Behavioural satisfaction and equivalence in concrete model categories*

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Abstract. We use the well-known framework of *concrete categories* to show how much of standard universal algebra may be done in an abstract and still rather intuitive way. This is used to recast the unifying view of behavioural semantics of specifications based on behavioural satisfaction and, respectively, on behavioural equivalence of models abstracting away from many particular features of standard algebras. We also give an explicit representation of behavioural equivalence between models in terms of behavioural correspondences.

1 Introduction

Behavioural semantics for specifications plays a crucial role in the formalisation of the development process, where a specification need not be implemented exactly but only so that the required system *behaviour* is achieved — the idea goes back to [GGM76], [Hoa72]; see e.g. [ST95] for the context in which we view it now. There have been two basic approaches to behavioural semantics of specifications. One introduces a new *behavioural satisfaction* of formulae, based on the interpretation of equality as an internal indistinguishability relation in each algebra defined so that two elements are considered equal if they are indistinguishable to the user of the data type given by the algebra, see e.g. [NO87]. The other is based on an external *behavioural equivalence* of algebras, where two algebras are considered equivalent if they cannot be distinguished by any computation the user can perform, see e.g. [ST87]. There have also been attempts to unify the two views [Rei85], recently concluded in [BHW95].

One goal of this paper is to show how the above ideas work in the framework of an arbitrary category of models, rather than just in that of standard algebras. The structure of an arbitrary category does not seem convenient enough for this though, and we assume that model categories we work with are concrete, that is, that models come equipped with the concept of their *carriers*. The technicalities of the version of concrete categories we find most appropriate for our purposes are presented in Sect. 2.

This technical machinery is used in Sect. 3 to present the concepts of behavioural satisfaction and equivalence and to study the relationship between the two approaches to behavioural semantics of specifications. We recast here the main results of [BHW95], throwing away the ballast of unnecessary specific properties of standard algebras and relying only on some abstract properties of

* A full version of this paper is available as [BT95b].

model categories. This yields a more general and, we believe, more clear view of this work. The basic results are presented for an arbitrary behavioural indistinguishability relation and the abstract behavioural equivalence it determines, and hinge on the crucial assumption that the indistinguishability structure is *(weakly) regular*. We specialise them further to the most typical case, where the indistinguishability relation is given as indistinguishability w.r.t. a set of *observable sorts*. It turns out that the resulting *observational indistinguishability* is regular, which allows us to simplify the main results for this case.

In the standard algebraic framework, a *correspondence* between two algebras can always be built to witness their behavioural equivalence. A similar role is played here by abstract *behavioural correspondences* between models, introduced and studied in Sect. 4. Again, this works especially nicely for the case of *observational equivalence* w.r.t. a set of observable sorts, characterised by a particularly simple concept of *observational correspondence*. Our abstract observational correspondences retain the most crucial properties of correspondences from the standard algebraic framework [Sch87], and seem to yield equally elegant and useful concepts in other frameworks we study.

By providing a more abstract view of behavioural semantics we indirectly propose in this paper a way of introducing related concepts for logical systems and specification formalisms based on various, not necessarily quite standard definitions of a model. This is illustrated in [BT95b], where we present the concepts and results introduced throughout the paper in the frameworks based on, respectively, standard many-sorted algebras, partial algebras, and regular algebras. To our knowledge, behavioural semantics for specifications of partial and of regular algebras has never been studied in the literature from this point of view. Even though the results are close enough to the standard ones, we were surprised to encounter quite a few technical intricacies which made this work interesting. Perhaps more importantly, the specialisations of the general concepts to partial and regular algebras serve as an indication of a more general line of proceeding which we believe is applicable to many other similar frameworks. For instance, these ideas should work as well for order-sorted and for error algebras. Another framework with behavioural semantics likely to fit into the mould presented here is that of higher-order logic, as studied in [HS95]. Of course, there also exist frameworks where some of our basic assumptions are not satisfied, and some expected facts and constructions break down. As shown in [BT95b], this is the case for instance for continuous algebras — with some relief we have realised that for regular algebras, which may replace continuous algebras for most practical purposes, the required properties hold and our constructions work as expected [BT95a].

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2 *S*-concrete categories

In this section we will recall the concept of a concrete category [MacL71]. The basic intuition to follow is that objects of a concrete category are sets with some extra structure, or in other words that each object comes equipped with a *carrier set*. Similarly, morphisms of a concrete category are functions between the carrier sets of objects that preserve the object structure. We will explore this intuition to present a number of "algebraic" concepts (subalgebra, congruence, quotient algebra, etc). In fact, what follows is rather easy and may be well-known folklore in the theory of concrete categories, but we were not able to locate appropriate references with just the formulations we need.

However, unlike in the standard approaches to concrete category theory, we will deal with objects whose carriers are many-sorted sets: this slight generalisation follows the common practice of working with many-sorted (or heterogeneous) algebras in the theory of algebraic specification.

Let S be an arbitrary set (of *sorts*) fixed throughout this section.

S-sorted sets

By an S-sorted set we mean any family $X = \langle X_s \rangle_{s\in S}$ of sets. We generalise all the standard set-theoretic notions and notations to S-sorted sets. For example, for $X = \langle X_s \rangle_{s\in S}$ and $Y = \langle Y_s \rangle_{s\in S}$: X is a subset of Y, written $X \subseteq Y$, if $X_s \subseteq Y_s$ for all $s \in S$; Cartesian product of X and Y is defined as $X \times Y = \langle X_s \times Y_s \rangle_{s\in S}$; an S-sorted relation from X to Y is $R \subseteq X \times Y$; its domain $\mathsf{dom}(R) \subseteq X$ is $\mathsf{dom}(R) = \langle \mathsf{dom}(R_s) \rangle_{s\in S}$, where $\mathsf{dom}(R_s) = \{x \in X_s \mid \text{for some } y \in Y_s, x\, R_s\, y\}$; an S-sorted function $f\colon X \to Y$ is $f = \langle f_s\colon X_s \to Y_s \rangle_{s\in S}$; a kernel $\mathsf{ker}(f) \subseteq X \times X$ of a function $f\colon X \to Y$ is $\langle \mathsf{ker}(f_s) \rangle_{s\in S}$, where $\mathsf{ker}(f_s) = \{\langle x, x' \rangle \mid f_s(x) = f_s(x')\}$; etc. These are just examples; we will feel free to use any standard concepts and notations. Moreover, the explicit qualification by the set S of sorts and by specific sorts $s \in S$ will often be omitted. For example, we write $x \in X$ meaning $x \in X_s$ if $s \in S$ is clear (or unimportant); for $R \subseteq X \times Y$ and $x \in X_s$, $y \in Y_s$, we will write $x\,R\,y$ meaning $x\,R_s\,y$; for $f\colon X \to Y$ and $x \in X$ we will write $f(x)$ meaning $f_s(x)$ for the appropriate $s \in S$; etc.

