Poset Properties of Complex Traces*

Paul GASTIN
Université PARIS 6
LITP, Institut Blaise Pascal
4, place Jussieu
75 252 PARIS CEDEX 05
FRANCE
gastin@litp.ibp.fr

Antoine PETIT
Université PARIS SUD
LRI, URA CNRS 410
Bât. 490
91 405 ORSAY CEDEX
FRANCE
petit@lri.lri.fr

Abstract: A. Mazurkiewicz introduced the monoid of traces to provide a very natural semantics for concurrent processes. In order to describe the behaviours of non sequential processes which never terminate, the notion of infinite traces was needed. The monoid $\mathbb{C}$ of complex traces has been introduced by V. Diekert as a quotient of the monoid $\mathbb{G}$ of infinite dependence graphs by a suitable congruence. In the monoid $\mathbb{C}$, the subset $\mathbb{R}$ of real traces (i.e. the set of infinite dependence graphs where all vertices have a finite past) is of first importance. The PoSet properties of the set $\mathbb{R}$ of real traces have already been studied: $\mathbb{R}$ is a coherently complete Scott-Domain and is Prime Algebraic. Moreover the compact and the prime traces in $\mathbb{R}$ have been characterized. The aim of this paper is to study generalizations of all these results to the set of complex traces $\mathbb{C}$ and to the set of dependence graphs $\mathbb{G}$. We show that a subset of $\mathbb{G}$ admits a least upper bound if and only if this set is coherent and countable, and we prove that the least upper bound is in fact the union of the elements of the set. Finally $\mathbb{G}$ is not a CPO but is simply bounded complete. The compact and the prime graphs in $\mathbb{G}$ are characterized and we prove that each graph is the least upper bound of its compact (resp. its prime) lower bounds. Therefore, up to the restriction to countable sets, $\mathbb{G}$ is a coherently complete Scott-Domain and is Prime Algebraic. We define very naturally two orders on $\mathbb{C}$: the product order and the prefix order. We show that $\mathbb{C}$ with each order is coherently complete (in particular $\mathbb{C}$ is a CPO with the empty trace as least element) and we characterize the least upper bound (the greatest lower bound resp.) of a subset of $\mathbb{C}$ when it exists. But contrary to the case of $\mathbb{R}$, we prove that in general a complex trace is not the least upper bound of its compact (its prime resp.) lower bounds.

1. Introduction

The concept of traces has been introduced by A. Mazurkiewicz [Maz77] as a suitable semantics for non-sequential processes. There are two equivalent ways to see a trace. A trace can be considered as the set of all possible sequential observations of a concurrent process. The semantics of each sequential observation is given, in a classical way, by a finite word. Hence, a trace is an equivalence class of words. An equivalent point of view is to see a trace as a finite labelled acyclic graph where edges represent dependence of actions. Such a graph is called a dependence graph and the semantics is that an execution has to respect the induced partial order. Whatever the chosen approach, the definition of the concatenation of two traces is easy and natural and traces form a monoid. This monoid has independently been introduced and studied by combinatorists [CF69] for other purposes. But it has particularly been studied as a model for concurrent systems since the pioneer work of Mazurkiewicz. Let us refer, for instance, surveys [Maz86, AR88, Per89] or the monograph [Die90] where also extensive bibliographies on the subject are given.
In order to describe the behaviours of non sequential processes which never terminate, the notion of infinite traces has been introduced. Finite dependence graphs admit a convenient

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generalization to infinite dependence graphs where a natural concatenation can be defined. Unfortunately, these graphs are not abstract enough for the required purpose. V. Diekert has defined the monoid $\mathbb{C}$ of complex traces as the quotient of the monoid $\mathbb{G}$ of infinite dependence graphs by a suitable congruence. The monoid $\mathbb{C}$ satisfies many nice properties [Die91]: it reduces to the free infinitary monoid (the free commutative infinitary monoid resp.) when the dependence relation is "full" ("empty" resp.); furthermore, for a natural ultrametric, $\mathbb{C}$ is a compact space and is the completion of the monoid of finite traces.

In this monoid, the subset $\mathbb{R}$ of real traces is of first importance. Intuitively, a real trace is an infinite dependence graph where all actions can be performed within a finite time (i.e. all actions have a finite past). Previously to the definition of $\mathbb{C}$, real traces where already introduced and intensively studied. Topological properties, links with event structures and recognizable and rational languages have in particular been investigated [BMP89, Gas90, GPZ91, GR91].

In this paper we focus on the PoSet properties of the sets $\mathbb{R}$, $\mathbb{G}$ and $\mathbb{C}$.

The PoSet properties of the set $\mathbb{R}$ of real traces have already been studied. It is the natural order completion of the PoSet $\mathbb{M}$ of finite traces [Kwi90,Kwi91]. In $\mathbb{R}$, the prefix relation turns out to be an order relation. With this relation any coherent subset of $\mathbb{R}$ (any pair of elements of the subset admits an upper bound in $\mathbb{R}$) admits a least upper bound. In other words the PoSet $\mathbb{R}$ of real traces is coherently complete and thus, in particular, a CPO [GR91]. In a CPO the compact and primes elements play an essential role. In the PoSet $\mathbb{R}$, the compact elements are exactly the finite traces whereas the prime elements are the finite traces with one maximal vertex only. Moreover any real trace is the least upper bound of its compact (prime resp.) lower bounds. In other words the CPO $\mathbb{R}$ is a Scott-Domain (is Prime Algebraic resp.).

The aim of this paper is to study generalizations of all these results to the set of complex traces $\mathbb{C}$ and to the set of dependence graphs $\mathbb{G}$.

First we show that a subset of $\mathbb{G}$ admits a least upper bound if and only if this set is coherent and countable. In this case we prove that the least upper bound is in fact the union of the graphs of the subset. Due to this restriction to countable sets, $\mathbb{G}$ is neither a CPO nor coherently complete but $\mathbb{G}$ is simply bounded complete. Then we characterize the compact (prime resp.) elements of $\mathbb{G}$ by showing that a dependence graph is compact (prime resp.) if and only if it is the past of some finite set of vertices (the past of one vertex resp.). At last, a dependence graph is the least upper bound of its compact lower bounds (its prime lower bounds resp.). Therefore, up to the restriction to countable sets, $\mathbb{G}$ is a coherently complete Scott-Domain and is Prime Algebraic. Note that we prove these results on $\mathbb{G}$ using extensions of the methods used in [GR91]. As a by-product we obtain of course the results on $\mathbb{R}$.

