Decidability of the Star Problem in $A^* \times \{b\}^*$ (†)

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Abstract: We adress in this paper the decidability of the Star Problem in trace monoids: Let L be a recognizable trace language, is L^* recognizable ? We prove that this problem is decidable when the trace monoid is a direct product of free monoids $A^* \times \{b\}^*$. This result shows, for the first time and contrary to a possible intuition, that the Star Problem is of distinct nature than the Recognizability Problem: Let L be a rational trace language, is L recognizable ?

Abstract: Ce papier est une contribution à l'étude de la décidabilité du Problème de l'Etoile dans les monoïdes de traces: Soit L un langage trace reconnaissable, L^* est-il reconnaissable ? On prouve que ce problème est décidable lorsque le monoïde de traces est un produit cartésien de deux monoïdes libres dont l'un n'a qu'un générateur: $A^* \times \{b\}^*$. Ceci montre que, contrairement à une possible intuition, le Problème de l'Etoile est de nature différente du Problème de la Reconnaissabilité: Soit L un langage trace rationnel, L est-il reconnaissable ?

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1. Introduction

Free partially commutative monoids have been introduced by Cartier and Foata [CF69] for combinatoric purposes. They have been intensively studied in computer science since the work of Mazurkiewicz [Maz77] who proposed these monoids, under the name of trace monoids, to modelize the behaviour of concurrent processes.
Let X be a finite alphabet, let $D \subseteq X \times X$ be a reflexive and symmetric dependence relation and let $I = X \times X \setminus D$ be the independence relation. The trace monoid $\mathbb{M}(X,D)$, or simply $\mathbb{M}$, is the quotient of the free monoid X^* by the congruence generated by $\{(ab,ba) / (a,b) \in I\}$. An element of $\mathbb{M}$ is called a trace and a subset of $\mathbb{M}$ a trace language. Let us refer, for instance, surveys [AR88, Maz89, Per89] or the monograph [Die90] for more details concerning trace theory.

As in any monoid, the families of rational and recognizable trace languages play a central role in trace theory.
The family of rational languages, denoted by Rat $\mathbb{M}$, is the least family of trace languages which contains the finite sets and which is closed under union, concatenation and iteration.
A trace language $L \subseteq \mathbb{M}$ is said to be recognizable if there exists an $\mathbb{M}$-automaton which accepts L. The family of recognizable trace languages is denoted by Rec $\mathbb{M}$. The family Rec $\mathbb{M}$ is closed under union, intersection, complementation and concatenation [Fli74, CP85]. But, as soon as the alphabet contains at least two independent letters, the family Rec $\mathbb{M}$ is not closed under iteration.
Hence, a very natural question concerns the decidability of the Star Problem (SP for short):

SP Instance: A recognizable trace language L
Question: is L^* recognizable ?

Since every recognizable trace language is rational, the Star Problem is clearly related to the Recognizability Problem (RP for short):

RP Instance: A rational trace language L
Question: is L recognizable ?

Indeed, if the RP is decidable then so is the SP. By Kleene's theorem, the RP is decidable in free monoids. Ginsburg and Spanier have shown that the RP is decidable in free commutative monoids [GS66a,b]. Recently, Sakarovitch characterizes exactly those trace monoids in which the RP is decidable [Sak92]: the RP is decidable if and only if the reflexive closure of the independence relation is transitive, i.e., if the trace monoid is a free product of free commutative monoids.

Sufficient conditions ensuring the recognizability of L^* can be found in [CM85, FR85, Mét86, Has91]. In the case where the set L is finite, necessary conditions have also been proposed in [Roz90, MR91]. But up to now, apart from the case of a free product of free commutative monoids, the decidability of the SP (even reduced to finite languages) remains open.
Considering the difficulty of the SP, we address in this paper the decidability of the SP in the particular case where the trace monoid is a direct product of two free monoids $A^* \times B^*$. Note that this is not a special case of Sakarovitch's characterization as soon as one alphabet contains at least two letters.
First we show, and it seems surprisingly to never have been noticed before, that the decidability of the SP in a direct product of free monoids reduces to consider alphabets of size at most two. If both alphabets are of size one, the trace monoid is in fact a commutative monoid and the RP, whence the SP, is decidable [GS66, Goh85].

