The Kleene-Schützenberger theorem for formal power series in partially commuting variables

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Abstract. Kleene's theorem on the coincidence of regular and rational languages in free monoids has been generalized by Schützenberger to a description of the recognizable formal power series in non-commuting variables over arbitrary semirings, and by Ochmański to a characterization of the recognizable languages in trace monoids. We will describe the recognizable formal power series over arbitrary semirings and in partially commuting variables, i.e. over trace monoids. We prove that the recognizable series are certain rational power series, which can be constructed from the polynomials by using the operations sum, product and a restricted star which is applied only to series for which the elements in the support all have the same connected alphabet. The converse is true if the underlying semi-ring is commutative. Moreover, if in addition the semiring is idempotent then the same result holds with a star restricted to series for which the elements in the support have connected (possibly different) alphabets. It is shown that these assumptions over the semiring are necessary. This provides a joint generalization of Kleene's, Schützenberger's and Ochmański's theorems.

1 Introduction

In the theory of automata and formal languages, Kleene's foundational theorem on the coincidence of regular and rational languages in free monoids has been extended in many ways. Schützenberger [18] investigated formal power series over arbitrary semirings (e.g., like the natural numbers) and the free monoid, i.e. in noncommuting variables, and showed that the recognizable formal power series coincide with the rational ones. This was the starting point for a large amount of work on formal power series, cf. [17,12,2,11] for surveys. The concept of recognizable formal power series has also been defined for arbitrary monoids instead of the free monoid, but it was clear and has been stressed by several authors (cf., e.g. [17]) that in general then the recognizable and the rational series do not coincide.

On the other hand, Mazurkiewicz [13,14] introduced an important mathematical model for the behavior of concurrent systems: trace monoids (or free partially commutative monoids), see also [1,4–6] for their well-developed theory.

They are monoids whose generators are partially commutative. Again, their recognizable languages do not coincide with the rational ones, but by Ochmański's theorem [15] they coincide with the c-rational languages where the iteration is restricted to connected languages.

It is the aim of this paper to investigate recognizable formal power series over trace monoids, thereby obtaining a generalization of both Schützenberger's and Ochmański's results.

A trace alphabet (Σ, I) consists of a finite alphabet Σ and an irreflexive symmetric relation I indicating when two elements a, b of Σ commute, e.g. can occur independently of each other in a given concurrent system. A trace monoid $\mathbb{M}$ is therefore defined as the quotient of the free monoid Σ^* modulo the congruence generated by the relations $ab \sim ba$ if $a\ I\ b$. Now let K be an arbitrary semiring, and let $K\langle\langle\mathbb{M}\rangle\rangle$ be the collection of all formal power series $S = \sum_{m \in \mathbb{M}} (S, m) \cdot m$. These can also be viewed as series with entries (S, m) from K in which certain of the variables (= elements of Σ) are allowed to commute, as indicated by the relation I. The main result of this paper is a description of the recognizable formal power series in $K\langle\langle\mathbb{M}\rangle\rangle$.

Sums and (Cauchy) products of two formal power series are defined in the natural way, and the star $S^* = \sum_n S^n$ of a series S is defined in $K\langle\langle\mathbb{M}\rangle\rangle$ if S is proper, i.e. $(S, 1) = 0$. The *rational* series in $K\langle\langle\mathbb{M}\rangle\rangle$ are then defined as the smallest collection of series containing all the polynomials and being closed under the operations sum, product, and star (the latter applied only to proper series).

It is known that in general the recognizable series in $K\langle\langle\mathbb{M}\rangle\rangle$ form a proper subclass of the rational ones. We therefore define the subclasses of *c-rational* and *mc-rational* series. For this, we introduce some more notation (see section 2 for background and details). An element $m \in \mathbb{M}$ is called *connected*, if it cannot be written as a product of two non-trivial commuting elements having disjoint sets of generators from Σ. The *support* of the series S is the set of all $m \in \mathbb{M}$ with non-zero entry, i.e., $(S, m) \neq 0$. We say that a series S is *connected*, if each element of its support is connected, and S is *mono-alphabetic*, if all elements of its support have the same set of generators. The c-rational series are obtained from the polynomials by allowing the operations sum, product, and star, but the latter applied only to proper and connected series. The mc-rational series are constructed in the same way, but using star only for series which are proper, mono-alphabetic and connected. In view of Ochmański's result, one might expect that the recognizable series in $K\langle\langle\mathbb{M}\rangle\rangle$ coincide with the c-rational ones. However, we will show that this fails in general even for the semiring $(\mathbb{N}, +, \times)$. Our main result is the following:

Theorem 1. *Let $\mathbb{M}$ be a trace monoid and K a semiring.*
(a) Each recognizable series in $K\langle\langle\mathbb{M}\rangle\rangle$ is mc-rational.
(b) If K is commutative, each mc-rational series in $K\langle\langle\mathbb{M}\rangle\rangle$ is recognizable.
(c) If K is commutative and idempotent, each c-rational series in $K\langle\langle\mathbb{M}\rangle\rangle$ is recognizable.

The closure under product of recognizable series in partially commutative variables was already investigated by Fliess. He obtained two partial results. He first proved this closure when the semiring K is commutative and the trace monoid $\mathbb{M}$ is a direct product of free monoids [9]. Note that a trace monoid is a direct product of free monoids if and only if the complement of the independence relation I is transitive, hence this first result deals only with a rather restricted class of trace monoids. Then, Fliess proved the closure under product for arbitrary trace monoids but only for very specific semirings K (strong Fatou semirings, or the Boolean semiring) [10]. By Theorem 1(b), the closure holds for all commutative semirings and trace monoids. We show by example that the commutativity of K is needed for this.

In Section 2 we recall the definitions and notations concerning formal power series and Mazurkiewicz traces. Theorem 1(b,c) is proved in section 3. There we also show that if the star S^* of a recognizable proper series S is connected, then it is also recognizable. This gives another closure property of the recognizable series under the star-operation. Part (a) of Theorem 1 is proved in section 4, and in section 5 we give examples showing that the hypotheses on the semiring in Theorem 1 are necessary. Finally in the conclusion we summarize our results and discuss the relationship with Schützenberger's and Ochmański's theorems.

Ochmański's characterization of the recognizable languages in trace monoids has been generalized to the larger class of concurrency monoids in [7]. It can be suspected that a version of Theorem 1 can also be shown for formal power series over concurrency monoids, but with a technically even more complicated proof. Besides, it seems a very interesting research road to investigate which other results from the theory of formal power series over non-commuting variables can be extended to series over partially commuting variables, i.e. over trace monoids.

An extended abstract of this work appeared in [8].

2 Background

Here we recall the necessary notation and background for formal power series and of trace theory. For more details, we refer the reader to [17,2,4,6].

Let M be any monoid and $K = (K, +, \cdot, 0, 1)$ any semiring, i.e., $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot x = x \cdot 0 = 0$ for each $x \in K$. If multiplication is commutative, we say that K is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. For instance, the Boolean semiring $\mathbb{B} = (\{0, 1\}, +, \cdot, 0, 1)$ is both commutative and idempotent. The semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ is commutative but not idempotent. The semiring $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{1\})$ of languages over some alphabet Σ is idempotent but not commutative. The semiring $\mathbb{N}^{n\times n}$ of $(n \times n)$-matrices is neither commutative nor idempotent. Other semirings useful in computer science (and also in optimization problems of operation research [19]) are the min-plus (or max-plus or min-max) semirings over the integers or the reals. For instance the min-plus semiring over the reals $(\mathbb{R} \cup \{\infty\}, min, +, \infty, 0)$ is both commutative and idempotent.

A formal power series is a mapping

$$\begin{aligned} S : M &\longrightarrow K \\ m &\longmapsto (S, m) \end{aligned}$$

It is usually denoted as a formal sum $S = \sum_{m \in M} (S, m).m$. The set $supp(S) = \{m \in M \mid (S, m) \neq 0\}$ is called the *support* of S, and if it is finite, then S is called a *polynomial*. The collection of all formal power series is denoted by $K\langle\langle M\rangle\rangle$, and its subset of all polynomials by $K\langle M\rangle$. We consider elements of K also as polynomials in the natural way, having a non-zero entry only at $1 \in M$. If $L \subseteq M$, we define the *characteristic series* of L by $1_L = \sum_{m \in L} 1 \cdot m$.

Let $n \geq 1$ and $[n] = \{1, \ldots, n\}$. We let $K^{n \times n}$ be the monoid of all $(n \times n)$-matrices over K (with matrix multiplication as usual). A series $S \in K\langle\langle M\rangle\rangle$ is called *recognizable*, if there exists an integer $n \geq 1$, a monoid morphism $\mu : M \longrightarrow K^{n \times n}$ and vectors $\lambda \in K^{1 \times n}, \gamma \in K^{n \times 1}$ such that

$$(S, m) = \lambda \cdot (\mu m) \cdot \gamma = \sum_{i,j \in [n]} \lambda_i (\mu m)_{ij} \gamma_j$$

for each $m \in M$. In this case, the triple (λ, μ, γ) is called a *representation* of dimension n of the series S, and we often shortly write $S = (\lambda, \mu, \gamma)$ to denote this. If $i, j \in [n]$, we also abbreviate $(\mu m)_{ij} =: \mu m_{ij}$. We let $K^{rec}\langle\langle M\rangle\rangle$ denote the set of all recognizable formal power series.

From now on, we assume that M is a finitely generated monoid which carries a length function, that is a morphism ℓ from M to $(\mathbb{N}, +)$ such that $\ell(m) = 0$ iff $m = 1$. This ensures in particular that each element $m \in M$ has only finitely many factorizations $m = m_1 \cdot m_2$ and allows us to define the Cauchy product of two series and the star of a proper series. These operations can also be defined in more general settings but such generalizations are not needed in the present paper.

With componentwise addition, $K\langle\langle M\rangle\rangle$ becomes a commutative monoid: the sum of two series $S, S' \in K\langle\langle M\rangle\rangle$ is defined for $m \in M$ by

$$(S + S', m) = (S, m) + (S', m).$$

Now, the *(Cauchy) product* of two series S, S' in $K\langle\langle M\rangle\rangle$ is the series defined for $m \in M$ by

$$(S \cdot S', m) = \sum_{m = m_1 \cdot m_2} (S, m_1) \cdot (S, m_2).$$

With these two operations, $K\langle\langle M\rangle\rangle$ is a semiring. The powers $S^n (n \geq 0)$ are defined in the natural way. We call S *proper*, if $(S, 1) = 0$, and then we put, in the natural way, $S^* = \sum_{n \geq 0} S^n$, the *star* (or iteration) of S, and $S^+ = \sum_{n \geq 1} S^n$. We let $K^{rat}\langle\langle M\rangle\rangle$ denote the smallest subset of $K\langle\langle M\rangle\rangle$ which contains all polynomials and is closed under the operations sum, product and star, where the latter is only applied to proper series. Its elements are called *rational* formal power series. Now Schützenberger's theorem states the following equivalence between recognizable and rational series over the free monoid.

Theorem 2 (Schützenberger [18]). *Let Σ be any finite set and K any semiring. Then*

$$K^{rec}\langle\langle \Sigma^* \rangle\rangle = K^{rat}\langle\langle \Sigma^* \rangle\rangle.$$

From this, Kleene's theorem on the coincidence of regular and rational languages follows by considering the Boolean semiring $\mathbb{B}$ and noting that a language $L \subseteq \Sigma^*$ is recognizable (rational resp.) iff its characteristic series $1_L \in \mathbb{B}\langle\langle \Sigma^* \rangle\rangle$ is recognizable (rational resp.).

Later we will also need the *Hadamard product* $S \odot T$ of two series $S, T \in K\langle\langle M \rangle\rangle$. It is defined by $(S \odot T, m) = (S, m) \cdot (T, m)$ for all $m \in M$.

Note that when we use the Boolean semiring $\mathbb{B}$ then each series $S \in \mathbb{B}\langle\langle M \rangle\rangle$ is the characteristic series of some language $L \subseteq M$. Therefore, series may be identified with languages. Moreover, the rational operations $+, \cdot, ^*$ on series correspond to the rational operations $\cup, \cdot, ^*$ on languages. Also, the Hadamard product is nothing but the intersection:

$$\begin{aligned} 1_L + 1_{L'} &= 1_{L \cup L'} \\ 1_L \odot 1_{L'} &= 1_{L \cap L'} \\ 1_L \cdot 1_{L'} &= 1_{L \cdot L'} \\ (1_L)^* &= 1_{L^*} \end{aligned}$$

Next we recall basic notions from trace theory. A pair (Σ, I) is called a *trace alphabet*, if Σ is a finite set and I is an irreflexive symmetric binary *independence* relation on Σ. Let $\sim$ denote the smallest congruence on Σ^* containing $\{(ab, ba) \mid a \; I \; b\}$. The quotient monoid $\mathbb{M} = \mathbb{M}(\Sigma, I) := \Sigma^* / \sim$ is called the *trace monoid* (or free partially commutative monoid) over (Σ, I) and its elements are called traces. Note that a trace monoid is finitely generated and carries a length function.

If $w \in \Sigma^*$, we let $[w]$ denote the equivalence class of w in $\mathbb{M}$. Also, let $\alpha(w)$ be the set of all letters of Σ occurring in w, called the *alphabet* of w. Since equivalent words have the same alphabet, we may put $\alpha([w]) = \alpha(w)$.

We say that two subsets A, B of Σ are independent and we write $A \; I \; B$ if $A \times B \subseteq I$. We also say that two words w, w' of Σ^* are independent, denoted by $w \; I \; w'$, if $\alpha(w) \; I \; \alpha(w')$. Similarly, we define $[w] \; I \; [w']$, $w \; I \; A$, etc.

A subset $C \subseteq \Sigma$ is called *connected*, if it cannot be split $C = A \cup B$ into two non-empty independent subsets. Again, w and $[w]$ are connected, if $\alpha(w)$ is connected. A language $L \subseteq \mathbb{M}$ or $L \subseteq \Sigma^*$ is called *connected*, if each of its elements is connected, and *mono-alphabetic*, if $\alpha(m) = \alpha(m')$ for all $m, m' \in L$.

Then the collection of all *c-rational* languages in $\mathbb{M}$ (respectively, in Σ^*) is defined as the smallest set of languages of $\mathbb{M}$ (respectively, of Σ^*) containing all finite languages and which is closed under the operations union, product and star, where the latter is applied only to connected languages. The following characterizes the recognizable languages of $\mathbb{M}$ (recall that a language $L \subseteq \mathbb{M}$ is recognizable iff it is accepted by some finite $\mathbb{M}$-automaton, or, equivalently, iff its syntactic monoid is finite).

Theorem 3 (Ochmański, [15,4,6]). *Let (Σ, I) be any trace alphabet and $\mathbb{M}$ its trace monoid. Then a language $L \subseteq \mathbb{M}$ is recognizable iff it is c-rational.*

Again, one should note that Kleene's theorem mentioned above is a special case of Theorem 3 since when the independence relation is empty, the trace monoid $\mathbb{M}(\Sigma, \emptyset)$ is the free monoid Σ^* and in this case all languages are connected, hence rational sets are also c-rational.

