Blocking a transition in a Free Choice net
and what it tells about its throughput*

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Abstract

In a live and bounded Free Choice Petri net, pick a non-conflicting transition. Then there exists a unique reachable marking in which no transition is enabled except the selected one. For a routed live and bounded Free Choice net, this property is true for any transition of the net. Consider now a live and bounded stochastic routed Free Choice net, and assume that the routings and the firing times are independent and identically distributed. Using the above results, we prove the existence of asymptotic firing throughputs for all transitions in the net. Furthermore the vector of the throughputs at the different transitions is explicitly computable up to a multiplicative constant.

Keywords: Petri net, Free Choice net, routed Petri net, stochastic Petri net, stability, monotone-separable framework.

1 Introduction

The paper is made of three parts, each of which considers a different kind of Petri nets. In the first part, we look at *classical* untimed Petri nets as studied in [18, 25]; more precisely, we study live and bounded *Free Choice nets (FCN)*. Using standard Petri net techniques, we show that after blocking a non-conflicting transition b, there exists a unique reachable marking M_b where no transition can fire but the blocked one. We call M_b the *blocking marking* associated with b.

In the second part, we look at *routed Petri nets*, where each place with several output transitions is equipped with a routing function for the successive tokens entering the place. More precisely, we consider live and bounded routed Free Choice nets with *equitable* routings. In this case, there exists a unique blocking marking for any transition, even a conflicting one. Furthermore all the firing sequences avoiding the blocked transition and leading to the blocking marking have the same Parikh vector (i.e., the same letter content).

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Introducing routings in a Petri net is, in some sense, an impoverishment since it removes the non-determinacy in the evolution: routing resolves all conflicts. On the other hand, it provides the right framework for an important enrichment of the model: the introduction of time.

In the last section, we consider live and bounded timed routed Free Choice nets in a stochastic setting. We assume the routings (at the places with several output transitions) to be random and the firing of a transition to take some random amount of time. The successive routings at a place and the successive firing times of a transition form sequences of i.i.d. r.v. (independent and identically distributed random variables). Using the so-called 'monotone-separable framework' (see [6, 10, 14]), we prove a *first order* limit theorem: each transition in the net fires with an asymptotic rate. The ratio between the rates at two different transitions is explicitly computable and depends only on the routing probabilities and not on the firing times. At the end of Section 5, we briefly discuss two types of extensions: *(i)*- first order results under stationary assumptions for the routings and the firing times; *(ii)*- second order results, that is, the existence of a unique stationary regime for the marking process.

First order results under stationary assumptions for the firing times were already known for the class of unbounded *Single-Input Free Choice nets* [7] (a subclass of FCN) and for bounded and unbounded Jackson networks [5, 8] (a subclass of Single-Input FCN). Here, we consider bounded FCN with general topology, thus generalizing from the Jackson setting and allowing for synchronization and splitting of streams. The probabilistic setting, however, is mainly limited to the i.i.d. case.

2 Preliminaries on Petri Nets

A *Petri net* is a 4-tuple $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, M)$, where $(\mathcal{P}, \mathcal{T}, \mathcal{F})$ is a finite bipartite directed graph with set of nodes $\mathcal{P} \cup \mathcal{T}$ and set of arcs $\mathcal{F} \subset (\mathcal{P} \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{P})$, and where M belongs to $\mathbb{N}^{\mathcal{P}}$. To avoid trivial cases, we assume that the sets $\mathcal{P}$ and $\mathcal{T}$ are non-empty. The elements of $\mathcal{P}$ are called *places*, those of $\mathcal{T}$ *transitions*; an element of $\mathbb{N}^{\mathcal{P}}$ is a *marking*, and M is the *initial marking*. To emphasize the role of the initial marking, we sometimes denote the Petri net $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, M)$ by $(\mathcal{N}, M)$.

We apply the standard terminology of graph theory to Petri nets. We assume throughout (and without loss of generality) all Petri nets considered to be connected.

A Petri net $\mathcal{N}' = (\mathcal{P}', \mathcal{T}', \mathcal{F}', M')$ is a *subnet* of $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, M)$, written $\mathcal{N}' = \mathcal{N}[\mathcal{P}' \cup \mathcal{T}']$, if

$$\mathcal{P}' \subset \mathcal{P}, \mathcal{T}' \subset \mathcal{T}, \mathcal{F}' = \mathcal{F} \cap \big(\ (\mathcal{P}' \times \mathcal{T}') \cup (\mathcal{T}' \times \mathcal{P}') \ \big) \ ,$$

and M' is the restriction of M to $\mathcal{P}'$, i.e. $M' = M|_{\mathcal{N}'}$. If X is a subset of $\mathcal{P} \cup \mathcal{T}$, the *subnet generated by* X is the subnet $\mathcal{N}[X]$. We use the classic graphical representation for Petri nets: circles for places, rectangles for transitions, and tokens for markings; see for example Figure 1. We write $x \to y$ if $(x, y) \in \mathcal{F}$, and denote by

$${}^\bullet x = \{y \ : \ y \to x\}, \quad \text{and} \quad x^\bullet = \{y \ : \ x \to y\} \,,$$

the sets of input/output nodes of a node x. The *incidence matrix* $N \in \{-1, 0, 1\}^{\mathcal{P} \times \mathcal{T}}$ of $\mathcal{N}$ is defined by $N(p, t) = 1$ if $(t \to p, p \not\to t)$, $N(p, t) = -1$ if $(p \to t, t \not\to p)$, and $N(p, t) = 0$ otherwise.

Let $\mathcal{T}^*$ be the free monoid over $\mathcal{T}$, that is, the set of finite words over $\mathcal{T}$ equipped with the concatenation product. We denote the empty word by e. Let $\mathcal{T}^{\mathbb{N}}$ be the set of infinite words over the alphabet $\mathcal{T}$. Consider a (finite or infinite) word u; we denote by $|u|$ its length (in $\mathbb{N} \cup \{\infty\}$) and, for $a \in \mathcal{T}$, by $|u|_a$ the number of occurrences of a in u. The prefix of length k of u ($k \in \mathbb{N}$, $k \leqslant |u|$) is denoted by $u_{[k]}$. Further, let $\vec{u} \in (\mathbb{N} \cup \{\infty\})^{\mathcal{T}}$ denote the *Parikh vector* or *commutative image* of u, that is, $\vec{u} = (|u|_a)_{a \in \mathcal{T}}$.

In a Petri net, the marking evolves with the *firing* of transitions. A transition a is *enabled* in the marking M if for all place p in ${}^\bullet a$, $M(p) > 0$; an enabled transition a can *fire*; the *firing* of a transforms the marking M into $M' = M + N \cdot \vec{a}$, written $M \xrightarrow{a} M'$. We say that a word $u \in \mathcal{T}^*$ is a *firing sequence* of $(\mathcal{N}, M)$ if for all $k \leqslant |u|$, we have $M + N \cdot \vec{u}_{[k]} \geqslant (0, \ldots, 0)$; we say that u transforms M into $M' = M + N \cdot \vec{u}$, in which case we write $M \xrightarrow{u} M'$. An infinite word over $\mathcal{T}$ is an *infinite firing sequence* if all its prefixes are firing sequences. The notation $M \xrightarrow{u}$ means that u is a (infinite) firing sequence of $(\mathcal{N}, M)$. A marking M_2 is *reachable* from a marking M_1 if there exists a firing sequence $u \in \mathcal{T}^*$ such that $M_1 \xrightarrow{u} M_2$. The set of *reachable markings* of $(\mathcal{N}, M)$ is $R(\mathcal{N}, M) = \{M' \ : \ \exists u \in \mathcal{T}^*, M \xrightarrow{u} M'\}$. We write $R(M)$ instead of $R(\mathcal{N}, M)$ when there is no risk of confusion.

