Well-Founded Recursive Relations

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Abstract. We give a short constructive proof of the fact that certain binary relations $>$ are well-founded, given a lifting $\gg$ à la Ferreira-Zantema and a well-founded relation $\triangleright$. This construction generalizes several variants of the recursive path ordering on terms and of the Knuth-Bendix ordering. It also applies to other domains, of graphs, of infinite terms, of word and tree automata notably. We then extend this construction further; the resulting family of well-founded relations generalizes Jouannaud and Rubio's higher-order recursive path orderings.

Keywords: Termination, well-foundedness, path orderings, Knuth-Bendix orderings, λ-calculus, higher-order path orderings, graphs, automata.

1 Introduction

The use of well-founded orderings is a well-established technique to show that term rewrite systems terminate [4]. On the other hand, the tradition in λ-calculus circles, exemplified by the Tait-Girard technique [10], is to show termination by structural induction on terms, backed by auxiliary well-founded inductions. In fact, the recursive path ordering can be proved terminating by structural induction on terms, as noticed in [14].

Prompted by [12], we wrote a direct inductive proof of the termination of the recursive path ordering $\succ_{rpo}$ based on a well-founded precedence $\succ$ [4], which turned out to be surprisingly short. Our point is that this proof generalizes considerably, while remaining short and constructive, and still using only elementary principles of logic. Chasing generalizations and simplifications, we arrived at Theorem 1, which is the core of this paper. We state it and prove it in Section 2. Here the use of the Coq proof assistant [1] helped us in reassuring ourselves that the proof was indeed correct, but more importantly helped us delineate useful generalizations and useless assumptions.

The rest of the paper examines applications and extensions of Theorem 1. First, we shall see in Section 3 that it subsumes Ferreira and Zantema's Theorem [8], and therefore most path orderings of the literature.

Theorem 1 is more general still, if only because it does not depend on term structure. That is, this construction works equally well on other kinds of algebras. We illustrate this by sketching a few well-founded relations resembling the recursive path ordering on graphs, on infinite terms, and on word and tree automata in Section 4.

Theorem 1 seems however helpless in establishing that simply-typed λ-terms terminate. We extend Theorem 1 by augmenting its proof with two ingredients that are crucial in the classical Tait-Girard proof of strong normalization [10], and come up with a suitable generalization of Jouannaud and Rubio's higher-order recursive path ordering (horpo) [13] in Section 5, Theorem 2. This provides for variants of the horpo that also generalize semantic path orderings and the general path ordering, just like Theorem 1 did in the first-order case. (Again, we checked the proof in Coq.)

We conclude in Section 6 by pondering over yet unexploited features of Theorem 1 and Theorem 2, and possible generalizations.

Related work. Proving the termination of term rewrite systems by structural induction on terms is not new. This is classical in the λ-calculus. Similar techniques are extensively used in [7], where several proofs of termination consist in showing that for every substitution σ mapping variables to terminating terms, the term $t\sigma$ terminates by induction on some well-founded measure on t and σ. Jouannaud and Rubio [14] also notice that recursive path orderings can be shown well-founded by the same technique. However, our proof of Theorem 1 is in fact simpler: it does not consider substitutions, and proceeds directly on t. Naturally, Theorem 1 does not consider the higher-order case. Theorem 2 does, and this requires both the use of substitutions as above and replacing strong normalization by the stronger, and more complex notion of reducibility [10], a.k.a. computabiliy [13]. We shall demystify the latter notion in Section 5: reducibility is just ordinary strong normalization, albeit of a richer reduction relation.

On first-order term algebras, the closest result to our first theorem (Theorem 1) is Ferreira and Zantema's Theorem [8], which is almost Theorem 1 specialized to the case of terms. We shall indeed rederive the latter from ours. Theorem 2 can then be seen as the higher-order, abstract version of Ferreira and Zantema's Theorem.

We stress the fact that our results are in no way tied to term structure, and that this allows to design syntactic well-founded path orderings on graphs, infinite terms and automata. We are not aware of any previous termination method resembling recursive path orderings on algebras other than terms.

Finally, Paul-André Melliès [16] suggested a deep connection between Kruskal's Theorem and the termination of recursive path orderings, including Ferreira and Zantema's Theorem. We have not managed to reverse the duality and deduce Kruskal's Theorem from Theorem 1: the problem is that Melliès' definition of a well-founded relation R is different from ours, namely that in any sequence $(s_i)_{i\in\mathbb{N}}$ there is $i < j$ such that not $s_i\ R\ s_j$. Furthermore, this difference seems essential.

2 The first termination theorem and its proof

Let T be any set, and $\triangleright$, $>$ and $\gg$ be three binary relations on T. For short, we write $u \triangleleft t$ for $t \triangleright u$, and $\geq$ for the reflexive closure of $>$. Assume that $>$ has the following property:

Property 1. For every $s, t \in T$, if $s > t$ then either:

(*i*) for some $u \in T$, $s \triangleright u$ and $u \geq t$;
(*ii*) or $s \gg t$ and for every $u \triangleleft t$, $s > u$.

Remark 1. Condition (*ii*) resembles Ferreira and Zantema's [8] condition that $\gg$ be a *term lifting*. We shall see the precise connection in Section 3, Corollary 1.

Remark 2. Although the notation suggests it, none of $\rhd$, $>$, $\gg$ are required to be transitive. Taking them to be orderings, as is standard in the literature, is an orthogonal issue to termination.

Remark 3. A more general definition would be to take $\geq$ as primitive and define $>$ by $s > t$ if and only if $s \geq t$ and $t \not\geq^* s$, where $\geq^*$ is the reflexive transitive closure of $\geq$. We feel that this would only make the presentation more clumsy, while the additional generality can be obtained by reasoning over $T/(\geq^* \cap \leq^*)$.

Remark 4. A general way of obtaining $>$ satisfying Property 1, which we shall use in the sequel, is as follows. Let $\rhd$ be given, and let $R \mapsto \gg_R$ be any monotonic function from binary relations to binary relations. By *monotonic*, we mean that $R \subseteq R'$ implies $\gg_R \subseteq \gg_{R'}$. Let $>$ be the greatest binary relation on T such that Property 1 holds, where $\gg$ abbreviates $\gg_>$. This is well-defined by Tarski's Fixpoint Theorem on the complete lattice of binary relations over T. (In fact, $>$ is then the greatest binary relation such that $s > t$ *if and only if* (*i*) or (*ii*).)