The goal of this paper is a common generalization of Theorems 2 and 3, that is, a characterization of the recognizable formal power series in $K\langle\langle\mathbb{M}\rangle\rangle$ where K is a semiring and $\mathbb{M}$ a trace monoid. Let $S \in K\langle\langle\mathbb{M}\rangle\rangle$. We say that S is *connected*, if $supp(S)$ is a connected language in $\mathbb{M}$, and *mono-alphabetic*, if $supp(S)$ is mono-alphabetic. In the latter case, we put $\alpha(S) = \alpha(m)$ if $S \neq 0$ and $m \in supp(S)$. Now let $K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle$ (mono-alphabetic-connected rational) be the smallest subset of $K\langle\langle\mathbb{M}\rangle\rangle$ which contains all polynomials and is closed under the operations sum, product and star, where the latter gets applied only to proper, mono-alphabetic and connected series. Similarly, we let $K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle$ (connected rational) be the collection of series obtained from the polynomials by allowing the operations sum, product and star, where now star is applied to all proper and connected series. Similarly, we define connected series in $K\langle\langle\Sigma^*\rangle\rangle$ and the collection of *mc-rational* series in $K\langle\langle\Sigma^*\rangle\rangle$.

The following is an important factorization result in trace theory.

Lemma 4 ([3,4]). *Let $u, v, z_1, \ldots, z_n \in \mathbb{M}$. Then, $uv = z_1 \ldots z_n$ if and only if there are $r_i, s_i \in \mathbb{M}$ such that $u = r_1 \ldots r_n, v = s_1 \ldots s_n, z_i = r_i s_i$ for each $i = 1, \ldots, n$ and $s_i \; I \; r_j$ for all $1 \leq i < j \leq n$.*

Note that, if M is any monoid, any function $\mu : \Sigma \longrightarrow M$ extends uniquely to a morphism from the free monoid Σ^* to M. Now, the function μ extends (uniquely) to a morphism from $\mathbb{M}(\Sigma, I)$ to M iff $\mu(a) \cdot \mu(b) = \mu(b) \cdot \mu(a)$ for all $(a, b) \in I$; this difference explains partly why the present calculations are more complicated than the classical ones.

It is very useful to see a representation (λ, μ, γ) of a series S in $K\langle\langle\Sigma^*\rangle\rangle$ as an automaton $\mathcal{A}$ with multiplicities in the semiring K. More precisely, if the representation is of dimension n then the set of states of the automaton $\mathcal{A}$ is $[n]$. Now, for all states $p, q \in [n]$ and letter $a \in \Sigma$, there is an a-transition with multiplicity (cost) $\mu a_{p,q}$ from state p to state q in $\mathcal{A}$. Note that, since μ is a morphism, we have for each word $w = a_1 \cdots a_k$ and states $p_0, p_k \in [n]$

$$\mu w_{p_0 p_k} = \sum_{p_1, \cdots, p_{k-1}} (\mu a_1)_{p_0 p_1} \cdots (\mu a_k)_{p_{k-1} p_k}.$$

That is, the costs are multiplied along a path $p_0, a_1, p_1, \ldots, p_{k-1}, a_k, p_k$ and summed among all possible paths. The vector λ and γ give the costs for entering and leaving the automaton in the various states. Finally, the value of the series S over $w = a_1 \cdots a_k$ is the cost of w in the automaton $\mathcal{A}$:

$$(S, w) = \sum_{p_0, \cdots, p_k} \lambda_{p_0} (\mu a_1)_{p_0 p_1} \cdots (\mu a_k)_{p_{k-1} p_k} \gamma_{p_k}.$$

Note that if we use the Boolean semiring $\mathbb{B}$, this definition reduces to classical non-deterministic automata: a state p is initial (final resp.) iff $\lambda_p = 1$ ($\gamma_p = 1$ resp.) and there is an a-transition between states p and q iff $\mu a_{pq} = 1$. The language accepted by the non-deterministic automaton is then precisely the support of the series.

The intuition is the same for a series over the trace monoid $\mathbb{M}(\Sigma, I)$. The additional requirement is the so-called I-diamond property for the automaton: if $(a, b) \in I$ and $i, k \in [n]$ then

$$(\mu ab)_{ik} = \sum_{j \in [n]} \mu a_{ij} \mu b_{jk} = \sum_{j \in [n]} \mu b_{ij} \mu a_{jk} = (\mu ba)_{ik}$$

This requirement ensures that two equivalent words $u \sim v$ have the same *cost* in the automaton $\mathcal{A}$ and allows us to see the automaton as a trace automaton.

This intuition of recognizable series is very useful and will be used throughout the paper to explain the various constructions.

3 Mc-rational series are recognizable

In this section, let (Σ, I) be a trace alphabet and $\mathbb{M} = \mathbb{M}(\Sigma, I)$ its trace monoid. We will prove Theorem 1(b,c). This will require a more particular notion of representations which we introduce first.

Definition 5. Let $S = (\lambda, \mu, \gamma) \in K\langle\langle \mathbb{M} \rangle\rangle$ be a recognizable series with $\mu : \mathbb{M} \longrightarrow K^{n \times n}$. The representation (λ, μ, γ) is *alphabetic*, if there exist two functions $\overleftarrow{\alpha}, \overrightarrow{\alpha} : [n] \longrightarrow \mathcal{P}(\Sigma)$ such that for all $u \in \mathbb{M}$, the following three conditions are satisfied:
(1) Whenever $\mu u_{ij} \neq 0$, then $\overleftarrow{\alpha}(j) = \overleftarrow{\alpha}(i) \cup \alpha(u)$ and $\overrightarrow{\alpha}(i) = \overrightarrow{\alpha}(j) \cup \alpha(u)$;
(2) whenever $\lambda_i \neq 0$, then $\overleftarrow{\alpha}(i) = \emptyset$;
(3) whenever $\gamma_j \neq 0$, then $\overrightarrow{\alpha}(j) = \emptyset$.
We call $(\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ an alphabetic representation of S. We say that k is *initial*, if $\overleftarrow{\alpha}(k) = \emptyset$, and k is *final*, if $\overrightarrow{\alpha}(k) = \emptyset$.

In this definition, the intuition is that $\overleftarrow{\alpha}(k)$ describes the *past alphabet* of state k. Each trace $u \in \mathbb{M}$ which may go from some initial state i to some state k with a non-zero cost ($\mu u_{ik} \neq 0$) must verify $\alpha(u) = \overleftarrow{\alpha}(k)$. Similarly, $\overrightarrow{\alpha}(k)$ describes the *future alphabet* of state k: if $\mu u_{kj} \neq 0$ and j is a final state then $\alpha(u) = \overrightarrow{\alpha}(k)$. Note also that if $u \neq 1$, then i initial implies $\mu u_{ki} = 0$, and j final implies $\mu u_{jk} = 0$, for any k.

Proposition 6. *Let $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be a recognizable series. Then there exists an alphabetic representation of S.*

Proof. Let $S = (\lambda', \mu', \gamma')$ with $\mu' : \mathbb{M} \longrightarrow K^{n' \times n'}$. Let $n = n' \cdot 2^{|\Sigma|} \cdot 2^{|\Sigma|}$. Subsequently, we identify $[n]$ with $[n'] \times \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma)$. We define $\mu : \mathbb{M} \longrightarrow K^{n \times n}$

and $\lambda \in K^{1\times n}$, $\gamma \in K^{n\times 1}$ by

$$(\mu u)_{(i,\overleftarrow{X},\overrightarrow{X})(j,\overleftarrow{Y},\overrightarrow{Y})} = \begin{cases} \mu' u_{ij} & \text{if } \overleftarrow{Y} = \overleftarrow{X} \cup \alpha(u) \text{ and } \overrightarrow{X} = \overrightarrow{Y} \cup \alpha(u) \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{(i,\overleftarrow{X},\overrightarrow{X})} = \begin{cases} \lambda'_i & \text{if } \overleftarrow{X} = \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_{(i,\overleftarrow{X},\overrightarrow{X})} = \begin{cases} \gamma'_i & \text{if } \overrightarrow{X} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Also, we put $\overleftarrow{\alpha}(i,\overleftarrow{X},\overrightarrow{X}) = \overleftarrow{X}$ and $\overrightarrow{\alpha}(i,\overleftarrow{X},\overrightarrow{X}) = \overrightarrow{X}$.

We first show that μ is a morphism. Indeed, we have

$$\begin{aligned}
(\mu uv)_{(i,\overleftarrow{X},\overrightarrow{X})(j,\overleftarrow{Y},\overrightarrow{Y})} &= \begin{cases} (\mu' uv)_{ij} & \text{if } \overleftarrow{Y} = \overleftarrow{X} \cup \alpha(uv), \overrightarrow{X} = \overrightarrow{Y} \cup \alpha(uv) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \sum_k \mu' u_{ik} \mu' v_{kj} & \text{if } \overleftarrow{Y} = \overleftarrow{X} \cup \alpha(uv), \overrightarrow{X} = \overrightarrow{Y} \cup \alpha(uv) \\ 0 & \text{otherwise} \end{cases} \\
&= \sum_k \mu u_{(i,\overleftarrow{X},\overrightarrow{X})(k,\overleftarrow{X}\cup\alpha(u),\overrightarrow{Y}\cup\alpha(v))} \mu v_{(k,\overleftarrow{X}\cup\alpha(u),\overrightarrow{Y}\cup\alpha(v))(j,\overleftarrow{Y},\overrightarrow{Y})} \\
&= \sum_{(k,\overleftarrow{Z},\overrightarrow{Z})} \mu u_{(i,\overleftarrow{X},\overrightarrow{X})(k,\overleftarrow{Z},\overrightarrow{Z})} \mu v_{(k,\overleftarrow{Z},\overrightarrow{Z})(j,\overleftarrow{Y},\overrightarrow{Y})} \\
&= (\mu u \cdot \mu v)_{(i,\overleftarrow{X},\overrightarrow{X})(j,\overleftarrow{Y},\overrightarrow{Y})}.
\end{aligned}$$

It is clear that (λ, μ, γ) is alphabetic. Finally, observe

$$\begin{aligned}
\lambda(\mu u)\gamma = \sum_{i,\overrightarrow{X},j,\overleftarrow{Y}} \lambda'_i (\mu u)_{(i,\emptyset,\overrightarrow{X})(j,\overleftarrow{Y},\emptyset)} \gamma'_j &= \sum_{i,j} \lambda'_i (\mu u)_{(i,\emptyset,\alpha(u))(j,\alpha(u),\emptyset)} \gamma'_j \\
&= \sum_{i,j} \lambda'_i \mu' u_{ij} \gamma'_j = (S, u).
\end{aligned}$$

□

First we want to show that the product of two recognizable series in $K\langle\langle \mathbb{M} \rangle\rangle$ is again recognizable. As stated in the introduction, this was already proved by Fliess in two special cases.

When the semiring is commutative, and the trace monoid is a direct product of free monoids $\mathbb{M} = \Sigma_1^* \times \cdots \times \Sigma_n^*$, Fliess [9, Thm. 1] proved that the set of recognizable series over $\mathbb{M}$ is the tensor product of the sets of recognizable series over the free monoids Σ_i^*:

$$K^{rec}\langle\langle \Sigma_1^* \times \cdots \times \Sigma_n^* \rangle\rangle = \bigotimes_{i=1}^{n} K^{rec}\langle\langle \Sigma_i^* \rangle\rangle.$$

In other words, a series over $\mathbb{M} = \Sigma_1^* \times \cdots \times \Sigma_n^*$ is recognizable if and only if it is a finite sum of series of the form $S_1 \cdot S_2 \cdots S_n$ where each S_i is a recognizable series over the free monoid Σ_i^*. With this characterization, the closure under product of recognizable series is quite straightforward. Fliess proved his result using a tensor product of the representations of the series S_i.

Second, Fliess [10, Prop. 2.2.14 and 2.2.15] proved the closure under product for arbitrary trace monoids but for strong Fatou semirings or for the Boolean semiring only. Here the technique is completely different. A weak Fatou semiring K admits a fraction field F and a series $S \in K\langle\langle \mathbb{M} \rangle\rangle$ is recognizable iff it is recognizable considered as a series over the field F. A strong Fatou semiring must satisfy a slightly stronger property and allows Fliess to use linear algebra to prove his result.

We will prove the closure under product of recognizable series for arbitrary trace monoids and for all commutative semirings. In some sense, our automata theoretic proof is in the spirit of the tensor product of the representations. Indeed, it is somewhat more involved because we are not in the simple case of direct product of free monoids. We will not use the full notion of alphabetic representation, since it can be based either on the past alphabets (the function $\overleftarrow{\alpha}$) or the future alphabets, only. The full notion of alphabetic representation will come into use when we deal with iteration.

Theorem 7. *Let K be a commutative semiring and let $S_1, S_2 \in K\langle\langle \mathbb{M} \rangle\rangle$ be two recognizable series. Then their product $S = S_1 \cdot S_2$ is also recognizable.*

Proof. Let $(\lambda^1, \mu^1, \gamma^1)$ be a representation of S_1 of dimension n_1 and let $(\lambda^2, \mu^2, \gamma^2; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ be an alphabetic representation of S_2 of dimension n_2 (Proposition 6) and let $n = n_1 \cdot n_2$. Subsequently we identify $[n]$ with $[n_1] \times [n_2]$. Next, we define $\mu : \Sigma^* \longrightarrow K^{n \times n}$ by

$$\mu(a)_{(i_1,i_2)(j_1,j_2)} = \delta_{i_2,j_2} I(a, i_2) \mu^1(a)_{i_1,j_1} + \delta_{i_1,j_1} \mu^2(a)_{i_2,j_2}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad I(u, i) = \begin{cases} 1 & \text{if } u \; I \; \overleftarrow{\alpha}(i) \\ 0 & \text{otherwise} \end{cases}$$

Note that $I(a, j_2)\mu^2(a)_{i_2,j_2} = 0$, hence at most one of the two terms is non-zero.

Intuitively, the new automaton $\mathcal{A}$ for S is a cartesian product of the automata $\mathcal{A}_1$ and $\mathcal{A}_2$ for S_1 and S_2. Each letter has two ways to act on $\mathcal{A}$. Either it is performed on $\mathcal{A}_2$ and the state of $\mathcal{A}_1$ remains unchanged ($\delta_{i_1,j_1}\mu^2(a)_{i_2,j_2}$). Or it acts on $\mathcal{A}_1$ leaving $\mathcal{A}_2$ unchanged provided it is independent from all letters that already occurred on $\mathcal{A}_2$ ($\delta_{i_2,j_2} I(a, i_2)\mu^1(a)_{i_1,j_1}$). Note that when the independence relation is empty, we obtain a sequential product of $\mathcal{A}_1$ and $\mathcal{A}_2$ since as soon as $\mathcal{A}_2$ has been started, $\mathcal{A}_1$ is blocked forever. Note also that without the restriction imposed by $I(a, i_2)$ we would get a shuffle product of $\mathcal{A}_1$ and $\mathcal{A}_2$.