The Petri net $(\mathcal{N}, M)$ is *live* if: $\forall M' \in R(M), \forall a \in \mathcal{T}, \exists M'' \in R(M'), M'' \xrightarrow{a}$. A simple consequence of this definition is that a live Petri net admits infinite firing sequences. The Petri net is *bounded* if: $\exists K \in \mathbb{N}, \forall M' \in R(M), \forall p \in \mathcal{P}, M'_p \leqslant K$.

A Petri net $\mathcal{N} = (\mathcal{P}, \mathcal{T}, F, M)$ is a

- *T-net* (or *event graph*, or *marked graph*) if: $\forall p \in \mathcal{P}, \quad |{}^\bullet p| = |p^\bullet| = 1$;
- *S-net* (or *state machine*) if: $\forall q \in \mathcal{T}, \quad |{}^\bullet q| = |q^\bullet| = 1$;
- *Free Choice net* (FCN) if: $\forall (p, q) \in \mathcal{F} \cap (\mathcal{P} \times \mathcal{T}), \quad p^\bullet = \{q\} \ \vee \ {}^\bullet q = \{p\}$.

An equivalent definition for a FCN is: $\forall q_1, q_2 \in \mathcal{T}, q_1 \neq q_2, \ (p \in {}^\bullet q_1 \cap {}^\bullet q_2) \Rightarrow ({}^\bullet q_1 = {}^\bullet q_2 = \{p\})$. Obviously, every T-net is a FCN and every S-net is a FCN as well.
In this paper, we study the class of live and bounded Free Choice nets. The membership of a given Petri net to this class can be checked in polynomial time (in the size of the net), see for instance [18], Chapter 6.

We use the notation $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. We denote by $x \leqslant y$ the coordinate-wise ordering of $\mathbb{R}^k$, and write $x < y$ if $x \leqslant y$ and $x \neq y$.

3 Blocking a Transition in a Free Choice net

Let $(\mathcal{N}, M)$ be a Petri net. A transition a is a *non-conflicting* transition if for all $p \in {}^\bullet a$, $|p^\bullet| = 1$; otherwise a is a *conflicting* transition. We set $R_q(M)$ (resp. $R'_q(M)$) to be the set of markings reachable from M (resp. reachable from M without firing transition q) and in which no transition is enabled except q:

$$\begin{aligned} R_q(M) &= \left\{ M' : \ M' \in R(M), \ \left(\tilde{q} \in \mathcal{T}, \ M' \xrightarrow{\tilde{q}} \ \Rightarrow \ \tilde{q} = q \right) \right\} \\ R'_q(M) &= \left\{ M' : \ M' \in R_q(M), \ \exists \sigma \in (\mathcal{T} - \{q\})^*, M \xrightarrow{\sigma} M' \right\} . \end{aligned} \tag{3.1}$$

As previously, we extend the notation to $R_q(\mathcal{N}, M)$ (resp. $R'_q(\mathcal{N}, M)$) when there is a possibility for ambiguity.

The next theorem is the heart of the article.

Theorem 3.1 (Blocking one transition). *Let $(\mathcal{N}, M_0)$ be a live and bounded Free Choice net. If b is a non-conflicting transition, then there exists a unique reachable marking M_b in which the only enabled transition is b. Furthermore, M_b can be reached from any reachable marking and without firing transition b.*

Using the above notations, the result can be rephrased as: $\forall M \in R(M_0),\ R_b(M) = R'_b(M) = \{M_b\}$. As the proof of Theorem 3.1 is rather long, we postpone it to Appendix A.3.

We call M_b the *blocking marking* associated with b. Note that a blocking marking is a *home state*, meaning that it is reachable from any reachable marking.

Example 3.2. *To illustrate Theorem 3.1, consider the live and bounded Free Choice net represented on the left of Figure 1.*

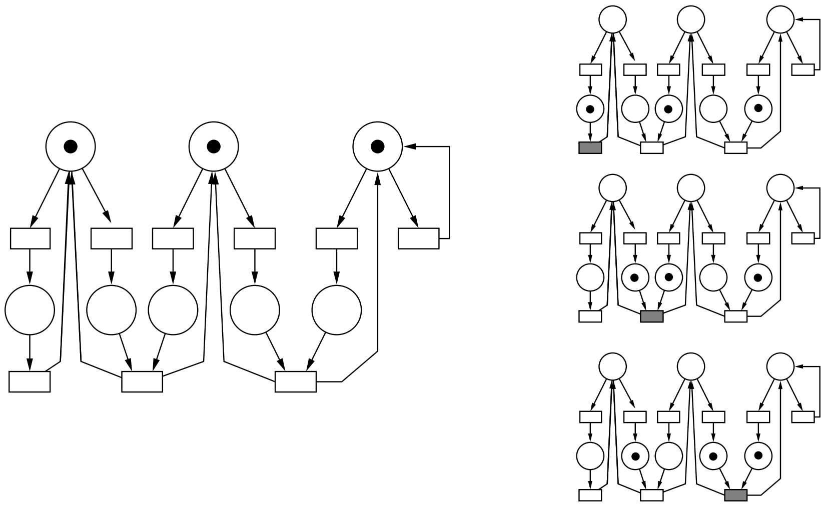


Figure 1: Blocking markings associated with the non-conflicting transitions.

The blocking markings associated with the three non-conflicting transitions have been represented on the right of the figure.

Now the natural question is: do there always exist non-conflicting transitions? The answer is given in the next lemma.

Lemma 3.3. *Let $\mathcal{N}$ be a live and bounded Free Choice net. If $\mathcal{N}$ is not a S-net, then it contains non-conflicting transitions.*

Proof. The net $\mathcal{N}$ is strongly connected (Theorem A.3), hence each node has at least one predecessor and one successor. Due to the Free Choice property, a sufficient condition for a transition a to be non-conflicting is that $|{}^\bullet a| > 1$. Assume that all transitions a are such that $|{}^\bullet a| = 1$. Since $\mathcal{N}$ is not a S-net, there exists at least one transition t such that $|t^\bullet| > 1$. If we have $M \xrightarrow{a} M'$, $a \in \mathcal{T}$, then $\sum_p M'_p = \sum_p M_p + |a^\bullet| - |{}^\bullet a|$. Since $|{}^\bullet a| = 1$ for all a in $\mathcal{T}$, the total number of tokens never decreases. On the other hand, if we have $M \xrightarrow{t} M'$, then $\sum_p M'_p \geqslant \sum_p M_p + 1$. Since the net is live, there exists an infinite firing sequence $\sigma \in \mathcal{T}^{\mathbb{N}}$ such that t occurs an infinite number of times in σ. We deduce that the total number of tokens along the markings reached by σ is unbounded. This is a contradiction. $\square$

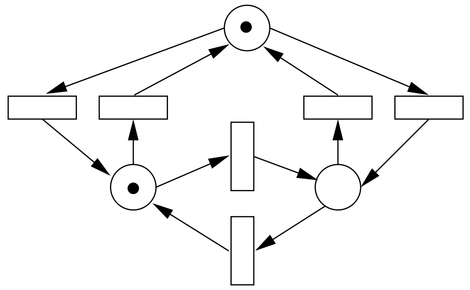
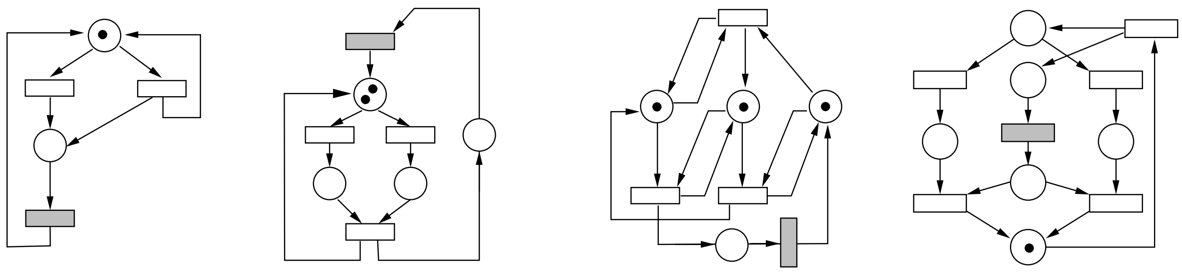


Figure 2: A live and bounded S-net without any non-conflicting transition.