It turns out that a complex trace is simply a real trace together with an alphabetic information [Die91]. Hence $\mathbb{C}$ can be seen as a subset of $\mathbb{R}\times\mathcal{P}(X)$. Then we introduce very naturally the product order on $\mathbb{C}$. We use it intensively to derive properties of the prefix order on $\mathbb{C}$. We show that $\mathbb{C}$ with each order is coherently complete (in particular $\mathbb{C}$ is a CPO with the empty trace as least element) and we characterize the least upper bound (the greatest lower bound resp.) of a subset of $\mathbb{C}$ when it exists. Contrary to the case of $\mathbb{R}$, we prove that in general a complex trace is not the least upper bound of its compact (prime resp.) lower bounds. More precisely, for the product order, the compacts are exactly the finite traces. Hence an upper bound of compacts is always a real trace. For the prefix order there exist compact infinite traces but, in general, there exist complex traces which are strictly greater than the least upper bound of their compact lower bounds.

2. Preliminaries

2.1. Dependence Graphs and Traces

In this subsection, we recall definitions and results on traces which are relevant in this paper. For more detailed discussions, the reader is referred to [Maz86, AR88, Per89, Die90] as concerns finite traces, to [Gas90, Kwi90, Kwi91, GR91] as concerns real traces and to [Die91, DGP91] for complex traces.

In the following (X,D) denotes a finite dependent alphabet, i.e. X is a finite alphabet together with a reflexive and symmetric dependence relation $D \subseteq X \times X$. We denote by $I = X \times X \setminus D$ the independence relation which is the complement of D.
A dependence graph over (X,D) is (an isomorphism class of) a labelled acyclic graph $[V,E,\lambda]$ where V is the (at most) countable set of vertices, $E \subseteq V \times V$ is the set of arcs, $\lambda : V \to X$ is the labelling and which satisfies:

i) the induced partial order (V,E^*) is well-founded: there does not exist an infinite descending chain i.e. an infinite sequence (x_i) such that $(x_{i+1}, x_i) \in E$ for all i.
ii) edges are between dependent vertices:

$$\forall\, x,y \in V, \qquad (\lambda(x),\lambda(y)) \in D \Leftrightarrow x = y \text{ or } (x,y) \in E \text{ or } (y,x) \in E$$

Then, for any a in X, the restriction to the set of vertices with label a is well-ordered. This allows to think of dependence graphs by standard representations where the vertices are pairs (a,i) with $a \in X$ and i is a countable ordinal. The set of dependence graphs is denoted by $\mathbb{G}(X,D)$ or simply by $\mathbb{G}$ when there is no confusion. It is a monoid for the operation $[V_1,E_1,\lambda_1]\cdot[V_2,E_2,\lambda_2] = [V,E,\lambda]$ where $[V,E,\lambda]$ is the disjoint union of $[V_1,E_1,\lambda_1]$ and $[V_2,E_2,\lambda_2]$ together with new arcs (p_1,p_2) for all vertices $p_1 \in V_1$, $p_2 \in V_2$ such that $(\lambda_1(p_1),\lambda_2(p_2)) \in D$ (see Figure 2.1). The neutral element is the empty graph $1 = [\emptyset,\emptyset,\emptyset]$.

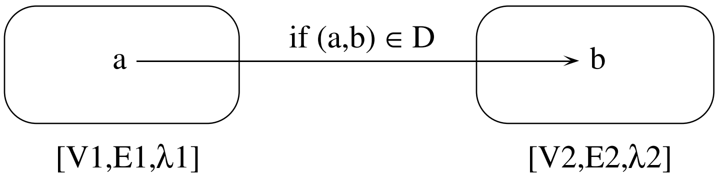

Figure 2.1.

The concatenation is generalized to an infinite product as follows. Let (g_i) be any sequence of dependence graphs. Then $g = g_1 g_2 \ldots g_i \ldots \in \mathbb{G}$ is defined as the disjoint union of the g_i with the obvious new arcs from g_i to g_j for i<j between vertices have dependent labels. Note that concatenation and infinite product are associative operations.
Using these operations, we relate words and dependence graphs. We define the mapping $\varphi: X^\infty \to \mathbb{G}$ by $\varphi(a) = [\{p\},\emptyset,p \mapsto a]$ for all a in X and $\varphi(a_1 a_2 \ldots) = \varphi(a_1)\varphi(a_2)\ldots$ for all finite or infinite word $a_1 a_2 \ldots$ in X^∞. Note that φ is not injective, for instance $\varphi(ab) = \varphi(ba)$ iff $(a,b) \in I$. We define an equivalence ~ on X^∞ by $u \sim v$ if $\varphi(u) = \varphi(v)$. It turns out that φ is a surjective morphism from the free monoid X^* onto the monoid $\mathbb{FG}(X,D)$ of finite dependence graphs. The monoid of finite traces introduced by A. Mazurkiewicz [Maz77] is precisely $X^*/\!\sim\, = \mathbb{FG}(X,D)$ which is denoted by $\mathbb{M}(X,D)$ or simply by $\mathbb{M}$ throughout.

In a similar way, we will define the monoid of infinite traces from the monoid $\mathbb{G}$. But it turns out that $\varphi : X^\infty \to \mathbb{G}$ is neither surjective nor a morphism. The set $\varphi(X^\infty) = X^\infty/\!\sim$ of dependence graphs which can be represented by words is denoted by $\mathbb{R}(X,D)$ or simply by $\mathbb{R}$. Its elements are called real dependence graphs or real traces. $\mathbb{R}$ is characterized as follows: a dependence graph $[V,E,\lambda]$ is a real one iff the past $\downarrow p = \{q \in V / (q,p) \in E^*\}$ is finite for every vertex p of V. For instance, the dependence graph of Figure 2.2 is not a real dependence graph. Note however that it can be obtained as the concatenation of two real dependence graphs, namely $\varphi(acbcca^{\omega})$ and $\varphi(ab^{\omega})$.

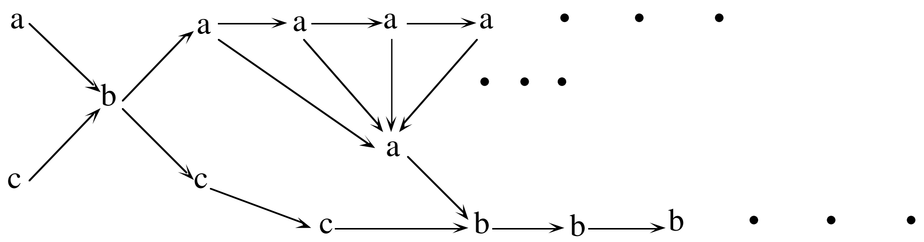

Figure 2.2.

More generally, each element $g = [V,E,\lambda] \in \mathbb{G}$ splits into its real part Re(g) and its transfinite part Tr(g) which are the restrictions of the dependence graph g to $\{p \in V / \downarrow p$ is finite$\}$ and $\{p \in V / \downarrow p$ is infinite$\}$ respectively. Obviously, Re(g) and Tr(g) are dependence graphs and, in $\mathbb{G}$, the equation $g = Re(g) \cdot Tr(g)$ holds. In fact, $\mathbb{R}$ is the set of dependence graphs with an empty transfinite part.