In the case of a direct product of two free monoids, we give a new (decidable) necessary condition and a new (decidable) sufficient condition for the iteration of a trace language to be recognizable. Then, we prove that the SP is decidable in $A^*\times B^*$ if B contains one letter only. Since the RP is undecidable in such a monoid, our result shows for the first time that, contrary to a possible intuition, the two problems are of distinct natures.

2. Preliminaries

Let t be a trace. We denote by alph(t) the set of letters which occur in t. The trace t is said to be connected if the restriction of the graph (X,D) to alph(t) is connected. A trace language L is connected if it contains connected traces only. Using this notion of connected languages, Métivier [Mét86], Ochmanski [Och85] and Clerbout / Latteux [CL87] (in the more general framework of semi-commutations) proved independently the following theorem : If L is a recognizable connected trace language then L^* is recognizable, too.

Let $L \subseteq \mathbb{M}$, we denote by C (N resp.) the set of connected (non connected resp.) traces of L. Suppose now that $\mathbb{M} = A^*\times B^*$ and let X be either A or B. Throughout, we identify A^* and B^* to the submonoids $A^*\times\{1\}$ and $\{1\}\times B^*$ of $\mathbb{M}$. We denote by π_X the projection from $\mathbb{M}$ onto X^*, moreover we set $C_X = \pi_X(C)$, $N_X = \pi_X(N)$ and $L_X = \pi_X(L)$. Note that, if L is recognizable then so are the sets C_A, C_B, C, N, N_A, N_B (for instance, Mezei's theorem [Ber79] may be used for the recognizability of the last two sets).
For a language L of any monoid and an integer n, we adopt the following abbreviations:

$$L^{\leqslant n} = L^0 \cup L^1 \cup \ldots \cup L^n \text{ and } L^{\geqslant n} = L^n \cup L^{n+1} \cup \ldots$$

Throughout the paper, we will use in a crucial way the Finite Power Property (FPP for short): a language L has the FPP if there exists some integer n such that $L^* = L^{\leqslant n}$. Hashiguchi [Has 79] and Simon [Sim78] have shown that in free monoids the FPP is decidable for recognizable languages.
More precisely, we will apply the FPP to languages of the form $C^* \cup N$. Note that for all n, $(C^* \cup N)^{\leqslant n} \subseteq C^*(NC^*)^{\leqslant n} \subseteq (C^* \cup N)^{\leqslant 2n+1}$. Hence, the language $C^* \cup N$ has the FPP if and only if there exists some integer n such that $(C^* \cup N)^* = C^*(NC^*)^{\leqslant n}$.

3. The Star Problem

3.1. Generalities

In this paper, in order to prove that the iteration of a given recognizable language L is recognizable too, we will always prove that $C^* \cup N$ has the FPP. The result follows then from the easy lemma.

Lemma 3.1. Let L be a recognizable language of any trace monoid $\mathbb{M}$. If $C^* \cup N$ has the FPP then L^* is recognizable, too.

Proof. Let $n \in \mathbb{N}$ be such that $L^* = C^*(NC^*)^{\leqslant n}$. Since L is recognizable, so are C and N. Since C is connected, we deduce that C^* is recognizable, too (see Section 2). We conclude using the closure of Rec $\mathbb{M}$ under union and concatenation. □

Note that we do not know neither whether the condition of the lemma above is necessary nor whether it is decidable.

In the remainder, we will restrict our study to the particular case where the trace monoid is a direct product of two free monoids. First, we prove that only alphabets with at most two letters need to be considered.

Proposition 3.2. Let A,B,C be three alphabets with $|A| \geqslant 2$ and $B = \{b,c\}$. The SP is decidable in $A^*\times C^*$ if and only if it is decidable in $B^*\times C^*$.

Proof. Assume that $A = \{a_1,\ldots,a_n\}$ and let $f : A^* \to B^*$ be the classical coding defined by $f(a_i) = bc^i$ for $1 \leqslant i \leqslant n$. We define the morphism $g : A^*\times C^* \to B^*\times C^*$ by $g(u,v) = (f(u),v)$. The proposition is a trivial consequence of the following claim:

a language $L \subseteq A^*\times C^*$ is recognizable if and only if $g(L)$ is recognizable.