On the other hand, it is possible for an S-net to contain only conflicting transitions. An example is displayed in Figure 2; there exists no marking in which only one transition is enabled.

It is worth noting that none of the three assumptions in Theorem 3.1 (liveness, boundedness, Free Choice property) can be dropped. Figure 3 displays four nets which are respectively non-live, unbounded and not Free Choice for the last two. When blocking the transition in grey in these nets, several blocking markings may be reached. More precisely, for each net in Figure 3, we have $|R_b(M_0)| \geq 2$ and $|R'_b(M_0)| \geq 2$. For the net on the left, we even have $|R_b(M_0)| = |R'_b(M_0)| = \infty$.



Live FCN, unbounded   Bounded FCN, non-live   Live and bounded, not free-choice

Figure 3: Several nets with non-unique blocking markings.

In the following we use Theorem 3.1 in the context of stochastic Petri nets, where the blocking marking is used as a regeneration point. We believe that the result may also be of interest in verification or in fault management, with the blocking of a transition modelling some breakdown in the system.

4 Blocking a Transition in a Routed Free Choice net

In a live and bounded Free Choice net, some but not all transitions lead to a blocking marking, see Theorem 3.1. Furthermore, given any transition (even non-conflicting) there exist in general infinite firing sequences not containing it. This is for instance the case in the net of Figure 1. In the present section, we introduce routed Free Choice nets and we show that there exists a blocking marking associated with any transition and that there is no infinite firing sequence avoiding a given transition.

A *routed Petri net* is a pair $(\mathcal{N}, u)$ where $\mathcal{N}$ is a Petri net (set of places $\mathcal{P}$) and $u = (u_p)_{p \in \mathcal{P}}$, u_p being a function from $\mathbb{N}^*$ to $p^\bullet$. For the places such that $|p^\bullet| \leqslant 1$, the function u_p is trivial. Below, it will be convenient to consider u_p as defined either on all the places or only on the places with several successors, depending on the context. We call u the *routing (function)*. To insist on the value of the initial marking M, we denote the routed Petri net by $(\mathcal{N}, M, u)$.

A routed Petri net $(\mathcal{N}, M, u)$ evolves as a Petri net except for the definition of an *enabled* transition. A transition t is *enabled* in $(\mathcal{N}, u)$ if it is enabled in $\mathcal{N}$ and if in each input place at least one of the tokens currently present is *assigned* to t by u. The assignment is defined as follows: (1) in the initial marking of place p, the number of tokens assigned to transition $t \in p^\bullet$ is equal to $\sum_{i=1}^{M_p} \mathbf{1}_{\{u_p(i)=t\}}$ (where $\mathbf{1}_A$ is the indicator function of A); (2) the n-th token to enter place p during an evolution of the net is assigned to transition $u_p(n+M_p)$, where the numbering of tokens entering p is done according to the "logical time" induced by the firing sequence.

Modulo the new definition of enabling of a transition, the definitions of *firing*, *firing sequence*, *reachable marking*, *liveness*, *boundedness* and *blocking transition* remain unchanged. We also say that a firing or a firing sequence is *compatible with* u. Let $(\mathcal{N}, M, u)$ be a routed Petri net and let us consider $M \xrightarrow{\sigma} M'$; the resulting routed Petri net is $(\mathcal{N}, M', u')$ where the routing u' is defined as follows. In the marking M', the number of tokens of place p assigned to transition $t \in p^\bullet$ is equal to

$$\sum_{i=1}^{M'_p} \mathbf{1}_{\{u'_p(i)=t\}} = \sum_{i=1}^{K} \mathbf{1}_{\{u_p(i)=t\}} - |\sigma|_t, \quad K = M_p + \sum_{t \in {}^\bullet p} |\sigma|_t\,; \tag{4.1}$$

and the n-th token to enter place p is assigned to $u'_p(n+M'_p) = u_p(n + M_p + \sum_{t \in {}^\bullet p} |\sigma|_t)$. For simplicity and with some abuse, we use the notation $(\mathcal{N}, M', u)$ instead of $(\mathcal{N}, M', u')$. We keep or adapt the notations of Section 2. For instance, the reachable markings of $(\mathcal{N}, M', u)$ are denoted by $R(M', u)$ (or $R(\mathcal{N}, M', u)$). We also use the notations $R_b(M, u)$ and $R'_b(M, u)$ for the analogs of the quantities defined in (3.1). For details on the semantics of routed Petri nets, see [19].

Clearly, we have $R(\mathcal{N}, M, u) \subset R(\mathcal{N}, M)$; hence, if $\mathcal{N}$ is bounded, so is $(\mathcal{N}, u)$. The converse is obviously false. The liveness of $\mathcal{N}$ or $(\mathcal{N}, u)$ does not imply the liveness of the other. For instance, the Petri net on the left of Figure 4 is live but its routed version is live only for the routing $ababa \cdots$ (a being the transition on the left and b the one on the right). For the Petri net on the right of the same figure, the routed version is live for the routing $ababa \cdots$ but the (unrouted) net is not live.
We need an additional definition: the routing u is *equitable* if

$$\forall p \in \mathcal{P}, \forall t \in p^\bullet, \quad \sum_{i \in \mathbb{N}^*} \mathbf{1}_{\{u_p(i)=t\}} = \infty\,. \tag{4.2}$$

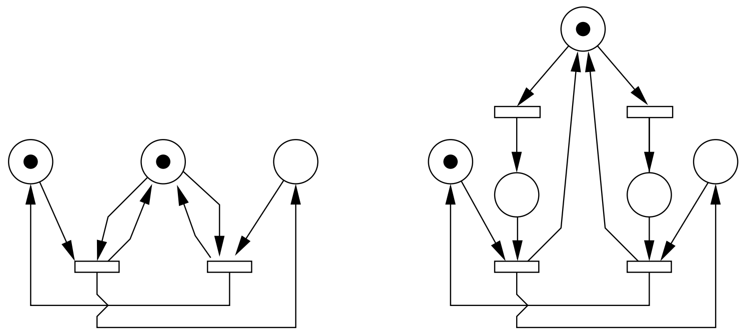


Figure 4: Compare the liveness of the routed and unrouted versions of the above Petri nets.

In words, a place that receives an infinite number of tokens assigns an infinite number of them to each of its output transitions. The next two results establish the relation between the unrouted and routed behaviors of a net.