Let us talk about termination. An element $s \in T$ is *accessible* in the binary relation R, a.k.a., s is in the *well-founded part* of R, if and only if every decreasing sequence $s_0 \hat{=} s \ R \ s_1 \ R \ s_2 \ R \ \ldots \ R \ s_k \ R \ \ldots$ starting from s is finite. The relation R is *well-founded*, or *terminating* (over T) if and only if every $s \in T$ is accessible in R.

We shall use a slightly more general notion than accessibility. Let S be any subset of T. Say that S *bars* s in R if and only if every infinite sequence $s_0 \hat{=} s \ R \ s_1 \ R \ s_2 \ R \ \ldots \ R \ s_k \ R \ \ldots$ starting from s meets S, that is, $s_k \in S$ for some $k \geq 0$. The proper, constructive characterization of bars is the following principle, due to Brouwer:

Proposition 1 (Bar induction). *For every property P on T, if:*

1. *every $s \in S$ satisfies $P(s)$,*
2. *and for every $s \in T$, if $P(t)$ for every t such that $s \ R \ t$, then $P(s)$,*

then every $s \in T$ barred by S in R satisfies $P(s)$.

In fact, the set B of all terms s barred by S in R is *defined* as the smallest such that $P \hat{=} \lambda s \cdot s \in B$ satisfies 1 and 2 above.

Remark 5. An element s is accessible in R if and only if the empty set bars s in R.

Our first theorem is the following:

Theorem 1. *Let SN be the set of all $s \in T$ that are accessible in $>$, and $\underline{SN}$ be the set of all s such that, if every $u \lhd s$ is in SN, then s is in SN. Assume that the following conditions hold:*

(iii) *$\rhd$ is well-founded on T;*
(iv) *for every $s \in T$, if every $u \lhd s$ is in SN, then $\underline{SN}$ bars s in $\gg$.*

Then $>$ is well-founded on T. Equivalently, $SN = T$.

Proof. We are going to prove this in excruciating detail: the proof is short but subtle.

First observe that SN satisfies the following properties:

$$\text{for every } s \in SN, \text{ if } s > t \text{ then } t \in SN \tag{1}$$

$$\text{for every } s \in T, \text{ if every } t \text{ such that } s > t \text{ is in } SN, \text{ then } s \in SN \tag{2}$$

We shall show that every $s \in T$ is in SN, by well-founded induction on $\rhd$, using (*iii*). I.e., let our first induction hypothesis be:

$$\text{For every } u \lhd s,\ u \in SN \tag{3}$$

and let us show that $s \in SN$. To show the latter, it is enough to show $s \in \underline{SN}$, that is, if every $u \lhd s$ is in SN then $s \in SN$. Indeed $s \in SN$ will follow easily, using (3).

Now let us show $s \in \underline{SN}$ under assumption (3). By (3) and (*iv*), $\underline{SN}$ bars s in $\gg$, so we may prove $s \in \underline{SN}$ by bar induction. Following Proposition 1, we have:

- (Base case) $s \in \underline{SN}$: obvious.
- (Inductive step) Assume that for every t such that $s \gg t$, $t \in \underline{SN}$ holds, and prove $s \in \underline{SN}$. Expanding the definition of $\underline{SN}$, we must show that $s \in SN$ under the assumptions:

$$\text{For every } t \text{ such that } s \gg t \text{ and such that every } u \lhd t \text{ is in } SN, t \in SN \tag{4}$$

$$\text{For every } u \lhd s,\ u \in SN \tag{5}$$

Using (2), it is enough to show that whenever $s > t$, $t \in SN$. We show this by well-founded induction on t ordered by $\rhd$—which is well-founded by (*iii*). Our new induction hypothesis is therefore:

$$\text{For every } u \lhd t, \text{ if } s > u \text{ then } u \in SN \tag{6}$$

Now since $s > t$, use Property 1, which yields two cases:

(*i*) For some $u \lhd s$, $u \geq t$. By (5), $u \in SN$. By (1), $t \in SN$.

(*ii*) Or $s \gg t$ and for every $u \lhd t$, $s > u$. By (6) every such u is in SN. To sum up, $s \gg t$ and for every $u \lhd t$, $u \in SN$. By (4) $t \in SN$. ☐

Remark 6. Condition (*iv*) is a bit hard to apply. Defining $\overline{SN}$ as the set of all $s \in T$ such that every $u \lhd s$ is in SN, a less general but simpler condition is:

(*v*) for every $s \in T$, if every $u \lhd s$ is in SN, then s is accessible in $\gg_{|\overline{SN}}$.

where $R_{|A}$ denotes R restricted to A. Indeed (*v*) implies (*iv*): assume that s is such that every $u \lhd s$ is in SN, and show that s is barred by $\underline{SN}$ in $\gg$ by induction on $\gg_{|\overline{SN}}$, which is legal by (*v*). It is enough to show that every t such that $s \gg t$ is barred by $\underline{SN}$ in $\gg$. If $t \in \overline{SN}$, apply the induction hypothesis. Otherwise $t \notin \overline{SN}$ clearly implies $t \in \underline{SN}$. (Note that this argument is *not* constructive.)

3 Path orderings

In this section, let T be the set of all first-order terms on a given signature and a given set of variables. Let $\blacktriangleright$ denote the *immediate superterm relation*, defined as the smallest such that $f(s_1, \ldots, s_m) \blacktriangleright s_i$ for each n-ary function symbol f and every i, $1 \leq i \leq m$.

Path orderings. Path orderings are obtained by setting $\lhd \hat{=} \blacktriangleleft$. Condition (*iii*) is then satisfied. The relation $>$ obtained by Remark 4 is then such that $s > t$ if and only if either:

(*i*) $s = f(s_1, \ldots, s_m)$ and $s_i \geq t$ for some i, $1 \leq i \leq m$;
(*ii*) or $s \gg_> t$ and either t is a variable, or $t = g(t_1, \ldots, t_n)$ and $s > t_j$ for all j, $1 \leq j \leq n$.