A relation $\simeq\, \subseteq X \times X$ is a partial equivalence if (each component of) $\simeq$ is transitive and symmetric. If $\simeq\, \subseteq X \times X$ is a partial equivalence then $\mathsf{dom}(\simeq) = \langle \{x \mid x \simeq_s x\} \rangle_{s\in S}$. A partial equivalence $\simeq\, \subseteq X \times X$ is an equivalence if it is reflexive; every partial equivalence is an equivalence on its domain. A quotient of X by an equivalence $\simeq\, \subseteq X \times X$ is $X/{\simeq} = \{[x]_\simeq \mid x \in X\}$, where $[x]_\simeq = \{x' \in X \mid x \simeq x'\}$.

S-sorted sets with S-sorted functions as morphisms form the category $\mathbf{Set}^S$ of S-sorted sets.

S-concrete categories

By an *S-concrete category* we mean a category $\mathbf{K}$ together with a *concretization functor* $|\_|\colon \mathbf{K} \to \mathbf{Set}^S$ such that $|\_|$ is faithful, i.e., for all parallel morphisms $f, g\colon A \to B$ in $\mathbf{K}$, if $|f| = |g|$ then $f = g$.

When dealing with S-concrete categories, we will often omit the indicator S and refer to a concrete category using the name of the category only, thus leaving the concretization functor implicit.

As we have mentioned earlier, the idea is that objects of an S-concrete category come equipped with a notion of their S-sorted carrier set, given by the concretization functor. Similarly, morphisms of the concrete category are essentially some functions between the carrier sets of objects.

Throughout this section, let $\mathbf{K}$ be an S-concrete category with concretization functor $|\_|\colon \mathbf{K} \to \mathbf{Set}^S$.

Since functors preserve isomorphisms, if $f\colon A \to B$ is an isomorphism in $\mathbf{K}$ then $|f|$ is a bijective function. An isomorphism $f\colon A \to B$ in $\mathbf{K}$ will be called *exact* if $|f|$ is an identity.

$\mathbf{K}$ *admits renaming of elements* of objects if the concretization functor $|\_|$ creates isomorphisms, that is, for each object[3] $A \in |\mathbf{K}|$ and bijective function $i\colon |A| \to X$, there is an object $B \in |\mathbf{K}|$ and an isomorphism $\widehat{i}\colon A \to B$ such that $|\widehat{i}| = i$ (and $|B| = X$).

$\mathbf{K}$ *has limits*, or *is complete*, if limits of all diagrams in $\mathbf{K}$ exist, and moreover, the concretization functor $|\_|$ is continuous, i.e., it maps limits in $\mathbf{K}$ to limits in $\mathbf{Set}^S$. Similarly, we say that $\mathbf{K}$ *has finite (directed) limits*, or *is finitely (directed) complete*, if limits of all finite (directed[4]) diagrams exist in $\mathbf{K}$ and are preserved by the concretization functor $|\_|$.

If $\mathbf{K}$ is (finitely) complete and admits renaming of elements then we can always choose a product in $\mathbf{K}$ so that the carrier of the product is the Cartesian product of the carriers of the components and (the underlying functions of) the product projections are the usual Cartesian product projections. Similarly, if $\mathbf{K}$ is finitely directed complete and admits renaming of elements then any equaliser in $\mathbf{K}$ may be chosen so that its underlying function is an inclusion, any pullback in $\mathbf{K}$ may be chosen so that the carrier of the pullback object is a subset of the Cartesian product of the carriers of the component objects and the pullback morphisms are the corresponding projections, etc.

Subobjects

Let $A \in |\mathbf{K}|$ be an object of $\mathbf{K}$. A *subobject* of A is an object $B \in |\mathbf{K}|$ together with a morphism $f\colon B \to A$ such that $|f|\colon |B| \to |A|$ is an inclusion. Notice that this requirement determines $f\colon B \to A$ unambiguously; we will write f as $\iota_{B \hookrightarrow A}$.

A subobject B of A is *full* if for any morphism $f\colon C \to A$ such that $|f|(|C|) \subseteq |B|$, where $|f|(|C|) \subseteq |A|$ is the image of $|C|$ under $|f|\colon |C| \to |A|$, there exists a (necessarily unique) morphism $\widehat{f}\colon C \to B$ such that[5] $\widehat{f};\iota_{B \hookrightarrow A} = f$. Then any (full) subobject B' of A with $|B'| \subseteq |B|$ is a (full) subobject of B, and any (full) subobject of a (full) subobject of A is a (full) subobject of A.

A full subobject B of $A \in |\mathbf{K}|$ is *compatible with* $\mathbf{K}$*-morphisms* if for each morphism $h\colon C \to A$, there is a full subobject $h^{-1}(B)$ of C with $|h^{-1}(B)| = |h|^{-1}(|B|)$, where $|h|^{-1}(|B|) = \{c \in C \mid |h|(c) \in |B|\}$ is the coimage of $|B|$ w.r.t. the function $|h|\colon |C| \to |A|$. $h^{-1}(B)$ will be called the *coimage of B under h*. Notice that if $|h|(|C|) \subseteq |B|$ then $h^{-1}(B) = C$.

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[3] The notation $|\mathbf{K}|$ for the class of objects of a category $\mathbf{K}$ should not be surprising. The slight confusion with the notation for concretization functors is deliberate and justified: the category $\mathbf{Cat}$ of all categories forms a (very large — as customary, we will disregard the usual fundational problems) concrete category, and $|\_|\colon \mathbf{Cat} \to \mathbf{Set}$ as used here is the related concretization functor.

[4] A diagram Δ is *directed* if for any nodes A and B in Δ there exists C with edges $f\colon A \to C$ and $g\colon B \to C$ in Δ.

[5] Composition in any category is denoted by ; (semicolon) and written in the diagrammatic order.

Proposition 1. *If* $\mathbf{K}$ *is finitely directed complete and admits renaming of elements then any full subobject in* $\mathbf{K}$ *is compatible with* $\mathbf{K}$*-morphisms.*

Given an object $A \in |\mathbf{K}|$ and a set $X \subseteq |A|$, a *full subobject of* A *generated by* X is a full subobject B of A such that $X \subseteq |B|$ and such that for any subobject C of A either $|B| \subseteq |C|$ or $X \not\subseteq |C|$. If it exists, it is unique up to an exact isomorphism, and any object exactly isomorphic to a full subobject of A generated by X is a full subobject of A generated by X. This allows us to speak of "the" full subobject of $A \in |\mathbf{K}|$ generated by $X \subseteq |A|$; it will be denoted by $\langle X \rangle_A$ (if it exists). $\mathbf{K}$ *has full subobjects* if for each $A \in |\mathbf{K}|$ and $X \subseteq |A|$ the full subobject of A generated by X exists.

For any morphism $f : A \to B$, we write $f(A)$ for $\langle |f|(|A|) \rangle_B$, the full subobject of B generated by the image of $|f|$ (if it exists) and call it the *epimorphic image* of f.