Lemma 4.1. *Let $\mathcal{N}$ be a Petri net. The following statements are equivalent:*
1. *$(\mathcal{N}, u)$ is bounded for any routing u;*
2. *$\mathcal{N}$ is bounded.*

Proof. Clearly, 2. implies 1. Assume that $(\mathcal{N}, M_0)$ is unbounded. Classically, this implies that there exists $M_1 \in R(M_0)$ and $M_2 \in R(M_1)$ such that $M_2 > M_1$. This is proved using a construction by Karp and Miller, see [21] or Chapter 4 in [26]. Consequently, there exists a sequence of reachable markings $(M_i)_{i \in \mathbb{N}^*}$ and a firing sequence σ such that $M_i \xrightarrow{\sigma} M_{i+1}$ and such that the total number of tokens of M_i is strictly increasing. Let σ_0 be such that $M_0 \xrightarrow{\sigma_0} M_1$ and let τ be the infinite sequence defined by $\tau = \sigma_0 \sigma \sigma \cdots$. Choose a posteriori a routing u compatible with τ. Clearly, $(\mathcal{N}, u)$ is unbounded and we have proved that non-2. implies non-1. $\square$

Lemma 4.2. *Let $\mathcal{N}$ be a Free Choice net. The following propositions are equivalent:*
1. *$(\mathcal{N}, u)$ is live for any equitable routing u;*
2. *$\mathcal{N}$ is live.*

Proof. First note that if $(\mathcal{N}, u)$ is live then clearly u must be equitable. Let us prove that 1. implies 2. Let M_0 be the initial marking and consider $M \in R(\mathcal{N}, M_0)$ and an arbitrary transition q of $\mathcal{N}$. Clearly there exists en equitable routing u such that $M \in R(\mathcal{N}, M_0, u)$. Since $(\mathcal{N}, M_0, u)$ is live, $(\mathcal{N}, M, u)$ is also live and there is a firing sequence of $(\mathcal{N}, M, u)$ which enables q. The same sequence enables q in $(\mathcal{N}, M)$.
Now let us prove that 2. implies 1. We assume that there exists an equitable routing u such that $(\mathcal{N}, u)$ is not live. There thus exists a transition q which is never enabled in $(\mathcal{N}, u)$, after some firing sequence σ. Set $X = \{q\}$. By equitability of the routing u, this implies that ${}^\bullet q$ contains a place p which receives only a finite number of tokens after σ. Then the transitions in ${}^\bullet p$ fire at most a finite number of times after σ. Set $X = X \cup \{p\} \cup {}^\bullet p$. For each one of the new transitions in X, we use the argument first applied to q and repeat the construction recursively. Since the net is finite, this construction terminates and we end up with a set of nodes X. The set $X \cap \mathcal{P}$ is non-empty and a siphon (see Section A.2). By construction, there is a finite firing sequence leading to an empty marking in the siphon $X \cap \mathcal{P}$. We deduce that the siphon cannot

contain an initially marked trap, hence $\mathcal{N}$ cannot be live by Commoner's Theorem A.9 (this is where we need the Free Choice assumption). $\square$

Lemma 4.3. *Let $\mathcal{N}$ be a live and bounded Petri net and let u be an equitable routing. For any infinite firing sequence σ of the routed net $(\mathcal{N}, u)$ and for any transition t, we have $|\sigma|_t = \infty$.*

Proof. We say that a transition q is σ*-live* if $|\sigma|_q = \infty$ and σ*-starved* otherwise. We are going to prove that all transitions are σ-live. Obviously, since σ is infinite, it is not possible for all transitions to be σ-starved. Assume there exists a transition s which is σ-live and a transition t which is σ-starved. Since $\mathcal{N}$ is strongly connected by Theorem A.3, there are places $p_1, \dots, p_n$ and transitions $q_1, \dots, q_{n-1}$ such that $s = q_0 \to p_1 \to q_1 \to \cdots \to q_{n-1} \to p_n \to q_n = t$. There exists an index i such that q_i is σ-live and q_{i+1} is σ-starved. Since u is equitable, an infinite number of tokens going through p_{i+1} are routed towards q_{i+1}. By assumption, q_{i+1} consumes only finitely many of them under σ, which implies that the marking of p_{i+1} is unbounded. This is a contradiction. $\square$

Using the above lemma, we obtain for routed Free Choice nets a stronger version of Theorem 3.1: all transitions yield a blocking marking, provided the routing is equitable.

Theorem 4.4. *Let $(\mathcal{N}, M_0)$ be a live and bounded Free Choice net. For any transition b, there exists a blocking marking M_b such that for every equitable routing u and all $M \in R(M_0, u)$, we have $R_b(M, u) = R'_b(M, u) = \{M_b\}$.*

The proof is postponed to the Appendix, where it will be carried out for a class of nets slightly more general than FCN. Here, we now prove some additional results on routed Petri nets to be used in Section 5.

Lemma 4.5. *Consider a live and bounded routed Free Choice net $(\mathcal{N}, M_0, u)$. Let b be a transition and M_b the associated blocking marking. For any $n \in \mathbb{N}$, there exists a firing sequence σ of $(\mathcal{N}, M_0, u)$ such that $|\sigma|_b = n$ and $M_0 \xrightarrow{\sigma} M_b$. If σ and σ' are firing sequences of $(\mathcal{N}, M_0, u)$ such that $|\sigma|_b = |\sigma'|_b$, $M_0 \xrightarrow{\sigma} M_b$, and $M_0 \xrightarrow{\sigma'} M_b$, then we have $\vec{\sigma} = \vec{\sigma}'$. If τ and σ are firing sequences such that $|\tau|_b \le |\sigma|_b$, and $M_0 \xrightarrow{\sigma} M_b$, then we have $\vec{\tau} \leqslant \vec{\sigma}$.*

Proof. The existence of σ such that $|\sigma|_b = n$ and $M_0 \xrightarrow{\sigma} M_b$ follows by induction from Theorem 4.4.
We give the proof of the remaining points in the case $\sigma \in (\mathcal{T} - \{b\})^*$. The general case can be argued in a similar way. The argument is basically the same as for Part 2. of the proof of Theorem 3.1, see the appendix. Let u_1 and u_2 be two firing sequences of $(\mathcal{N}, M_0, u)$ such that $\vec{u}_1 = \vec{\sigma}, \vec{u}_2 = \vec{\sigma}'$, and with the longest possible common prefix. We set $u_1 = xv_1$ and $u_2 = xv_2$ where x is the common prefix. If $v_1 = v_2 = e$, then obviously $\vec{\sigma} = \vec{\sigma}'$. Assume that $v_1 \neq e$, and let a be the first letter of v_1. Let $\tilde{M}$ be such that $M_0 \xrightarrow{x} \tilde{M}$. Since $|u_1|_a > 0$, we deduce that $a \neq b$. The transition a is enabled in $\tilde{M}$. Furthermore, by definition, a is not enabled in M_b. However, in a routed net, once a transition is enabled, the only way to disable it is by firing it. This implies that the firing sequence v_2 must contain a; so, set $v_2 = yaz$ with $|y|_a = 0$. Since a is enabled in $\tilde{M}$, it follows that ayz is a firing sequence and $\tilde{M} \xrightarrow{ayz} M_b$. To summarize, we have found two firing sequences u_1 and $u'_2 = xayz$ leading to M_b, with respective Parikh vectors $\vec{\sigma}$ and $\vec{\sigma}'$ and with a common prefix at least equal to xa. This is a contradiction.
Now let us consider a firing sequence $\tau \in (\mathcal{T} - \{b\})^*$ and let M' be such that $M_0 \xrightarrow{\tau} M'$. By Theorem 4.4, there exists a firing sequence θ of $(\mathcal{N}, M', u)$ such that $\theta \in (\mathcal{T} - \{b\})^*$ and $M' \xrightarrow{\theta} M_b$. Applying the first part of the proof, we get that $\vec{\tau} + \vec{\theta} = \vec{\sigma}$. $\square$

A *deadlock* is a reachable marking in which no transition is enabled.