In this form, $>$ starts looking more like the recursive path ordering and its variants. And indeed, Theorem 1 has the following corollaries:

- Dershowitz' original *recursive path ordering* [3] on the well-founded precedence $\succ$ is well-founded : define $s \gg_R t$ if and only if s is of the form $f(s_1, \ldots, s_m)$, t is of the form $g(t_1, \ldots, t_n)$, and either $f \succ g$ or $f = g$ and $\{s_1, \ldots, s_m\}\ R_{mul}\ \{t_1, \ldots, t_n\}$;
- similarly Kamin and Lévy's *lexicographic path ordering* [15] : $s \gg_R t$ if and only if $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and either $f \succ g$, or $f = g$, $m = n$, and $(s_1, \ldots, s_n)\ R_{lex}\ (t_1, \ldots, t_n)$ (meaning $s_1 = t_1$, …, $s_{k-1} = t_{k-1}$ and $s_k\ R\ t_k$ for some k, $1 \leq k \leq n$);
- The recursive path ordering with status (easy exercise);
- Plaisted's *semantic path ordering* [4] : given a well-founded quasi-ordering $\succeq$ on terms with strict part $\succ$ and equivalence $\approx$, $s \gg_R t$ if and only if either $s \succ t$, or $s \approx t$ and $\{s_1, \ldots, s_m\}\ R_{mul}\ \{t_1, \ldots, t_n\}$ (recall that the *strict part* of a preordering $\succeq$ is $\succ \hat{=} \succeq \setminus \preceq$, while its *equivalence* is $\approx \hat{=} \succeq \cap \preceq$);
- Dershowitz and Hoot's *general path ordering* [5] (easy justification omitted).

One way to show at once that these orderings are well-founded is to show that Ferreira and Zantema's result [8] is a consequence of Theorem 1. We silently assume here that all terms are ground, each variable x being considered as a nullary function symbol.

Corollary 1 (Ferreira-Zantema). *Let $>$ be a partial order on a set of first-order terms. A* term lifting *$>^\Lambda$ is a strict ordering on terms such that for each set A of terms, if $>_{|A}$ is well-founded, then $>^\Lambda_{|\overline{A}}$ is well-founded, where $\overline{A}$ is the set of terms $f(s_1, \ldots, s_m)$ where $s_i \in A$ for all i, $1 \leq i \leq m$.*

Assume that $>$ has the subterm property (i.e., $s \rhd t$ implies $s > t$). Also, assume that $s \hat{=} f(s_1, \ldots, s_m) > t$ implies either that $s_i \geq t$ for some i, $1 \leq i \leq m$, or $s >^\Lambda t$.

Then $>$ is well-founded.

Proof. Define $\gg$ as $>^\Lambda$, $>$ and $\rhd$ are already defined.

We must first show that $s > t$ implies (*i*) or (*ii*) (see Property 1). So assume $s > t$. By assumption, there are two cases. Case 1: $s_i \geq t$ for some i; then (*i*) holds with $u \hat{=} s_i$. Case 2: $s >^\Lambda t$, so $s \gg t$. In this case every $u \lhd t$ is of the form t_j, $1 \leq j \leq n$; since $>$ has the subterm property, $t > t_j$; since $>$ is transitive and $s > t$, it obtains $s > t_j$, therefore (*ii*) holds.

Condition (*iii*) is trivial. Let us show (*iv*), or rather (*v*) (Remark 6): let s be such that every $u \lhd s$ is in SN, i.e., $s \in \overline{SN}$. Let A be SN. Then $\overline{A}$ is $\overline{SN}$: by assumption $>^\Lambda_{|\overline{A}}$, i.e., $\gg_{|\overline{SN}}$ is well-founded. So s is accessible in $\gg_{|\overline{SN}}$. Then apply Theorem 1. $\square$

Knuth-Bendix orderings. Theorem 1 also generalizes Dershowitz' version of the Knuth-Bendix ordering [4]. Let $\succeq_{\mathrm{T}}$ and $\succeq_{\mathrm{F}}$ be two preorderings, respectively on terms and on function symbols, with strict parts $\succ_{\mathrm{T}}$ and $\succ_{\mathrm{F}}$, and equivalences $\approx_{\mathrm{T}}$ and $\approx_{\mathrm{F}}$. Choose $\triangleright \hat{=} \emptyset$ and let $s \gg_R t$ if and only if (a) $s \succ_{\mathrm{T}} t$, or (b) $s \approx_{\mathrm{T}} t$, $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and either $f \succ_{\mathrm{F}} g$ or $f \approx_{\mathrm{F}} g$, $m = n$ and $(s_1, \ldots, s_m)\ R_{lex}\ (t_1, \ldots, t_n)$. Remark 4 builds a binary relation $>$: this is the Knuth-Bendix ordering $\succ_{kbo}$ of [4]. (Note that since $\triangleright = \emptyset$, $s > t$ if and only if $s \gg_{>} t$.) Theorem 1 then allows us to retrieve:

Corollary 2. *If $\blacktriangleright \subseteq \succ_{\mathrm{T}}$, and $\succ_{\mathrm{T}}$ and $\succ_{\mathrm{F}}$ are well-founded, then $\succ_{kbo}$ is well-founded.*

Proof. Property (*iii*) is trivial. Let us show (*iv*), and consider any chain $s = s_0 \gg_{\succ_{kbo}} s_1 \gg_{\succ_{kbo}} \ldots \gg_{\succ_{kbo}} s_k \gg_{\succ_{kbo}} \ldots$ For every $k \geq 0$, for every immediate subterm u of s_k, $u \prec_{\mathrm{T}} s_k$, since $\blacktriangleright \subseteq \succ_{\mathrm{T}}$. Since $\gg_{\succ_{kbo}} \subseteq \succeq_{\mathrm{T}}$ and $\succeq_{\mathrm{T}}$ is transitive, $u \prec_{\mathrm{T}} s$, so $u \in SN$ by assumption. In particular, the whole chain is inside $\overrightarrow{SN}^{\blacktriangleright}$, the set of terms whose immediate subterms all are in SN. Clearly $\gg_{\succ_{kbo}}$ restricted to $\overrightarrow{SN}^{\blacktriangleright}$ is well-founded, since it is a lexicographic product of $\succeq_{\mathrm{T}}$, $\succeq_{\mathrm{F}}$, and for each function symbol f, of a lexicographic extension of $\succ_{kbo}$ restricted to SN. So the chain is finite, hence s is accessible in $\gg$. This proves (*v*) (Remark 6). $\square$

Monotonicity, stability. To show that a rewrite system terminates, it is important that the relation $>$ be *monotonic*, i.e., $s > t$ implies $f(\ldots, s, \ldots) > f(\ldots, t, \ldots)$—this notation meaning $f(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) > f(s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n)$; and that it be *stable*, i.e., $s > t$ implies $s\sigma > t\sigma$ for every substitution σ. We state conditions under which these properties hold.