Two dependence graphs $g,h \in \mathbb{G}$ are said to be *practically distinguishable* if there exists a (finite) real trace s such that $g \cdot s$ and $h \cdot s$ have different real parts, i.e., if $Re(g \cdot s) \neq Re(h \cdot s)$. In particular, two traces with different real parts are practically distinguishable. It turns out [Die91] that practical-undistinguishability is the coarsest congruence of $\mathbb{G}$ which respects real parts. The quotient of $\mathbb{G}(X,D)$ by this congruence yields the monoid of *complex traces* $\mathbb{C}(X,D)$ (denoted also by $\mathbb{C}$ when there is no confusion).

In order to give an explicit characterization of the congruence "practical-undistinguishability", we introduce the following notations. For a subset $A \subseteq X$, define $D(A) = \{a \in X / (a,b) \in D$ for some $b \in A\}$ and $I(A) = X \setminus D(A)$. For $g = [V,E,\lambda] \in \mathbb{G}$, let $alph(g) = \lambda(V)$ be the alphabet of g and let $alphinf(g) = \{a \in X / |\lambda^{-1}(a)|$ is infinite$\} \cup alph(Tr(g))$ be the alphabet at infinity of g. Finally, let $Im(g) = D(alphinf(g))$ be the imaginary part of g.

Two dependence graphs $g,h \in \mathbb{G}$ are practically undistinguishable if and only if $Re(g) = Re(h)$ and $Im(g) = Im(h)$ [Die91]. Hence a complex trace is a pair $(Re(g),Im(g)) = (r,D(A))$ for some $g \in \mathbb{G}$.

Remark that two real traces are practically undistinguishable if and only if they are equal. Thus, $\mathbb{R}$ is embedded in $\mathbb{C}$. More precisely, a real trace $r \in \mathbb{R}$ is identified with its image $(r,Im(r))$ in $\mathbb{C}$.

Even if the concatenation in $\mathbb{C}$ is of course defined through the practical-undistinguishability congruence, it will be convenient to have a direct formula of the concatenation of two complex traces $(r,D(A))$ and $(s,D(B))$. To this purpose we extract from the real part s, the maximal real prefix $\mu_A(s)$ of s containing letters from I(A) only. Formally, for any real trace $s = [V,E,\lambda]$, $\mu_A(s) = \{p \in V / alph(\downarrow p) \subseteq I(A)\}$. Since $\mathbb{R}$ is left-cancellative, there is a unique real trace $suff_A(s)$ such that $s = \mu_A(s) \cdot suff_A(s)$. With these notations, the concatenation in $\mathbb{C}$ is simply given by [Die91]:

$$(r,D(A))(s,D(B)) = (r\mu_A(s),D(A \cup B \cup alph(suff_A(s)))).$$

2.2 PoSets.

In this subsection, we recall the definitions concerning Partial Order Sets (PoSets) which will be used in this paper. We are mainly concerned by completeness of the PoSet, that is

sufficient conditions ensuring the existence of least upper bounds. We will also study the algebraicity of the PoSet which states that each member of the PoSet can be approximated, by means of least upper bounds, using “simple” (compact or prime) elements of the PoSet. For more details concerning continuous order, the reader is referred to [COMP, Law88].
Let $(Z, \leqslant)$ be a PoSet and let Y be a subset of Z. Y is (upper-)*bounded* if there exists $z \in Z$ such that $y \leqslant z$ for all $y \in Y$. Y is *directed* if each pair $\{x,y\} \subseteq Y$ admits an upper-bound *in Y*. Y is *coherent* if each pair $\{x,y\} \subseteq Y$ admits an upper-bound *in Z*. The PoSet $(Z, \leqslant)$ is *bounded-complete* (a *complete partial order (CPO)*, *coherently complete* resp.) if any bounded subset (directed subset, coherent subset resp.) of Z admits a least upper bound. Frequently, the existence of a least element is also required for a CPO. The coherent completeness is clearly the strongest property since any bounded set (directed set resp.) is coherent.
Whenever it exists, the least upper bound (the greatest lower bound resp.) of a subset $Y \subseteq Z$ is denoted by $\sqcup Y$ ($\sqcap Y$ resp.).
Let $(Z, \leqslant)$ be a PoSet. An element $x \in Z$ is *compact* (*prime* resp.) if for any directed subset (any subset resp.) Y of Z which admits a least upper-bound and such that $x \leqslant \sqcup Y$, there exists $y \in Y$ such that $x \leqslant y$. For $x \in Z$, we denote by C(x) (P(x) resp.) the set of compact (prime resp.) elements lower than x. A PoSet $(Z, \leqslant)$ is *algebraic (prime algebraic* resp.) if $x = \sqcup C(x)$ ($x = \sqcup P(x)$ resp.) for all $x \in Z$. An algebraic CPO is called a (Scott-)Domain. Frequently, one restricts oneself to domains with a countable set of compacts.

3. The PoSet of (Real) Dependence Graphs

In this section, we study some properties of the PoSets $\mathbb{G}$ and $\mathbb{R}$. We show that some of the results obtained in [GR91] on real traces can easily be extended to the whole set of dependence graphs.
Since the induced partial order of a dependence graph is well-founded, for any a in X, the restriction to the set of vertices with label a is well-ordered. Hence, for a dependence graph g, we will use the standard representation $[V_g,E_g,\lambda]$ where the vertices are pairs (a,i) with $a \in X$ and i is a countable ordinal and the labelling is defined by $\lambda((a,i)) = a$. For any countable ordinal α, we denote by $\mathbb{G}_\alpha$ the subset of $\mathbb{G}$ which consists of graphs g such that, $(a,i) \in V_g \Rightarrow i \leq \alpha$, for all letter $a \in X$.
Let us consider the prefix relation $\leqslant$ in $\mathbb{G}$ defined by $f \leqslant h$ if $f{\cdot}g = h$ for some g in $\mathbb{G}$. Since $\mathbb{G}$ is left-cancellative, $f{\cdot}g = h$ and $h{\cdot}g' = f$ implies $g{\cdot}g' = 1$ and therefore $g = g' = 1$, whence $f = h$. Hence the prefix relation in $\mathbb{G}$ is anti-symmetric and since it is clearly reflexive and transitive, is a partial order relation.
The following lemma is easy to prove but allows a good understanding of the prefix order relation. It will be used several times in the sequel of this section. Recall that a subgraph of a graph $[V,E,\lambda]$ is a graph $[V',E',\lambda]$ such that $V' \subseteq V$ and $E' = E \cap (V'{\times}V')$.