We first wish to show that μ factors to a morphism from $\mathbb{M}$. For this, let $a, b \in \Sigma$ with $a \; I \; b$, and $(i_1, i_2), (k_1, k_2) \in [n]$. Then,

$$\begin{aligned}
(\mu(a) \cdot \mu(b))_{(i_1,i_2)(k_1,k_2)} = \sum_{(j_1,j_2)} & \delta_{i_2,j_2} I(a,i_2)\mu^1(a)_{i_1,j_1}\delta_{j_2,k_2} I(b,k_2)\mu^1(b)_{j_1,k_1} \\
& + \delta_{i_2,j_2} I(a,i_2)\mu^1(a)_{i_1,j_1}\delta_{j_1,k_1}\mu^2(b)_{j_2,k_2} \\
& + \delta_{i_1,j_1}\mu^2(a)_{i_2,j_2}\delta_{j_2,k_2} I(b,k_2)\mu^1(b)_{j_1,k_1} \\
& + \delta_{i_1,j_1}\mu^2(a)_{i_2,j_2}\delta_{j_1,k_1}\mu^2(b)_{j_2,k_2} \\
= \; & \delta_{i_2,k_2} I(ab,k_2)\mu^1(ab)_{i_1,k_1} \\
& + I(a,i_2)\mu^1(a)_{i_1,k_1}\mu^2(b)_{i_2,k_2} \\
& + I(b,k_2)\mu^2(a)_{i_2,k_2}\mu^1(b)_{i_1,k_1} \\
& + \delta_{i_1,k_1}\mu^2(ab)_{i_2,k_2}
\end{aligned}$$

If $\mu^2(b)_{i_2,k_2} \neq 0$ then $\overleftarrow{\alpha}(k_2) = \overleftarrow{\alpha}(i_2) \cup \{b\}$. Since a and b are independent, it follows that $I(a,i_2)\mu^2(b)_{i_2,k_2} = I(a,k_2)\mu^2(b)_{i_2,k_2}$. The result follows since μ^1 and μ^2 are morphisms from $\mathbb{M}$ (hence, $\mu^1(ab) = \mu^1(ba)$ and $\mu^2(ab) = \mu^2(ba)$) and K is commutative.

We will also write μ for its factorization to a morphism $\mu : \mathbb{M} \longrightarrow K^{n \times n}$. Next we claim that this factorization is given by the explicit formula

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} I(u,i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$$

For this, we proceed by induction on the length of w. If $w = 1$, the result is clear. Next we compute

$$\begin{aligned}
\mu(wa)_{(i_1,i_2)(k_1,k_2)} &= \sum_{j_1,j_2} \mu(w)_{(i_1,i_2)(j_1,j_2)}\mu(a)_{(j_1,j_2)(k_1,k_2)} \\
&= \sum_{\substack{w=uv \\ j_1,j_2}} I(u,i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2} \\
&\qquad\qquad (\delta_{j_2,k_2} I(a,k_2)\mu^1(a)_{j_1,k_1} + \delta_{j_1,k_1}\mu^2(a)_{j_2,k_2}) \\
&= \sum_{w=uv} I(a,k_2) I(u,i_2)\mu^1(ua)_{i_1,k_1}\mu^2(v)_{i_2,k_2} \\
&\qquad\qquad + I(u,i_2)\mu^1(u)_{i_1,k_1}\mu^2(va)_{i_2,k_2}
\end{aligned}$$

In the last equality, we have used the commutativity of K. Now, by Lemma 4, $wa = u'v'$ iff $w = uv$ and either $u' = u$ and $v' = va$ or $u' = ua$ and $v' = v$ and $a \; I \; v$. Moreover, if $\mu^2(v)_{i_2,k_2} \neq 0$ then $\overleftarrow{\alpha}(k_2) = \overleftarrow{\alpha}(i_2) \cup \alpha(v)$. Therefore, we obtain

$$\mu(wa)_{(i_1,i_2)(k_1,k_2)} = \sum_{wa=u'v'} I(u',i_2)\mu^1(u')_{i_1,k_1}\mu^2(v')_{i_2,k_2}$$

which proves our claim.

Finally, define $\lambda \in K^{1\times n}$ by $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$ and $\gamma \in K^{n\times 1}$ by $\gamma_{(k_1,k_2)} = \gamma^1_{k_1}\gamma^2_{k_2}$. We claim that $S = (\lambda, \mu, \gamma)$. For any $w \in \mathbb{M}$ we compute

$$\begin{aligned}
(S_1S_2, w) &= \sum_{w=uv} (S_1, u)(S_2, v) = \sum_{w=uv} \lambda^1\mu^1(u)\gamma^1\lambda^2\mu^2(v)\gamma^2 \\
&= \sum_{w=uv} \left(\sum_{i_1,k_1} \lambda^1_{i_1}\mu^1(u)_{i_1k_1}\gamma^1_{k_1} \cdot \sum_{i_2,k_2} \lambda^2_{i_2}\mu^2(v)_{i_2k_2}\gamma^2_{k_2} \right) \\
&= \sum_{i_1,i_2,k_1,k_2} \lambda^1_{i_1}\lambda^2_{i_2} \left(\sum_{w=uv} \mu^1(u)_{i_1k_1}\mu^2(v)_{i_2k_2} \right) \gamma^1_{k_1}\gamma^2_{k_2},
\end{aligned}$$

using commutativity of K. Now if $\overleftarrow{\alpha}(i_2) \neq \emptyset$, then $\lambda^2_{i_2} = 0$. Hence in the last sum over $w = uv$ we may add the factor $I(u, i_2)$ without changing the value of the whole sum. Therefore, using our computation of μ, we obtain

$$(S_1S_2, w) = \sum_{(i_1,i_2),(k_1,k_2)} \lambda_{(i_1,i_2)}\mu(w)_{(i_1,i_2)(k_1,k_2)}\gamma_{(k_1,k_2)} = \lambda\mu(w)\gamma,$$

which proves the theorem. □

The following result will not be used subsequently, but could be of independent interest. It shows that a recognizable series has an alphabetic representation $(\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ with an even more specific form. Intuitively, we can get rid of the initial and final costs carried by λ and γ and use instead subsets of initial and final sets as in classical automata.

Proposition 8. *Let $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be recognizable. Then there exists an alphabetic representation $(\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ of S such that i is initial iff $\lambda_i = 1$, and j is final iff $\gamma_j = 1$, for any i, j.*

Proof. Let $(\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ be an alphabetic representation of S, with $\mu : \mathbb{M} \longrightarrow K^{n\times n}$. We define $(\lambda', \mu', \gamma')$ by putting

$$\mu' u_{ij} = \begin{cases} \mu u_{ij} & \text{if } i \text{ not initial, } j \text{ not final} \\ \lambda_i \mu u_{ij} & \text{if } i \text{ initial, } j \text{ not final} \\ \mu u_{ij}\gamma_j & \text{if } i \text{ not initial, } j \text{ final} \\ \lambda_i \mu u_{ij}\gamma_j & \text{if } i \text{ initial, } j \text{ final} \end{cases}$$

Also, let

$$\lambda'_i = \begin{cases} 1 & \text{if } i \text{ initial} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma'_j = \begin{cases} 1 & \text{if } j \text{ final} \\ 0 & \text{otherwise} \end{cases}$$

Now assume $u, v \neq 1$. Observe that if k is initial, then $\mu' u_{ik} = 0$ for any i, and if k is final, then $\mu' v_{kj} = 0$ for any j. Using this, we compute

$$\begin{aligned}
(\mu' u \mu' v)_{ij} &= \sum_k \mu' u_{ik} \mu' v_{kj} \\
&= \begin{cases} \sum_k \mu u_{ik} \mu v_{kj} = (\mu uv)_{ij} & \text{if } i \text{ not initial, } j \text{ not final} \\ \sum_k \lambda_i \mu u_{ik} \mu v_{kj} = \lambda_i (\mu uv)_{ij} & \text{if } i \text{ initial, } j \text{ not final} \\ \sum_k \mu u_{ik} \mu v_{kj} \gamma_j = (\mu uv)_{ij} \gamma_j & \text{if } i \text{ not initial, } j \text{ final} \\ \sum_k \lambda_i \mu u_{ik} \mu v_{kj} \gamma_j = \lambda_i (\mu uv)_{ij} \gamma_j & \text{if } i \text{ initial, } j \text{ final} \end{cases} \\
&= (\mu' uv)_{ij}.
\end{aligned}$$

Hence μ' is a morphism. Clearly $(\lambda', \mu', \gamma')$ is alphabetic and has the required form. Finally, note that

$$\lambda' \cdot \mu' u \cdot \gamma' = \sum_{\substack{i \text{ initial} \\ j \text{ final}}} (\mu' u)_{ij} = \sum_{\substack{i \text{ initial} \\ j \text{ final}}} \lambda_i \mu u_{ij} \gamma_j = \sum_{i,j} \lambda_i \mu u_{ij} \gamma_j = \lambda(\mu u)\gamma.$$

$\square$

For mono-alphabetic series, we get even more specific alphabetic representations with only one initial state and one final state. Note that it is not possible to have an alphabetic representation with this property if the series is not mono-alphabetic.

Proposition 9. *Let $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be recognizable, proper and mono-alphabetic with $\alpha(S) = A$. Then there exists an alphabetic representation of S of dimension n of the form $(e_1, \mu, e_n; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ where $e_1 = (1, 0, \ldots, 0) \in K^{1\times n}$, $e_n = (0, \ldots, 0, 1)^t \in K^{n\times 1}$ and $\overrightarrow{\alpha}(1) = \overleftarrow{\alpha}(n) = A$.*

Proof. We proceed similarly to the previous proposition. Let $S = (\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ be an alphabetic representation of S. If $S = 0$, the result is clear. So let $S \neq 0$. Thus $A \neq \emptyset$. We put

$$\begin{aligned}
I &= \{i \in [n] : \overleftarrow{\alpha}(i) = \emptyset, \overrightarrow{\alpha}(i) = A\}, \\
J &= \{j \in [n] : \overrightarrow{\alpha}(j) = \emptyset, \overleftarrow{\alpha}(j) = A\}.
\end{aligned}$$

Obviously, I contains only initial and J only final states and $I \cap J = \emptyset$.

We first show that for all $u \in \mathbb{M}$, $(S, u) = \sum_{i \in I, j \in J} \lambda_i \mu u_{ij} \gamma_j$. Indeed, let $u \in \mathbb{M}$. If $\lambda_i \mu u_{ij} \gamma_j \neq 0$, then $\overleftarrow{\alpha}(i) = \overrightarrow{\alpha}(j) = \emptyset$ and $\overrightarrow{\alpha}(i) = \overleftarrow{\alpha}(j) = \alpha(u)$. Therefore, if $\alpha(u) = A$ then $\lambda_i \mu u_{ij} \gamma_j \neq 0$ implies $i \in I$ and $j \in J$ and we get $(S, u) = \sum_{i \in I, j \in J} \lambda_i \mu u_{ij} \gamma_j$. Now, if $\alpha(u) \neq A$ then $\lambda_i \mu u_{ij} \gamma_j = 0$ for all $i \in I$ and $j \in J$ and we obtain $(S, u) = 0 = \sum_{i \in I, j \in J} \lambda_i \mu u_{ij} \gamma_j$.

Since $S \neq 0$, I and J must be nonempty and we may assume without loss of generality that $1 \in I$ and $n \in J$. Now we define $\mu' : \mathbb{M} \longrightarrow K^{n\times n}$ by

$$\mu' u_{ij} = \begin{cases} \mu u_{ij} & \text{if } i \neq 1, j \neq n \\ \sum_{p\in I} \lambda_p \mu u_{pj} & \text{if } i = 1, j \neq n \\ \sum_{q\in J} \mu u_{iq} \gamma_q & \text{if } i \neq 1, j = n \\ \sum_{p\in I, q\in J} \lambda_p \mu u_{pq} \gamma_q & \text{if } i = 1, j = n \end{cases}$$

Now let $u, v \neq 1$. Note that, since μ is alphabetic and 1 is initial and n is final, we have $\mu u_{i1} = 0 = \mu' u_{i1}$ for any i and $\mu v_{nj} = 0 = \mu' v_{nj}$ for any j. Using this, we compute

$$\begin{aligned}
(\mu' u \mu' v)_{ij} &= \sum_k \mu' u_{ik} \mu' v_{kj} = \sum_{1<k<n} \mu' u_{ik} \mu' v_{kj} \\
&= \begin{cases} \sum_k \mu u_{ik} \mu v_{kj} = (\mu uv)_{ij} & \text{if } i \neq 1, j \neq n \\ \sum_k \sum_{p\in I} \lambda_p \mu u_{pk} \mu v_{kj} = \sum_{p\in I} \lambda_p (\mu uv)_{pj} & \text{if } i = 1, j \neq n \\ \sum_k \sum_{q\in J} \mu u_{ik} \mu v_{kq} \gamma_q = \sum_{q\in J} (\mu uv)_{iq} \gamma_q & \text{if } i \neq 1, j = n \\ \sum_k \sum_{p\in I, q\in J} \lambda_p \mu u_{pk} \mu v_{kq} \gamma_q = \sum_{p\in I, q\in J} \lambda_p (\mu uv)_{pq} \gamma_q & \text{if } i = 1, j = n \end{cases} \\
&= (\mu' uv)_{ij}.
\end{aligned}$$

Hence μ' is a morphism. Moreover, we have seen that

$$(S, u) = \sum_{i\in I, j\in J} \lambda_i \mu u_{ij} \gamma_j = e_1 \mu' u_{ij} e_n.$$

Therefore, $S = (e_1, \mu', e_n)$ and it remains to show that $(e_1, \mu', e_n; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ is alphabetic. This follows easily from the definition of μ' and $(\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ alphabetic. For instance, if $\mu' u_{1n} = \sum_{p\in I, q\in J} \lambda_p \mu u_{pq} \gamma_q \neq 0$ then $\mu u_{pq} \neq 0$ for some $p \in I$ and $q \in J$ and it follows $\overrightarrow{\alpha}(1) = \overrightarrow{\alpha}(p) = \overrightarrow{\alpha}(q) \cup \alpha(u) = \overrightarrow{\alpha}(n) \cup \alpha(u)$ and $\overleftarrow{\alpha}(n) = \overleftarrow{\alpha}(q) = \overleftarrow{\alpha}(p) \cup \alpha(u) = \overleftarrow{\alpha}(1) \cup \alpha(u)$. The other cases are similar. $\square$

We will now prove the following essential closure property of recognizable series.

Theorem 10. *Let K be a commutative semiring and let $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be a proper, connected, mono-alphabetic and recognizable series. Then, S^* is recognizable.*

The proof of this theorem is based on a rather involved construction. Let $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be a proper, recognizable, connected and mono-alphabetic series with $\alpha(S) = A$. Let $S = (e_1, \mu, e_n; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ be an alphabetic representation with $\overrightarrow{\alpha}(1) = \overleftarrow{\alpha}(n) = A$ (Proposition 9).

If we want to generalize the proof of the product to recognize some power S^m we could run simultaneously m copies of the automaton $\mathcal{A}$ for S. Actually, with m copies of $\mathcal{A}$, we can easily recognize $S^{\leq m} = \sum_{0\leq k\leq m} S^k$. We cannot use exactly the same technic for S^* since it would require an unbounded number of copies of $\mathcal{A}$ and the representation would not be of finite dimension. Fortunately, when the series S is proper connected and mono-alphabetic we will show that there is a bound ($|A|$) on the number of simultaneously active copies of $\mathcal{A}$ (by active we mean neither in its initial nor in its final state). Using this fact, we will be able to construct an automaton $\mathcal{A}^*$ for S^* keeping in memory only the bounded sequence of active copies of the automaton $\mathcal{A}$ for S.