Lemma 1. *Let $>$ be defined as in Remark 4. Assume that:*

(vi) whenever $s\ R\ t$, $f(\ldots, s, \ldots) \gg_R f(\ldots, t, \ldots)$;
(vii) whenever $f(\ldots, t, \ldots) \triangleright u$, then $u = t$ or $f(\ldots, s, \ldots) \triangleright u$ for every $s \in T$.

Then $>$ is monotonic.

Proof. Assume that $s > t$. By (*vi*) $f(\ldots, s, \ldots) \gg_{>} f(\ldots, t, \ldots)$. It remains to show that for every $u \triangleleft f(\ldots, t, \ldots)$, $f(\ldots, s, \ldots) > u$. By (*vii*) every such u is such that $f(\ldots, s, \ldots) \triangleright u$—therefore $f(\ldots, s, \ldots) > u$—, or $t = u$—then $f(\ldots, s, \ldots) \triangleright s > t = u$, so $f(\ldots, s, \ldots) > u$. $\square$

Lemma 2. *Let $>$ be defined as in Remark 4. Recall in particular that $\gg_R$ is monotonic in R. Assume that:*

(viii) $\triangleright$ is stable;
(ix) $\gg_R$ is stable whenever R is;
(x) whenever $s \gg_R x$, where x is a variable, then $s \triangleright^* x$;
(xi) if $u \triangleleft t'\sigma$ and t' is not a variable, then for some u', $u = u'\sigma$ and $u' \triangleleft t'$.

Then $>$ is stable.

Proof. Let $>'$ be defined by $s >' t$ if and only if for some terms s', t' and for some substitution σ, $s = s'\sigma$, $t = t'\sigma$, and $s' > t'$. We claim that $s >' t$ implies either:

(i') for some $u \lhd s$, $u \geq' t$;
(ii') or $s \gg_{>'} t$ and for every $u \lhd t$, $s >' u$.

Since $> \subseteq >'$ and $>$ is the greatest relation satisfying (*i*) or (*ii*), it will follow that $> = >'$, therefore that $>$ is indeed stable.

So assume $s >' t$, i.e., $s' > t'$ and $s = s'\sigma$, $t = t'\sigma$. Then either (*i*) $s' \rhd u' \geq t'$, in which case $s'\sigma \rhd u'\sigma$ by (*viii*) and $u'\sigma \geq' t'\sigma$ by definition, so (i') holds; or (*ii*) $s' \gg_{>} t'$ and for every $u' \lhd t'$, $s' > u'$. In the latter case, since $R \mapsto \gg_R$ is monotonic and $> \subseteq >'$, $s' \gg_{>'} t'$; since $>'$ is stable, by (*ix*) $\gg_{>'}$ is, too, so $s \gg_{>'} t$. On the other hand, we claim that for every $u \lhd t$, $s >' u$; (ii') will follow. Distinguish two cases:

1. If t' is not a variable, by (*xi*) there is a term u' such that $u = u'\sigma$ and $u' \lhd t'$. But by assumption $u' \lhd t'$ implies $s' > u'$, so $s >' u$.
2. If t' is a variable x, by (*x*) $s' \rhd^* x$. By (*viii*) $s = s'\sigma \rhd^* x\sigma = t \rhd u$. So $s \rhd^+ u$. It follows that $s > u$, so trivially $s >' u$. $\square$

Remark 7. Conditions (*vii*), (*viii*) and (*xi*) are automatically verified when $\rhd$ is the empty relation or $\blacktriangleright$, as above. In the recursive and lexicographic cases, as well as the recursive path ordering with status, (*vi*) holds: we retrieve the fact that these orderings are monotonic. In the same cases, (*ix*) and (*x*) hold—the latter because $s \gg_R x$ is always false—, so these orderings are also stable, as is already known.

4 Well-founded orderings on graphs, infinite terms, automata

Theorem 1 does not need to operate on terms. As an application of this remark, let T be the set of all finite edge-labeled rooted graphs—*graphs* for short. Recall that a *graph* G is a 6-tuple $(V, E, \partial_0, \partial_1, F, v_0)$, where V is a finite set of so-called *vertices*, E is a finite set of so-called *edges*, ∂_0 and ∂_1 are functions from E to V—$\partial_0 e$ is the *source* vertex of edge e, $\partial_1 e$ is its *target*—, F is a map from E to *labels* in some fixed set Σ, $v_0 \in V$ is called the *root* of G. We abbreviate the fact that e is an edge with source v_0 and target v_1, and label f, by the notation $e : v_0 \xrightarrow{f} v_1$. We let $root(G)$ be the root of G, and if v is any vertex in V, we let G/v be the graph G with new root v, i.e., $(V, E, \partial_0, \partial_1, F, v)$.