Quotients

Let $A \in |\mathbf{K}|$ be an object of $\mathbf{K}$. A *quotient* of A is an object $B \in |\mathbf{K}|$ together with an epimorphism $\pi_{A/B} : A \to B$ (the epimorphism, called the *quotient projection*, will often be left implicit) such that for any morphism $f : A \to C$ with $\mathsf{ker}(|\pi_{A/B}|) \subseteq \mathsf{ker}(|f|)$ there is a (necessarily unique) morphism $\overline{f} : B \to C$ such that $\pi_{A/B} ; \overline{f} = f$.

If B and B' are quotients of $A \in |\mathbf{K}|$ such that $\mathsf{ker}(|\pi_{A/B}|) \subseteq \mathsf{ker}(|\pi_{A/B'}|)$ then B' is a quotient of B. Unfortunately, in general a quotient of a quotient of A need not be a quotient of A. If, however, this is the case for all objects $A \in |\mathbf{K}|$, we say that *quotients compose* in $\mathbf{K}$.

A quotient B of A is *surjective* if $|\pi_{A/B}|$ is surjective. A quotient of a surjective quotient of $A \in |\mathbf{K}|$ is a quotient of A as well.

Pullbacks preserve surjective quotients in $\mathbf{K}$ if for any pullback in $\mathbf{K}$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "q'"'] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

if q is a surjective quotient projection then q' is a quotient projection.

Given an object $A \in |\mathbf{K}|$ and an equivalence relation $\simeq \subseteq |A| \times |A|$, a *quotient of* A *by* $\simeq$ is a quotient B of A such that $\simeq \subseteq \mathsf{ker}(|\pi_{A/B}|)$ and such that for any morphism $f : A \to C$ either $\mathsf{ker}(|\pi_{A/B}|) \subseteq \mathsf{ker}(|f|)$ or $\simeq \not\subseteq \mathsf{ker}(|f|)$.

The quotient of $A \in |\mathbf{K}|$ by an equivalence $\simeq \subseteq |A| \times |A|$, if it exists, is unique up to an isomorphism, and any object isomorphic to a quotient of A by $\simeq$ is a quotient of A by $\simeq$. This allows us to speak of "the" quotient of A by $\simeq$; it will be denoted by $A/{\simeq}$ (if it exists). The morphism $\pi_{A/(A/\simeq)} : A \to A/{\simeq}$ will be written as $\pi_{A/\simeq} : A \to A/{\simeq}$.

We say that *quotients are fully compatible with subobjects* in $\mathbf{K}$ if the image of any full subobject under any quotient projection exists and is a quotient of the subobject, that is, if for any object $A \in |\mathbf{K}|$, full subobject A' of A, and quotient B of A with quotient projection $\pi_{A/B} : A \to B$, the full subobject $\pi_{A/B}(A')$ exists and is a quotient of A' with the natural quotient projection.

Congruences

Given an object $A \in |\mathbf{K}|$, by a *congruence* on A we mean any equivalence relation $\cong \subseteq |A| \times |A|$ such that for some morphism $f\colon A \to B$, $\cong = \mathsf{ker}(|f|)$.

If the quotient of A by an equivalence $\simeq \subseteq |A| \times |A|$ exists then $\mathsf{ker}(|\pi_{A/\simeq}|)$ is the least congruence containing $\simeq$.

$\mathbf{K}$ *has (surjective) quotients* if for any $A \in |\mathbf{K}|$ and congruence $\cong$ on A, there exists a quotient of A by $\cong$ (which is surjective). Then for any object $A \in |\mathbf{K}|$ and equivalence relation $\simeq \subseteq |A| \times |A|$, the quotient $A/\simeq$ exists if and only if there exists the least congruence relation on A that includes $\simeq$.

A *partial congruence* relation on an object $A \in |\mathbf{K}|$ is a partial equivalence $\cong \subseteq |A| \times |A|$ such that the full subobject of A generated by $\mathsf{dom}(\cong) \subseteq |A|$ exists and $\mathsf{dom}(\cong) = |\langle \mathsf{dom}(\cong)\rangle_A|$, and $\cong$ is a congruence on $\langle \mathsf{dom}(\cong)\rangle_A$. If this is the case, we will write $A_\cong$ for $\langle \mathsf{dom}(\cong)\rangle_A$ and $A/\cong$ for the quotient of $A_\cong$ by $\cong$ (which is a congruence on $A_\cong$). As before, we will refer to the latter as the quotient of A by $\cong$.

A family $\cong = \langle \cong_A \subseteq |A|\times|A|\rangle_{A\in|\mathbf{K}|}$ of partial congruence relations on objects of $\mathbf{K}$ is *domain-uniform* if for each morphism $f\colon A \to B$, $|f(A_{\cong_A})| \subseteq |B_{\cong_B}|$, that is, if the image under any morphism of the domain of a partial congruence on the source of the morphism (exists and) is a subobject of the domain of the partial congruence on the target of the morphism.

Proposition 2. *Suppose that* $\mathbf{K}$ *has subobjects and surjective quotients and let* $\cong = \langle \cong_A \subseteq |A| \times |A|\rangle_{A\in|\mathbf{K}|}$ *be a domain-uniform family of partial congruences on objects of* $\mathbf{K}$*. Then for each* $A \in |\mathbf{K}|$*, for any quotient* B *of* $A_{\cong_A}$*,* $\cong_B$ *is in fact a total congruence on* B*.*

A family $\cong = \langle \cong_A \subseteq |A| \times |A|\rangle_{A\in|\mathbf{K}|}$ of partial congruences is *preserved by isomorphisms* if for each isomorphism $i\colon A \to B$ in $\mathbf{K}$, for all $a, a' \in |A|$, if $a \cong_A a'$ then $|i|(a) \cong_B |i|(a')$. This is a stronger property than that of *isomorphism compatibility* [BHW95], where just an isomorphism between $A/\cong$ and $B/\cong$ is required to exist for any isomorphic $A, B \in |\mathbf{K}|$.

Proposition 3. *Let* $\cong$ *be preserved by isomorphisms. Consider an isomorphism* $i\colon A \to B$*. Then the quotient* $A/\cong_A$ *exists if and only if the quotient* $B/\cong_B$ *exists and if this is the case then there exists a (necessarily unique) isomorphism* $i/\cong\colon A/\cong \to B/\cong$ *such that* $\pi_{A/\cong}; (i/\cong) = i; \pi_{B/\cong}$*.*

A family $\cong = \langle \cong_A \subseteq |A| \times |A|\rangle_{A\in|\mathbf{K}|}$ of partial congruences is *regular* if for each $A \in |\mathbf{K}|$ the partial congruence relation $\cong_{A/\cong_A}$ on the quotient $A/\cong_A$ is the identity. It follows then that the quotient projection $\pi_{(A/\cong_A)/\cong_{A/\cong_A}}\colon A/\cong_A \to (A/\cong_A)/\cong_{A/\cong_A}$ is an isomorphism. A weaker property is *weak regularity*, where we require that for any $A \in |\mathbf{K}|$, $A/\cong_A$ and $(A/\cong_A)/\cong_{A/\cong_A}$ are isomorphic (but the quotient projection need not be even bijective then).