Let $m \geq 1$. We identify $[n^m]$ with the set $[n]^m$ of all m-tuples with entries from $[n]$. We use $\tilde{\imath}$ as abbreviation for such an m-tuple $(i_1, \ldots, i_m)$, similarly $\tilde{\jmath}, \tilde{k}$. Now we define functions $\mu^0, \ldots, \mu^m : \Sigma^* \longrightarrow K^{n^m \times n^m}$ by

$$\mu^0 a_{\tilde{\imath}\tilde{\jmath}} = \begin{cases} \mu a_{i_1 n} & \text{if } \tilde{\jmath} = (i_2, \ldots, i_m, 1) \\ 0 & \text{otherwise} \end{cases}$$

and for $1 \leq p \leq m$,

$$\mu^p a_{\tilde{\imath}\tilde{\jmath}} = \begin{cases} \mu a_{i_p j_p} & \text{if } j_l = i_l \text{ for all } l \neq p \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, μ^p ($1 \leq p \leq m$) describes an action on the p-th active copy of $\mathcal{A}$ while the other copies remain unchanged. The case of μ^0 is special. It is used only when the first active automaton reaches its final state (n) and thus becomes inactive. In this case, we do not need to keep this automaton in the sequence of possibly active automata since it will never act again. Instead, we enable a new copy of $\mathcal{A}$ at the end of the sequence. This new copy may become active afterwards. The shift in the sequence performed by μ^0 describes precisely this situation.

Also, we will restrict the accessible states to legal states. A legal state must satisfy several conditions but the main one is that for all $p < q$, all actions that have occurred on the q-th active copy are independent from all actions that remain to be done on the p-th copy. In order to describe these legal states we introduce

$$H_{\tilde{\imath}} = \begin{cases} 1 & \text{if } \overrightarrow{\alpha}(i_p) \cup \overleftarrow{\alpha}(i_p) = A = \alpha(S) \text{ for all } p,\ \overrightarrow{\alpha}(i_1) \neq \emptyset \text{ and} \\ & \quad \overrightarrow{\alpha}(i_p)\ I\ \overleftarrow{\alpha}(i_q) \text{ for all } p < q \\ 0 & \text{otherwise} \end{cases}$$

Let $H \in K^{n^m \times n^m}$ be given by $H_{\tilde{\imath}\tilde{\jmath}} = H_{\tilde{\imath}} \cdot H_{\tilde{\jmath}}$, and define $\mu^* : \Sigma^* \longrightarrow K^{n^m \times n^m}$ by $\mu^* = H \odot (\mu^0 + \cdots + \mu^m)$, where $\odot$ denotes the componentwise product of matrices: $(X \odot Y)_{ij} = X_{ij} \cdot Y_{ij}$. Therefore,

$$\mu^* w_{\tilde{\imath}\tilde{\jmath}} = H_{\tilde{\imath}} H_{\tilde{\jmath}} (\mu^0 w_{\tilde{\imath}\tilde{\jmath}} + \cdots + \mu^m w_{\tilde{\imath}\tilde{\jmath}})$$

Theorem 10 results clearly from the following two essential results.

Proposition 11. *Let K be a commutative semiring and assume that $m \geq |A|$. Then $\mu^*(ab) = \mu^*(ba)$ for all $a, b \in \Sigma$ such that $a\ I\ b$.*

Hence μ^* factorizes to a morphism from $\mathbb{M}$ to $K^{n^m \times n^m}$, and we have:

Proposition 12. *Let K be a commutative semiring and assume that $m \geq |A|$. Then $S^* = (\lambda_{\tilde{1}}, \mu^*, \gamma_{\tilde{1}})$ where $\lambda_{\tilde{1}}, \gamma_{\tilde{1}}$ are the row respectively column vectors which have a 1 only at entry $\tilde{1} = (1, \ldots, 1)$, and 0 otherwise.*

In order to prove these two propositions, we start with some technical results. The first lemma shows that if a state $\tilde{\imath}$ is legal ($H_{\tilde{\imath}} \neq 0$) then all copies greater than $|A|$ are inactive (they are still in their initial state). Hence there are at most $|A|$ simultaneously active copies of $\mathcal{A}$ and there is no need to keep in the sequence of possibly active automata more than $|A|$ copies of $\mathcal{A}$. This explains partly why Propositions 11 and 12 hold for all $m \geq |A|$.

Lemma 13. *Let $H_{\tilde{\imath}} \neq 0$. Then:*

1. $\overleftarrow{\alpha}(i_m) \subseteq \cdots \subseteq \overleftarrow{\alpha}(i_2) \subseteq \overleftarrow{\alpha}(i_1)$
2. $\emptyset \neq \overrightarrow{\alpha}(i_1) \subseteq \overrightarrow{\alpha}(i_2) \subseteq \cdots \subseteq \overrightarrow{\alpha}(i_m)$.
3. *If $q \geq |A|$, then $\overleftarrow{\alpha}(i_q) = \emptyset$.*

Proof. Here, if $X \subseteq \Sigma$, we put $D(X) = \{a \in \Sigma \mid \neg(a \ I \ X)\}$, the letters *dependent* of some element of X. Since $H_{\tilde{\imath}} \neq 0$, for all $p < q$ we have $A \setminus \overleftarrow{\alpha}(i_p) \subseteq \overrightarrow{\alpha}(i_p) \ I \ \overleftarrow{\alpha}(i_q) \supseteq A \setminus \overrightarrow{\alpha}(i_q)$, hence $A \cap D(\overleftarrow{\alpha}(i_q)) \subseteq \overleftarrow{\alpha}(i_p)$ and $D(\overrightarrow{\alpha}(i_p)) \cap A \subseteq \overrightarrow{\alpha}(i_q)$. Therefore,

$$\overleftarrow{\alpha}(i_m) \subseteq D(\overleftarrow{\alpha}(i_m)) \cap A \subseteq \overleftarrow{\alpha}(i_{m-1}) \subseteq \cdots \subseteq \overleftarrow{\alpha}(i_1) \subseteq D(\overleftarrow{\alpha}(i_1)) \cap A,$$

and

$$\overrightarrow{\alpha}(i_1) \subseteq D(\overrightarrow{\alpha}(i_1)) \cap A \subseteq \overrightarrow{\alpha}(i_2) \subseteq D(\overrightarrow{\alpha}(i_2)) \cap A \subseteq \cdots \subseteq \overrightarrow{\alpha}(i_m).$$

This proves (1) and (2). For (3), note that since A is connected, if $\emptyset \neq B \subset A$, then $B \subset D(B) \cap A$. Now we have $\emptyset \neq \overrightarrow{\alpha}(i_1) \ I \ \overleftarrow{\alpha}(i_2)$, hence $|\overleftarrow{\alpha}(i_2)| \leq |A| - 2$ since A is connected. Therefore, if $\overleftarrow{\alpha}(i_q) \neq \emptyset$, from the first chain of inclusions above we deduce $\emptyset \neq \overleftarrow{\alpha}(i_q) \subset \overleftarrow{\alpha}(i_{q-1}) \subset \cdots \subset \overleftarrow{\alpha}(i_2)$ and hence $|\overleftarrow{\alpha}(i_2)| \geq q - 1$. This contradicts the assumption that $q \geq |A|$. □

The second lemma shows that if we are going from one legal state $\tilde{\imath}$ to another legal state $\tilde{\jmath}$ using two independent actions a and b then each intermediary state $\tilde{k}$ which gives a non zero contribution is also legal.

Lemma 14. *Let $a, b \in \Sigma$ with $a \ I \ b$. For any $\tilde{\imath}, \tilde{\jmath}$ and $p, q \geq 0$, the inequality $H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^p a_{\tilde{\imath}\tilde{k}} \mu^q b_{\tilde{k}\tilde{\jmath}} \neq 0$ implies $H_{\tilde{k}} = 1$.*

Proof. We first consider the case $p = 0$. From $\mu^0 a_{\tilde{\imath}\tilde{k}} \neq 0$ we obtain $\tilde{k} = (i_2, \ldots, i_m, 1)$. Using $H_{\tilde{\imath}} \neq 0$ we deduce $\overrightarrow{\alpha}(k_1) = \overrightarrow{\alpha}(i_2) \neq \emptyset$ by Lemma 13. Moreover, $\overrightarrow{\alpha}(k_m) \cup \overleftarrow{\alpha}(k_m) = A \cup \emptyset = A$ and for all $l < m$, $\overleftarrow{\alpha}(k_m) = \emptyset \ I \ \overrightarrow{\alpha}(k_l)$. Therefore, $H_{\tilde{k}} = 1$.

Assume now that $p \geq 1$. Since $H_{\tilde{\imath}}\mu^p a_{\tilde{\imath}\tilde{k}} \neq 0$, then for all $l \neq p$ we have $k_l = i_l$ and $\overleftarrow{\alpha}(k_p) = \overleftarrow{\alpha}(i_p) \cup \{a\}$ and $\overrightarrow{\alpha}(i_p) = \overrightarrow{\alpha}(k_p) \cup \{a\}$. Hence for all $l \neq p$, $\overrightarrow{\alpha}(k_l) \cup \overleftarrow{\alpha}(k_l) = \overrightarrow{\alpha}(i_l) \cup \overleftarrow{\alpha}(i_l) = A$ and

$$\overleftarrow{\alpha}(k_p) \cup \overrightarrow{\alpha}(k_p) = \overleftarrow{\alpha}(i_p) \cup \{a\} \cup \overrightarrow{\alpha}(k_p) = \overleftarrow{\alpha}(i_p) \cup \overrightarrow{\alpha}(i_p) = A$$

which proves the first condition for $H_{\tilde{k}} = 1$.

Assume in addition that $q = 0$. From $\mu^0 b_{\tilde{k}\tilde{\jmath}} \neq 0$ we obtain $\tilde{\jmath} = (k_2, \ldots, k_m, 1)$ and $\overrightarrow{\alpha}(k_1) = \overrightarrow{\alpha}(n) \cup \{b\} = \{b\} \neq \emptyset$. Now, from $H_{\tilde{\jmath}} \neq 0$, we obtain $\overrightarrow{\alpha}(k_l) = \overrightarrow{\alpha}(j_{l-1})\ I\ \overleftarrow{\alpha}(j_{l'-1}) = \overleftarrow{\alpha}(k_{l'})$ for all $1 < l < l'$. Finally, let $l > 1$. From $\mu^p a_{\tilde{\imath}\tilde{k}} \neq 0$ we have $\overleftarrow{\alpha}(k_l) \subseteq \overleftarrow{\alpha}(i_l) \cup \{a\}$ and $\{b\} = \overrightarrow{\alpha}(k_1) \subseteq \overrightarrow{\alpha}(i_1)$. Therefore, using $H_{\tilde{\imath}} \neq 0$ and $a\ I\ b$, it follows $\overrightarrow{\alpha}(k_1)\ I\ \overleftarrow{\alpha}(k_l)$ which concludes the proof of $H_{\tilde{k}} = 1$ when $p \geq 1$ and $q = 0$.

It remains to consider the case $p \geq 1$ and $q \geq 1$. First, we have $\emptyset \neq \overrightarrow{\alpha}(j_1) \subseteq \overrightarrow{\alpha}(k_1)$. Second, we have

$$\begin{aligned} &\overrightarrow{\alpha}(k_l) = \overrightarrow{\alpha}(j_l)\ I\ \overleftarrow{\alpha}(j_{l'}) \supseteq \overleftarrow{\alpha}(k_{l'}) && \text{for all } q \neq l < l' \\ &\overrightarrow{\alpha}(k_l) \subseteq \overrightarrow{\alpha}(i_l)\ I\ \overleftarrow{\alpha}(i_{l'}) = \overleftarrow{\alpha}(k_{l'}) && \text{for all } l < l' \neq p \end{aligned}$$

and it remains to show that $\overrightarrow{\alpha}(k_q)\ I\ \overleftarrow{\alpha}(k_p)$ if $q < p$. Using $a\ I\ b$, this follows from

$$\begin{aligned} \overrightarrow{\alpha}(k_q) &= \overrightarrow{\alpha}(j_q) \cup \{b\} = \overrightarrow{\alpha}(i_q)\ I\ \overleftarrow{\alpha}(i_p) \\ \overleftarrow{\alpha}(k_p) &= \overleftarrow{\alpha}(i_p) \cup \{a\} = \overleftarrow{\alpha}(j_p)\ I\ \overrightarrow{\alpha}(j_q). \end{aligned}$$

□

The next result describes how the I-diamond property occurs in the automaton $\mathcal{A}^*$ defined by μ^*. For this, we give the commutativity properties of the functions $H \odot \mu^p a$.

Lemma 15. *Assume that K is commutative and let $a, b \in \Sigma$ with $a\ I\ b$. Then:*

$$\begin{aligned} &(H \odot \mu^q b) = 0 && \textit{if } q \geq |A| && (1) \\ &(H \odot \mu^0 a)(H \odot \mu^0 b) = 0 && && (2) \\ &(H \odot \mu^1 a)(H \odot \mu^0 b) = (H \odot \mu^1 b)(H \odot \mu^0 a) && && (3) \\ &(H \odot \mu^p a)(H \odot \mu^q b) = (H \odot \mu^q b)(H \odot \mu^p a) && \textit{for all } p, q \geq 1 && (4) \\ &(H \odot \mu^p a)(H \odot \mu^0 b) = (H \odot \mu^0 b)(H \odot \mu^{p-1} a) && \textit{for all } p > 1 && (5) \end{aligned}$$

Proof. (1) Let $q \geq |A|$ and assume that $\mu^q b_{\tilde{k}\tilde{\jmath}} \neq 0$. Then $\overleftarrow{\alpha}(j_q) = \overleftarrow{\alpha}(k_q) \cup \{b\} \neq \emptyset$, hence $H_{\tilde{\jmath}} = 0$ by Lemma 13, and so $(H \odot \mu^q b) = 0$.

For the other cases, observe first that for any $p, q \geq 0$, by Lemma 14 we have

$$(H \odot \mu^p a)(H \odot \mu^q b)_{\tilde{\imath}\tilde{\jmath}} = \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} H_{\tilde{k}} \mu^p a_{\tilde{\imath}\tilde{k}} \mu^q b_{\tilde{k}\tilde{\jmath}} = \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^p a_{\tilde{\imath}\tilde{k}} \mu^q b_{\tilde{k}\tilde{\jmath}}.$$

(2) Suppose that $\mu^0 a_{\tilde{\imath}\tilde{k}} \mu^0 b_{\tilde{k}\tilde{\jmath}} \neq 0$. Then $\vec{\alpha}(i_1) = \vec{\alpha}(n) \cup \{a\} = \{a\}$ and $\vec{\alpha}(i_2) = \vec{\alpha}(k_1) = \vec{\alpha}(n) \cup \{b\} = \{b\}$. Thus $H_{\tilde{\imath}} = 0$ by Lemma 13, and we obtain $(H \odot \mu^0 a)(H \odot \mu^0 b) = 0$.

(3) We have $(H \odot \mu^1 a)(H \odot \mu^0 b)_{\tilde{\imath}\tilde{\jmath}} = \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^1 a_{\tilde{\imath}\tilde{k}} \mu^0 b_{\tilde{k}\tilde{\jmath}}$. This sum equals $\sum_{k_1} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu a_{i_1 k_1} \mu b_{k_1 n} = H_{\tilde{\imath}} H_{\tilde{\jmath}} (\mu ab)_{i_1 n}$ if $\tilde{\jmath} = (i_2, \ldots, i_m, 1)$, and 0 otherwise. In any case, by symmetry and $\mu ab = \mu ba$, this is the value of $(H \odot \mu^1 b)(H \odot \mu^0 a)_{\tilde{\imath}\tilde{\jmath}}$.