Lemma 3.1. Let f, h be two dependence graphs in $\mathbb{G}$. Then $f \leqslant h$ if and only if f is a downward closed subgraph of h, i.e.

$$\forall y \in V_f,\ (x,y) \in E_h \Rightarrow (x \in V_f \text{ and } (x,y) \in E_f) \qquad (1)$$

Proof. If $f \leqslant h$, there exists an element g in $\mathbb{G}$ such that $f{\cdot}g = h$. By definition of the concatenation in $\mathbb{G}$, f is obviously a subgraph of h which verifies (1).
Conversely, suppose that f is a subgraph of h which verifies (1). Let g be the subgraph of h defined by $V_g = V_h \setminus V_f$ and $E_g = E_h \cap (V_g{\times}V_g)$. By the condition (1) on f, there is no edge (x,y) in h with $x \in V_g$ and $y \in V_f$. Hence we get $f{\cdot}g = h$ and the lemma is proved. ◊

We now consider a coherent subset G of $\mathbb{G}$ and we will investigate the existence of the least upper bound of G. Let us first define the labelled graph $h = [V,E,\lambda] = \cup_{g\in G}\, g$ by $V = \cup_{g\in G} V_g$, $E = \cup_{g\in G} E_g$ and for all (a,i) in V, $\lambda((a,i)) = a$. We will prove that h is a dependence graph which is precisely the least upper bound of G. We first need a technical lemma on the paths in h and G.

Lemma 3.2. Let $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n$ be a finite path in $h = \cup_{g\in G}\, g$ and let g be an element of G such that $x_n \in V_g$. Then $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n$ is a finite path in g.

Proof. We prove this lemma by induction on the length n of the path. If n = 0, the result is obvious. Suppose now that the lemma is proved for paths of length n and let $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n \rightarrow x_{n+1}$ be a path in h and $x_{n+1} \in V_g$. Since $(x_n,x_{n+1}) \in E$, there exists some g' in G such that $(x_n,x_{n+1}) \in E_{g'}$. Since G is coherent, there exists an element g" in $\mathbb{G}$ such that $g \leqslant g''$ and $g' \leqslant g''$. Hence we have $(x_n,x_{n+1}) \in E_{g''}$, $x_{n+1} \in V_g$ and $g \leqslant g''$. By Lemma 3.1, we deduce $x_n \in V_g$ and $(x_n,x_{n+1}) \in E_g$. Now, by induction hypothesis, $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n$ is a finite path in g. Therefore $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n \rightarrow x_{n+1}$ is also a finite path in g and the lemma is proved. ◊

Proposition 3.3. Let α be a countable ordinal and let G be a coherent subset of $\mathbb{G}_\alpha$. Then G admits $\cup_{g\in G}\, g$ as least upper bound. (In particular $\mathbb{G}_\alpha$ is a coherently complete CPO).

Proof. We will first show that $h = \cup_{g\in G}\, g$ is a dependence graph.
Since G is included in $\mathbb{G}_\alpha$, the set V is a subset of $X\times\alpha$ and thus is countable.
Suppose that $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n = x_0$ is a cycle in h. Let g be in G such that $x_n \in V_g$. By Lemma 3.2, $x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n = x_0$ is a cycle in g which is in contradiction with $g \in \mathbb{G}$. Hence h is acyclic.
Let x, y be two distinct vertices in V such that $(\lambda(x),\lambda(y)) \in D$. Let g, g' be in G such that $x \in V_g$ and $y \in V_{g'}$. Since G is coherent, $g \leqslant g''$ and $g' \leqslant g''$ for some g" in $\mathbb{G}$. Since g" is a dependence graph, we get either $(x,y) \in E_{g''}$ or $(y,x) \in E_{g''}$. By Lemma 3.1, we deduce that either $x \in V_{g'}$ and $(x,y) \in E_{g'}$ or $y \in V_g$ and $(y,x) \in E_g$. Hence we have either $(x,y) \in E$ or $(y,x) \in E$. Conversely let x, y be in V such that $(x,y) \in E$. There exists some g in G such that $(x,y) \in E_g$ and thus $(\lambda(x),\lambda(y)) \in D$.
Assume that $\ldots x_n \rightarrow x_{n-1} \rightarrow \ldots \rightarrow x_0$ is a decreasing sequence in h. There exists g in $\mathbb{G}$ such that $x_0 \in g$, then by Lemma 3.2, $\ldots x_n \rightarrow x_{n-1} \rightarrow \ldots \rightarrow x_0$ is also a decreasing sequence in g. Since g is a dependence graph, such a sequence must be finite. Hence there is no infinite decreasing sequence in h.
Hence we have proved that h is a dependence graph.
Let now g be in G. By construction, g is a subgraph of h, moreover, from Lemma 3.2, g verifes the condition (1) of Lemma 3.1. Hence $g \leqslant h$ and h is an upper bound of G. Finally, from Lemma 3.1, any upper bound h' of G must contain any element g of G as a subgraph. Then h' must contain h as a subgraph. From Lemma 3.1 and 3.2, we deduce that h is in fact a prefix of h'. Therefore h is the least upper bound of G. ◊

Note that, by construction if G is a coherent subset of $\mathbb{G}_\alpha$, the least upper bound $\sqcup G = \cup_{g\in G}\, g$ is also an element of $\mathbb{G}_\alpha$. On the other hand, let G be a coherent subset of $\mathbb{R} \subseteq \mathbb{G}_\omega$ and let x be a vertex of $h = \cup_{g\in G}\, g$. We have $x \in g$ for some g in G. By Lemma 3.2, the past of x in h is also the past of x in g. Since $g \in \mathbb{R}$, this past is finite, therefore we obtain:

Theorem 3.4. [GR91]. The PoSet $\mathbb{R}$ of real traces is coherently complete.

Since the coherent completeness is the strongest completeness property, we deduce immediately that $\mathbb{R}$ is bounded complete and is a CPO with the empty trace as least element. From classical results of PoSet theory, we also get that every non empty subset of $\mathbb{R}$ admits a greatest lower bound.
Since a countable union of countable sets remains countable, every countable subset of $\mathbb{G}$ is included in $\mathbb{G}_\alpha$ for some countable ordinal α. Hence we get from Proposition 3.3.

Corollary 3.5. Any countable coherent subset G of $\mathbb{G}$ admits $\cup_{g\in G}$ g as least upper bound.

Every dependence graph $g = [V,E,\lambda]$ admits a countable number of prefixes f in $\mathbb{G}$. Indeed, let α be a countable ordinal such that $g \in \mathbb{G}_\alpha$. Let us consider an application μ from X into α, there is at most one prefix $f = [V_f,E_f,\lambda]$ of g such that $(a,i) \in V_f \Leftrightarrow i \leq \mu(a)$ for all letter a in X. Hence the number of prefix of g is bounded by the number of applications from X into α and is therefore countable. Therefore any bounded subset of $\mathbb{G}$ is countable (and indeed coherent). Thus from Corollary 3.5, we claim:

Proposition 3.6. i) The set $\mathbb{G}$ of dependence graphs is bounded complete.
ii) A subset G of $\mathbb{G}$ admits a least upper bound if and only if it is coherent and countable. This least upper bound is then $\cup_{g\in G}$ g.