First assume that L is recognizable. By Mezei's theorem [Ber79], $L = \cup_{1\leqslant i\leqslant k} R_iS_i$ with $R_i \in \mathrm{Rec}A^*$ and $S_i \in \mathrm{Rec}C^*$. Hence $g(L) = \cup_{1\leqslant i\leqslant k} f(R_i)\cdot S_i$ is recognizable. Conversely, assume that $g(L)$ is recognizable. Since g is an injective morphism, $L = g^{-1}(g(L))$ is recognizable too and the claim is proved. □

Since the SP is decidable in free commutative monoids, the proposition above shows that only two cases remain to be considered: $\{a,b\}^*\times\{c\}^*$ and $\{a,b\}^*\times\{c,d\}^*$. We will solve the first case in Section 4.

3.2. Properties in $A^*\times B^*$

Since any non-connected trace of $\mathbb{M} = A^*\times B^*$ contains at least one letter of A and one letter of B, a factorization of a trace $u \in \mathbb{M}$ contains at most $min\{|u|_A,|u|_B)$ non-connected traces (where $|u|_X$ denotes the number of occurrences from X in u). Therefore, for a trace language $L \subseteq \mathbb{M}$ and a trace $u \in L^*$, we have

Remark 3.3. $\quad u \in C^*(NC^*)^n \Rightarrow n \leqslant min\{|u|_A,|u|_B)$

The following proposition gives our strongest (decidable) necessary condition for the iteration of a trace language to be recognizable. A special case of this result, where L is finite and contains exactly one non connected trace, was previously proved by Ochmanski [Och90].

Proposition 3.4. Let L be a language of $\mathbb{M} = A^*\times B^*$. If L^* is recognizable then $C_A^* \cup N_A$ and $C_B^* \cup N_B$ have the FPP.

Proof. Let n be the number of states of the minimal automaton $\mathcal{A}$ recognizing L^*. Let $u \in (C_A^* \cup N_A)^* = L_A^*$. There exists $v \in B^*$ such that $uv \in L^*$. Using the iteration lemma in $\mathcal{A}$ we find a subword v' of v such that $uv' \in L^*$ and $|v'| < n$. Therefore, $uv' \in C^*(NC^*)^{<n}$ (Remark 3.3) and $u = \pi_A(uv') \in C_A^*(N_AC_A^*)^{<n}$. □

Unfortunately, this condition is not sufficient, even if L is a finite set. For instance, let $A = \{a,c\}$, $B = \{b,d\}$ and $L = \{ca,c,aa,b,d,ab,acd\}$. We have $C_A = \{1,ca,c,aa\}$, $C_B = \{1,b,d\}$, $N_A = \{a,ac\}$ and $N_B = \{b,d\}$. We can easily verify that $(C_A^* \cup N_A)^* = A^* = C_A^* \cup aC_A^* = C_A^* \cup N_AC_A^*$ and $(C_B^* \cup N_B)^* = C_B^*$. Therefore the necessary condition is fulfilled but L^* is not recognizable since $L^* \cap (ac)^*d^* = \{(ac)^nd^m \,/\, n \leqslant m\}$ which is not recognizable.

We proceed now with our weakest (decidable) sufficient condition which ensure that the iteration of a recognizable trace language is recognizable too. Again the special case of this result where L is finite and contains exactly one non connected trace was previously proved by Ochmanski in [Och90].

Proposition 3.5. Let L be a recognizable language of $\mathbb{M} = A^*\times B^*$. If $C_A^* \cup N_A$ and $C_B^* \cup N_B$ have the FPP and moreover $N = N_A N_B$ then L^* is recognizable.

Proof. Let $n \in \mathbb{N}$ be such that

$$(C_A^* \cup N_A)^* = C_A^*(N_A C_A^*)^{<n} \text{ and } (C_B^* \cup N_B)^* = C_B^*(N_B C_B^*)^{<n} \quad (1)$$

Let $w \in C^*(N C^*)^{n^3}$, we can write $w = w_1 w_2 \ldots w_{n^2}$ with $w_i \in C^*(NC^*)^n$ for $1 \leqslant i \leqslant n^2$. For $X = A,B$ and $1 \leqslant i \leqslant n^2$, define $w_{X,i} = \pi_X(w_i)$.
From (1), there exist integers $p_{X,i}$ such that $1 \leqslant p_{X,i} \leqslant n$ and $w_{X,i} \in C_X^*(N_X C_X^*)^{n-p_{X,i}}$. Since there are n^2 integers $p_{A,i}$ (resp. $p_{B,i}$), there exists some integer p_A (resp. p_B) such that $p_A = p_{A,i}$ (resp. $p_B = p_{B,i}$) for at least n different values of i.
Using $w_{A,i} \in C_A^*(N_A C_A^*)^{n-p_A}$ for p_B values of i and $w_{A,i} \in C_A^*(N_A C_A^*)^n$ for the other values of i, we obtain $\pi_A(w) \in C_A^*(N_A C_A^*)^{n^3-p_A p_B}$. In a similar way $\pi_B(w) \in C_B^*(N_B C_B^*)^{n^3-p_A p_B}$.
Since $C^* = C_A^* C_B^*$ and $N = N_A N_B$, we deduce that $w = \pi_A(w)\pi_B(w) \in C^*(N C^*)^{n^3-p_A p_B}$.
Therefore $L^* = C^*(N C^*)^* = C^*(N C^*)^{<n^3}$ and the result follows from Lemma 3.1. □