(4) Here we have $(H \odot \mu^p a)(H \odot \mu^q b)_{\tilde{\imath}\tilde{\jmath}} = \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^p a_{\tilde{\imath}\tilde{k}} \mu^q b_{\tilde{k}\tilde{\jmath}}$. First, if $p \neq q$, this sum equals $H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu a_{i_p j_p} \mu b_{i_q j_q}$ if $j_l = i_l$ for all $l \neq p, q$ and 0 otherwise. Second, if $p = q$, the sum equals $\sum_{k_p} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu a_{i_p k_p} \mu b_{k_p j_p} = H_{\tilde{\imath}} H_{\tilde{\jmath}} (\mu ab)_{i_p j_p}$ if $j_l = i_l$ for all $l \neq p$, and 0 otherwise. We obtain the result using the commutativity of K in the first case and using $\mu ab = \mu ba$ in the second case.

(5) Here we have

$$
\begin{aligned}
(H \odot \mu^p a)(H \odot \mu^0 b)_{\tilde{\imath}\tilde{\jmath}} &= \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^p a_{\tilde{\imath}\tilde{k}} \mu^0 b_{\tilde{k}\tilde{\jmath}} \\
&= \begin{cases} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu a_{i_p j_{p-1}} \mu b_{i_1 n} & \text{if } \tilde{\jmath} = (i_2 \ldots i_{p-1} j_{p-1} i_{p+1} \ldots i_m 1) \\ 0 & \text{otherwise} \end{cases} \\
&= \sum_{\tilde{k}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \mu^0 b_{\tilde{\imath}\tilde{k}} \mu^{p-1} a_{\tilde{k}\tilde{\jmath}} = (H \odot \mu^0 b)(H \odot \mu^{p-1} a)_{\tilde{\imath}\tilde{\jmath}}
\end{aligned}
$$

where the last but one equality is obtained using the commutativity of K. □

Note that Lemma 15 (1) shows that the $|A|$-th copy of the automaton $\mathcal{A}$ is actually never used in $\mathcal{A}^*$. Hence one should be able to modify the construction in order to use $|A| - 1$ copies only. We have decided against it because we find the construction clearer with (at least) $|A|$ copies of $\mathcal{A}$.

We are now able to show that μ^* factorizes to a morphism from $\mathbb{M}$ to $K^{n^m \times n^m}$ if $m \geq |A|$.

Proof of Proposition 11. Recall that $m \geq |A|$ and $a\ I\ b$. Hence, using Lemma 15, we obtain

$$
\begin{aligned}
\mu^* ab = (\mu^* a)(\mu^* b) &= \sum_{p,q} (H \odot \mu^p a)(H \odot \mu^q b) \\
&= \sum_{p,q \geq 1} (H \odot \mu^p a)(H \odot \mu^q b) + \sum_{p > 1} (H \odot \mu^p a)(H \odot \mu^0 b) \\
&\quad + (H \odot \mu^1 a)(H \odot \mu^0 b) \ + \ (H \odot \mu^0 a)(H \odot \mu^0 b) \\
&\quad + \sum_{1 \leq q < m} (H \odot \mu^0 a)(H \odot \mu^q b) + (H \odot \mu^0 a)(H \odot \mu^m b) \\
&= \sum_{p,q \geq 1} (H \odot \mu^q b)(H \odot \mu^p a) + \sum_{1 \leq p < m} (H \odot \mu^0 b)(H \odot \mu^p a) \\
&\quad + (H \odot \mu^1 b)(H \odot \mu^0 a) \ + \ (H \odot \mu^0 b)(H \odot \mu^0 a) \\
&\quad + \sum_{q > 1} (H \odot \mu^q b)(H \odot \mu^0 a) + (H \odot \mu^0 b)(H \odot \mu^m a) \\
&= \mu^* ba.
\end{aligned}
$$

□

As mentioned before, by this result we will consider μ^* also as a morphism from $\mathbb{M}$ into $K^{n^m \times n^m}$.

Our goal now will be to give an explicit formula for $\mu^* u_{\tilde{\imath}\tilde{\jmath}}$ describing the action of a trace u on the automaton $\mathcal{A}^*$. Actually, we will give this description only when $\tilde{\imath}$ is the initial state $\tilde{1} = (1, \ldots, 1)$.

Definition 16. Let $\nu : \mathbb{M} \longrightarrow K^{n^m}$ be given by

$$\nu u_{\tilde{\jmath}} = \sum_{u = u_0 \ldots u_m} (S^*, u_0)(\mu u_1)_{1 j_1} \ldots (\mu u_m)_{1 j_m} \quad (u \in \mathbb{M}).$$

This definition describes the expected behavior of the automaton $\mathcal{A}^*$ over some trace u. Since the series S is mono-alphabetic and connected, whenever some copy of $\mathcal{A}$ reaches its final state, all preceding copies must be already terminated. This gives the factor (S^*, u_0) which stands for all completed copies. Then, at most m copies may be started without being done and they are described by the product $(\mu u_1)_{1 j_1} \ldots (\mu u_m)_{1 j_m}$.

The next proposition shows that if $\tilde{\jmath}$ is a legal state then indeed the automaton $\mathcal{A}^*$ defined by μ^* behaves as expected.

Proposition 17. *Let K be a commutative semiring and assume that $m \geq |A|$. Then, for all $u \in \mathbb{M}$ we have $\nu u_{\tilde{\jmath}} \cdot H_{\tilde{\jmath}} = \mu^* u_{\tilde{1}\tilde{\jmath}}$ where $\tilde{1} = (1, \ldots, 1)$.*

Proof. We proceed by induction on the length of u. Clearly,

$$\mu^* 1_{\tilde{1}\tilde{\jmath}} = \begin{cases} 1 & \text{if } \ \tilde{\jmath} = \tilde{1} \\ 0 & \text{otherwise} \end{cases}$$

$$\nu 1_{\tilde{\jmath}} H_{\tilde{\jmath}} = (S^*, 1)(\mu 1)_{1 j_1} \dots (\mu 1)_{1 j_m} H_{\tilde{\jmath}} = \begin{cases} 1 & \text{if } \tilde{\jmath} = \tilde{1} \\ 0 & \text{otherwise} \end{cases}$$

For the induction step, consider a trace of the form ua. By Lemma 4, we have $ua = u'_0 \dots u'_m$ iff there are traces u_i, v_i such that

$$u = u_0 \dots u_m, a = v_0 \dots v_m \text{ and } u'_i = u_i v_i, v_i \; I \; u_{i+1} \dots u_m \quad \text{for all } i.$$

This is the case iff for some $p \geq 0$ and some trace u_p we have

$$u'_p = u_p a, a \; I \; u'_{p+1} \dots u'_m, u = u'_0 \dots u'_{p-1} u_p u'_{p+1} \dots u'_m.$$

Using this, we obtain

$$\begin{aligned}
(\nu ua)_{\tilde{\jmath}} H_{\tilde{\jmath}} &= \sum_{ua = u'_0 \dots u'_m} (S^*, u'_0)(\mu u'_1)_{1 j_1} \dots (\mu u'_m)_{1 j_m} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \dots u_m \\ a I u_1 \dots u_m}} (S^*, u_0 a)(\mu u_1)_{1 j_1} \dots (\mu u_m)_{1 j_m} H_{\tilde{\jmath}} \\
&\quad + \sum_{p \geq 1} \sum_{\substack{u = u_0 \dots u_m \\ a I u_{p+1} \dots u_m}} (S^*, u_0)(\mu u_1)_{1 j_1} \dots (\mu u_p a)_{1 j_p} \dots (\mu u_m)_{1 j_m} H_{\tilde{\jmath}}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\mu^* ua)_{\tilde{1}\tilde{\jmath}} = \sum_{\tilde{\imath}} (\mu^* u)_{\tilde{1}\tilde{\imath}} (\mu^* a)_{\tilde{\imath}\tilde{\jmath}} &= \sum_{p,\, \tilde{\imath}} (\nu u)_{\tilde{\imath}} H_{\tilde{\imath}} (\mu^p a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\jmath}} \\
&= \sum_{\tilde{\imath}} \nu u_{\tilde{\imath}} (\mu^0 a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} + \sum_{p \geq 1} \sum_{\tilde{\imath}} \nu u_{\tilde{\imath}} (\mu^p a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}}
\end{aligned}$$

We will show that

$$\sum_{\tilde{\imath}} \nu u_{\tilde{\imath}} (\mu^0 a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} = \sum_{\substack{u = u_0 \dots u_m \\ a I u_1 \dots u_m}} (S^*, u_0 a)(\mu u_1)_{1 j_1} \dots (\mu u_m)_{1 j_m} H_{\tilde{\jmath}} \tag{6}$$

$$\sum_{\tilde{\imath}} \nu u_{\tilde{\imath}} (\mu^p a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} = \sum_{\substack{u = u_0 \dots u_m \\ a I u_{p+1} \dots u_m}} (S^*, u_0)(\mu u_1)_{1 j_1} \dots (\mu u_p a)_{1 j_p} \dots (\mu u_m)_{1 j_m} H_{\tilde{\jmath}} \tag{7}$$

for all $p \geq 1$. This will conclude the proof.

We start with two auxiliary results. First, if $(\mu u_1)_{1 i_1} \dots (\mu u_m)_{1 i_m} \mu^p a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} \neq 0$ then either $p = 0$ and $a \; I \; u_2 \cdots u_m$ or $p > 0$ and $a \; I \; u_{p+1} \cdots u_m$. Indeed, if $p = 0$ then $\overrightarrow{\alpha}(i_1) = \overrightarrow{\alpha}(n) \cup \{a\} = \{a\} \; I \; \overleftarrow{\alpha}(i_2) \cup \dots \cup \overleftarrow{\alpha}(i_m) = \alpha(u_2 \cdots u_m)$. Now, if $p > 0$, then $a \in \overrightarrow{\alpha}(i_p) \; I \; \overleftarrow{\alpha}(i_{p+1}) \cup \dots \cup \overleftarrow{\alpha}(i_m) = \alpha(u_{p+1} \cdots u_m)$.

Second, if $(\mu u_1)_{1 i_1} \dots (\mu u_m)_{1 i_m} \mu^p a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\jmath}} \neq 0$ and either $p = 0$ and $a \; I \; u_2 \cdots u_m$ or $p > 0$ and $a \; I \; u_{p+1} \cdots u_m$, then $H_{\tilde{\imath}} = 1$.

Indeed, if $p = 0$, we have $\tilde{\jmath} = (i_2 \dots i_m 1)$, $\overrightarrow{\alpha}(i_1) = \{a\} \neq \emptyset$ and $\overleftarrow{\alpha}(i_1) \cup \{a\} = \overleftarrow{\alpha}(n) = A$ and $\overleftarrow{\alpha}(i_l) \cup \overrightarrow{\alpha}(i_l) = \overleftarrow{\alpha}(j_{l-1}) \cup \overrightarrow{\alpha}(j_{l-1}) = A$ for all $l > 1$. Then,

$\overrightarrow{\alpha}(i_l) = \overrightarrow{\alpha}(j_{l-1}) \; I \; \overleftarrow{\alpha}(j_{l'-1}) = \overleftarrow{\alpha}(i_{l'})$ for all $1 < l < l'$, and $\overrightarrow{\alpha}(i_1) = \{a\} \; I$ $\alpha(u_2 \cdots u_m) = \overleftarrow{\alpha}(i_2) \cup \cdots \cup \overleftarrow{\alpha}(i_m)$. Therefore $H_{\tilde{\imath}} = 1$.

Now, if $p > 0$, we obtain $j_l = i_l$ for all $l \neq p$, $\overrightarrow{\alpha}(i_p) = \overrightarrow{\alpha}(j_p) \cup \{a\}$ and $\overleftarrow{\alpha}(j_p) = \overleftarrow{\alpha}(i_p) \cup \{a\}$. Hence we get $\overrightarrow{\alpha}(i_1) \supseteq \overrightarrow{\alpha}(j_1) \neq \emptyset$, $\overrightarrow{\alpha}(i_p) \cup \overleftarrow{\alpha}(i_p) = \overrightarrow{\alpha}(j_p) \cup \overleftarrow{\alpha}(j_p) = A$, and $\overleftarrow{\alpha}(i_p) \subseteq \overleftarrow{\alpha}(j_p) \; I \; \overrightarrow{\alpha}(j_l) = \overrightarrow{\alpha}(i_l)$ for all $l < p$. Moreover, for all $l > p$ we have $\overrightarrow{\alpha}(j_p) \; I \; \overleftarrow{\alpha}(j_l) = \overleftarrow{\alpha}(i_l)$ and $a \; I \; \overleftarrow{\alpha}(i_l)$, hence $\overrightarrow{\alpha}(i_p) \; I \; \overleftarrow{\alpha}(i_l)$ and again $H_{\tilde{\imath}} = 1$.

Using these observations and the commutativity of K now we compute

$$
\begin{aligned}
&\sum_{\tilde{\imath}} \nu u_{\tilde{\imath}} (\mu^0 a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \\
&= \sum_{u = u_0 \ldots u_m,\ \tilde{\imath}} (S^*, u_0)(\mu u_1)_{1 i_1} \ldots (\mu u_m)_{1 i_m} \mu^0 a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ \tilde{\imath},\ a I u_2 \ldots u_m}} (S^*, u_0)(\mu u_1)_{1 i_1} \ldots (\mu u_m)_{1 i_m} \mu^0 a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ a I u_2 \ldots u_m}} (S^*, u_0) \left(\sum_{i_1} (\mu u_1)_{1 i_1} (\mu a)_{i_1 n} \right) (\mu u_2)_{1 j_1} \ldots (\mu u_m)_{1 j_{m-1}} (\mu 1)_{1 j_m} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0' u_2 \ldots u_m \\ a I u_2 \ldots u_m}} \left(\sum_{u_0' = u_0 u_1} (S^*, u_0)(S, u_1 a) \right) (\mu u_2)_{1 j_1} \ldots (\mu u_m)_{1 j_{m-1}} (\mu 1)_{1 j_m} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0' u_1' \ldots u_{m-1}' \\ a I u_1' \ldots u_{m-1}'}} (S^*, u_0' a)(\mu u_1')_{1 j_1} \ldots (\mu u_{m-1}')_{1 j_{m-1}} (\mu 1)_{1 j_m} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ a I u_1 \ldots u_m}} (S^*, u_0 a)(\mu u_1)_{1 j_1} \ldots (\mu u_{m-1})_{1 j_{m-1}} (\mu u_m)_{1 j_m} H_{\tilde{\jmath}}
\end{aligned}
$$

where the last equality holds since $(\mu u_m)_{1 j_m} H_{\tilde{\jmath}} \neq 0$ implies by Lemma 13 $\alpha(u_m) = \overleftarrow{\alpha}(j_m) = \emptyset$ which in turn yields $u_m = 1$. We have proved (6).

Next for $p \geq 1$ we compute similarly

$$
\begin{aligned}
\sum_{\tilde{\imath}} (\nu u)_{\tilde{\imath}} (\mu^p a)_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} &= \sum_{u = u_0 \ldots u_m,\ \tilde{\imath}} (S^*, u_0)(\mu u_1)_{1 i_1} \ldots (\mu u_m)_{1 i_m} \mu^p a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\imath}} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ \tilde{\imath},\ a I u_{p+1} \ldots u_m}} (S^*, u_0)(\mu u_1)_{1 i_1} \ldots (\mu u_m)_{1 i_m} \mu^p a_{\tilde{\imath}\tilde{\jmath}} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ a I u_{p+1} \ldots u_m}} (S^*, u_0)(\mu u_1)_{1 j_1} \ldots \left(\sum_{i_p} (\mu u_p)_{1 i_p} (\mu a)_{i_p j_p} \right) \ldots (\mu u_m)_{1 j_m} H_{\tilde{\jmath}} \\
&= \sum_{\substack{u = u_0 \ldots u_m \\ a I u_{p+1} \ldots u_m}} (S^*, u_0)(\mu u_1)_{1 j_1} \ldots (\mu u_p a)_{1 j_p} \ldots (\mu u_m)_{1 p_m} H_{\tilde{\jmath}}.
\end{aligned}
$$

We have proved (7) and hence the proposition. $\square$

It is now easy to show that the automaton $\mathcal{A}^*$ computes indeed the series S^*.