Lemma 4.6. *Let* $(\mathcal{N}, M_0, u)$ *be a routed Petri net admitting a deadlock* M_d*. Then* M_d *is the unique deadlock of* $(\mathcal{N}, M_0, u)$*. If* σ *and* σ' *are firing sequences of* $(\mathcal{N}, M_0, u)$ *such that* $M_0 \xrightarrow{\sigma} M_d, M_0 \xrightarrow{\sigma'} M_d$*, then we have* $\vec{\sigma} = \vec{\sigma}'$*. Furthermore if* τ *is a firing sequence of* $(\mathcal{N}, M_0, u)$*, then* $\vec{\tau} \leqslant \vec{\sigma}$*.*

The proof mimics the one of the second point in Lemma 4.5 (which does not require using Theorem 4.4 and is valid for any routed Petri net).

5 Stationarity in Stochastic Routed Free Choice nets

5.1 Stochastic routed Petri net

A *timed routed Petri net* is a routed Petri net with *firing times* associated with transitions. (Here we do not consider *holding times* associated with places for simplicity. As usual, this restriction is done without loss of generality. Indeed a timed Petri net with firing and holding times can be transformed into an expanded Petri net with only firing times.) The firing semantic is defined as follows. The timed evolution of the marking starts at instant 0 in the initial marking. Let a be a transition with firing time $\sigma_a \in \mathbb{R}_+$ and which becomes enabled at instant t. Then,

1. at instant t, the firing of a *begins*: one token is *frozen* in each of the input places of a. A frozen token can not get involved in any other enabling or firing;
2. at instant $t + \sigma_a$, the firing of a *ends*: the frozen tokens are removed and one token is added in each of the output places of a.

Obviously, this semantic makes sense only if a given token can not enable several transitions simultaneously. In a routed Petri net, this is the case. With this semantic, an enabled transition immediately starts its firing, we say that the evolution is *as soon as possible*. Timed routed Petri nets were first studied in [3].

The firing times at a given transition may not be the same from firing to firing. In general, the firing times at transition a are given by a function $\sigma_a : \mathbb{N}^* \to \mathbb{R}_+$, the real $\sigma_a(n)$ is the firing time for the n-th firing at transition a. The numbering of the firings is done according to the instant of initiation of the firing (the "physical time"). Let u be the routing; recall that $u_p(n)$ is the transition to which u assigns the n-th token to enter place p. Here again, we assume that the numbering of the tokens entering place p is done according to the "physical time" (as opposed to the untimed case, where the numbering was done according to the "logical time" induced by the underlying firing sequence).

Let $(\Omega, \mathcal{S}, P)$ be a probability space. From now on, all random variables are defined with respect to this space. A *stochastic routed Petri net* is a timed routed Petri net where the routings and the firing times are random variables. More precisely, a *stochastic routed Petri net* is a quadruple $(\mathcal{N}, M, u, \sigma)$ where $(\mathcal{N}, M)$ is a Petri net (places $\mathcal{P}$ and transitions $\mathcal{T}$), where $u = [(u_p(n))_{n\in\mathbb{N}^*}, p \in \mathcal{P}]$ are the routing sequences, and where $\sigma = [(\sigma_a(n))_{n\in\mathbb{N}^*}, a \in \mathcal{T}]$ are the firing time sequences. Furthermore, we assume that

- for each place p, $(u_p(n))_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. r.v. (the so-called *Bernoulli routing*);
- for each transition a, $(\sigma_a(n))_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. r.v. and $E(\sigma_a(1)) < \infty$;

- the sequences $(u_p(n))_{n\in\mathbb{N}^*}$ and $(\sigma_a(n))_{n\in\mathbb{N}^*}$ are mutually independent.

For details and other approaches concerning stochastic Petri nets, see for instance [1, 12].

By the Borel-Cantelli Lemma, we have for any place p and any transition $t \in p^\bullet$:

$$P\{\sum_{i=1}^{+\infty} \mathbf{1}_{\{u_p(i)=t\}} = +\infty\ \} = \begin{cases} 1 & \text{if } P\{u_p(1)=t\} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $\forall p \in \mathcal{P}, \forall t \in p^\bullet,\ P\{u_p(1)=t\} > 0$, the random routing is said to be *equitable* (since it is equitable in the sense of (4.2) for almost all $\omega \in \Omega$).

5.2 Existence of asymptotic throughputs

This section is devoted to the proof of the following result.

Theorem 5.1. *Consider a live and bounded stochastic routed Free Choice net with an equitable routing. For any transition b, there exists a constant $\gamma_b \in \mathbb{R}_+$ such that*

$$\lim_{n\to\infty} \frac{X_b(n)}{n} = \lim_{t\to\infty} \frac{t}{\mathcal{X}_b(t)} = \gamma_b \quad \textit{a.s. and in } L_1\ ,$$

where $X_b(n), n \in \mathbb{N}^*$, is the instant of completion of the n-th firing at transition b and where $\mathcal{X}_b(t), t \in \mathbb{R}_+$, is the number of firings completed at transition b up to time t.

The quantity γ_b^{-1} is the asymptotic *throughput* at transition b. To prove Theorem 5.1, we need some preparations.

Let $\mathfrak{N} = (\mathcal{N}, M, u, \sigma)$ with $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, M)$ be a *live and bounded stochastic routed Free Choice net with an equitable random routing* (SRFC in the following). We select a transition b and we denote by M_b the associated blocking marking.

Lemma 5.2. *Assume that $\sigma_b(n) = +\infty$ for $n \in \mathbb{N}^*$, the other firing times and the routings being unchanged. Let τ be the first instant of the evolution when the marking reaches M_b ($\tau = \infty$ if M_b is never attained). The r.v. τ is a.s. finite and integrable.*

Proof. According to Theorem 4.4, we have $R_b'(\mathcal{N}, M, u) = \{M_b\}$ which means precisely that there exists a firing sequence x such that $|x|_b = 0$ and $M \xrightarrow{x} M_b$. Define

$$T = \sum_{a\in\mathcal{T}-\{b\}} \sum_{i=1}^{|x|_a} \sigma_a(i)\ .$$

Let us consider the timed evolution of the Petri net and let v be the firing sequence up to a given instant $t \in \mathbb{R}_+$. Since $\sigma_b(n) = +\infty$, we have $|v|_b = 0$. According to Lemma 4.5, this implies that $\vec{v} \leq \vec{x}$. Due to the as soon as possible firing semantics, $\mathfrak{N}$ is non-idling: at all instant at least one transition is firing. Furthermore, if the marking is different from M_b, there is always at least one transition other than b which is firing. We deduce that if $t \geq T$, then we must have $\vec{v} = \vec{x}$; in other words, we have $\tau \leq T$. This shows in particular that τ is a.s. finite.
To prove that τ is integrable, we need a further argument. A consequence of Lemma 4.5 is that $\vec{x}$ depends only on the routings and not on the timings in the SRFC. This implies in particular that the r.v. $\vec{x}$ is independent of the random sequences $(\sigma_a(n))_n, a \in \mathcal{T}$, and hence

$$E(T) \quad = \sum_{a\in\mathcal{T}-\{b\}} E(|x|_a)E(\sigma_a(1)). \tag{5.1}$$

We specialize the SRFC to the case where all the firing times are exponentially distributed with parameter 1, i.e. $P\{\sigma_a(1) > z\} = \exp(-z)$. Let M_t be the marking at instant t. The process $(M_t)_t$ is a continuous time Markov chain with state space $R(M)$. Let T_n be the instants of jumps of M_t and set $M_n = M_{T_n}$. Then $(M_n)_n$ is a discrete time Markov chain and $\sum_a |x|_a$ is precisely the time needed by the chain to reach the marking M_b starting from M. Using elementary Markov chain theory, we get that $E(\sum_a |x|_a) < \infty$. Using (5.1), this yields the integrability of τ. □

