M_1'$, there is a $M_2 \xrightarrow{t_2} M_2'$ s.t. ...", by induction over the pair $(|M_1|, \Delta(M_1, M_2))$ ordered lexicographically.

So let us assume $M_1 \xrightarrow{t_1} M_1'$ and $M_1 \overline{B} M_2$. Write $n$ for $\Delta(M_1, M_2)$. We distinguish four cases:
1. $n = 0$: then $M_2 = M_1$ and we are done.
2. $n = 1$ and $M_1 \in C(t_1)$: then we are exactly in the situation where the weak transfer property applies.
3. $n > 1$: then there is a $M_3 \in \mathcal{A}$ s.t. $M_1 \overline{B} M_3 \overline{B} M_2$ and $\Delta(M_i, M_3) < n$ for $i = 1, 2$. By ind. hyp., we can transfer $M_1 \xrightarrow{t_1} M_1'$ to some $M_3 \xrightarrow{t_3} M_3'$ that we can then transfer to some $M_2 \xrightarrow{t_2} M_2'$. We have $M_1' \overline{B} M_3' \overline{B} M_2'$, hence $M_1' \overline{B} M_2'$.
4. $M_1 \notin C(t_1)$: then there is a $M_{1,1} \in C(t_1)$ s.t. $M_{1,1} \subset M_1$. We decompose $M_1$ as $M_{1,1} + M_{1,2}$ and $M_2$ as $M_{2,1} + M_{2,2}$ s.t. $M_{1,i} \overline{B} M_{2,i}$ for $i = 1, 2$. $M_{1,1} \xrightarrow{t_1} M_{1,1}'$ and the ind. hyp. transfers this into a $M_{2,1} \xrightarrow{t_2} M_{2,1}'$ with $M_{1,1}' \overline{B} M_{2,1}'$. We complete into $M_2 \xrightarrow{t_2} M_2' \stackrel{\text{def}}{=} M_{2,1}' + M_{2,2}$ and have $M_1' \overline{B} M_2'$. □

We have the immediate corollary:

**Theorem 8.8.** *The largest reflexive and symmetric $B$ having the weak transfer property exists and is $B(N, \mathcal{A})$.*

Finiteness of $C(t)$ implies that checking whether a given $B$ has the weak transfer property only involves a finite number of checks. Therefore the following abstract algorithm computes $B(N, \mathcal{A})$:

**Algorithm 8.9.** *input* a labeled Petri net $N$, and a succ-closed set $\mathcal{A}$ of relevant markings.

*output* $B(N, \mathcal{A})$.

*step1* Let $B = E(N, \mathcal{A})$, "the largest place equivalence under which $\mathcal{A}$ is closed".

*step2* Check if $B$ has the weak transfer property:

- If it has, then $B$ is $B(N, \mathcal{A})$.
- Otherwise, there is a $t \in T$, a $M \in C(t)$, a $p \in M$, a $q \in P$ with $pBq$ s.t. $M \xrightarrow{t} M'$ can not be imitated from $M - p + q$. Then remove the pairs $(p, q)$ and $(q, p)$ from $B$ and return to step 2.

□

This algorithm is obviously correct and stops after a finite number of steps. Its complexity depends on the size of $C(t)$, and the cost of computing $C(t)$ and $E(N, \mathcal{A})$.

Note that it is possible to run the algorithme starting with any equivalence $E$ included in $E(N, \mathcal{A})$ (i.e. any $E$ such that $\mathcal{A}$ is $E$-closed), and obtain the largest place bisimulation included in $E$, a safe approximation of $B(N, \mathcal{A})$.

## 9 Place bisimulation in practice

*Algorithmical aspects.* When we want to use Algorithm 8.9 in practice, we face two problems: we need some effective way of computing $C(t)$ for $t \in T$, and we need to compute $E(N, \mathcal{A})$ that will be the first upper approximation of $B(N, \mathcal{A})$.

Whether this can be done depends on how $\mathcal{A}$ is given. In the simple case where $\mathcal{A}$ is finite, then $C(t)$ and $E(N, \mathcal{A})$ are computable in a direct way. More interestingly, when $\mathcal{A}$ is given by a Presburger formula (i.e., $\mathcal{A}$ is a semilinear set) then $C(t)$ and $E(N, \mathcal{A})$ can then be defined by Presburger formulas, and can be effectively computed: indeed assume $P = \{p_1, \ldots, p_k\}$ and $\mathcal{A}$ is given by the Presburger formula $\psi(n_1, \ldots, n_k)$. Then $(p_i, p_j) \in E(N, \mathcal{A})$ iff $\psi(n_1, \ldots, n_k) \wedge n_i > 0$ entails $\psi(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_k)$.

*Does it work ?* As far as we could judge by dealing with a few examples, the equivalences we compute with Algorithm 8.9 are not much of an improvement over the classical notion of place bisimulation where all markings are considered as relevant.

The reason seems to be that, when using Algorithm 8.9, we first have to give a succ-closed set $\mathcal{A}$. If this is a large set, then it contains many irrelevant markings and this defeats the purpose of our study. If $\mathcal{A}$ is as small as possible (e.g., it contains only the reachable markings) then this may impact the equivalence $E(N, \mathcal{A})$ computed in *step1* and make it quite drastically limited even before it is refined into a place bisimulation.

In the end, we can only argue the usefulness of our definition with ad-hoc examples, so that it seems like it is mostly a theoretical contribution to place bisimulation. Our hope is that this contribution can be combined smoothly with the ideas from [APS94] where silent moves are abstracted from, and that this combination does work well in practice.

## 10 Conclusion

This paper proposed a new point of view on Petri net reductions based on the bisimilarity notion, and that do not aim at correctness for all markings.

We started with a net $N$ and looked at reductions that lead to bisimilar configurations for any of the markings in some set $\mathcal{A}$. It turns out restrictions must be imposed on $\mathcal{A}$ in order to get a smooth theory.

Since the largest correct reduction is in general not computable, place bisimulations is an elegant close approximation. We gave an abstract algorithm computing the largest place bisimulation when $\mathcal{A}$ is given in a sufficiently effective way.

These results shed a new light on earlier works on place bisimulation. Having reductions in mind from the start help understand why place bisimulation cannot cope easily with arbitrary set of markings. Indeed, we end up with quite strong restrictions on the set $\mathcal{A}$ which, ideally, we would like to consist of the reachable markings only.

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