3 Behavioural and abstractor semantics of specifications

The usual interpretation of a set of axioms viewed as a specification, yielding the class of algebras that satisfy the axioms, is too restrictive to directly model the intuitive concept of implementation. The goal is to define the semantics of any specification so that the class of algebras it yields corresponds to all intuitively admissible realisations of the specification, which exhibit the *behaviour* prescribed by the specification rather than satisfying all the axioms literally. One approach to *behavioural semantics* of specifications this idea implies is to re-define the notion of satisfaction of logical sentences in algebras; another approach is to define an equivalence relation between algebras and re-define the semantics of specifications so that it yields the class that includes each algebra equivalent to an algebra that satisfies (in the usual sense) the axioms of the specification. We will sketch both approaches in this section in the rather abstract setting of an arbitrary concrete category of models, an arbitrary set of sentences and an arbitrary satisfaction relation between models and sentences.

Throughout this section, let S be a set and $\mathbf{Mod}$ be an S-concrete category with the concretization functor $|\_\_|: \mathbf{Mod} \to \mathbf{Set}^S$. We assume that $\mathbf{Mod}$ is finitely directed complete, admits renaming of elements, has full subobjects and surjective quotients. Let then $\mathbf{Sen}$ be a set and $\models \subseteq |\mathbf{Mod}| \times \mathbf{Sen}$ a relation. We will think of $\mathbf{Mod}$ as a (concrete) category of models, of $\mathbf{Sen}$ as a set of sentences, and of $\models$ as a satisfaction relation between models in $|\mathbf{Mod}|$ and sentences in $\mathbf{Sen}$. We will refer to this as a (concrete) *satisfaction frame* $\mathbf{Sat} = \langle \mathbf{Mod}, \mathbf{Sen}, \models, S, |\_\_| \rangle$. We assume that $\mathbf{Sat}$ satisfies the *abstraction condition*: for all $A, B \in |\mathbf{Mod}|$, if A and B are isomorphic in $\mathbf{Mod}$ then for each $\varphi \in \mathbf{Sen}$, $A \models \varphi$ if and only if $B \models \varphi$.

By a *presentation* in the satisfaction frame $\mathbf{Sat}$ we mean any set of sentences $\Phi \subseteq \mathbf{Sen}$. For any presentation $\Phi \subseteq \mathbf{Sen}$ and model $M \in |\mathbf{Mod}|$, we will write $M \models \Phi$ if $M \models \varphi$ for all $\varphi \in \Phi$. Then $Mod[\Phi] = \{M \in |\mathbf{Mod}| \mid M \models \Phi\}$ is the *class of models of* Φ. Given $\mathcal{M} \subseteq |\mathbf{Mod}|$ and $\varphi \in \mathbf{Sen}$, we write $\mathcal{M} \models \varphi$ if $M \models \varphi$ for all $M \in \mathcal{M}$. Then $Th(\mathcal{M}) = \{\varphi \in \mathbf{Sen} \mid \mathcal{M} \models \varphi\}$ is the *theory* of $\mathcal{M}$. Finally, for $\Phi \subseteq \mathbf{Sen}$, we also write $\Phi \models \varphi$ for $\varphi \in Th(Mod[\Phi])$, and $Th(\Phi)$ for $\{\varphi \in \mathbf{Sen} \mid \Phi \models \varphi\} = Th(Mod[\Phi])$.

Behavioural semantics

The first approach to behavioural semantics of presentations is to base the interpretation of sentences in models on some notion of behavioural *indistinguishability relation* in place of the usual identity of data. The exact definition of this indistinguishability relation varies from one framework to another: the common feature is that the indistinguishability relation between elements of models is compatible with the structure of the models considered.

Let the satisfaction frame $\mathbf{Sat}$ be equipped with an *indistinguishability structure*, that is, for each model $M \in |\mathbf{Mod}|$, a partial congruence relation $\approx_M \subseteq |M| \times |M|$. We will write $\approx$ for $\langle \approx_M \rangle_{M \in |\mathbf{Mod}|}$ and then, for each model $M \in |\mathbf{Mod}|$, $M_\approx$ for the full subobject of M with the carrier $|M_\approx| = \mathsf{dom}(\approx_M)$ that coincides with the domain of the partial congruence $\approx_M$. We will often omit the subscript M in $\approx_M$.

For each model $M \in |\mathbf{Mod}|$ and sentence $\varphi \in \mathbf{Sen}$, M *behaviourally satisfies* φ, written $M \models^\approx \varphi$, if $M/{\approx} \models \varphi$. Given a presentation $\Phi \subseteq \mathbf{Sen}$ and a model $M \in |\mathbf{Mod}|$, $M \models^\approx \Phi$ if $M \models^\approx \varphi$ for all $\varphi \in \Phi$. Then, the *behavioural semantics* of Φ yields $Mod_\approx[\Phi] = \{M \in |\mathbf{Mod}| \mid M \models^\approx \Phi\}$. Given $\mathcal{M} \subseteq |\mathbf{Mod}|$ and $\varphi \in \mathbf{Sen}$, we write $\mathcal{M} \models^\approx \varphi$ if $M \models^\approx \varphi$ for all $M \in \mathcal{M}$. Then $Th_\approx(\mathcal{M}) = \{\varphi \in \mathbf{Sen} \mid \mathcal{M} \models^\approx \varphi\}$ is the *behavioural theory* of $\mathcal{M}$. Finally, for $\Phi \subseteq \mathbf{Sen}$, we also write $\Phi \models^\approx \varphi$ for $\varphi \in Th_\approx(Mod_\approx[\Phi])$, and $Th_\approx(\Phi)$ for $\{\varphi \in \mathbf{Sen} \mid \Phi \models^\approx \varphi\} = Th_\approx(Mod_\approx[\Phi])$.

The intuitive idea behind the above definitions is that for any model $M \in |\mathbf{Mod}|$, its *abstract behaviour* is given by the quotient $M/\cong$, and then a model behaviourally satisfies a sentence if and only if its abstract behaviour satisfies it in the usual sense.

This intuition underlies the role of the behavioural quotient operator on models. The image and coimage functions on classes of models induced by this operator deserve special notation. For any class $\mathcal{M} \subseteq |\mathbf{Mod}|$, $\mathcal{M}/\!\approx\, = \{M/\!\approx\, \mid M \in \mathcal{M}\}$ is the class of all behavioural quotients of models in $\mathcal{M}$. (For each model $M \in \mathcal{M}$, all quotients of M by $\approx$ are in $\mathcal{M}/\!\approx$, so that $\mathcal{M}/\!\approx$ is closed under isomorphism.) Dually, $\mathsf{Beh}_{\approx}(\mathcal{M}) = \{M \in |\mathbf{Mod}| \mid M/\!\approx\, \in \mathcal{M}\}$ is the class of models with (some of) their behavioural quotients in $\mathcal{M}$. Using this notation, the definition of the behavioural semantics of a presentation may now be rewritten as $Mod_{\approx}[\Phi] = \mathsf{Beh}_{\approx}(Mod[\Phi])$.