From now on, we assume without loss of generality that $M = M_b$, that is, the initial marking is the blocking marking. Let us define

$$K = \max\{k \ : \ M \xrightarrow{b^k}\} . \tag{5.2}$$

By construction, we have $K \geq 1$. We now introduce an auxiliary construction, the *Open*

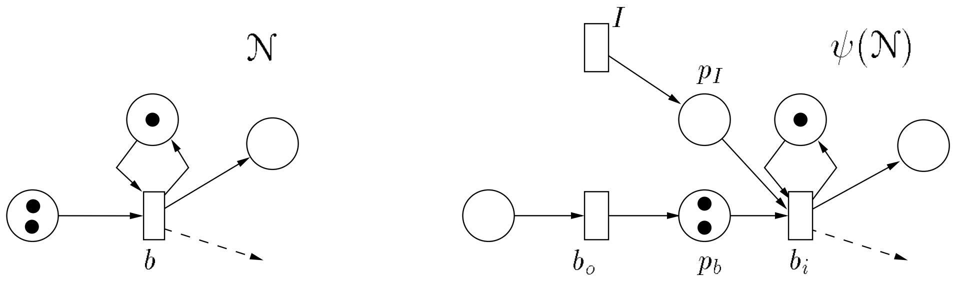


Figure 5: Open Expansion of a Free Choice net.

Expansion of an SRFC, which is characteriesd by an input transition I without input places and a splitting of b into an immediate transition b_o and a transition b_i that inherits the firing duration of b.

Definition 5.3. *The* Open Expansion *associated with* $\mathfrak{N}$ *and* b *is the stochastic routed Free Choice net* $\psi(\mathfrak{N}) = (\psi(\mathcal{N}), \psi(M), \psi(u), \psi(\sigma))$ *with* $\psi(\mathcal{N}) = (\psi(\mathcal{P}), \psi(\mathcal{T}), \psi(\mathcal{F}), \psi(M))$ *and*

- $\psi(\mathcal{P}) \quad = \quad \mathcal{P} \ \cup \ \{p_b, p_I\}$
- $\psi(\mathcal{T}) \quad = \quad (\mathcal{T} - \{b\}) \ \cup \ \{I, b_i, b_o\}$
- $$\begin{aligned}\psi(\mathcal{F}) \quad = \quad & (\mathcal{F} - \{(p,b) \in \mathcal{F}, (b,p) \in \mathcal{F}\}) \ \cup \\ & \{(p, b_o) \ : \ (p,b) \in \mathcal{F}, (b,p) \notin \mathcal{F}\} \ \cup \ \{(b_i, p) \ : \ (b,p) \in \mathcal{F}, (p,b) \notin \mathcal{F}\} \ \cup \\ & \{(b_i, p), (p, b_i) \ : (p,b) \in \mathcal{F}, (b,p) \in \mathcal{F}\} \ \cup \ \{(I, p_I), (p_I, b_i), (b_o, p_b), (p_b, b_i)\}\end{aligned}$$
- $$\psi(M)_p \quad = \quad \begin{cases} M_p & : \ p \in \mathcal{P} - ({}^\bullet b) \\ M_p - K + K\mathbf{1}_{\{p \in b^\bullet\}} & : \ p \in ({}^\bullet b) \\ K & : \ p = p_b \\ 0 & : \ p = p_I \end{cases}$$
- $$\psi(\sigma)_a(n) \quad = \quad \begin{cases} \sigma_a(n) & : \ a \in (\mathcal{T} - \{b\}) \\ \sigma_b(n) & : \ a = b_i \\ 0 & : \ a = b_o \end{cases}$$
- $\psi(u)_p(n) \quad = \quad u_p(n)$.

The construction is illustrated in Figure 5. Note that $\psi(\mathcal{N})$ is neither live nor bounded. The marking $\psi(M)$ is a deadlock for the Petri net $\psi(\mathcal{N})$ (no transition is enabled).
In the definition of $\psi(\mathfrak{N})$, we have not specified the value of $(\sigma_I(n))_n$. This is on purpose. Assume first that transition I fires an infinite number of times at instant 0 ($\forall n, \sigma_I(n) = 0$). Then this *saturated* version of the net $\psi(\mathfrak{N})$ behaves exactly as $\mathfrak{N}$ (the firing times of $t \in \mathcal{T} - \{b\}$ are the same in the two nets and the firing times of b_i in $\psi(\mathfrak{N})$ are equal to the firing times of b in $\mathfrak{N}$). We are going to use this remark below.
Assume now that I fires a finite number of times at positive instants. Then we can view $\psi(\mathfrak{N})$ as a mapping of the instants of (completion of) firings of I into the instants of (completion of) firings of b_o. Let us make this point more precise.

Let $\mathcal{B}$ be the borelian σ-field of $\mathbb{R}_+$. A *(positive finite) counting measure* is a measure a on $(\mathbb{R}_+, \mathcal{B})$ such that $a(C) \in \mathbb{N}$ for all $C \in \mathcal{B}$. For instance, $a([0,T])$ can be interpreted as the number of events of a certain type occurring between times 0 and T; this will be used below. We denote by $\mathcal{M}_f$ the set of counting measures. Given a set E, we denote by $\mathcal{M}_f(E)$ the set of all couples (m, ξ) where $m \in \mathcal{M}_f$ and $\xi = (\xi_1, \ldots, \xi_k)$, $\xi_i \in E, k = m(\mathbb{R}_+)$. The elements of $\mathcal{M}_f(E)$ are called *marked counting measures*.

Consider $\psi(\mathfrak{N})_{[1]} = \psi(\mathfrak{N})$. Assume that transition I fires only once. According to Lemma 4.5, transition b_o will also fire once, and according to Lemma 5.2, the net will end up in the marking $\psi(M)$ after an a.s. finite time τ. We define the random vector

$$\xi_1 = [(u_p(1), \ldots, u_p(k_p)), p \in \psi(\mathcal{P});\ (\sigma_a(1), \ldots, \sigma_a(n_a)), a \in \psi(\mathcal{T}) - \{I\}],$$

where n_a is the number of firings of transition a up to time τ, and k_p is the number of tokens which have been routed at place p up to time τ. Let us set $\psi(u)_{[2]} = [(\psi(u)_p(k + k_p))_{k\in\mathbb{N}^*}, p \in \psi(\mathcal{P})]$ and $\psi(\sigma)_{[2]} = [(\psi(\sigma)_a(n + n_a))_{n\in\mathbb{N}^*}, a \in \psi(\mathcal{T}) - \{I\}]$. Consider now $\psi(\mathfrak{N})_{[2]} = (\psi(\mathcal{N}), \psi(M), \psi(u)_{[2]}, \psi(\sigma)_{[2]})$, still with the assumption that I fires only once. We define the random vector ξ_2 associated with $\psi(\mathfrak{N})_{[2]}$ in the same way as we defined the random vector ξ_1 associated with $\psi(\mathfrak{N})_{[1]}$. By iterating the construction, we define $(\xi_n)_{n\in\mathbb{N}^*}$. Obviously the sequence $(\xi_n)_{n\in\mathbb{N}^*}$ is i.i.d.