Here are a few natural candidates for the $\rhd$ relation, resembling the immediate superterm relation on terms:

- the *post-edge-erasure* relation $\rhd_+$: if $G = (V, E, \partial_0, \partial_1, F, v_0)$, $G \rhd_+ G'$ if and only if there is an edge $e : v_0 \xrightarrow{f} v_1$ in E such that G' is isomorphic to $(V, E \setminus \{e\}, \partial_0, \partial_1, F, v_1)$—i.e., we erase e and change the root to the target of e;
- the *pre-edge-erasure* relation $\rhd_-$: with the same notations, $G \rhd_- G'$ if and only if G' is isomorphic to $(V, E \setminus \{e\}, \partial_0, \partial_1, F, v_0)$—i.e., we erase e and leave the root on the source of e;
- the *edge-collapsing* relation $\rhd_0$: with the same notations again, $G \rhd_0 G'$ if and only if G' is isomorphic to $(V/\sim, E \setminus \{e\}, e \mapsto |\partial_0 e|_\sim, e \mapsto |\partial_1 e|_\sim, F, |v_0|_\sim)$, where $\sim$ is the equivalence relation $v_0 \sim v_1$, and $|v|_\sim$ is the equivalence class of v under $\sim$—i.e., we equate v_0 and v_1 and remove e;

– the *garbage-collection* relation $\rhd_{gc}$: $G \rhd_{gc} G'$ if and only if G' is obtained from G by removing vertices and edges that are unreachable from $root(G)$.

Any union of any of these relations is well-founded, since all of them decrease the size of graphs, measured as the number of vertices plus the number of edges. Define the reflexive transitive closure $\sqsupseteq_{minor}$ of $\rhd_+ \cup \rhd_- \cup \rhd_0 \cup \rhd_{gc}$. It is natural to say that G' is a *minor* of G whenever $G \sqsupseteq_{minor} G'$ [18], as then G' is obtained by taking a subgraph of some graph obtained from G by collapsing edges.

For every graph G, let $top(G)$ be the multiset of all pairs $(f, G/v_1)$, where e : $root(G) \xrightarrow{f} v_1$ ranges over edges in G with source $root(G)$. We may then define an analogue of the recursive path ordering by the construction of Remark 4 again. Take any precedence $\succ$ on Σ, let $\rhd$ be any union of relations $\rhd_+$, $\rhd_-$, $\rhd_0$, $\rhd_{gc}$, and define $\gg_R$ by, say, $G \gg_R G'$ if and only if $top(G)((\succ, R)_{lex})_{mul} top(G')$. Condition (*iii*) is trivial. Condition (*iv*), in fact (*v*) (Remark 6) also holds. So Theorem 1 allows us to conclude that $>$ is well-founded.

When $\rhd$ is exactly the minor relation $\sqsupseteq_{minor}$, another plausible line of proof to show that $>$ is well-founded would have been by adapting Dershowitz' proof of the termination of the recursive path ordering [3], replacing any use of Kruskal's Theorem by a suitable variant of Robertson and Seymour's Theorem ([18], p. 305). The latter indeed states that (non-rooted, non-labeled) graphs under the embedding-by-minors ordering are well-quasi-ordered. However, the proof of Robertson and Seymour's Theorem is considerably more involved than that of Theorem 1. Moreover, there is no hope of using it to establish that $>$ is well-founded when $\rhd$ is not $\sqsupseteq_{minor}$.

Remark 8. Here Tarski's Fixpoint Theorem is needed in Remark 4. Contrarily to Section 3, there is no unique relation $>$ such that $s > t$ if and only if (*i*) or (*ii*) holds, since graphs may contain loops. Choosing the greatest allows us to get a comparison predicate $>$ that makes the most pairs of graphs comparable. We conjecture that simple loop-checking mechanisms will provide an algorithm for deciding $>$.

Remark 9. It is immediate to adapt the above definitions to non-oriented graphs, where edges are just unordered pairs $\{v_0, v_1\}$ of distinct edges.

Remark 10. It is easy to extend the above definitions to ordered *multigraphs*, where labeled edges are rewrite rules of the form $f(v_1, \ldots, v_n) \leftarrow v_0$, where v_0, v_1, …, v_n are vertices and the label f is an n-ary function symbol. (This generalizes edges $v_0 \xrightarrow{f} v_1$ when f is unary.) We may let $top(G)$, for any multigraph G, be the multiset of all $f(v_1, \ldots, v_n)$ such that $f(v_1, \ldots, v_n) \leftarrow v_0$ is an edge in G with $v_0 = root(G)$, and define $\gg_R$ by comparing such multisets by the multiset extension of the lexicographic product of a precedence $\succ$ on function symbols and R, n times, as in the lexicographic path ordering. (It is of course possible to have function symbols with multiset status, where arguments $v_1, \ldots, v_n$ are compared by the multiset extension of R, and in general to use any kind of extraction and termination function as in [5].)

Such multigraphs are exactly non-deterministic finite tree automata with one final state (the root) [9]; in this context the edge $f(v_1, \ldots, v_n) \leftarrow v_0$ is usually written $f(v_1, \ldots, v_n) \to v_0$. The $>$ relations might have applications in showing that certain sequences of tree automata are ultimately stationary, as needed in building widenings [2] in tree automata-based abstract interpretations such as [17, 11].

Remark 11. Multigraphs where, for each vertex v_0, there is exactly one edge of the form $f(v_1, \ldots, v_n) \leftarrow v_0$ are exactly regular infinite terms, as used in Prolog for example. The construction of this section therefore provides well-founded relations that may be of use in extending completion and narrowing-based automated deduction tools to the case of regular infinite terms.

5 Higher-order path orderings

Theorem 1 seems to be insufficient to show that every simply-typed λ-term terminates. While the proof of Theorem 1 proceeds by showing that every $s \in T$ is in SN directly, in the classical strong normalization proof of the simply-typed λ-calculus [10] one shows that for every $s \in T$, $s\sigma$ is *reducible* of its type, where σ is any substitution mapping variables to reducible terms of the correct types, and reducibility is a new property that implies termination. Trying to integrate these notions into Theorem 1, we obtain the following. (For the sake of comparison, we have taken similar numbering conventions as in Section 2, with a $\circ$ superscript; e.g., condition (*i*) becomes (*i*)$^\circ$.)

Let T be any set, and $\triangleright$, $\blacktriangleright$, $>$ and $\gg$ be four binary relations on T. Assume that:

Property 2. For every $s, t \in T$, if $s > t$ then either:

(*i*)$^\circ$ for some $u \in T$, $s \blacktriangleright u$ and $u \geq t$;
(*ii*)$^\circ$ or $s \gg t$ and for every $u \triangleleft t$, either $s > u$ or for some $v \triangleleft s$, $v \geq u$.