Proposition 4. *The two operators $\mathsf{Beh}_{\approx}(\_)$ and $(\_)/\!\approx$ on model classes closed under isomorphism form a Galois connection; if $\mathcal{M}, \mathcal{N} \subseteq |\mathbf{Mod}|$ are classes of models closed under isomorphism then:*

- *if $\mathcal{M} \subseteq \mathcal{N}$ then $\mathsf{Beh}_{\approx}(\mathcal{M}) \subseteq \mathsf{Beh}_{\approx}(\mathcal{N})$ and $\mathcal{M}/\!\approx\, \subseteq \mathcal{N}/\!\approx$, and*
- *$\mathcal{M}/\!\approx\, \subseteq \mathcal{N}$ if and only if $\mathcal{M} \subseteq \mathsf{Beh}_{\approx}(\mathcal{N})$.*

A class $\mathcal{M} \subseteq |\mathbf{Mod}|$ is *behaviourally closed*[6] if it contains the behaviours of its elements, that is, if $\mathcal{M}/\!\approx\, \subseteq \mathcal{M}$. A presentation $\Phi \subseteq \mathbf{Sen}$ is *behaviourally closed* if $Mod[\Phi]$ is so.

Proposition 5. *Any presentation $\Phi \subseteq \mathbf{Sen}$ is behaviourally closed if and only if $Mod[\Phi] \subseteq Mod_{\approx}[\Phi]$.*

Abstractor semantics

Another approach to behavioural semantics is to leave the satisfaction of sentences as it is given in the satisfaction frame, but instead change the interpretation of presentations so that their model classes are closed under a behavioural *abstract equivalence* relation between models. Again, the exact definition of this abstract equivalence varies from one framework to another.

Let the satisfaction frame $\mathbf{Sat}$ be equipped with an *abstract equivalence* structure, that is, an equivalence relation $\equiv\, \subseteq |\mathbf{Mod}| \times |\mathbf{Mod}|$.

For any class $\mathcal{M} \subseteq |\mathbf{Mod}|$, the *abstractor closure* of $\mathcal{M}$ is the class $\mathsf{Abs}_{\equiv}(\mathcal{M}) = \{M' \in |\mathbf{Mod}| \mid M' \equiv M, \text{ for some } M \in \mathcal{M}\}$ of models. For any presentation $\Phi \subseteq \mathbf{Sen}$, its *abstractor semantics* is $Mod_{\equiv}[\Phi] = \mathsf{Abs}_{\equiv}(Mod[\Phi])$.

Relating behavioural and abstractor semantics

Under an arbitrary choice of the indistinguishability and abstract equivalence structures for a satisfaction frame there is little chance that the two semantics of specifications would be related. However, in typical examples the two structures are very closely linked to one another and the following condition holds: for all models $M, N \in |\mathbf{Mod}|$, $M \equiv N$ if and only if $M/\!\approx$ and $N/\!\approx$ are isomorphic. If this is the case, we say that $\equiv$ is *factorized* by $\approx$ and write $\equiv$ as $\equiv_{\approx}$. The idea is

[6] In the framework of standard algebras, this was called behavioural consistency in [BHW95], but the terminology proposed here seems more adequate.

that when the abstract behaviour of a model $M \in |\mathbf{Mod}|$ is given by the quotient $M/{\approx}$, two models $M, N \in |\mathbf{Mod}|$ are abstractly *behaviourally equivalent* if they exhibit the same (up to isomorphism) abstract behaviour.

Throughout the rest of this section let $\approx$ be an indistinguishability structure on $\mathbf{Sat}$ and let $\equiv\, = \,\equiv_{\approx}$ be the abstract equivalence structure factorized by $\approx$.

Proposition 6. *For $M, N \in |\mathbf{Mod}|$, if $M \equiv N$ then M and N are logically equivalent w.r.t. behavioural satisfaction, that is, for each sentence $\varphi \in \mathbf{Sen}$, $M \models^{\approx} \varphi$ if and only if $N \models^{\approx} \varphi$.*

Lemma 7. *The family $\approx$ is weakly regular if and only if for each $M \in |\mathbf{Mod}|$, $M \equiv M/{\approx}$.*

The main theorems of this section link the behavioural and abstractor semantics of presentations:

Theorem 8. *If $\approx$ is weakly regular then for any class $\mathcal{M} \subseteq |\mathbf{Mod}|$ of models, $\mathsf{Beh}_{\approx}(\mathcal{M}) \subseteq \mathsf{Abs}_{\equiv}(\mathcal{M})$; in particular, for any presentation $\Phi \subseteq \mathbf{Sen}$, $Mod_{\approx}[\Phi] \subseteq \mathsf{Abs}_{\equiv}(Mod[\Phi])$.*

Theorem 9. *If $\approx$ is weakly regular then for any class $\mathcal{M} \subseteq |\mathbf{Mod}|$ of models closed under isomorphism, $\mathsf{Beh}_{\approx}(\mathcal{M}) = \mathsf{Abs}_{\equiv}(\mathcal{M})$ if and only if $\mathcal{M}$ is behaviourally closed; in particular, for any presentation $\Phi \subseteq \mathbf{Sen}$, $Mod_{\approx}[\Phi] = \mathsf{Abs}_{\equiv}(Mod[\Phi])$ if and only if Φ is behaviourally closed.*

Another link between behavioural and abstractor semantics may be based on the concept of full abstraction (cf. [Mil77, Plo77]): a model $M \in |\mathbf{Mod}|$ is *fully abstract w.r.t.* $\approx$ (or briefly: M is $\approx$-fully abstract) if $\approx_M$ is the identity relation on $|M|$. We will also use a weaker version of the same idea: a model $M \in |\mathbf{Mod}|$ is *weakly fully abstract w.r.t.* $\approx$ (M is $\approx$-weakly fully abstract) if M and $M/{\approx}$ are isomorphic. Then, for any class $\mathcal{M} \subseteq |\mathbf{Mod}|$, we put $\mathsf{FA}_{\approx}(\mathcal{M}) = \{M \in \mathcal{M} \mid M \text{ is fully abstract w.r.t. } \approx\}$ and $\mathsf{WFA}_{\approx}(\mathcal{M}) = \{M \in \mathcal{M} \mid M \text{ is weakly fully abstract w.r.t. } \approx\}$.