Proof of Proposition 12. For all $u \in \mathbb{M}$ we have

$$\begin{aligned}\lambda_{\tilde{1}}(\mu^* u)\gamma_{\tilde{1}} &= \mu^* u_{\tilde{1}\tilde{1}} = \nu u_{\tilde{1}} H_{\tilde{1}} = \nu u_{\tilde{1}} \\ &= \sum_{u=u_0\ldots u_m} (S^*, u_0)(\mu u_1)_{11} \ldots (\mu u_m)_{11} = (S^*, u),\end{aligned}$$

using that since 1 is initial, we have $(\mu w)_{k1} = 0$ for all k and $w \neq 1$. $\square$

Now we can prove the main result of this section, which restates Theorem 1(b).

Theorem 18. *Let $\mathbb{M}$ be a trace monoid and K any commutative semiring. Then each mc-rational series in $K\langle\langle\mathbb{M}\rangle\rangle$ is recognizable:*

$$K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle \subseteq K^{rec}\langle\langle\mathbb{M}\rangle\rangle$$

Proof. It is easy to see that series of the form $k \cdot 1$ with $k \in K$ or $1 \cdot a$ with $a \in \Sigma$ are recognizable. Now suppose S_1, S_2 are two recognizable series with representations $(\lambda^i, \mu^i, \gamma^i)$ of dimensions n_i for $(i = 1, 2)$. Let $n = n_1 + n_2$, and define $\mu : \mathbb{M} \longrightarrow K^{n\times n}$ and $\lambda \in K^{1\times n}, \gamma \in K^{n\times 1}$ by the *diagonal sum*:

$$\mu = \begin{pmatrix} \mu^1 & 0 \\ 0 & \mu^2 \end{pmatrix} \qquad \lambda = (\lambda^1 \lambda^2) \qquad \gamma = \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}.$$

More precisely,

$$\mu(m)_{ij} = \begin{cases} (\mu^1 m)_{ij} & \text{if } 1 \leq i, j \leq n_1 \\ (\mu^2 m)_{(i-n_1)(j-n_1)} & \text{if } n_1 < i, j \leq n \ , \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda_i = \begin{cases} (\lambda^1)_i & \text{if } 1 \leq i \leq n_1 \\ (\lambda^2)_{(i-n_1)} & \text{if } n_1 < i \leq n \end{cases} \qquad \gamma_j = \begin{cases} (\gamma^1)_j & \text{if } 1 \leq j \leq n_1 \\ (\gamma^2)_{(j-n_1)} & \text{if } n_1 < j \leq n \end{cases}$$

Then (λ, μ, γ) is a representation of $S_1 + S_2$. The closure of $K^{rec}\langle\langle\mathbb{M}\rangle\rangle$ under product and our restricted version of star follows from Theorems 7 and 10. $\square$

Next we wish to derive further closure properties of $K^{rec}\langle\langle\mathbb{M}\rangle\rangle$. For this, we start by studying the restriction of a series to entries with a given alphabet.

Definition 19. Let $S \in K\langle\langle\mathbb{M}\rangle\rangle$ or $S \in K\langle\langle\Sigma^*\rangle\rangle$ and $A \subseteq \Sigma$. Then the restriction of S to A is the series S_A defined by

$$(S_A, w) = \begin{cases} (S, w) & \text{if } \alpha(w) = A \\ 0 & \text{otherwise} \end{cases}$$

First we show that the restriction preserves both recognizability and mc-rationality of series.

Proposition 20. *Let $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be recognizable. Then S_A is also recognizable.*

Proof. Let $S = (\lambda, \mu, \gamma; \overleftarrow{\alpha}, \overrightarrow{\alpha})$ be an alphabetic representation of S. We proceed similarly to the proof of Proposition 9. We define

$$I = \{i \in [n] : \overleftarrow{\alpha}(i) = \emptyset, \overrightarrow{\alpha}(i) = A\},$$
$$J = \{j \in [n] : \overrightarrow{\alpha}(j) = \emptyset, \overleftarrow{\alpha}(j) = A\},$$

and we let

$$\lambda'_i = \begin{cases} \lambda_i & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma'_j = \begin{cases} \gamma_j & \text{if } j \in J \\ 0 & \text{otherwise} \end{cases}$$

We claim that $S_A = (\lambda', \mu, \gamma'; \overleftarrow{\alpha}, \overrightarrow{\alpha})$. Indeed, let $u \in \mathbb{M}$. If $\lambda_i \mu u_{ij} \gamma_j \neq 0$, then $\overleftarrow{\alpha}(i) = \overrightarrow{\alpha}(j) = \emptyset$ and $\overrightarrow{\alpha}(i) = \overleftarrow{\alpha}(j) = \alpha(u)$. Therefore, if $\alpha(u) = A$ then $\lambda_i \mu u_{ij} \gamma_j \neq 0$ implies $i \in I$ and $j \in J$ and we get $(S_A, u) = (S, u) = \sum_{i,j} \lambda'_i \mu u_{ij} \gamma'_j$. Now, if $\alpha(u) \neq A$ then $\lambda_i \mu u_{ij} \gamma_j = 0$ for all $i \in I$ and $j \in J$ and we obtain $(S_A, u) = 0 = \sum_{i,j} \lambda'_i \mu u_{ij} \gamma'_j$. □

Proposition 21. *Let $S \in K\langle\langle \Sigma^* \rangle\rangle$ or $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be mc-rational. Then S_A is also mc-rational.*

Proof. We proceed by induction on the structure of the given mc-rational expression. It is clear that $(\lambda S)_A = \lambda S_A$, $(S\lambda)_A = S_A \lambda$, and $(S+T)_A = S_A + T_A$. For the product, note that $S = \sum_{B \subseteq \Sigma} S_B$, $T = \sum_{C \subseteq \Sigma} T_C$ and $(S_B T_C)_A = S_B T_C$ if $A = B \cup C$ and $(S_B T_C)_A = 0$ otherwise. Therefore, $(ST)_A = \sum_{B \cup C = A} S_B T_C$. Finally, we consider S^* where S is proper, mono-alphabetic, and connected. Let $B = \alpha(S)$. Clearly, $(S^*)_\emptyset = 1$ is mc-rational. Now if $w \neq 1$ and $(S^*, w) \neq 0$, we have $w = w_1 \dots w_n$ with $n \geq 1$ and elements w_i such that $(S, w_i) \neq 0$ for each i. Hence $\alpha(w) = B$. This shows that if $\emptyset \neq A \neq B$, then $(S^*)_A = 0$, and $(S^*)_B = S^+$. In any case, $(S^*)_A$ is mc-rational. □

The following lemma generalizes a result of Pighizzini [16] for trace languages.

Lemma 22. *Let $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be proper and $A \subseteq \Sigma$ be nonempty. Then $(S^*)_A = Z^+ X$ where $X = \sum_{B \subset A} (S^*)_B$ and $Z = (X \cdot S)_A$.*

Proof. Let $w \in \mathbb{M}$. Note that $(X, w) = (S^*, w)$ if $\alpha(w) \subset A$, and $(X, w) = 0$ otherwise. Now if $(Z^+ X, w) \neq 0$, we can write $w = w_1 \dots w_n w_{n+1}$ with $n \geq 1$

such that $(Z, w_1) \dots (Z, w_n)(X, w_{n+1}) \neq 0$. Then $\alpha(w) = A$. Therefore, $\alpha(w) \neq A$ implies $((S^*)_A, w) = 0 = (Z^+X, w)$. Assume now that $\alpha(w) = A$. Then

$$
\begin{aligned}
(Z^+X, w) &= \sum_{\substack{w=w_1\dots w_n w_{n+1}\\ n\geq 1}} (Z, w_1)\dots(Z, w_n)(X, w_{n+1}) \\
&= \sum_{\substack{w=u_1v_1\dots u_n v_n u_{n+1}\\ n\geq 1, \alpha(u_i v_i)=A}} (X, u_1)(S, v_1)\dots(X, u_n)(S, v_n)(X, u_{n+1}) \\
&= \sum_{\substack{w=u_1v_1\dots u_n v_n u_{n+1}\\ n\geq 1, \alpha(u_i)\subset A, \alpha(u_i v_i)=A}} (S^*, u_1)(S, v_1)\dots(S^*, u_n)(S, v_n)(S^*, u_{n+1}) \\
&= \sum (S, u_{1,1})\dots(S, u_{1,i_1})(S, v_1)\dots(S, u_{n,1})\dots(S, u_{n,i_n})(S, v_n)\cdot \\
&\qquad \cdot(S, u_{n+1,1})\dots(S, u_{n+1,i_{n+1}});
\end{aligned}
$$

here the last sum ranges over all decompositions

$$w = u_{1,1}\dots u_{1,i_1}v_1\dots u_{n,1}\dots u_{n,i_n}v_n u_{n+1,1}\dots u_{n+1,i_{n+1}}$$

such that $n \geq 1$, $i_1, \dots, i_{n+1} \geq 0$, $\alpha(u_{j,1}\dots u_{j,i_j}) \subset A$ and $\alpha(u_{j,1}\dots u_{j,i_j}v_j) = A$ for all j. Since $\alpha(w) = A$, this sum equals

$$\sum_{w=w_1\dots w_m,\ m\geq 1} (S, w_1)\dots(S, w_m) = (S^+, w) = (S^*, w).$$

□

Next we derive another sufficient condition which implies that the star of a recognizable series is again recognizable and, also, that the star of an mc-rational series is again mc-rational.

Theorem 23.

1. *Let K be a commutative semiring and $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be proper and recognizable such that S^* is connected. Then S^* is recognizable.*
2. *Let K be any semiring and $S \in K\langle\langle \Sigma^* \rangle\rangle$ or $S \in K\langle\langle \mathbb{M} \rangle\rangle$ be proper and mc-rational such that S^* is connected. Then S^* is mc-rational.*

Proof. We prove (1) and (2) in a similar way. Note that $S^* = \sum_{A \text{ connected}} (S^*)_A$. We prove by induction on $|A|$ that $(S^*)_A$ is recognizable (resp., mc-rational) if A is connected. Clearly, $(S^*)_\emptyset = 1$ is recognizable and mc-rational. Now let $A \neq \emptyset$ be connected and assume that $(S^*)_B$ is recognizable (resp., mc-rational) for all connected subsets $B \subset A$. By Lemma 22 and connectedness of S^* we obtain $(S^*)_A = Z^+X$ with $X = \sum_{B\subset A} (S^*)_B = \sum_{B\subset A, B \text{ connected}} (S^*)_B$ and $Z = (X \cdot S)_A$. Then X is recognizable (resp., mc-rational) by induction hypothesis, and so Z is also recognizable (resp., mc-rational). Hence Z is proper (because S is proper), mono-alphabetic and connected. Therefore $Z^+ = Z \cdot Z^*$ is recognizable by Theorems 7 and 10 (resp., is mc-rational), and our claim follows. $\square$

When the semiring satisfies $k + k' = 0 \Longrightarrow k = k' = 0$ (e.g. for positive semirings), then the condition S^* connected is stronger than S connected. This latter condition is actually sufficient to obtain the closure properties stated in Theorem 23 when the semiring is commutative and idempotent. For this, the following lemma is crucial.

Lemma 24. *Let K be a commutative and idempotent semiring. Let $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be a connected series and let $B, C \subseteq \Sigma$ be independent subsets of the alphabet. Then, $(S^*)_{B\cup C} = (S^*)_B \cdot (S^*)_C$.*

Proof. Let $w \in \mathbb{M}$ be such that $\alpha(w) = B \cup C$. We start with a few remarks. Assume that $w = w_1 \cdots w_n$ with w_i connected. Since B and C are independent, we deduce that for all $1 \leq i \leq n$, either $\alpha(w_i) \subseteq B$ or $\alpha(w_i) \subseteq C$. Let $i_1, \dots, i_p$ (resp. $j_1, \dots, j_q$) be the subsequence of $1, \dots, n$ corresponding to the w_k with $\alpha(w_k) \subseteq B$ (resp. $\alpha(w_k) \subseteq C$). Using again the independence of B and C, we obtain $w = w_{i_1} \cdots w_{i_p} w_{j_1} \cdots w_{j_q}$. Now, since K is commutative, we also have

$$(S, w_1) \cdots (S, w_n) = (S, w_{i_1}) \cdots (S, w_{i_p})(S, w_{j_1}) \cdots (S, w_{j_q}).$$

Using these remarks and the idempotence of K, we derive

$$\begin{aligned}
(S^*, w) &= \sum_{w = w_1 \cdots w_n} (S, w_1) \cdots (S, w_n) \\
&= \sum_{\substack{w = w_1 \cdots w_n \\ w_i \text{ connected}}} (S, w_1) \cdots (S, w_n) \\
&= \sum_{\substack{w = u_1 \cdots u_p v_1 \cdots v_q \\ \alpha(u_i) \subseteq B, \alpha(v_j) \subseteq C \\ u_i, v_j \text{ connected}}} (S, u_1) \cdots (S, u_p)(S, v_1) \cdots (S, v_q) \\
&= \sum_{\substack{w = uv \\ \alpha(u) = B \\ \alpha(v) = C}} \left(\sum_{u = u_1 \cdots u_p} (S, u_1) \cdots (S, u_p) \right) \left(\sum_{v = v_1 \cdots v_q} (S, v_1) \cdots (S, v_q) \right) \\
&= \sum_{w = uv} ((S^*)_B, u)((S^*)_C, v) \\
&= ((S^*)_B \cdot (S^*)_C, w).
\end{aligned}$$

$\square$

Theorem 25. *Let K be a commutative and idempotent semiring and let S be a proper and connected series in $K\langle\langle\mathbb{M}\rangle\rangle$.*

1. *If S is recognizable then S^* is recognizable.*
2. *If S is mc-rational then S^* is mc-rational.*

Proof. The proofs of (1) and (2) are quite similar. Note first that $S^* = \sum_{A\subseteq\Sigma}(S^*)_A$, hence it is sufficient to show that if S is recognizable (resp. mc-rational) then so is $(S^*)_A$. We proceed by induction on the size of $A \subseteq \Sigma$.

Clearly, $(S^*)_\emptyset = 1$ is both recognizable and mc-rational. Now, assume $A \neq \emptyset$ and let $A_1, \dots, A_n$ be the connected components of A: $A = A_1 \cup \cdots \cup A_n$ and $A_i \; I \; A_j$ for $i \neq j$. By Lemma 24, we obtain $(S^*)_A = (S^*)_{A_1} \cdots (S^*)_{A_n}$ and we are reduced to the case A connected. Now, using Lemma 22 we obtain $(S^*)_A = Z^+X$ where $X = \sum_{B\subset A}(S^*)_B$ and $Z = (X \cdot S)_A$. Then X is recognizable (resp. mc-rational) by induction hypothesis and it follows that Z is also recognizable (resp. mc-rational). Since we have assumed $A \neq \emptyset$ connected, Z is a proper, mono-alphabetic and connected series and using Theorem 10 we deduce that $(S^*)_A = Z \cdot Z^* \cdot X$ is recognizable (resp. mc-rational) which concludes the proof. □

It is now very easy to derive the following fundamental results.