Hence, similarly to the case of subsets of $\mathbb{R}$, every non empty subset of $\mathbb{G}$ admits a greatest lower bound. Note that, since the set $\{a^i$ / i is a countable ordinal$\}$ is directed but obviously not bounded in $\mathbb{G}$, the set $\mathbb{G}$ is not a CPO.

The two following results characterize the compact and the prime elements of $\mathbb{G}$. They extend results on $\mathbb{R}$ presented in [GR91]. Recall that for $g = [V,E,\lambda] \in \mathbb{G}$ and $P \subseteq V$, we denote by $\downarrow P \in \mathbb{G}$ the prefix of g which is the restriction of g to the past of P: $\{q \in V$ / $(q,p) \in E^*$ for some $p \in P\}$.

Proposition 3.7. Let α be a countable ordinal and let $f = [V,E,\lambda]$ be a dependence graph of $\mathbb{G}_\alpha$. Then f is compact in $\mathbb{G}_\alpha$ if and only if $f = \downarrow P$ for some finite subset P of V.

Proof. First, suppose that $f = \downarrow P$ for some finite subset P of V. Let G be a directed subset of $\mathbb{G}_\alpha$ such that $f \leqslant \sqcup G = \cup_{g\in G}$ g (Proposition 3.3). Since P is finite and G is directed, there exists some g in G such that $P \subseteq V_g$. Since f and g are both bounded by $\sqcup G$, we deduce from Lemma 3.1 that the past of P in f equals the past of P in $\sqcup G$, equals the past of P in g. Using again Lemma 3.1, we obtain $f = \downarrow P \leqslant g$.
Conversely, suppose that $f = [V,E,\lambda]$ is compact. The set $Y = \{\downarrow P$ / P is a finite subset of V$\}$ is clearly directed since $\sqcup\{\downarrow P,\downarrow P'\} = \downarrow(P \cup P')$. From Proposition 3.6, Y admits f as least upper bound. Since f is compact, we deduce that $f \leqslant \downarrow P$ for some $P \subseteq V$ and thus $f = \downarrow P$. ◊

Proposition 3.8. Let $f = [V,E,\lambda]$ be a dependence graph in $\mathbb{G}$. Then f is prime if and only if $f = \downarrow p$ for some p in V.

Proof. First, suppose that $f = \downarrow p$ for some p in V. Let G be a subset of $\mathbb{G}$ such that $\sqcup G$ exists. By Proposition 3.6, $\sqcup G = \cup_{g\in G}$ g. Hence if $f \leqslant \sqcup G$, $p \in V_g$ for some g in G. Since f and g are both bounded by $\sqcup G$, we deduce from Lemma 3.1 that the past of p in f equals the past of p in $\sqcup G$, equals the past of p in g. Using again Lemma 3.1, we obtain $f = \downarrow p \leqslant g$.

Conversely, suppose that f = [V,E,λ] is prime. From Proposition 3.6, the set $\{\downarrow p \,/\, p \in V\}$ admits clearly f as least upper bound. Since f is prime, we obtain $f \leqslant \downarrow p$ for some $p \in V$ and thus $f = \downarrow p$. ◊

The following proposition asserts that any dependence graph g can be reconstructed from the compact (prime resp.) elements which are lower than it. Note that for any dependence graph g, C(g) (P(g) resp.) is bounded by g and then admits a least upper bound (Proposition 3.6).

Proposition 3.9. Let g be a dependence graph of $\mathbb{G}$. Then $g = \sqcup C(g) = \sqcup P(g)$.

Proof. Let g = [V,E,λ]. Since $P(g) \subseteq C(g)$, we have obviously $\sqcup P(g) \leqslant \sqcup C(g) \leqslant g$. Conversely, let $p \in V$, from Proposition 3.8, $\downarrow p \in P(g)$. Hence $g = \cup_{p \in V} \downarrow p \leqslant \sqcup P(g)$. ◊

In particular, a real trace is compact if and only if it is finite. Moreover this finite trace is prime if and only if it admits a unique maximal vertex. Hence, the set of compact real traces and the set of prime real traces are countable. Thus, from Propositions 3.7 and 3.8, we deduce immediately.

Corollary 3.10 [GR91]. The PoSet $\mathbb{R}$ of real traces is a Scott Domain and is Prime Algebraic with a countable set of compact (prime resp.) elements.

4. The PoSet of Complex Traces

As for real traces and dependence graphs, the partial order that will be considered on $\mathbb{C}$ is the prefix relation defined for all $x,y \in \mathbb{C}$ by:

$$x \leqslant y \text{ if } y = x \cdot z \text{ for some } z \in \mathbb{C}.$$

In the previous section, we have shown that the prefix relation on $\mathbb{G}$ is a partial order using two properties of the monoid $\mathbb{G}$: the left-cancellativity and the fact that $x \cdot y = 1 \Rightarrow x = y = 1$. Since $\mathbb{C}$ is not left-cancellative ($a^{\omega} \cdot a = a^{\omega} \cdot 1$) we cannot use the same arguments. To prove that the prefix relation on $\mathbb{C}$ is a partial order (and many other properties in this section) we use an auxiliary partial order on $\mathbb{C}$. Since $\mathbb{C} \subseteq \mathbb{R} \times \mathcal{P}(X)$, we naturally consider the partial order on $\mathbb{C}$ induced by the product order on $\mathbb{R} \times \mathcal{P}(X)$. More precisely, the product order on $\mathbb{C}$ is defined for all $x,y \in \mathbb{C}$ by:

$$x \sqsubseteq y \text{ if } \mathrm{Re}(x) \leqslant \mathrm{Re}(y) \text{ and } \mathrm{Im}(x) \subseteq \mathrm{Im}(y).$$

Since the prefix order on $\mathbb{R}$ and the subset order on $\mathcal{P}(X)$ are well-founded partial orders, we deduce easily that the product order on $\mathbb{C}$ is a well-founded partial order as well.
As in any monoid, the prefix relation on $\mathbb{C}$ is reflexive and transitive. Now, from the definition of the concatenation in $\mathbb{C}$, it follows directly that the prefix relation on $\mathbb{C}$ is coarser than the product order : $\forall\, x,y \in \mathbb{C},\ x \leqslant y \Rightarrow x \sqsubseteq y$ (note that, in general, the converse implication does not hold). Hence, the prefix relation on $\mathbb{C}$ is also a well-founded partial order.