For regular families of partial congruences preserved by isomorphisms, the two concepts coincide:

Lemma 10. *If $\approx$ is a regular family of partial congruence relations preserved by isomorphisms then any $\approx$-weakly fully abstract model is $\approx$-fully abstract and vice versa.*

Proposition 11. *If $\approx$ is weakly regular then for each model $M \in |\mathbf{Mod}|$, its quotient $M/{\approx}$ is $\approx$-weakly fully abstract.*

Proposition 12. *If $M \in |\mathbf{Mod}|$ is $\approx$-weakly fully abstract then for any sentence $\varphi \in \mathbf{Sen}$, $M \models^{\approx} \varphi$ if and only if $M \models \varphi$.*

Theorem 13. *If $\approx$ is weakly regular then for any class $\mathcal{M} \subseteq |\mathbf{Mod}|$ of models closed under isomorphism, $\mathsf{Beh}_{\approx}(\mathcal{M}) = \mathsf{Abs}_{\equiv}(\mathsf{WFA}_{\approx}(\mathcal{M}))$; in particular, for any presentation $\Phi \subseteq \mathbf{Sen}$ we have $Mod_{\approx}[\Phi] = \mathsf{Abs}_{\equiv}(\mathsf{WFA}_{\approx}(Mod[\Phi]))$.*

Corollary 14. *If $\approx$ is regular and preserved by isomorphisms then for any class $\mathcal{M} \subseteq |\mathbf{Mod}|$ of models closed under isomorphism,* $\mathsf{Beh}_{\approx}(\mathcal{M}) = \mathsf{Abs}_{\equiv}(\mathsf{FA}_{\approx}(\mathcal{M}))$*; in particular, for any presentation $\Phi \subseteq \mathbf{Sen}$,* $Mod_{\approx}[\Phi] = \mathsf{Abs}_{\equiv}(\mathsf{FA}_{\approx}(Mod[\Phi]))$*.*

Finally, the following theorem gives a characterisation of behavioural theories of presentations under behavioural and abstractor semantics in terms of the standard theories of some model classes:

Theorem 15. *Let $\approx$ be weakly regular, and $\mathcal{M} \subseteq |\mathbf{Mod}|$ be a class of models closed under isomorphism. Then $Th_{\approx}(\mathsf{Beh}_{\approx}(\mathcal{M})) = Th(\mathsf{WFA}_{\approx}(\mathcal{M}))$ and $Th_{\approx}(\mathsf{Abs}_{\equiv}(\mathcal{M})) = Th(\mathcal{M}/\approx)$. In particular, for any presentation $\Phi \subseteq \mathbf{Sen}$, $Th_{\approx}(\Phi) = Th(\mathsf{WFA}_{\approx}(Mod[\Phi]))$ and $Th_{\approx}(\mathsf{Abs}_{\equiv}(Mod[\Phi])) = Th(Mod[\Phi]/\approx)$.*

Corollary 16. *Let $\approx$ be regular and preserved by isomorphisms, and $\mathcal{M} \subseteq |\mathbf{Mod}|$ be a class of models closed under isomorphism. Then $Th_{\approx}(\mathsf{Beh}_{\approx}(\mathcal{M})) = Th(\mathsf{FA}_{\approx}(\mathcal{M}))$. In particular, for any presentation $\Phi \subseteq \mathbf{Sen}$ we have $Th_{\approx}(\Phi) = Th(\mathsf{FA}_{\approx}(Mod[\Phi]))$.*

Observational semantics

In the above we have defined two behavioural semantics for specifications, based respectively on a family of indistinguishability relations and on an abstract equivalence in a concrete model category. The two semantics were linked with one another when the abstract equivalence is factorized by the indistinguishability structure, additionally assumed to be (weakly) regular. In most situations of practical importance we know much more about the indistinguishability structure of the model category considered, to be discussed below.

An *observability structure* on **Sat** is given by a choice of a set $OBS \subseteq S$ of *observable sorts*. Then for each model $M \in |\mathbf{Mod}|$, an *observable indistinguishability relation* $\sim_M^{\mathrm{OBS}} \subseteq |M| \times |M|$ is a partial congruence relation on M such that:

- $\mathsf{dom}(\sim_M^{OBS}) = \langle |M|_{OBS} \rangle_M$, where $|M|_{OBS} \in \mathbf{Set}^S$ is given by $(|M|_{OBS})_s = |M|_s$ for $s \in OBS$ and $(|M|_{OBS})_s = \emptyset$ for $s \notin OBS$; that is, the domain of the observational indistinguishability relation on a model is the least full subobject of this model that includes the carriers of observable sorts; we will write $\langle |M|_{OBS} \rangle_M$ as M_{OBS}.[7]
- $\sim_M^{OBS} \subseteq |M_{OBS}| \times |M_{OBS}|$ is the largest congruence on M_{OBS} such that for each observable sort $o \in OBS$, $(\sim_M^{OBS})_o$ is the identity on $|M|_o$.

Intuitively, OBS contains the sorts of data which are directly observable by the user of data types represented by the models and then for a model $M \in |\mathbf{K}|$, $|M_{OBS}|$ contains the elements that can be generated by the operations from the directly observable data, and $\sim_M^{OBS}$ identifies those elements of $|M_{OBS}|$ that cannot be distinguished by observing them via the observable sorts only, using the operations available in M.

The choice of $OBS \subseteq S$ yields the observational indistinguishability structure on **Sat**, written as $\sim^{OBS} = \langle \sim_M^{OBS} \rangle_{M \in |\mathbf{Mod}|}$, if for each $M \in |\mathbf{Mod}|$ there exists the largest congruence on M_{OBS} that is the identity on sorts in OBS. In the following we will assume that this is indeed the case.

[7] Note the notational distinction: $|M|_{OBS} \neq |M_{OBS}|$ in general.

Any morphism between models that is bijective on observable sorts will be called an *observational morphism*, any congruence relation on a model that is the identity on observable sorts will be called an *observational congruence*, any (full) subobject such that the subobject inclusion is an observational morphism will be called an *observational (full) subobject*, etc. Notice that all isomorphisms are observational morphisms.

The abstract characterisation of observational indistinguishability is sufficient (and in fact rather convenient, see the proofs in [BT95b]) to prove some important properties of the observational indistinguishability structure:

Lemma 17. *The observational indistinguishability structure $\sim^{\mathrm{OBS}}$ is domain-uniform, regular and preserved by isomorphisms.*

The *observational abstract equivalence* is $\equiv_{OBS} = \equiv_{\sim OBS}$, the abstract equivalence factorized by $\sim^{OBS}$. For $M, N \in |\mathbf{Mod}|$, we say that M and N are *observationally equivalent* if $M \equiv_{OBS} N$.

Corollary 18. *For any class $\mathcal{M} \subseteq |\mathbf{Mod}|$ of models closed under isomorphism, and so in particular for any presentation $\Phi \subseteq \mathbf{Sen}$, we have:*