Theorem 26. *Let K be a commutative and idempotent semiring.*

1. *A series in $K\langle\langle\mathbb{M}\rangle\rangle$ is mc-rational iff it is c-rational.*
2. *Each c-rational series in $K\langle\langle\mathbb{M}\rangle\rangle$ is recognizable.*

$$K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle = K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle \subseteq K^{rec}\langle\langle\mathbb{M}\rangle\rangle$$

Proof. (1) It is clear that an mc-rational series is also c-rational. For the converse, one only has to show that the star of an mc-rational connected proper series S is still mc-rational which is precisely Theorem 25 (2).

(2) follows directly from (1) and Theorem 18. □

4 Recognizable series are mc-rational

Throughout this section, let K be an arbitrary (possibly non-commutative) semiring and (Σ, I) a trace alphabet. We will prove that all recognizable series in $K\langle\langle\mathbb{M}\rangle\rangle$ are mc-rational. This uses the concept of lexicographic normal forms of traces and LNF-representations of series which we introduce first. For this, fix any linear order $\leq$ on Σ. We extend this to the lexicographic linear order, also denoted by $\leq$, on Σ^*. We say that a word w is the lexicographic normal form of $[w]$, if it is the smallest element of $[w]$ with respect to $\leq$. Then $\mathrm{LNF} \subseteq \Sigma^*$ is the set of all words which are lexicographic normal forms. Note that LNF is a recognizable word language which is both closed under prefixes and suffixes. Now let $\mathcal{A}_{\mathrm{LNF}} = (Q, \Sigma, \delta, q_0, Q)$ be the minimal (reduced) automaton for LNF.

Definition 27. We will call a morphism $\mu : \Sigma^* \longrightarrow K^{n\times n}$ an LNF-morphism, if there exists a function $\pi : [n] \longrightarrow Q$ such that for all $a \in \Sigma$ and all $i, j \in [n]$, $\mu a_{ij} \neq 0$ implies $\pi(i) \xrightarrow{a} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$. Then any representation (λ, μ, γ) with an LNF-morphism μ of a series $S \in K\langle\langle\Sigma^*\rangle\rangle$ will be called an LNF-representation of S.

Proposition 28. *A morphism $\mu : \Sigma^* \longrightarrow K^{n\times n}$ is an* LNF-*morphism, iff there exists a function $\pi : [n] \longrightarrow Q$ such that for all $w \in \Sigma^*$ and all $i, j \in [n]$, $\mu w_{ij} \neq 0$ implies $\pi(i) \xrightarrow{w} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$.*

Proof. We proceed by induction on the length of w. If $w = 1$, clearly $\mu 1_{ij} \neq 0$ implies $i = j$, and so $\pi(i) \xrightarrow{1} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$. For the induction step, if $0 \neq (\mu wa)_{ij} = \sum_k \mu w_{ik} \mu a_{kj}$, choose k such that $\mu w_{ik} \neq 0$ and $\mu a_{kj} \neq 0$. By induction hypothesis, then $\pi(i) \xrightarrow{w} \pi(k) \xrightarrow{a} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$. □

Intuitively, an LNF-representation is simply a synchronized product of an automaton with multiplicities with the automaton $\mathcal{A}_{\mathrm{LNF}}$. This is precisely the proof of the next proposition.

Proposition 29. *Let $S' \in K\langle\langle\Sigma^*\rangle\rangle$ be recognizable. Then $S = S' \odot 1_{\mathrm{LNF}}$ has an* LNF-*representation.*

Proof. Let $S' = (\lambda', \mu', \gamma')$ where $\mu' : \Sigma^* \longrightarrow K^{n'\times n'}$. Let $n = n' \cdot |Q|$. We identify $[n]$ with $[n'] \times Q$. Define λ, γ, π and $\mu : \Sigma^* \longrightarrow K^{n\times n}$ by

$$\lambda_{(i,p)} = \begin{cases} \lambda'_i & \text{if } p = q_0 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_{(i,p)} = \gamma'_i \qquad \pi_{(i,p)} = p$$

$$\mu a_{(i,p)(j,q)} = \begin{cases} \mu' a_{ij} & \text{if } p \xrightarrow{a} q \text{ in } \mathcal{A}_{\mathrm{LNF}} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, μ is an LNF-morphism. Next we prove the following claim:

$$\mu w_{(i,p)(j,q)} = \begin{cases} \mu' w_{ij} & \text{if } p \xrightarrow{w} q \text{ in } \mathcal{A}_{\mathrm{LNF}} \\ 0 & \text{otherwise} \end{cases}$$

For this we proceed by induction. If $w = 1$, we have

$$\mu 1_{(i,p)(j,q)} = \begin{cases} 1 & \text{if } i = j, p = q \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \mu' 1_{ij} & \text{if } p \xrightarrow{1} q \text{ in } \mathcal{A}_{\mathrm{LNF}} \\ 0 & \text{otherwise} \end{cases}$$

Next observe that $(\mu wa)_{(i,p)(j,q)} = \sum_{k,r} \mu w_{(i,p)(k,r)} \mu a_{(k,r)(j,q)}$. Since $\mathcal{A}_{\mathrm{LNF}}$ is deterministic, there is at most one state r such that $p \xrightarrow{w} r \xrightarrow{a} q$ in $\mathcal{A}_{\mathrm{LNF}}$. Hence

$$\mu wa_{(i,p)(j,q)} = \begin{cases} \sum_k \mu' w_{ik} \mu' a_{kj} & \text{if } p \xrightarrow{wa} q \text{ in } \mathcal{A}_{\mathrm{LNF}} \\ 0 & \text{otherwise} \end{cases}$$

which proves our claim.

Now let $S = (\lambda, \mu, \gamma)$. We show that $S = S' \odot 1_{\mathrm{LNF}}$. Indeed, for any $w \in \Sigma^*$ we have

$$
\begin{aligned}
(S, w) &= \sum_{(i,p),(j,q)} \lambda_{(i,p)} (\mu w)_{(i,p)(j,q)} \gamma_{(j,q)} = \sum_{i,j,q} \lambda'_i \mu w_{(i,q_0)(j,q)} \gamma'_j \\
&= \begin{cases} \sum_{i,j} \lambda'_i \mu' w_{ij} \gamma'_j & \text{if } q_0 \xrightarrow{w} \text{ in } \mathcal{A}_{\mathrm{LNF}} \text{ (i.e., } w \in \mathrm{LNF}) \\ 0 & \text{otherwise} \end{cases} \\
&= (S' \odot 1_{\mathrm{LNF}})(w),
\end{aligned}
$$

as claimed. □

It is not hard to see that only words which admit a run through $\mathcal{A}_{\mathrm{LNF}}$, that is words in LNF, will have a non-zero value with an LNF-representation. The converse is an easy consequence of Proposition 29.

Corollary 30. *A power series $S \in K\langle\langle \Sigma^* \rangle\rangle$ has an* LNF*-representation iff $S \in K^{rec}\langle\langle \Sigma^* \rangle\rangle$ and $supp(S) \subseteq \mathrm{LNF}$.*

Proof. Let $S = (\lambda, \mu, \gamma) \in K\langle\langle \Sigma^* \rangle\rangle$ with an LNF-morphism μ. Let $w \in Supp(S)$. Then $(S, w) = \sum_{i,j} \lambda_i \mu w_{ij} \gamma_j \neq 0$, so there are i, j such that $\mu w_{ij} \neq 0$ and hence $\pi(i) \xrightarrow{w} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$. Since $\mathcal{A}_{\mathrm{LNF}}$ is minimal, $\pi(i)$ is reachable from q_0, so $q_0 \xrightarrow{u} \pi(i)$ for some word u in $\mathcal{A}_{\mathrm{LNF}}$. Hence, $q_0 \xrightarrow{uw} \pi(j)$ in $\mathcal{A}_{\mathrm{LNF}}$ and $uw \in \mathrm{LNF}$. Therefore, $w \in \mathrm{LNF}$ which shows that $Supp(S) \subseteq \mathrm{LNF}$.

The converse is clear from Proposition 29. □

Next we note that for any $n \geq 1$ there is a canonical isomorphism Φ between the semiring of $(n \times n)$-matrices $K\langle\langle \Sigma^* \rangle\rangle^{n \times n}$ and the semiring of formal power series $K^{n \times n}\langle\langle \Sigma^* \rangle\rangle$, given by $(\Phi(A), w) = ((A_{ij}, w))_{1 \leq i,j \leq n}$ if $A = (A_{ij})_{1 \leq i,j \leq n} \in K\langle\langle \Sigma^* \rangle\rangle^{n \times n}$. Subsequently, we will often identify A with its image $\Phi(A)$. The following lemma is well-known (cf., e.g. [2, pp.5,14]).

Lemma 31. *Let $A \in K\langle\langle \Sigma^* \rangle\rangle^{n \times n}$ be proper.*

1. *Let $C \in K\langle\langle \Sigma^* \rangle\rangle^{n \times p}$, then the equation $B = AB + C$ has a unique solution, which is $B = A^*C$.*
2. *Let $C \in K\langle\langle \Sigma^* \rangle\rangle^{p \times n}$, then the equation $B = BA + C$ has a unique solution, which is $B = CA^*$.*

Next, we give a slight variation of another classical result (cf., e.g. [2, pp.14,15]). We need this variation since the form of M^* given in [2] cannot be used to carry out the proof of Proposition 34.

Lemma 32. *Let $M \in K\langle\langle \Sigma^* \rangle\rangle^{n \times n}$ be proper. Consider a decomposition of M into blocks (where A, D are square matrices)*

$$
M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}
$$

Then

$$M^* = \begin{pmatrix} A^* + A^*C(D+BA^*C)^*BA^* & A^*C(D+BA^*C)^* \\ (D+BA^*C)^*BA^* & (D+BA^*C)^* \end{pmatrix}.$$

Proof. Let

$$M^* = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

be the decomposition of M^* into blocks. Since $M^* = 1 + MM^*$, we have

$$\begin{cases} \gamma = A\gamma + C\delta \\ \delta = 1 + B\gamma + D\delta \end{cases}$$

Therefore $\gamma = A^*C\delta$ by Lemma 31 and $\delta = 1 + BA^*C\delta + D\delta = 1 + (D+BA^*C)\delta$. We deduce $\delta = (D+BA^*C)^*$, again by Lemma 31, and $\gamma = A^*C(D+BA^*C)^*$.

Now also $M^* = 1 + M^*M$, hence

$$\begin{cases} \alpha = 1 + \alpha A + \gamma B \\ \beta = \beta A + \delta B \end{cases}$$

Using the previous results, $\beta = \beta A + (D+BA^*C)^*B$, hence $\beta = (D+BA^*C)^*BA^*$ by Lemma 31 and $\alpha = 1 + A^*C(D+BA^*C)^*B + \alpha A$, so $\alpha = A^* + A^*C(D + BA^*C)^*BA^*$ again by Lemma 31. □

We will also use the following result.

Lemma 33 (Ochmański, [15,4]). *Let $w \in \Sigma^*$ be a word such that $w, w^2 \in$ LNF. Then w is connected.*

In Schützenberger's theorem, to prove that each recognizable series is rational, one shows first that the entries of the matrix

$$\left(\sum_{a\in\Sigma} \mu a \cdot a\right)^* \in K\langle\langle\Sigma^*\rangle\rangle^{n\times n}$$

are rational series. In our case, we need to have mc-rational series. This is not true for an arbitrary morphism $\mu : \mathbb{M} \longrightarrow K^{n\times n}$ but we are able to obtain the desired result when we restrict to LNF-morphisms.

Proposition 34. *Let $\mu : \Sigma^* \longrightarrow K^{n\times n}$ be an* LNF*-morphism, and let*

$$M = \sum_{a\in\Sigma} \mu a \cdot a \in K^{n\times n}\langle\Sigma^*\rangle.$$

Then the entries of $M^* \in K\langle\langle\Sigma^*\rangle\rangle^{n\times n}$ are mc-rational series.

Proof. We first show, by induction on the length of w, that $(M^*, w) = \mu w$ for any word w. Indeed, clearly $(M^*, 1) = 1 = \mu 1$ and $(M^*, wa) = (1+M^*M, wa) = (M^*M, wa) = (M^*, w)(M, a) = \mu w \cdot \mu a = \mu(wa)$.

Now, we prove the proposition by induction on n. First, assume that $n = 1$. Then $M \in K\langle \Sigma^* \rangle$ is proper and mc-rational. Now, let $w \in \Sigma^*$. If $(M^*, w) = \mu w \neq 0$, since μ is an LNF-morphism, we have a path $\pi(1) \xrightarrow{w} \pi(1)$ in $\mathcal{A}_{\mathrm{LNF}}$. Therefore, $w, w^2 \in \mathrm{LNF}$ and by Ochmański's lemma 33, w is connected. Hence M^* is connected and so, by Theorem 23, mc-rational.

Now let $n > 1$. Define $\mu' : \Sigma^* \longrightarrow K^{(n-1)\times(n-1)}$ by putting $\mu' a_{ij} = \mu a_{ij}$ for all $i, j < n$, and let $A = \sum_{a \in \Sigma} \mu' a \cdot a \in K^{(n-1)\times(n-1)}\langle \Sigma^* \rangle$.
Clearly, μ' is an LNF-morphism. Hence, by induction hypothesis, A^* has mc-rational entries. We have

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

and by Lemma 32,

$$M^* = \begin{pmatrix} A^* + A^*C(D + BA^*C)^*BA^* & A^*C(D + BA^*C)^* \\ (D + BA^*C)^*BA^* & (D + BA^*C)^* \end{pmatrix}.$$

Let $T = D + BA^*C$. Note that $T \in K\langle\langle \Sigma^* \rangle\rangle$ and by induction hypothesis T is mc-rational. It remains to show that T^* is mc-rational.

Indeed, if $w \in \Sigma^*$ and $|w| \neq 1$, then $(M, w) = 0$ and hence $(A, w) = 0$, $(B, w) = 0$, $(C, w) = 0$, $(D, w) = 0$. Therefore, if $w = a_1 \dots a_k$, we have

$$(T, w) = \begin{cases} 0 & \text{if } k = 0 \\ (D, a_1) & \text{if } k = 1 \\ (B, a_1)(A, a_2) \dots (A, a_{k-1})(C, a_k) & \text{otherwise} \end{cases}$$

It follows that T is proper. Next we claim that if $(T, w) \neq 0$, then $\pi(n) \xrightarrow{w} \pi(n)$ in $\mathcal{A}_{\mathrm{LNF}}$.

Indeed, let $w = a_1 \dots a_k \in supp(T)$. First, if $k = 1$, it follows that $(D, a_1) = (\mu a_1)_{n,n} \neq 0$. Since μ is an LNF-morphism, we obtain $\pi(n) \xrightarrow{a_1} \pi(n)$ in $\mathcal{A}_{\mathrm{LNF}}$. Secondly, if $k \geq 2$, there are $i_1, \dots, i_{k-1} \leq n - 1$ such that

$$(\mu a_1)_{n,i_1}(\mu a_2)_{i_1,i_2} \dots (\mu a_{k-1})_{i_{k-2},i_{k-1}}(\mu a_k)_{i_{k-1},n} \neq 0.$$

Since μ is an LNF-morphism, we obtain

$$\pi(n) \xrightarrow{a_1} \pi(i_1) \xrightarrow{a_2} \pi(i_2) \longrightarrow \dots \xrightarrow{a_{k-1}} \pi(i_{k-1}) \xrightarrow{a_k} \pi(n)$$

in $\mathcal{A}_{\mathrm{LNF}}$. This proves our claim.