Consider again the SRFC $\psi(\mathfrak{N})$ now with the assumption that transition I fires a finite number of times, say k. According to Lemma 4.5, the transition b_o will also fire k times, and according to Lemma 5.2 the net will end up in the marking $\psi(M)$ after an a.s. finite time τ_k. It follows from Lemma 4.5 that the set of firings and routings used up to time τ_k is precisely the union of the ones in $\xi_1, \ldots, \xi_k$ (although the order in which they are used may differ from the one induced by $\xi_1, \ldots, \xi_k$). Assume furthermore that the instants of firings of I are deterministic and given by a counting measure $a \in \mathcal{M}_f$ and set $\xi = (\xi_1, \ldots, \xi_k)$. Then (a, ξ) belongs to $\mathcal{M}_f(E)$ for an appropriate set E. Now let us set $\Phi(a, \xi) = (b, \xi)$ where b is the counting measure of the instants of completions of the firings of b_o. This defines a mapping $\Phi : \mathcal{M}_f(E) \to \mathcal{M}_f(E)$.

We will now need some operations and relations on counting measures. For $a \in \mathcal{M}_f$, set $|a| = a(\mathbb{R}_+)$, the number of points of the counting measure. For $\alpha = (a, \mu) \in \mathcal{M}_f(E)$, set $|\alpha| = |a|$. For $a \in \mathcal{M}_f$, define the smallest point $\min(a) = \inf\{t : a(\{t\}) \geq 1\}$ and the largest point $\max(a) = \sup\{t : a(\{t\}) \geq 1\}$. For $\alpha = (a, \mu) \in \mathcal{M}_f(E)$, set $\max(\alpha) = \max(a)$ and $\min(\alpha) = \min(a)$. For $a, b \in \mathcal{M}_f$, define $a + b \in \mathcal{M}_f$ by $(a + b)(C) = a(C) + b(C)$. For $\alpha, \beta \in \mathcal{M}_f(E), \alpha = (a, \mu), \beta = (b, \nu), \max(a) < \min(b)$, let $\alpha + \beta \in \mathcal{M}_f(E)$ be given by $\alpha + \beta = (a + b, (\mu, \nu))$. For $a \in \mathcal{M}_f, t \in \mathbb{R}_+$, define $a + t \in \mathcal{M}_f$ by $(a + t)(C) = a(C - t)$, and if

$\alpha = (a, \xi) \in \mathcal{M}_f(E), t \in \mathbb{R}_+$, set $\alpha + t = (a + t, \xi)$. Define a partial order on $\mathcal{M}_f$ as follows. For $a, b \in \mathcal{M}_f$,

$$a \leq b \ \text{ if } \ \forall x \in \mathbb{R}_+, \ a([x, \infty)) \leq b([x, \infty)) \,.$$

Similarly, define a partial order on $\mathcal{M}_f(E)$ as follows: For $\alpha, \beta \in \mathcal{M}_f(E)$, $\alpha = (a, \mu), \beta = (b, \nu)$,

$$\alpha \leq \beta \ \text{ if } \ a \leq b \text{ and } \mu_{|a|} = \nu_{|b|}, \ \mu_{|a|-1} = \nu_{|b|-1}, \ldots, \ \mu_1 = \nu_{|b|-|a|+1} \,.$$

The mapping $\Phi : \mathcal{M}_f(E) \longrightarrow \mathcal{M}_f(E)$ is *monotone-separable*, i.e., satisfies the following properties:

1. **Causality**: $\alpha \in \mathcal{M}_f(E) \implies |\Phi(\alpha)| = |\alpha|$ and $\Phi(\alpha) \geq \alpha$;
2. **Homogeneity**: $\alpha \in \mathcal{M}_f(E), x \in \mathbb{R}_+ \implies \Phi(\alpha + x) = \Phi(\alpha) + x$;
3. **Monotonicity**: $\alpha, \beta \in \mathcal{M}_f(E), \ \alpha \leq \beta \implies \Phi(\alpha) \leq \Phi(\beta)$;
4. **Separability**: $\alpha, \beta \in \mathcal{M}_f(E), \max(\Phi(\alpha)) \leq \min(\beta) \implies \Phi(\alpha + \beta) = \Phi(\alpha) + \Phi(\beta)$.

The monotone-separable framework has been introduced in [6]. Actually, the setting used here is the one proposed in [14] and differs slightly from the one in [6]. The above properties of Φ are proved in a slightly different and more restrictive setting in [7], Section 5. However, the arguments remain essentially the same. Consequently, we provide only an outline of the proof.

The argument is based on the equations satisfied by the *daters* associated with the net. For $a \in \psi(\mathcal{T}), n \in \mathbb{N}^*$, let $X_a(n)$ be the n-th instant of completion of a firing at transition a with $X_a(n) = +\infty$ if a fires strictly less than n times. It is also convenient to set $X_a(n) = 0$ for $n \leq 0$. The variables $X_a(n)$ are called the *daters* associated with the SRFC.

Assume that I fires k times, the instants of firings being $0 \leq x_1 \leq \cdots \leq x_k$. Given a transition a and a place $p \in {}^\bullet a$, we define $\nu_{pa}(n) = \min\{k \ : \ \sum_{i=1}^k \mathbf{1}_{\{u_p(i)=a\}} = n\}$. The daters satisfy the following recursive equations, see [3] for a proof:

$$\begin{aligned}
&X_I(1) = x_1, \ldots, X_I(k) = x_k, \quad X_I(n) = \infty, \forall n > k \,; \\
&\forall a \in \psi(\mathcal{T}) - \{I\}, \quad X_a(n) = \left\{ \max_{p \in {}^\bullet a} \left[\min_{(n_i, i \in {}^\bullet p) \ : \ M_p + \sum_{i \in {}^\bullet p} n_i = \nu_{pa}(n)} \quad \max_{i \in {}^\bullet p} X_i(n_i) \right] \right\} + \sigma_a(n) \,.
\end{aligned}$$

Playing with the above equations, it is not difficult (although tedious) to prove that the operator Φ is monotone-separable.

Assume that I fires exactly k times with all the firings occurring at instant 0. The corresponding marked counting measure is $\alpha_k = ((0, \ldots, 0) \ ; \ (\xi_1, \ldots, \xi_k))$. Given that Φ is monotone-separable and that $(\xi_n)_{n \in \mathbb{N}^*}$ is i.i.d., we obtain using directly the results in [6, 14] that there exists $\gamma_b \in \mathbb{R}_+$ such that $\lim_n \max(\Phi(\alpha_n))/n = \gamma_b$ a.s. and in L_1.

We have seen above that the firings of b_i in the saturated version of $\psi(\mathfrak{N})$ coincide with the ones of b in $\mathfrak{N}$. More precisely, consider $k > K$ (we recall that K is defined in (5.2)) and let $b_1 \leq \cdots \leq (b_k = \max(\Phi(\alpha_k)))$ be the points of the counting measure of $\Phi(\alpha_k)$. The net $\psi(\mathfrak{N})$ with input α_k coincides with $\mathfrak{N}$ up to the instant b_{k-K}. Now it follows from Lemma 5.2 that $E[b_k - b_{k-K}] < \infty$. This implies in a straightforward way that $\lim_k X_b(k)/k = \lim_k \max(\Phi(\alpha_k))/k = \gamma_b$ a.s. and in L_1. This concludes the proof of Theorem 5.1.