- $\mathsf{Beh}_{\sim^{\mathrm{OBS}}}(\mathcal{M}) \subseteq \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(\mathcal{M})$,
  $Mod_{\sim^{\mathrm{OBS}}}[\Phi] \subseteq \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(Mod[\Phi])$,
- $\mathsf{Beh}_{\sim^{\mathrm{OBS}}}(\mathcal{M}) = \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(\mathcal{M})$ *iff* $\mathcal{M}$ *is behaviourally closed (w.r.t.* $\sim^{\mathrm{OBS}}$*),*
  $Mod_{\sim^{\mathrm{OBS}}}[\Phi] = \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(Mod[\Phi])$ *iff* Φ *is behaviourally closed (w.r.t.* $\sim^{\mathrm{OBS}}$*),*
- $\mathsf{Beh}_{\sim^{\mathrm{OBS}}}(\mathcal{M}) = \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(\mathsf{FA}_{\sim^{\mathrm{OBS}}}(\mathcal{M}))$,
  $Mod_{\sim^{\mathrm{OBS}}}[\Phi] = \mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(\mathsf{FA}_{\sim^{\mathrm{OBS}}}(Mod[\Phi]))$,
- $Th_{\sim^{\mathrm{OBS}}}(\mathsf{Beh}_{\sim^{\mathrm{OBS}}}(\mathcal{M})) = Th(\mathsf{FA}_{\sim^{\mathrm{OBS}}}(\mathcal{M}))$,
  $Th_{\sim^{\mathrm{OBS}}}(\Phi) = Th(\mathsf{FA}_{\sim^{\mathrm{OBS}}}(Mod[\Phi]))$,
- $Th_{\sim^{\mathrm{OBS}}}(\mathsf{Abs}_{\equiv}(\mathcal{M})) = Th(\mathcal{M}/{\sim^{\mathrm{OBS}}})$,
  $Th_{\sim^{\mathrm{OBS}}}(\mathsf{Abs}_{\equiv_{\mathrm{OBS}}}(Mod[\Phi])) = Th(Mod[\Phi]/{\sim^{\mathrm{OBS}}})$.

In the standard algebraic framework any algebra is observationally equivalent to any of its subalgebras containing all the observable values as well as to any of its quotients w.r.t. a relation that is an identity on observable sorts. A more convenient characterisation of observational equivalence of algebras then follows: two algebras are observationally equivalent if some of their observational subalgebras have a common observational quotient. Unfortunately, this does not carry over to an arbitrary behavioural equivalence structure. However, the following results lead to such a further characterisation of the observational equivalence of models in our general framework, identifying an extra technical assumption needed here.

Lemma 19. *Any model is observationally equivalent to each of its observational full subobjects.*

Lemma 20. *Consider a model $M \in |\mathbf{Mod}|$ and an observational quotient N of M_{OBS}. Then $M \equiv_{\mathrm{OBS}} N$.*

Theorem 21. *For all models $M, N \in |\mathbf{Mod}|$, $M \equiv_{\mathrm{OBS}} N$ if and only if M_{OBS} and N_{OBS} have isomorphic observational quotients.*

Lemma 22. *If quotients are fully compatible with subobjects in* **Mod** *then any model is observationally equivalent to each of its observational quotients.*

Theorem 23. *If quotients are fully compatible with subobjects in* **Mod** *then two models $M, N \in |\mathbf{Mod}|$ are observationally equivalent if and only if there exist observational full subobjects M' and N' of M and N, respectively, such that M' and N' have isomorphic observational quotients.*

4 Behavioural relations

In the previous section abstract equivalence relations have in fact been defined indirectly, as equivalences factorized by an indistinguishability structure. Even though in many particular cases it is possible to give a more specific "logical" characterisation (cf. [BT95b], also e.g. [ST87]) it seems interesting and important to be able to provide a notion of a "morphism" witnessing an abstract equivalence of models, which would in practice be a common tool in actual proofs of the abstract equivalence of specific models. In the standard algebraic framework this is given by the concept of a *correspondence* introduced in [Sch87]. We will aim here at its generalisation to the framework of an arbitrary concrete category with some indistinguishability structure.

Let $\mathbf{Sat} = \langle \mathbf{Mod}, \mathbf{Sen}, \models, S, |\_\_| \rangle$ be a concrete satisfaction frame with an indistinguishability structure $\approx$ and the abstract equivalence $\equiv\, = \,\equiv_{\approx}$ factorized by $\approx$. As before, we assume that **Mod** is finitely directed complete, admits renaming of elements, has full subobjects and surjective quotients. Since some of the constructions below are in fact given up to an isomorphism only, we also assume that $\approx$ is preserved by isomorphisms.

The starting point for our considerations in this section is the category **Rel** of relations between models. The objects of **Rel** are all models and the morphisms between them are relations on the carrier sets, that is, a morphism from $M \in |\mathbf{Mod}|$ to $N \in |\mathbf{Mod}|$ is a relation $R \subseteq |M| \times |N|$. The composition in **Rel** is the usual composition of relations.

A relation $R \subseteq |M| \times |N|$, where $M, N \in |\mathbf{Mod}|$, is *representable* if there exist a model $C \in |\mathbf{Mod}|$ and morphisms $h{:}\, C \to M$ and $h'{:}\, C \to N$ such that for all $a \in |M|$ and $b \in |N|$, aRb if and only if for some $c \in |C|$, $|h|(c) = a$ and $|h'|(c) = b$. We say then that the *span* $M \xleftarrow{h} C \xrightarrow{h'} N$ *represents* R.

A span $M \xleftarrow{h} C \xrightarrow{h'} N$ is *structural* if the full subobject $h(C) = \langle |h|(|C|) \rangle_M$ of M generated by the image of $|h|$ is a surjective quotient of C with projection $\pi_{C/h(C)}$ such that $\pi_{C/h(C)};\iota_{h(C)\hookrightarrow M} = h$. Note that then $h(C) = C/\mathsf{ker}(|h|)$. A structural span $M \xleftarrow{h} C \xrightarrow{h'} N$ is *behaviourally surjective* if $M_{\approx}$ (the full subobject of M on which the indistinguishability congruence $\approx_M$ is defined) is a full subobject of the surjective quotient $h(C)$ of C; it is *behaviourally exact* if $M_{\approx} = h(C)$; it is *behavioural* if in addition for all $c, c' \in |C|$ if $|h|(c) \approx_M |h|(c')$ then $|h'|(c) \approx_N |h'|(c')$.

A *correspondence* is a span $M \xleftarrow{h} C \xrightarrow{h'} N$ such that both $M \xleftarrow{h} C \xrightarrow{h'} N$ and $N \xleftarrow{h'} C \xrightarrow{h} M$ are structural; it is *behavioural* if both $M \xleftarrow{h} C \xrightarrow{h'} N$ and $N \xleftarrow{h'} C \xrightarrow{h} M$ are so. That is, a span $M \xleftarrow{h} C \xrightarrow{h'} N$ is a *behavioural* *correspondence* if $M_\approx$ is a surjective quotient of C by $\mathsf{ker}(|h|)$, $N_\approx$ is a surjective quotient of C by $\mathsf{ker}(|h'|)$, and for all $c, c' \in |C|$, $|h|(c) \approx_M |h|(c')$ if and only if $|h'|(c) \approx_N |h'|(c')$.

Theorem 24. *If there exists a behavioural correspondence* $M \xleftarrow{h} C \xrightarrow{h'} N$ *then* $M \equiv N$.

Theorem 25. *If pullbacks preserve surjective quotients in* $\mathbf{Mod}$ *then for any two models* $M, N \in |\mathbf{Mod}|$, $M \equiv N$ *if and only if there exists a behavioural correspondence* $M \xleftarrow{h} C \xrightarrow{h'} N$.

Observational relations

Suppose now that we have an observational structure on $\mathbf{Sat}$, with a set $OBS \subseteq S$ of observable sorts, and a partial congruence $\sim_M^{OBS}$ on each model $M \in |\mathbf{Mod}|$, where $\sim_M^{OBS}$ is the largest congruence on the full subobject $M_{OBS} = \langle |M|_{OBS} \rangle_M$ of M that is the identity on observable sorts of M.