Remark 12. Compared with Property 1, the main difference is the use of $\blacktriangleright$ instead of $\triangleright$ in the first alternative. In general we will have $\blacktriangleright \supseteq \triangleright$; in the typed λ-calculus for example, $\blacktriangleright$ will be the union of the immediate superterm relation and of β-contraction at the top. The added complication in (*ii*)$^\circ$ is inspired from [13].

Theorem 2. *Let $\triangleright^\circ$ be any binary relation on T. Let SN° be the set of all $s \in T$ that are accessible in $> \cup \triangleright^\circ$, $\overline{SN}^\circ \hat{=} \{s \in T | \forall u (\triangleleft \cup \triangleleft^\circ) s \cdot u \in SN^\circ\}$, $\underline{SN}^\circ \hat{=} \{s \in T | \text{if } s \in \overline{SN}^\circ \text{ then } s \in SN^\circ\}$.*

Let S be some set, $\sigma_0 \in S$, and $\_ \cdot \_$ be an (infix) map from $T \times S$ to T. Assume:

(iii)$^\circ$ $\triangleright$ is well-founded on T;
(iv)$^\circ$ for every $s \in T$, if for every $u \triangleleft s$, for every $\sigma \in S$, $u \cdot \sigma$ is in SN°, then for every $\sigma \in S$, $\underline{SN}^\circ$ bars $s \cdot \sigma$ in $\gg$;
(xii) for every $s \in T$, $s \cdot \sigma_0 = s$;
(xiii) for every $s \in T$, $\sigma \in S$, either $s \cdot \sigma \in SN^\circ$ or for every $u (\triangleleft \cup \triangleleft^\circ) s \cdot \sigma$, $u \in SN^\circ$ or there is a $v \triangleleft s$ and $\sigma' \in S$ such that $v \cdot \sigma' (\geq \cup \triangleright^\circ) u$;
(xiv) for every $s', t, u \in T$, if $s' \gg t \triangleright^\circ u$, then $t \triangleright u$ or $s' \triangleright^\circ v \geq t$ for some $v \in T$;
(xv) for every $s', u \in T$, if $s' \blacktriangleright u$ and for every $v (\triangleleft \cup \triangleleft^\circ) s'$, v is in SN°, then u is in SN°.

Then $T = SN^\circ$. In particular, $>$ is well-founded.

We prove Theorem 2 shortly. The proof is very similar to that of Theorem 1, and the reader is invited to proceed directly to applications following the proof. Meanwhile, observe that Theorem 1 is the special case of Theorem 2 where $\blacktriangleright \hat{=} \rhd$, S is any one-element set $\{\sigma_0\}$, $s \cdot \sigma_0 \hat{=} s$, and $\rhd^\circ$ is the empty relation.

Proof. We show that for every $s \in T$, for every $\sigma \in S$, $s \cdot \sigma$ is in SN°. Using (*xii*) and $\sigma \hat{=} \sigma_0$, it will follow that $s \in SN^\circ$. This is by induction on $\rhd$, which is legal by (*iii*)$^\circ$. So the following induction hypothesis is available:

$$\text{For every } u \lhd s, \text{ for every } \sigma \in S, u \cdot \sigma \in SN^\circ \tag{3°}$$

Fix σ: we wish to show $s \cdot \sigma \in SN^\circ$. We claim that this is entailed by: $(*)$ $s \cdot \sigma \in \underline{SN}^\circ$. Indeed, by (*xiii*), either $s \cdot \sigma \in SN^\circ$ and we are done, or for every $u(\lhd \cup \lhd^\circ)s \cdot \sigma$, $u \in SN^\circ$, or there is a $v \lhd s$ and $\sigma' \in S$ such that $v \cdot \sigma'(\geq \cup \rhd^\circ)u$; in the latter case $v \cdot \sigma' \in SN^\circ$ by (3)$^\circ$, so $u \in SN^\circ$ again: since every $u(\lhd \cup \lhd^\circ)s \cdot \sigma$ is in SN°, $(*)$ indeed implies $s \cdot \sigma \in SN^\circ$, by the definition of $\underline{SN}^\circ$.

It remains to show $(*)$. By (3)$^\circ$ and (*iv*)$^\circ$, $\underline{SN}^\circ$ bars $s \cdot \sigma$ in $\gg$. We show that $s \cdot \sigma \in \underline{SN}^\circ$ by showing that any s' barred by $\underline{SN}^\circ$ in $\gg$ is in fact in $\underline{SN}^\circ$, by bar induction. The base case is obvious. In the inductive case, assume that for every t such that $s' \gg t$, $t \in \underline{SN}^\circ$, and prove $s \in \underline{SN}^\circ$. Expanding the definition of $\underline{SN}^\circ$, we must show that $s' \in SN^\circ$ under the assumptions:

$$\text{For every } t, \text{ if } s' \gg t \text{ and every } u(\lhd \cup \lhd^\circ)t \text{ is in } SN^\circ, \text{ then } t \in SN^\circ \tag{4°}$$

$$\text{For every } u(\lhd \cup \lhd^\circ)s', u \in SN^\circ \tag{5°}$$

To show that $s' \in SN^\circ$, it suffices to show that every t such that $s'(> \cup \rhd^\circ)t$ is in SN°, which we show by induction on $\rhd$. So the following induction hypothesis is available:

$$\text{For every } u \lhd t, \text{ if } s'(> \cup \rhd^\circ)u \text{ then } u \in SN^\circ \tag{6°}$$

Since $s'(> \cup \rhd^\circ)t$, we distinguish three cases, two when $s' > t$, one when $s' \rhd^\circ t$:

- (*i*)$^\circ$ For some $u \blacktriangleleft s'$, $u \geq t$. By (*xv*) and (5)$^\circ$, $u \in SN^\circ$, so $t \in SN^\circ$.
- (*ii*)$^\circ$ Or $s' \gg t$ and for every $u \lhd t$, either $s' > u$ or $v \geq u$ for some $v \lhd s'$. For each $u \lhd t$, if $s' > u$ then by (6)$^\circ$ $u \in SN^\circ$; and if $s' \rhd v \geq u$, then by (5)$^\circ$ $v \in SN^\circ$ so $u \in SN^\circ$ again. To sum up: $(\dagger)$ for every $u \lhd t$, $u \in SN^\circ$. Moreover, for each $u \lhd^\circ t$, by (*xiv*) either $t \rhd u$ or $s' \rhd^\circ v \geq t$ for some v; if $t \rhd u$, by $(\dagger)$ $u \in SN^\circ$, and if $s' \rhd^\circ v \geq t$, then $v \in SN^\circ$ by (5)$^\circ$, so $t \in SN^\circ$, so $u \in SN^\circ$ since $t \rhd^\circ u$. Summing up and combining with $(\dagger)$, we get: for every $u(\lhd \cup \lhd^\circ)t$, $u \in SN^\circ$. By (4)$^\circ$ it obtains $t \in SN^\circ$.
- Or $s' \rhd^\circ t$. Then $t \in SN^\circ$ by (5)$^\circ$. □

Higher-order recursive path orderings. Theorem 2 entails that Jouannaud and Rubio's *higher-order path ordering* [13, 14] is well-founded. We also take the opportunity to generalize it.

Let $\mathcal{T}$ be an algebra of so-called *types*, including a $\rightarrow$ binary infix operator. That is, whenever τ_1 and τ_2 are types, so is $\tau_1 \rightarrow \tau_2$. Let $>_\mathcal{T}$ be a well-founded ordering on types such that $(\tau_1 \to \tau_2) >_{\mathcal{T}} \tau_1$ and $(\tau_1 \to \tau_2) >_{\mathcal{T}} \tau_2$. A *signature* Σ is any map from so-called *function symbols* f, g, ..., to *arities* $\tau_1, \ldots, \tau_n \Rightarrow \tau$, where $\tau_1, \ldots, \tau_n$, τ are types in $\mathcal{T}$. For each type $\tau \in \mathcal{T}$, let $\mathcal{X}_\tau$ be pairwise disjoint countably infinite sets of so-called *variables of type* τ x_τ, y_τ, ... We write x, y, ..., instead of the latter when the types are clear from context.

Recall that the language of *algebraic* λ*-terms* over Σ is the smallest collection of sets Λ^Σ_τ when τ ranges over types in $\mathcal{T}$, such that $\mathcal{X}_\tau \subseteq \Lambda^\Sigma_\tau$, such that $MN \in \Lambda^\Sigma_{\tau_2}$ whenever $M \in \Lambda^\Sigma_{\tau_1 \to \tau_2}$ and $N \in \Lambda^\Sigma_{\tau_1}$, such that $\lambda x_{\tau_1} \cdot M \in \Lambda^\Sigma_{\tau_1 \to \tau_2}$ whenever $M \in \Lambda^\Sigma_{\tau_2}$, and such that $f(M_1, \ldots, M_n) \in \Lambda^\Sigma_\tau$ whenever $f \in \operatorname{dom} \Sigma$, $\Sigma(f) = \tau_1, \ldots, \tau_n \Rightarrow \tau$, and $M_1 \in \Lambda^\Sigma_{\tau_1}$, ..., $M_n \in \Lambda^\Sigma_{\tau_n}$. We drop type subscripts when irrelevant or clear from context. We also consider that any two α-equivalent terms are equated, i.e., we reason on the set T of equivalence classes of terms in $\Lambda^\Sigma \hat{=} \bigcup_{\tau \in \mathcal{T}} \Lambda^\Sigma_\tau$ modulo α-renaming. By abuse of language, we shall say that $M >_{\mathcal{T}} N$ when $M \in \Lambda^\Sigma_\tau$, $N \in \Lambda^\Sigma_{\tau'}$ and $\tau >_{\mathcal{T}} \tau'$.

Let us build a relation $>$ on typed λ-terms by mimicking Remark 4:

Definition 1. *Let $\rhd$ be the immediate subterm relation (in particular $\lambda x_{\tau_1} \cdot M \rhd M[x := y_{\tau_1}]$ for every variable y_{τ_1}, where $M[x := N]$ denotes the capture-avoiding substitution of x for N in M). Let β be the smallest relation such that $(\lambda x \cdot M)N \; \beta \; M[x := N]$, and $\blacktriangleright$ be $\rhd \cup \beta$. Let $R \mapsto \gg_R$ be given and monotonic, and define $>$ as the largest binary relation such that $s > t$ implies $s \geq_{\mathcal{T}} t$ and also either (i)$^\circ$ or (ii)$^\circ$, where $\gg$ denotes $\gg_>$.*

Clearly $>$ satisfies Property 2. Moreover $s > t$ implies $s \geq_{\mathcal{T}} t$, by construction.

Remark 13. We get Jouannaud and Rubio's horpo [14] by using the following definition for $\gg_R$. Split function symbols in $\operatorname{dom} \Sigma$ into symbols with *multiset status* and symbols with *lexicographic status*. Let $\succ$ be any strict ordering on $\operatorname{dom} \Sigma$. Let $M \gg_R N$ if and only if either:

1. $M = f(M_1, \ldots, M_m)$, $N = g(N_1, \ldots, N_n)$ $(f, g \in \operatorname{dom} \Sigma)$ and $f \succ g$, or $f = g$ has multiset status and $\{M_1, \ldots, M_m\} \; R_{mul} \; \{N_1, \ldots, N_n\}$, or $f = g$ has lexicographic status, $m = n$, and $(M_1, \ldots, M_n) \; R_{lex} \; (N_1, \ldots, N_n)$;
2. or $M = f(M_1, \ldots, M_m)$, $N = N_1 N_2$ $(f \in \operatorname{dom} \Sigma)$;
3. or $M = M_1 M_2$, $N = N_1 N_2$ and $\{M_1, M_2\} \; R_{mul} \; \{N_1, N_2\}$;
4. or $M = \lambda x {\cdot} M_1$, $N = \lambda x {\cdot} N_1$ (where x is the same on both sides, up to α-renaming) and $M_1 \; R \; N_1$.