Again, this sufficient condition is not necessary, even for finite languages. For instance, let $A = \{a,c\}$, $B = \{b\}$ and $L = \{aa,c,ca,b,ab^2,acb\}$. Clearly, $C_A^* \cup N_A$ and $C_B^* \cup N_B$ have the FPP but $N \neq N_A N_B$. Nevertheless, using Proposition 4.1 we deduce easily that L^* is recognizable.

4. Decidability of the SP in $A^*\times\{b\}^*$

First, we give a new (decidable) sufficient condition which ensures the recognizability of the iteration of a recognizable trace language.

Proposition 4.1. Let L be a recognizable trace language of $\mathbb{M} = A^*\times\{b\}^*$. If $C_A^* \cup N_A$ has the FPP and $C_B \cap B^+ \neq \emptyset$ then L^* is recognizable too.

Proof. By hypothesis, there exist two integers $n,m > 0$ such that $L_A^* = C_A^*(N_A C_A^*)^{<n}$ and $b^m \in C_B$. Let k be the number of states of the minimal automaton $\mathcal{A}$ recognizing L. We claim that

$$\forall\ u \in L_A^*,\ \exists\ p < nk \text{ such that } ub^p \in C^*(NC^*)^{<n}$$

Let $u \in L_A^*$, using the FPP we write $u = u'_0 u_1 u'_1 \ldots u_j u'_j$ with $u'_i \in C_A^*$, $u_i \in N_A$ and $j < n$. For each $1 \leqslant i \leqslant j$, there exists $\lambda_i > 0$ such that $u_i b^{\lambda_i} \in N$. Using the iteration lemma in the automaton $\mathcal{A}$, we may assume that $\lambda_i \leqslant k$. Let $p = \lambda_1 + \ldots + \lambda_j \leqslant jk < nk$. We have $ub^p = ub^{\lambda_1+\ldots+\lambda_j} = u'_0 u_1 b^{\lambda_1} u'_1 \ldots u_j b^{\lambda_j} u'_j \in C_A^*(NC_A^*)^j \subseteq C_A^*(NC_A^*)^{<n}$ and the claim is proved.

We prove now that $C^* \cup N$ has the FPP, hence the result will follow from Lemma 3.1.
Let $w \in C^*(N C^*)^{(m+1)nk}$. We write $w = w_0 w_1 \ldots w_m$ with $w_i \in C^*(N C^*)^{nk}$ for $0 \leqslant i \leqslant m$. For each $0 \leqslant i \leqslant m$, applying the previous claim to $u_i = \pi_A(w_i)$, there exists some integer $p_i < nk$ such that $u_i b^{p_i} \in C^*(NC^*)^{<n}$. From Remark 3.3, $|w_i|_b \geqslant nk$ and hence $q_i = |w_i|_b - p_i > 0$. Among the $m+1$ integers, q_0, q_0+q_1, … , $q_0+\ldots+q_m$, two must be equal modulo m, that is, there exists $0 \leqslant i < j \leqslant m$ such that $q_{i+1}+\ldots+q_j = mq$ for some integer q. Therefore,

$$w_{i+1}\ldots w_j = (u_{i+1}b^{p_{i+1}})\ldots(u_j b^{p_j})b^{mq} \in C^*(NC^*)^{<(j-i)n} \subseteq C^*(NC^*)^{<(j-i)nk}$$

and thus $w \in C^*(NC^*)^{<(m+1)nk}$. □

As a first corollary we obtain the decidability of the SP for *finite sets*.

Corollary 4.2. Let L be a finite language of $A^*\times\{b\}^*$. L^* is recognizable if and only if $C_A{}^* \cup N_A$ and $C_B{}^* \cup N_B$ have the FPP.