Given two models $M, N \in |\mathbf{Mod}|$, a relation $R \subseteq |M| \times |N|$ is *observational* if it is bijective on observable sorts. A span $M \xleftarrow{h} C \xrightarrow{h'} N$ is *observational* if both h and h' are observational morphisms, i.e., $|h|$ and $|h'|$ are bijective on observable sorts. A correspondence $M \xleftarrow{h} C \xrightarrow{h'} N$ is *observational* if it is an observational span. Notice that observational spans are not required to be structural, behaviourally surjective, behaviourally exact, behavioural, etc., w.r.t. the observational indistinguishability structure; similarly, observational correspondences are correspondences, of course, but need not be behavioural.

Proposition 26. *Assume that for any object* $C \in |\mathbf{Mod}|$, *for any equivalence relation* $\simeq \subseteq |C| \times |C|$ *that is included in some congruence on* C, *there exists the least congruence on* C *that contains* $\simeq$. *Then a representable relation is observational if and only if it is represented by an observational span.*

The following results give a characterisation of observational equivalence via the existence of an observational correspondence between objects.

Theorem 27. *Let quotients be fully compatible with subobjects in* $\mathbf{Mod}$. *If there exists an observational correspondence* $M \xleftarrow{h} C \xrightarrow{h'} N$ *then* $M \equiv_{\mathrm{OBS}} N$.

Theorem 28. *Assume that in* $\mathbf{Mod}$ *pullbacks preserve surjective quotients and that quotients are fully compatible with subobjects. Then two models* $M, N \in |\mathbf{Mod}|$ *are observationally equivalent if and only if there exists an observational correspondence* $M \xleftarrow{h} C \xrightarrow{h'} N$ *between them.*

5 Final remarks

This paper presents an abstract study of two concepts used to give behavioural semantics to specifications: behavioural indistinguishability congruences (which may be and typically are partial) on individual models, leading to a new notion of behavioural satisfaction of axioms, and abstract behavioural equivalence of models, leading to a new semantics of specifications obtained as the closure of the usual semantics under the abstract equivalence. In the standard algebraic framework both these notions have been extensively studied and proved important for the methodology of algebraic specification and formal software development advocated in the area. We carry them further to a rather general framework of an abstract satisfaction frame.

A technical tool which we found useful to present the necessary algebraic concepts and background results is that of a concrete category. We recall this notion in Sect. 2, and then introduce the familiar concepts of subobject, relation between (the carriers of) objects, quotient and congruence in an arbitrary concrete category satisfying few mild technical assumptions. It should be stressed that this is only one possible way of proceeding: similar concepts may be introduced in the purely categorical framework, without requiring the category to be concrete. However, we believe that in concrete categories the notions introduced, while still rather abstract, remain comfortably close to the standard algebraic concepts, and therefore are perhaps more convincing and easier to work with. Nevertheless, it would be interesting to try to repeat the developments presented in this paper using a different representation of the underlying algebraic concepts, for instance modelling relations (and hence kernels, etc) between objects as spans (as in Sect. 4) and then defining quotients and subobjects w.r.t. some given factorization system for the category of models. [BD94] may provide a good starting point in this direction, although technicalities there are quite different. For instance, in [BD94] an extra assumption about the existence of a terminal "abstract behaviour" is used, which in typical algebraic frameworks means that the approach is applicable only to the (practically important) case of signatures where each operation has at most one non-observable argument.

In Sect. 3 we introduce the abstract definitions of behavioural satisfaction of sentences and then of behavioural and abstractor semantics for presentations. Following [BHW95] we link the two resulting semantics in Theorems 8, 9 and 13 and Corollary 14 assuming that the abstract equivalence is factorized by the behavioural indistinguishability given on the models, and that the indistinguishability structure is (weakly) regular. Theorem 15 and Corollary 16 give then a characterisation of behavioural theories of specifications in terms of standard theories of some classes of models, thus providing a basis for further development of proof techniques for behavioural consequence similar to those proposed in [BH95a, BH95b] for the standard algebraic framework. In a sense, these results are not really new: given the background algebraic concepts and facts from Sect. 2, the proofs just mimic the corresponding proofs in [BHW95] (although we explore a bit more carefully the weaker versions of full abstraction and regularity). However, by placing them explicitly in a more abstract setting we make them — both the results and their proofs — more clear and readable, and applicable in many other similar frameworks.

An interesting novelty which follows is the abstract characterisation of observational indistinguishability w.r.t. a set of observable sorts as the largest congruence that is defined on the submodel generated by the observable sorts and that is the identity on observable sorts. This is one of equivalent formulations of the standard observational indistinguishability relation, which also yields intuitively expected and justified concepts in other frameworks. We prove then that the observational indistinguishability and the observational equivalence it determines enjoy a number of useful properties; in particular, it is always regular, which allows us to drop the extra assumption on the theorems characterising links between behavioural and abstractor semantics of presentations in the observational case (Corollary 18).

A practically important and useful way to prove observational equivalence of two standard algebras is to define a correspondence between them, cf. [Sch87]. In Sect. 4 we generalise this concept to the abstract framework used in this paper, and show that under some technical assumptions (satisfied in the examples we study) two models are abstractly behaviourally equivalent if and only if there exists a behavioural correspondence between them. Somewhat restrictively, behavioural correspondences are required to match exactly the domains of the partial indistinguishability congruence on the models considered, and preserve and reflect this indistinguishability. This is not very satisfactory, and we do not know of any more elegant way to formulate the equivalent condition in this abstract setting. However, in the case of observational equivalence, a much nicer result is obtained: two models are observationally equivalent if and only if there exists an observational correspondence between them, where the abstract notion of an observational correspondence is as nice and permissive as in the standard case: it is an arbitrary relation closed under the "algebraic structure" of models which is bijective on observable sorts. This concept (and the result stated in Theorem 28) seems to yield a really convenient practical tool for proving behavioural equivalence in all the specific frameworks we study.

This work is very much in the spirit of the theory of institutions, as introduced by Goguen and Burstall [GB92] and used in a number of similar attempts to generalise and clarify some fundamental ideas and concepts of algebraic specification and program development. We follow a similar pattern in [BT96], assuming that model categories are concrete, which results in a concept of *concrete institution*. For example, a rather intuitive treatment of variables is possible in an arbitrary concrete institution, yielding a more explicit justification for the treatment of variables in institutions proposed in [Tar86, ST87]. We believe that this also sets up a general abstract framework to develop a practical methodology for proving behavioural consequences of presentations, following the results of [BH95a, BH95b] for the standard algebraic framework. This also opens the questions of *stability* [Sch87] of reduct functors and other basic constructions in concrete institutions. Using the characterisation of the abstract (observational) equivalence between models via the existence of behavioural (observational) correspondences, where the correspondences are given as spans in the model category as in Sect. 4, we can show that stability of a constructor is equivalent to its functoriality w.r.t. some model morphisms.

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