To apply Theorem 2, first define $M \rhd^\circ N$ by induction on the type of M, ordered by $>_{\mathcal{T}}$, as follows. $M \rhd^\circ N$ if only if M is an *abstraction* $\lambda x_{\tau_1} \cdot M_1$, of type $\tau_1 \to \tau_2$, and $N = M_1[x := N_1]$ for some N_1 in SN° of type τ_1. Note that since $> \cup \rhd^\circ \subseteq \geq_{\mathcal{T}}$, any $(> \cup \rhd^\circ)$-reduction starting from N_1 only involves λ-terms of types $\leq_{\mathcal{T}} \tau_1$, on which $\rhd^\circ$ is already defined by induction hypothesis: therefore "N_1 in SN° of type τ_1" is well-defined by induction hypothesis in the definition of $\rhd^\circ$ for terms M of type $\tau_1 \to \tau_2$.

Remark 14. This construction implies that SN°, the set of accessible terms in $> \cup \rhd^\circ$, is also the set of terms M that are in SN (the set of terms that are strongly normalizing for $>$), and such that, if M has type $\tau_1 \to \tau_2$, then whenever $M >^* \lambda x_{\tau_1} \cdot M_1$, for every $N_1 \in SN^\circ$ of type τ_1, $M_1[x := N_1]$ is in SN° of type τ_2. This is one of the classical definitions of reducibility (a.k.a., computability) [6]. In this sense, Theorem 2 is a reducibility argument, just like [13].

Second, let S be the set of all substitutions mapping variables x_τ to λ-terms in SN° of type τ, let σ_0 be the empty substitution, and let $M \cdot \sigma$ denote application of substitution σ to M. Conditions (*iii*)$^\circ$ and (*xii*) are obvious.

Condition (*xiii*) is justified as follows. Let s be a typed λ-term, $\sigma \in S$. If s is a variable by construction $s \cdot \sigma \in SN^\circ$. Otherwise, consider any $u(\lhd \cup \lhd^\circ) s \cdot \sigma$. If $u \lhd s \cdot \sigma$, then since s is not a variable, u is written $v \cdot \sigma$ for some immediate subterm v of s; i.e., (*xiii*) is proved with $\sigma' \hat{=} \sigma$, $v \cdot \sigma' = u$. If $u \lhd^\circ s \cdot \sigma$, then since s is not a variable, s must be written $\lambda x \cdot M_1$, with $M_1 \cdot \sigma[x := N_1] = u$ for some $N_1 \in SN^\circ$: take $v \hat{=} M_1$, and σ' be the substitution mapping x to N_1 and every other variable y to $y \cdot \sigma$ (taking x outside the domain of σ, by α-renaming).

Condition (*xv*) is justified as follows. Assume that: $(*)$ every $v(\lhd \cup \lhd^\circ)s'$ is in SN°. If $s' \blacktriangleright u$, then either $s' \rhd u$ and (*xv*) is clear; or $s' \ \beta \ u$. If $s' \ \beta \ u$, then s' is of the form $(\lambda x \cdot M)N$ and $u = M[x := N]$. By $(*)$ N is in SN° and $\lambda x \cdot M$ is in SN° too. So $M[x := N] \lhd^\circ \lambda x \cdot M$ is in SN°.

So only conditions (*iv*)$^\circ$ and (*xiv*) are not automatically verified. We get the following corollary to Theorem 2:

Lemma 3. *Let $>$ be as in Definition 1, and $\gg$ be $\gg_>$. Assume that:*

1. *for every λ-term M, if every immediate subterm of M is in SN°, then M is accessible in $\gg_{|\overline{SN}^\circ}$;*
2. *if $M \gg \lambda x \cdot M_1$, then M is of the form $\lambda x \cdot M_0$ and $M_0 \cdot \sigma > M_1 \cdot \sigma$ for every $\sigma \in S$.*

Then $>$ is well-founded.

Proof. Condition (*iv*)$^\circ$ follows from 1 by the same argument as in Remark 6. Let us prove (*xiv*). Assume $s' \gg t \rhd^\circ u$. By definition t must be of the form $\lambda x \cdot M_1$ and $u = M_1[x := N_1]$ with $N_1 \in SN^\circ$. By 2, $s' = \lambda x \cdot M_0$ and $M_0[x := N_1] > M_1[x := N_1]$. Take $v \hat{=} M_0[x := N_1]$, then $s' \rhd^\circ v > u$. $\square$

Lemma 1 still holds in the higher-order case, provided the following condition is added: whenever $s \ R \ t$ and $f(\ldots, s, \ldots)$ and $f(\ldots, t, \ldots)$ are well-typed, then $f(\ldots, s, \ldots) \geq_{\mathcal{T}} f(\ldots, t, \ldots)$; and provided in the notation $f(\ldots, s, \ldots)$, f is taken to be any function symbol, or application, or λ-abstraction. Lemma 2 holds without modification. It follows that Jouannaud and Rubio's horpo (Remark 13) is well-founded, monotonic and stable. By construction, it also includes β-reduction.

Remark 15. Lemma 3 can be seen as a suitable generalization of Ferreira and Zantema's result to the case of higher-order rewrite relations. Note that condition 2, which basically says that any term that is $\gg$ an abstraction must also be an abstraction, is fundamental. Jouannaud and Rubio's horpo in addition requires applications to be $\ll$ any term of the form $f(\ldots)$ with $f \in \Sigma$, and abstraction cannot be $\gg$ any non-abstraction term; these conditions are simply not needed.

6 Conclusion

We hope to have demonstrated that Theorem 1 and Theorem 2 have very general scopes. In term algebras, we retrieve all variants of path orderings, as well as the Knuth-Bendix orderings. The scope of Theorem 1, and naturally of Theorem 2 as well, exceeds term algebras, and we have sketched a few path ordering-like constructions on graphs, automata, and infinite terms. Theorem 2 then provides a general extension of Jouannaud and Rubio's higher-order path ordering.

There are at least two points that deserve further study. First, the proofs of Theorem 1 and Theorem 2 are intuitionistic, and therefore give rise to algorithms that might be used to implement reduction machines and study the complexity of reductions. Second, because Theorem 1 and Theorem 2 deal with general binary relations, we believe that a deeper understanding of the duality of [16] and of possible connections with Theorem 1 would be profitable. It has been pointed out by an anonymous referee that F. Veltman has produced a constructive version of Kruskal's Theorem, which may be useful in this endeavor.

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