Proof. The conditions are necessary by Proposition 3.4. Conversely, if $C_B \cap B^+ \neq \emptyset$, the result follows from Proposition 4.1. If $C_B \cap B^+ = \emptyset$, then $C_B{}^* \cup N_B = \{1\} \cup N_B$ has the FPP if and only if $N_B \subseteq \{1\}$. Since any non-connected trace contains letters from B, this implies $N = \emptyset$. Hence L is a connected language and L^* is recognizable (see Section 2). □

Unfortunately, for infinite sets, the FPP for $C_A{}^* \cup N_A$ and $C_B{}^* \cup N_B$ are not sufficient conditions. For instance, let $A = \{a,c\}$, $B = \{b\}$ and $L = \{a,c,ab\} \cup cb^+$. The FPP for $C_A{}^* \cup N_A$ and $C_B{}^* \cup N_B$ are easy to verify but L^* is not recognizable since $L^* \cap a^*b^* = \{a^ib^j / i \geqslant j\}$ which is not a recognizable trace language.

Therefore, we need a more involved decidable criterium in the case where $C_B \cap B^+ = \emptyset$. For this purpose, we split N_A in two parts: $N_A = F_A \cup I_A$ with $I_A = \{u \in N_A / ub^m \in N$ for infinitely many $m\}$ and $F_A = N_A \setminus I_A$. From Mezei's theorem, we obtain $L = \cup_{1\leqslant i\leqslant k} R_iS_i$ for some $R_i \in \mathrm{Rec}A^*$ and $S_i \in \mathrm{Rec}B^*$, and then

$$I_A = \bigcup_{1\leqslant i\leqslant k \ \&\ S_i \text{ infinite}} R_i$$

In particular, I_A and F_A are recognizable.

Proposition 4.3. Let L be an infinite recognizable language of $A^*\times\{b\}^*$ and such that $C_B \cap B^+ = \emptyset$. Then L^* is recognizable if and only if $C_A{}^* \cup N_A$ has the FPP and the following condition holds:

$$\exists\ s \geqslant 0 \text{ such that } C_A{}^*(N_AC_A{}^*)^s \subseteq L_A{}^*I_AL_A{}^* \qquad (\dagger)$$

Proof. We first show that the condition is necessary. Assume that L^* is recognizable. From Proposition 3.4, it remains to prove that (†) holds. Let s be the number of states of the minimal automaton $\mathcal{A}$ recognizing L^*.
Let $u \in C_A{}^*(N_AC_A{}^*)^s$, there exists $v \in B^*$ such that $uv \in C_A{}^*(NC_A{}^*)^s \subseteq L^*$. From Remark 3.3, $|v| \geqslant s$. Since there exists a loop in the "v-part" of the path of uv in $\mathcal{A}$, we have $uv(b^q)^* \subseteq L^*$ for some integer $q > 0$. Hence, $ub^m \in L^*$ for infinitely many integers m.
Since $C_B \cap B^+ = \emptyset$, using Remark 3.3 we obtain $ub^m \in C_A{}^*(NC_A{}^*)^{\leqslant|u|}$ for infinitely many integers m. It follows that u admits a factorization in $L_A{}^*I_AL_A{}^*$. Therefore, $C_A{}^*(N_AC_A{}^*)^s \subseteq L_A{}^*I_AL_A{}^*$.

Conversely, let us show that the condition is sufficient. The proof is similar to the one of Proposition 4.1.
Let n,s be integers such that $L_A{}^* = C_A{}^*(N_AC_A{}^*)^{<n}$ and $C_A{}^*(N_AC_A{}^*)^s \subseteq L_A{}^*I_AL_A{}^*$. Let $\mathcal{A}$ be the minimal automaton recognizing L, let k be the number of states of $\mathcal{A}$ and let m be the least common multiple of the length of elementary loops of $\mathcal{A}$. Note that, since L is infinite, we have $m > 0$. We claim that

$$\forall\ u \in C_A{}^*(N_AC_A{}^*)^{\geqslant s},\ \exists\ p < (2n+1)k \text{ such that } ub^p(b^m)^* \subseteq C^*(NC^*)^{<2n}$$

We have $C_A{}^*(N_AC_A{}^*)^{\geqslant s} \subseteq L_A{}^*I_AL_A{}^* = C_A{}^*(N_AC_A{}^*)^{<n}I_AC_A{}^*(N_AC_A{}^*)^{<n}$.
Let $u \in C_A{}^*(N_AC_A{}^*)^{\geqslant s}$, we can write $u = u_1u_2u_3$ with $u_1,u_3 \in C_A{}^*(N_AC_A{}^*)^{<n}$ and $u_2 \in I_A$.