Now assume $w \in supp(T^+)$. Then we can decompose $w = w_1 \dots w_k$ such that $(T, w_1) \dots (T, w_k) \neq 0$. By our claim just proved, we obtain $\pi(n) \xrightarrow{w} \pi(n)$ in $\mathcal{A}_{\mathrm{LNF}}$. Therefore $w, w^2 \in \mathrm{LNF}$ and w is connected by Ochmański's lemma. This shows that T^+ and so T^* are connected. By Theorem 23, it follows that T^* is mc-rational. □

We can now derive that a recognizable series over Σ^* is mc-rational when its support is contained in LNF.

Theorem 35. *Let $S \in K\langle\langle \Sigma^* \rangle\rangle$ be recognizable. Then $S \odot 1_{\mathrm{LNF}}$ is mc-rational.*

Proof. By Proposition 29 we can choose an LNF-representation (λ, μ, γ) of $S' = S \odot 1_{\mathrm{LNF}}$. Let $M = \sum_{a \in \Sigma} \mu a \cdot a$. We have seen in the proof of Proposition 34 that $(M^*, w) = \mu w$ for any word w.

Now, λ and γ are vectors with entries in K, and M^* has only mc-rational series as entries by Proposition 34. Hence $\lambda M^* \gamma \in K\langle\langle \Sigma^* \rangle\rangle$ is an mc-rational series. Finally, observe that for any word w,

$$(\lambda M^* \gamma, w) = (\sum_{i,j} \lambda_i (M^*)_{ij} \gamma_j, w) = \sum_{i,j} \lambda_i ((M^*)_{ij}, w) \gamma_j)$$

$$= \sum_{i,j} \lambda_i \mu w_{ij} \gamma_j = \lambda \mu w \gamma = (S', w).$$

Therefore $S \odot 1_{\mathrm{LNF}} = S' = \lambda M^* \gamma$ is mc-rational. □

Corollary 36. *Let $S \in K\langle\langle \Sigma^* \rangle\rangle$ be recognizable with supp$(S) \subseteq$ LNF. Then S is mc-rational.*

Note that, since the beginning of the section, we have only worked with series over words ($K\langle\langle \Sigma^* \rangle\rangle$). We need now to transfer the results to series over traces ($K\langle\langle \mathbb{M} \rangle\rangle$). For this, we will use classical semiring morphisms between $K\langle\langle \Sigma^* \rangle\rangle$ and $K\langle\langle \mathbb{M} \rangle\rangle$ derived from the canonical morphism $\varphi : \Sigma^* \longrightarrow \mathbb{M}$.

Let M, N be two monoids and $h : M \longrightarrow N$ be a morphism. Then $h^{-1} : K\langle\langle N \rangle\rangle \longrightarrow K\langle\langle M \rangle\rangle$ given by $(h^{-1}(S), w) = (S, h(w))$ $(w \in N)$ is a semiring morphism. Moreover, if $S = (\lambda, \mu, \gamma) \in K^{rec}\langle\langle N \rangle\rangle$, then $(h^{-1}(S), w) = (S, h(w)) = \lambda \mu h(w) \gamma$, hence (cf. [17, p.32])

$$h^{-1}(S) = (\lambda, \mu \circ h, \gamma) \in K^{rec}\langle\langle M \rangle\rangle.$$

Let $\varphi : \Sigma^* \longrightarrow \mathbb{M}$ be the canonical epimorphism. Then φ extends naturally to a mapping, denoted by Φ, from $K\langle\langle \Sigma^* \rangle\rangle$ to $K\langle\langle \mathbb{M} \rangle\rangle$ given by

$$\Phi(S) = \sum_{w \in \Sigma^*} (S, w) \varphi(w) = \sum_{t \in \mathbb{M}} \left(\sum_{w \in \varphi^{-1}(t)} (S, w) \right) .t.$$

As is well-known from general results (cf., e.g., [17, pp.13,14]), Φ is a semiring morphism and if S is proper, then $\Phi(S^*) = \Phi(S)^*$. Furthermore, if S is connected (respectively, mono-alphabetic), then $\Phi(S)$ is also connected (respectively, mono-alphabetic). From this, it is clear that if S is mc-rational, then $\Phi(S)$ is also mc-rational.

Now we prove Theorem 1(a).

Theorem 37. *Let $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be recognizable. Then S is mc-rational.*

Proof. Let $S = (\lambda, \mu, \gamma) \in K^{rec}\langle\langle\mathbb{M}\rangle\rangle$. As noted before, $\varphi^{-1}(S) \in K^{rec}\langle\langle\Sigma^*\rangle\rangle$. By Theorem 35, $\varphi^{-1}(S) \odot 1_{\mathrm{LNF}}$ is mc-rational. Hence also $\Phi(\varphi^{-1}(S) \odot 1_{\mathrm{LNF}})$ is mc-rational. Now for each $t \in \mathbb{M}$ we have

$$\begin{aligned}(\Phi(\varphi^{-1}(S) \odot 1_{\mathrm{LNF}}), t) &= \sum_{w \in \varphi^{-1}(t)} (\varphi^{-1}(S) \odot 1_{\mathrm{LNF}}, w) \\ &= \sum_{w \in \varphi^{-1}(t) \cap \mathrm{LNF}} (\varphi^{-1}(S), w) \\ &= \sum_{w \in \varphi^{-1}(t) \cap \mathrm{LNF}} (S, \varphi(w)) = (S, t).\end{aligned}$$

Therefore, $S = \Phi(\varphi^{-1}(S) \odot 1_{\mathrm{LNF}})$ is mc-rational. □

5 Sharpness of the results

Here we will give two examples to show that the assumptions in Theorems 7 and 10 (hence, in Theorem 1(b,c)) are necessary. First, we show that in Theorem 7 the commutativity of K is necessary.

Example 38. Consider the trace alphabet (Σ, I) with $\Sigma = \{a, b\}$ and $a\ I\ b$, and let $K = \mathbb{B}\langle\Sigma^*\rangle$. Let $S = \sum_n a^n.a^n, T = \sum_n b^n.b^n \in K\langle\langle\mathbb{M}\rangle\rangle$. Then S and T are recognizable. Indeed, let $\mu : \mathbb{M} \longrightarrow K$ be the morphism defined by $\mu(a) = a$ and $\mu(b) = 0$ and let $\lambda = \gamma = 1$, then $S = (\lambda, \mu, \gamma)$. However, we will show that $S \cdot T \in K\langle\langle\mathbb{M}\rangle\rangle$ is *not* recognizable. Indeed, suppose we had $S \cdot T = (\lambda, \mu, \gamma)$ where $\mu : \mathbb{M} \longrightarrow K^{n \times n}$. We have $(S \cdot T, t) = a^{|t|_a} b^{|t|_b}$ for each $t \in \mathbb{M}$. Let $m \geq n$. Then

$$\begin{aligned}(S \cdot T, (ab)^m) = a^m b^m &= \lambda \cdot (\mu ab)^m \cdot \gamma \\ &= \sum_{i_0, \ldots, i_m} \lambda_{i_0} (\mu ab)_{i_0 i_1} \ldots (\mu ab)_{i_{m-1} i_m} \gamma_{i_m}.\end{aligned}$$

Hence there exist $i_0, \ldots, i_m$ such that $a^m b^m = \lambda_{i_0} (\mu ab)_{i_0 i_1} \ldots (\mu ab)_{i_{m-1} i_m} \gamma_{i_m}$. Since $m \geq n$, there are $p < q$ with $i_p = i_q$. Let $r = q - p$, and put

$$\begin{aligned}u &= \lambda_{i_0} (\mu ab)_{i_0 i_1} \ldots (\mu ab)_{i_{p-1} i_p}, \\ v &= (\mu ab)_{i_p i_{p+1}} \ldots (\mu ab)_{i_{q-1} i_q}, \\ w &= (\mu ab)_{i_q i_{q+1}} \ldots (\mu ab)_{i_{m-1} i_m} \gamma_{i_m}.\end{aligned}$$

Then $uvw = a^m b^m$. Hence $uw \neq 0$, and from $(S \cdot T, (ab)^{m-r}) = a^{m-r} b^{m-r}$ we deduce $uw = a^{m-r} b^{m-r}$. Therefore $v = a^r b^r$. Similarly we obtain $uvvw = (S \cdot T, (ab)^{m+r}) = a^{m+r} b^{m+r}$. But $vv = a^r b^r a^r b^r$ and $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ is not commutative, which yields a contradiction.

Secondly, we want to show that in general $K^{rec}\langle\langle\mathbb{M}\rangle\rangle$ is properly contained in $K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle$. That is, we show that the star of a connected recognizable series may not be recognizable. (Thus by Theorem 23, the star of this series will not be connected.)

Example 39. Again consider the trace alphabet (Σ, I) with $\Sigma = \{a, b\}$ and $a\ I\ b$, and let $S = a + b \in \mathbb{N}\langle\mathbb{M}\rangle$. Then, obviously, S is a connected polynomial and $(S^*, t) = \binom{|t|_a+|t|_b}{|t|_a}$ for all $t \in \mathbb{M}$. Hence, $S^* = \sum_{n,m\in\mathbb{N}} \binom{n+m}{n} a^n b^m$. We will prove that S^* is not recognizable.

Assume on the contrary that S^* is recognizable. Then it is also recognizable in $\mathbb{Q}\langle\langle\mathbb{M}\rangle\rangle$. The *Hankel matrix* [10] of S^* is defined as the infinite matrix $H(S^*) = ((S^*, uv))_{u,v\in\mathbb{M}}$ whose rows and columns are indexed by the elements of $\mathbb{M}$ and whose entry with indices u and v is equal to (S^*, uv). Since S^* is recognizable, by Fliess' theorem [10, Thm. 2.1.1], its Hankel matrix $H(S^*)$ has finite rank $k \in \mathbb{N}$. In particular this implies that all sub-determinants of $H(S^*)$ of order $k+1$ must be 0. Now, consider the sub-determinant of order $k+1$

$$\left|((S^*, a^n b^m))_{0\le n,m\le k}\right| = \left|\left(\binom{n+m}{n}\right)_{0\le n,m\le k}\right|.$$

It is easy to show using the binomial formula $\binom{p+1}{q+1} = \binom{p}{q+1} + \binom{p}{q}$ and linear combinations of columns that this sub-determinant equals 1, a contradiction.

6 Conclusion

As a conclusion, we want to summarize the results obtained in the paper and also to indicate the relationship with the results of Schützenberger and Ochmański.

We have shown that, for an arbitrary semiring K, the following inclusions *hold and are strict*:

$$K^{rec}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{rat}\langle\langle\mathbb{M}\rangle\rangle$$

The sharpness of the first inclusion follows from Example 38: the family $K^{rec}\langle\langle\mathbb{M}\rangle\rangle$ may not be closed under product when the independence relation I is nonempty and K is not commutative. On the other hand, when the semiring K is commutative, we have:

$$K^{rec}\langle\langle\mathbb{M}\rangle\rangle = K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{rat}\langle\langle\mathbb{M}\rangle\rangle$$

Again, the second inclusion is strict since, even for commutative semirings, $K^{rec}\langle\langle\mathbb{M}\rangle\rangle$ may not be closed under star restricted to proper and connected series when the independence relation I is nonempty (Example 39). Now, if the semiring K is both commutative and idempotent then we have

$$K^{rec}\langle\langle\mathbb{M}\rangle\rangle = K^{mc-rat}\langle\langle\mathbb{M}\rangle\rangle = K^{c-rat}\langle\langle\mathbb{M}\rangle\rangle \subsetneq K^{rat}\langle\langle\mathbb{M}\rangle\rangle$$

The sharpness of the last inclusion is well-known already for the Boolean semiring $\mathbb{B}$ since the family $\mathbb{B}^{rec}\langle\langle\mathbb{M}\rangle\rangle$ is not closed under star when the independence relation I is nonempty. Indeed, if two letters a and b commute then the star of the recognizable language $\{ab\}$ is not recognizable.

We will now relate our results with Schützenberger's theorem (Theorem 2). Let Σ be any finite alphabet. If $I = \emptyset$, the trace monoid $\mathbb{M}(\Sigma, I)$ is isomorphic to Σ^*. From the above results we have

$$K^{rec}\langle\langle \Sigma^* \rangle\rangle \subseteq K^{mc-rat}\langle\langle \Sigma^* \rangle\rangle \subseteq K^{c-rat}\langle\langle \Sigma^* \rangle\rangle \subseteq K^{rat}\langle\langle \Sigma^* \rangle\rangle.$$

Now, using the inclusion $K^{rat}\langle\langle \Sigma^* \rangle\rangle \subseteq K^{rec}\langle\langle \Sigma^* \rangle\rangle$ of Schützenberger's theorem, we obtain

$$K^{rec}\langle\langle \Sigma^* \rangle\rangle = K^{mc-rat}\langle\langle \Sigma^* \rangle\rangle = K^{c-rat}\langle\langle \Sigma^* \rangle\rangle = K^{rat}\langle\langle \Sigma^* \rangle\rangle$$

which is in fact a strengthening of Schützenberger's theorem. Indeed, when the independence relation is empty, all words are connected and the c-rational series coincide clearly with the rational ones. But it is not so clear that we may restrict the star to mono-alphabetic and proper series and still obtain all rational series.

Now we show how to deduce and actually strengthen Ochmański's theorem (Theorem 3) from our results. The following can be proved in the same way as classically for the free monoid (cf. [17,2]).

Proposition 40. *Let $\mathbb{M}$ be any trace monoid and $L \subseteq \mathbb{M}$. Then, L is a recognizable language in $\mathbb{M}$ iff 1_L is a recognizable series in $\mathbb{B}\langle\langle \mathbb{M} \rangle\rangle$ iff 1_L is a recognizable series in $\mathbb{N}\langle\langle \mathbb{M} \rangle\rangle$.*

A similar result can be obtained for rational languages.

Proposition 41. *$L \subseteq \mathbb{M}$ is rational (resp. c-rational, mc-rational) iff $1_L \in \mathbb{B}\langle\langle \mathbb{M} \rangle\rangle$ is rational (resp. c-rational, mc-rational).*

Proof. Let $L, L' \subseteq \mathbb{M}$ be trace languages. One can easily verify that in $\mathbb{B}\langle\langle \mathbb{M} \rangle\rangle$ we have $1_{L \cup L'} = 1_L + 1_{L'}$, $1_{L \cdot L'} = 1_L \cdot 1_{L'}$ and $1_{L^*} = (1_L)^*$. Moreover, L is mono-alphabetic (resp. connected) iff 1_L is mono-alphabetic (resp. connected) and 1_L is a polynomial iff L is finite. Therefore, the result follows by structural induction. □

Since the boolean semiring $\mathbb{B}$ is both commutative and idempotent, we deduce from Theorem 1 that a series in $\mathbb{B}\langle\langle \mathbb{M} \rangle\rangle$ is recognizable iff it is c-rational iff it is mc-rational. Using Propositions 40 and 41, we deduce that a trace language $L \subseteq \mathbb{M}$ is recognizable iff it is c-rational iff it is mc-rational. The first equivalence is precisely Ochmański's theorem. The second one is a strengthening of a result by Pighizzini [16] which characterizes the recognizable languages as those languages obtained from finite sets of traces using union, concatenation, *restriction to subalphabet* and star restricted to mono-alphabetic and connected languages. Our results show that the restriction to subalphabet is superfluous.

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