In this section, the least upper bound of a coherent subset Y of $\mathbb{C}$ will be defined as a member of $\mathbb{R} \times \mathcal{P}(X)$ and then will be proved to be a complex trace. Hence, we need a characterization of the members of $\mathbb{R} \times \mathcal{P}(X)$ which are complex traces. To this purpose, we introduce the notion of consistent couples. A couple (A,B) of subsets of X is consistent if there exists a complex trace x such that Im(x) = D(B) and alphinf(Re(x)) = A. The following two lemmas state some useful properties of consistent couples.

Lemma 4.1. Let A,B be two subsets of X. The following conditions are equivalent:

i) (A,B) is consistent.
ii) There exists a finite sequence of letters $a_1,\ldots,a_k \in X$ such that for all $1 \le i \le k$, $a_i \in D(A \cup \{a_1,\ldots,a_{i-1}\})$ and $D(B) = D(A \cup \{a_1,\ldots,a_k\})$.
iii) (r,D(B)) is a complex trace for any real trace r such that alphinf(r) = A.

Proof. i) ⇒ ii) Let x be a complex trace such that Im(x) = D(B) and alphinf(Re(x)) = A. Let g be a dependence graph representing x. Let h be the subgraph of Tr(g) which consists of the first occurrence of each letter of Tr(g). It is easy to verify that g and Re(g)·h are practically undistinguishable, hence f = Re(g)·h represents x as well. Let $h = a_1\ldots a_k$ be a linearization of h. Since Tr(f) = h we obtain $a_i \in D(A \cup \{a_1,\ldots,a_{i-1}\})$ for all $1 \le i \le k$ and since f represents x we deduce $D(B) = Im(x) = D(A \cup \{a_1,\ldots,a_k\})$.
ii) ⇒ iii) Let r be a real trace such that alphinf(r) = A. Since $\mu_A(a_1\ldots a_k) = 1$, by definition of the concatenation in $\mathbb{C}$ we obtain $(r,D(A))\cdot(a_1\ldots a_k,\emptyset) = (r,D(A \cup \{a_1,\ldots,a_k\})) = (r,D(B))$ which is therefore a complex trace.
iii) ⇒ i) Clear. ◊

Lemma 4.2. Let (A,B) and (A',B') be two consistent couples. Then the couple (A ∪ A', B ∪ B') is consistent as well.

Proof. Let $a_1,\ldots,a_k$ and $a'_1,\ldots,a'_{k'}$ be the sequences of letters defined in Lemma 4.1 for the couples (A,B) and (A',B') respectively. Clearly, the sequence $a_1,\ldots,a_k, a'_1,\ldots,a'_{k'}$ satisfies the conditions of Lemma 4.1 ii) for the couple (A ∪ A', B ∪ B') which is therefore consistent. ◊

As for real traces, we will prove that the PoSet ($\mathbb{C}$, $\leqslant$) of complex traces satisfies the strongest completeness property. But first, we establish the same property for the product order.

Proposition 4.3. ($\mathbb{C}$,$\sqsubseteq$) is coherently complete.

Proof. We first introduce some notations. For any complex trace y, let A_y = alphinf(y) and B_y be such that Im(y) = D(B_y).
Let $Y \subseteq \mathbb{C}$ be $\sqsubseteq$-coherent. Clearly, Re(Y) is a coherent subset of $\mathbb{R}$. By Theorem 3.4, Re(Y) admits a least upper bound s. Let A = alphinf(s) and $B = A \cup (\cup_{y\in Y} B_y)$. We will prove that z = (s,D(B)) is the least upper bound of Y.
First, we show that z is a complex trace, that is, (A,B) is consistent. Let F be a finite subset of Y such that $\cup_{y\in F} B_y = \cup_{y\in Y} B_y$. Clearly $\cup_{y\in Y} A_y \subseteq A$, then $(A,B) = (A \cup (\cup_{y\in F} A_y), A \cup (\cup_{y\in F} B_y))$. Hence, by Lemma 4.2, we infer that (A,B) is consistent.
Clearly, z is an upper bound of Y. Conversely, let z' be an upper bound of Y. By definition of $\sqsubseteq$, we obtain $\sqcup \mathrm{Re}(Y) \leqslant \mathrm{Re}(z')$ and $\cup_{y\in Y} \mathrm{Im}(y) \subseteq \mathrm{Im}(z')$. Hence, $\mathrm{Re}(z) \leqslant \mathrm{Re}(z')$ and $\mathrm{Im}(z) = \mathrm{Im}(\mathrm{Re}(z)) \cup (\cup_{y\in Y} \mathrm{Im}(y)) \subseteq \mathrm{Im}(\mathrm{Re}(z')) \cup \mathrm{Im}(z') = \mathrm{Im}(z')$, that is $z \sqsubseteq z'$. ◊

The following remark, very often used in the sequel, allows to generalize this result to the prefix order. Let r, s be two real traces. Then, $r \leqslant s$ (in $\mathbb{C}$ or in $\mathbb{G}$) if and only if s = r·t for some real trace t such that alph(t) ⊆ I(alphinf(r)).

Theorem 4.4. ($\mathbb{C}$, $\leqslant$) is coherently complete.

Proof. Let $Y \subseteq \mathbb{C}$ be $\leqslant$-coherent. Since the prefix order is coarser than the product order, Y is also $\sqsubseteq$-coherent and by Proposition 4.3 it admits a least upper bound z = (s,D(B)) for the product order. We will prove that z is also the least upper bound for the prefix order. We use the notations introduced in the proof of Proposition 4.3.

First, we show that $z = (s,D(B))$ is an $\leqslant$-upper bound of Y. Let $y \in Y$, since $y \sqsubseteq z$ we have $Re(y) \leqslant s$ and thus there exists $t \in \mathbb{R}$ such that $s = Re(y)\cdot t$ and $alph(t) \subseteq I(A_y)$.

We claim that $alph(t) \subseteq I(B_y)$. Let $a \in alph(t)$ and let $r = {\downarrow}p$ where p is the $(|Re(y)|_a + 1)$-th vertex of $s = Re(y)\cdot t$ with label a. Since r is prime (Proposition 3.8) and $r \leqslant \sqcup Re(Y)$, there exists $x \in Y$ such that $r \leqslant Re(x)$. Now Y is $\leqslant$-coherent, hence there exists an upper bound for x and y, that is $x\cdot x' = y\cdot y'$ for some $x',y' \in \mathbb{C}$. Hence we have $Re(x\cdot x') = Re(x)\cdot\mu_{B_x}(Re(x')) = Re(y\cdot y') = Re(y)\cdot\mu_{B_y}(Re(y'))$. Since $|r|_a > |Re(y)|_a$ and $r \leqslant Re(x)$ we obtain $a \in alph(\mu_{B_y}(Re(y')))$. Therefore $a \in I(B_y)$ which proves the claim.