As in the claim of the proof of Proposition 4.1, we find two integers $p_1, p_3 < nk$ such that $u_1 b^{p_1}$, $u_3 b^{p_3} \in C^*(NC^*)^{<n}$.
Now, since $u_2 \in I_A$, we have $u_2 v_2 \in N$ for some $v_2 \in B^+$ with $|v_2| \geqslant k$. Applying the iteration lemma in $\mathcal{A}$, we deduce that $u_2 b^{p_2}(b^m)^* \subseteq N$ for some $p_2 < k$.
The claim follows with $p = p_1 + p_2 + p_3$.

We conclude the proof by showing that $C^* \cup N$ has the FPP.
Let $h = \max(s,(2n+1)k)$ and let $w = w_0 \dots w_m$ with $w_i \in C^*(NC^*)^h$ for $0 \leqslant i \leqslant m$. For each $0 \leqslant i \leqslant m$, let $u_i = \pi_A(w_i)$ and let $p_i < (2n+1)k$ be such that $u_i b^{p_i}(b^m)^* \subseteq C^*(NC^*)^{<2n}$.
Since $|w_i|_b \geqslant (2n+1)k$ (Remark 3.3), we have $q_i = |w_i|_b - p_i > 0$. As in the proof of Proposition 4.1, we find $0 \leqslant i < j \leqslant m$ such that $w_{i+1} \dots w_j = (u_{i+1} b^{p_{i+1}}) \dots (u_j b^{p_j}) b^{mq}$ for some integer q.
Hence, $w_{i+1} \dots w_j \in C^*(NC^*)^{<(j-i)2n} \subseteq C^*(NC^*)^{<(j-i)h}$ and we obtain $w \in C^*(NC^*)^{<(m+1)h}$. □

From Propositions 4.1 and 4.3, we obtain directly the following characterization.

Corollary 4.4. Let L be an infinite recognizable language of $A^* \times \{b\}^*$. L^* is recognizable if and only if $C_A{}^* \cup N_A$ has the FPP and either $C_B \cap B^+ \neq \emptyset$ or the condition (†) holds.

It remains to prove that this criterium is decidable.

Theorem 4.5. The Star Problem is decidable in $A^* \times \{b\}^*$.

Proof. Let L be a recognizable language of $A^* \times B^*$. Using the characterizations of Corollaries 4.2 and 4.4 it is sufficient to prove that the FPP for $C_A{}^* \cup N_A$ and $C_B{}^* \cup N_B$ and the condition (†) are decidable. Since C_A, C_B, N_A and N_B are recognizable, we already know, from Hashigushi and Simon [Has79, Sim78], that the FPP's are decidable.
Clearly, the set $K = A^* \setminus (L_A{}^* I_A L_A{}^*)$ is recognizable. Let n be the number of states of the minimal automaton $\mathcal{A}$ recognizing K.
We claim that (†) holds if and only if $C_A{}^*(N_A C_A{}^*)^n \subseteq L_A{}^* I_A L_A{}^*$. Since inclusion is decidable for recognizable sets in a free monoid, this shows that the condition (†) is decidable.
Assume that $C_A{}^*(N_A C_A{}^*)^n \not\subseteq L_A{}^* I_A L_A{}^*$. There exists a word $u \in K$ such that $u = u_0 v_1 u_1 \dots v_n u_n$ with $u_i \in C_A{}^*$ and $v_i \in N_A$. Let q_0 be the initial state of $\mathcal{A}$ and let $q_i = q_0 \cdot (u_0 v_1 \dots u_{i-1} v_i)$ for $1 \leqslant i \leqslant n$. We have $q_i = q_j$ for some $i < j$.
Hence $u_0 v_1 \dots u_{i-1} v_i (u_i v_{i+1} \dots u_{j-1} v_j)^* u_j v_{j+1} \dots v_n u_n \subseteq K$ and there are arbitrarily large integers s such that $C_A{}^*(N_A C_A{}^*)^s \cap K \neq \emptyset$. Therefore, the condition (†) does not hold. □

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