Let $A_y = \{a_1,\ldots,a_k\}$ and let $t' = (a_1\ldots a_k)^\omega$. Since $alphinf(t\cdot t') = alphinf(s)$, $(t\cdot t',D(B))$ is a complex trace and we have $y\cdot(t\cdot t',D(B)) = (Re(y)\cdot t,\ D(B_y) \cup D(t') \cup D(B)) = (s,D(B))$. Therefore, $y \leqslant (s,D(B)) = z$.

Second, we show that z is the least $\leqslant$-upper bound of Y. Let z' be an $\leqslant$-upper bound of Y. Since the prefix order is coarser than the product order, z' is also an $\sqsubseteq$-upper bound of Y and by Proposition 4.3 we have $z \sqsubseteq z'$. Hence $Re(z') = s\cdot t$ for some $t \in \mathbb{R}$ such that $alph(t) \subseteq I(A)$.

We claim that $alph(t) \subseteq I(B)$. For all $y \in Y$, there exists $y' \in \mathbb{C}$ such that $z' = y\cdot y'$. Hence, $Re(z') = Re(y)\cdot\mu_{B_y}(Re(y')) = s\cdot t$. Since $Re(y) \leqslant s$ and $\mathbb{R}$ is left-cancellative, we obtain $\mu_{B_y}(Re(y')) = s'\cdot t$ for some suffix s' of s. Then, $alph(t) \subseteq I(B_y)$ for all $y \in Y$ which proves the claim since $B = A \cup \cup_{y\in Y} B_y$.

As above, let $A = \{a_1,\ldots,a_k\}$ and let $t' = (a_1\ldots a_k)^\omega$. Since $alphinf(t\cdot t') = alphinf(Re(z'))$, $(t\cdot t',Im(z'))$ is a complex trace and $z\cdot(t\cdot t',Im(z')) = (s\cdot t,\ D(B) \cup D(t') \cup Im(z')) = z'$. Therefore, $z \leqslant z'$. ◊

From the proofs of Proposition 4.3 and Theorem 4.4, we obtain a characterization of the least upper bound of a coherent subset of $\mathbb{C}$. Such a characterization is indeed basic when we deal with least upper bounds.

Corollary 4.5. Let $Y \subseteq \mathbb{C}$ be $\leqslant$-coherent ($\sqsubseteq$-coherent resp.). Then,

$$Re(\sqcup Y) = \sqcup Re(Y) \quad \text{and} \quad Im(\sqcup Y) = Im(\sqcup Re(Y)) \cup (\cup_{y\in Y} Im(y)).$$

Since the coherent completeness is the strongest completeness property, we deduce immediately from Proposition 4.3 and Theorem 4.4 the

Corollary 4.6.
i) $(\mathbb{C}, \leqslant)$ and $(\mathbb{C},\sqsubseteq)$ are bounded complete.
ii) $(\mathbb{C}, \leqslant)$ and $(\mathbb{C},\sqsubseteq)$ are CPO with the empty trace as least element.

In any bounded complete PoSet, every non empty set admits a greatest lower bound. But this general existence result does not provide us with any characterization of the greatest lower bound. Indeed, such a characterization may be useful to deal with greatest lower bounds. It is the object of the next proposition.

Proposition 4.7. Every non empty subset Z of $\mathbb{C}$ admits a greatest lower bound $\sqcap Z$ for both partial order $\leqslant$ and $\sqsubseteq$. Moreover,

$Re(\sqcap Z) = \sqcap Re(Z) = s$ and

$Im(\sqcap Z) = D(B)$ where B is the greatest subset of X consistent with $A = alphinf(s)$ such that $D(B) \subseteq Im(z) \setminus alph(s^{-1}\cdot Re(z))$ for all $z \in Z$.

In order to illustrate the characterization of the imaginary part of a greatest lower bound, which is somewhat intricate, we give an example before the proof of this result. Let $(X,D) = a—b—c—d—e—f$ and let $Z = \{(b^n c(bf)^\omega, X) \mid n \in \mathbb{N}\}$. We have $s = \mathrm{Re}(\sqcap Z) = \sqcap \mathrm{Re}(Z) = f^\omega$ and for all $z \in Z$, $\mathrm{Im}(z) \setminus \mathrm{alph}(s^{-1}\cdot\mathrm{Re}(z)) = X \setminus \{b,c\} = \{a,d,e,f\}$. The greatest subset B of X consistent with $\mathrm{alphinf}(s) = \{f\}$ and such that $D(B) \subseteq \{a,d,e,f\}$ is $\{e,f\}$. Hence, $\sqcap Z = (f^\omega, D(\{e,f\}))$.

Proof. We give the proof for the prefix order only. The proof is even simpler for the product order. Let Z be a non empty subset of $\mathbb{C}$ and let $Y = \{y \in \mathbb{C} \mid y \leqslant z \text{ for all } z \in Z\}$. Since Y is bounded by any member of Z, it admits a least upper bound $\sqcup Y = (s,D(B))$ which is precisely the greatest lower bound of Z.
First, we claim that Re(Y) is the set of lower bounds of Re(Z) in $\mathbb{R}$. Clearly, each member of Re(Y) is a lower bound of Re(Z). Conversely, let r be a lower bound of Re(Z) in $\mathbb{R}$, we will prove that $(r,\mathrm{Im}(r))$ is a lower bound of Z in $\mathbb{C}$. Let $z \in Z$, we have $\mathrm{Re}(z) = r\cdot t$ for some real trace t such that $\mathrm{alph}(t) \cap \mathrm{Im}(r) = \emptyset$. Let $t' = (a_1\ldots a_k)^\omega$ where $\mathrm{alphinf}(r) = \{a_1,\ldots,a_k\}$. Then $(t\cdot t',\mathrm{Im}(z))$ is a complex trace since $\mathrm{alphinf}(t\cdot t') = \mathrm{alphinf}(\mathrm{Re}(z))$. Therefore, we have $(r,\mathrm{Im}(r))\cdot(t\cdot t',\mathrm{Im}(z)) = (r\cdot t, \mathrm{Im}(r) \cup D(t') \cup \mathrm{Im}(z)) = z$ which proves the claim.
Using this claim we obtain $s = \mathrm{Re}(\sqcap Z) = \mathrm{Re}(\sqcup Y) = \sqcup \mathrm{Re}(Y) = \sqcap \mathrm{Re}(Z)$.
Let B be the greatest subset of X consistent with $A = \mathrm{alphinf}(s)$ and such that for all $z \in Z$, $D(B) \subseteq \mathrm{Im}(z) \setminus \mathrm{alph}(s^{-1}\cdot\mathrm{Re}(z))$. We show that $(s,D(B))$ is a lower bound of Z. Let $t' = (a_1\ldots a_k)^\omega$ where $A = \{a_1,\ldots,a_k\}$ and let $z \in Z$. We have $\mathrm{Re}(z) = s\cdot t$ for some real trace t such that $\mathrm{alph}(t) \subseteq I(A)$. We have $\mathrm{alphinf}(t\cdot t') = \mathrm{alphinf}(\mathrm{Re}(z))$, hence $(t\cdot t',\mathrm{Im}(z))$ is a complex trace. Since $D(B) \cap \mathrm{alph}(t) = \emptyset$ we deduce $(s,D(B))\cdot(t\cdot t',\mathrm{Im}(z)) = z$ whence $(s,D(B)) \leqslant z$.
Let $\sqcap Z = x = (s,D(C))$, since x is a complex trace, (A,C) is consistent. Now for all $z \in Z$ we have $x \leqslant z$ which implies $D(C) \subseteq \mathrm{Im}(z)$ and $\mathrm{Re}(z) = s\cdot t$ for some real trace t such that $\mathrm{alph}(t) \cap D(C) = \emptyset$. Therefore, $D(C) \subseteq \mathrm{Im}(z) \setminus \mathrm{alph}(s^{-1}\cdot\mathrm{Re}(z))$ for all $z \in Z$.
By Lemma 4.2, the union of two subsets consistent with A is consistent with A as well. Hence we deduce that $C \subseteq B$. Since $x = (s,D(C))$ is the greatest lower bound of Z, we obtain $(s,D(B)) \leqslant (s,D(C))$. Therefore $\sqcap Z = (s,D(C)) = (s,D(B))$ which concludes the proof. ◊

We saw that $(\mathbb{C}, \leqslant)$ satisfies the strongest completeness property. Unfortunately, in general, neither $(\mathbb{C}, \leqslant)$ nor $(\mathbb{C},\sqsubseteq)$ is a Scott-Domain. As concerns the product order, it is a consequence of the following fact.

Lemma 4.8. A complex trace is $\sqsubseteq$-compact if and only if it is finite.

Proof. Let $x = (s,D(B))$ be an infinite trace, let $A = \mathrm{alphinf}(s)$ and let $s' = (a_1\ldots a_k)^\omega$ where $I(A) = \{a_1,\ldots,a_k\}$. Let $(s_n)_{n\geq 0}$ be an increasing sequence of finite traces such that $s = \sqcup\{s_n \mid n \geq 0\}$ and $\mathrm{alph}(s_0^{-1}\cdot s) = A$. For all $n \geq 0$, there exists a real trace t_n such that $s_{n+1} = s_n\cdot t_n$ and $\mathrm{alph}(t_n) \subseteq A$. Since $\mathrm{alph}(s')\times\mathrm{alph}(t_n) \subseteq I$, we have $s_{n+1}\cdot s' = s_n\cdot s'\cdot t_n$ whence $(s_n\cdot s')_{n\geq 0}$ is an increasing sequence of real traces. Using Corollary 4.5 we obtain $\sqcup\{s_n\cdot s' \mid n \geq 0\} = s\cdot s'$. From $\mathrm{Im}(s\cdot s') = D(A) \cup D(I(A)) = X$, we obtain $x \sqsubseteq s\cdot s'$. Hence x is not compact since for all $n \geq 0$, $x \not\sqsubseteq s_n\cdot s'$.
Conversely, let x be a finite trace such that $x \sqsubseteq \sqcup Y$ where Y is a $\sqsubseteq$-directed subset of $\mathbb{C}$. Clearly, Re(Y) is a directed subset of $\mathbb{R}$ and $x \leqslant \sqcup \mathrm{Re}(Y)$. From Proposition 3.7, $x \leqslant \mathrm{Re}(y)$ for some $y \in Y$. Therefore, $x \leqslant y$ which proves that x is $\sqsubseteq$-compact. ◊

For $x \in ₵$, we set $C_{\sqsubseteq}(x) = \{y \sqsubseteq x \mid y \text{ is } \sqsubseteq\text{-compact}\}$. From Corollary 4.5 and Lemma 4.8 we deduce that for all $x \in ₵$, $\sqcup C_{\sqsubseteq}(x)$ is a real trace. Therefore, non-real traces cannot be $\sqsubseteq$-approximated with $\sqsubseteq$-compact traces, that is, $(₵,\sqsubseteq)$ is not a Scott-Domain if there exist some non real traces. Note that such non real traces exist if and only if the dependence relation D is not transitive.

The situation is different for the prefix order. There may exist $\leqslant$-compact traces and $\leqslant$-prime traces which are not real traces and therefore are not finite. For instance, let $(X,D) = a — b — c$ and let $x = (a^\omega,D(a,b)) = (a^\omega,X)$. We show that x is $\leqslant$-prime. Let Y be a subset of ₵ such that $x \leqslant \sqcup Y$. Since $Im(x) = X$, x is $\leqslant$-maximal in ₵, whence $x = \sqcup Y$. Now, from Corollary 4.5, since $c \in Im(x) \setminus Im(Re(x))$, we deduce that there exists $y \in Y$ such that $c \in Im(y)$. Using Corollary 4.5 and $Im(y) \neq \emptyset$, we obtain $Re(y) = a^\omega$. Therefore, $Im(y) \supseteq D(a) \cup \{c\} = X$ and $y = (a^\omega,X) = x$ which proves that x is prime.
Nevertheless, the following example shows that $(₵, \leqslant)$ is not a Scott-Domain in general. Let $(X,D) = a — b — c — d$ and let $x = (a^\omega,D(\{a,b\}))$. We show that x is not $\leqslant$-compact. For all $n \geq 0$, let $y_n = (a^n d^\omega,D(c,d))$. Clearly, $y_{n+1} = y_n \cdot a$ for all $n \geq 0$. Hence, $(y_n)_{n\geq 0}$ is an $\leqslant$-increasing sequence and by Corollary 4.5, $y = \sqcup\{y_n \mid n \geq 0\} = ((ad)^\omega,X)$. We have $y = x \cdot d^\omega$ whence $x \leqslant y$. But for all $n \geq 0$, x is not a prefix of y_n since y_n has finitely many a's. Therefore, x is not compact. Now the set of prefixes of x is $\{a^n \mid n \geq 0\} \cup \{a^\omega, x\}$. For each $n \geq 0$, the trace a^n is $\leqslant$-prime. The trace a^ω is not $\leqslant$-compact since $a^\omega = \sqcup\{a^n \mid n \geq 0\}$. Hence the set of $\leqslant$-compact traces lower than x is $C_{\leqslant}(x) = \{a^n \mid n \geq 0\}$. Therefore $\sqcup C_{\leqslant}(x) = a^\omega \neq x$ which proves that $(₵, \leqslant)$ is not a Scott-Domain.

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