5.3 Computation of the asymptotic throughputs

The section is devoted to proving that the limits $(\gamma_a, a \in \mathcal{T})$ in Theorem 5.1 can be explicitly computed up to a multiplicative constant.

Proposition 5.4. *The assumptions and notations are the ones of Section 5.2 and Theorem 5.1. The constants $\lambda_a = \gamma_a^{-1}, a \in \mathcal{T}$, are the throughputs at the transitions. Let us define the matrix $R = (R_{ij})_{i,j\in\mathcal{T}}$ as follows:*

$$R_{ij} = \begin{cases} \frac{1}{|{}^\bullet j|}\sum_{p:i\to p\to j} P\{u_p(1) = j\} & \text{if } \exists p \in \mathcal{P}, i \to p \to j\ . \\ 0 & \text{otherwise}\ . \end{cases}$$

The matrix R is irreducible, its spectral radius is 1, and there is a unique vector $x = (x_a, a \in \mathcal{T}), x_a \in \mathbb{R}_+^*, \sum_a x_a = 1$, such that $xR = x$. The vector $(\lambda_a, a \in \mathcal{T})$ is proportional to x, i.e., there exists $c \in \mathbb{R}_+^* \cup \{\infty\}$ such that $\lambda_a = cx_a$ for all $a \in \mathcal{T}$.

Proof. If there exists a transition a such that $\lambda_a = \infty$, then clearly $\lambda = (\lambda_a, a \in \mathcal{T}) = (\infty, \dots, \infty)$ since the net is bounded. We assume first that the constants λ_a are finite (the constants γ_a are strictly positive).
We recall that for a transition a, the *counter* $\mathcal{X}_a(t)$ is the number of firings completed at transition a up to time t. We also define for all $a \in \mathcal{T}$ and $p \in {}^\bullet a$, the *counter* $\mathcal{Y}_{pa}(t)$ which counts the number of tokens assigned by the place p to the transition a up to time t. We have

$$\mathcal{X}_a(t) \le \mathcal{Y}_{pa}(t) \le \mathcal{X}_a(t) + \overline{M}_p\ , \tag{5.3}$$

where $\overline{M}_p$ is the maximal number of tokens in place p (which is finite since the net is bounded). We also have

$$\mathcal{Y}_{pa}(t) = \sum_{i=1}^{K(t)} \mathbf{1}_{\{u_p(i)=a\}}, \quad K(t) = M_p + \sum_{b\in{}^\bullet p} \mathcal{X}_b(t)\ . \tag{5.4}$$

Going to the limit in (5.3) and (5.4), we get

$$\lim_t \frac{\mathcal{X}_a(t)}{t} = \lim_t \frac{\mathcal{Y}_{pa}(t)}{t} = \lim_t \frac{\sum_{i=1}^{K(t)} \mathbf{1}_{\{u_p(i)=a\}}}{K(t)} \times \frac{K(t)}{t}\ .$$

Applying Theorem 5.1 and the Strong Law of Large Numbers, we obtain

$$\lambda_a = P\{u_p(1) = a\} \sum_{b\in{}^\bullet p} \lambda_b\ .$$

Since the above equality holds for any $p \in {}^\bullet a$, we deduce

$$\lambda_a = \frac{1}{|{}^\bullet a|} \sum_{p\in{}^\bullet a} P\{u_p(1) = a\} \sum_{b\in{}^\bullet p} \lambda_b\ .$$

The above equality can be rewritten as $\lambda = \lambda R$, where R is the matrix defined in the statement of the Proposition.
Since the Petri net is strongly connected, it follows straightforwardly that R is irreducible. The Perron-Frobenius Theorem (see for instance [13]) states that R has a unique (up to a multiple)

eigenvector with coefficients in $\mathbb{R}_+^*$, and that the associated eigenvalue is the spectral radius. We conclude that the spectral radius of R is 1, and that λ is defined up to a multiple by the equality $\lambda = \lambda R$.

It remains to consider the case where $(\lambda_a, a \in \mathcal{T}) = (\infty, \ldots, \infty)$. The only point to be proved is that R is of spectral radius 1. If this is the case, the statement of the Proposition holds with the constant $c = \infty$. However, the matrix R depends only on the routing characteristics and not on the firing times. Modify the stochastic routed net by setting all the firing times to be identically equal to 1. Then the new throughputs belong to $\mathbb{R}_+^*$. The first part of the proof applies, the vector of throughputs is a left eigenvector associated with the eigenvalue 1, and we conclude that the matrix R is indeed of spectral radius 1. $\square$

A consequence of Proposition 5.4 is that the ratio $\lambda_a/\lambda_b, a, b \in \mathcal{T}$, depends only on the routings of the models and not on the timings. On the other hand, the multiplicative constant c of Proposition 5.4 depends on the timings. A concrete application of Proposition 5.4 is proposed in Example 6.3.

The vector $\lambda = (\lambda_a, a \in \mathcal{T})$ is a strictly positive and real-valued T-invariant of the net, that is, a solution of $N\lambda = 0$, where N is the incidence matrix of the net. The vector λ is a particular T-invariant associated with the routing probabilities.

An interesting special case is the one of live and bounded stochastic routed T-nets. For this restricted model, Theorem 5.1 was proved in [2] (see also [4]) with the additional result that $(\lambda_a, a \in \mathcal{T}) = (\lambda, \ldots, \lambda)$. This is consistent with Proposition 5.4. Indeed, for a T-net, the matrix R is such that $(1, \ldots, 1) = (1, \ldots, 1)R$, which implies according to Proposition 5.4 that $(\lambda_a, a \in \mathcal{T}) = (\lambda, \ldots, \lambda)$. This is also consistent with Proposition A.4. It is well known that the value of λ is hard to compute or even to approximate in T-nets, see [4], Chapter 8. We conclude that for a general SRFC the multiplicative constant c of Proposition 5.4 must be even harder to compute or approximate. Note, however, that this constant can be computed for a fluid approximation of the net, when the firing times are all deterministic, by using dynamic programming and Howard-type algorithms, see [16].

Stationary assumptions. The monotone-separable framework is designed to deal with more general than i.i.d. stochastic assumptions. In our case, simply by using the results in [6, 14], we obtain the same results as in Theorem 5.1 under the following assumptions: the sequence $(\xi_n)_n$ is stationary and ergodic, and the r.v. τ defined in Lemma 5.2 is a.s. finite and integrable. Proposition 5.4 also holds under the generalized assumptions.
However, proving that τ is integrable is a potentially difficult point. In [9], closed Jackson networks, a subclass of live and bounded Free Choice nets, are studied. The stochastic assumptions are such that $(\xi_n)_n$ is stationary and ergodic, and the main difficulty consists in proving that $E(\tau) < \infty$.

5.4 Stationary regime for the marking.

The existence of asymptotic throughputs for all the transitions can be seen as a 'first order' result. A more precise, 'second order', result would be the existence and uniqueness of a stationary regime for the marking process; we discuss this type of result here.

The model is the same as in Theorem 5.1 and M_b is the blocking marking associated with a transition b. We make the following additional assumptions:
(i) in the marking M_b, the *enabling degree* of b is equal to 1, i.e., $\min_{p \in {}^\bullet b}(M_b)_p = 1$;

(ii) the distribution of σ_b is unbounded, i.e., $P\{\sigma_b(1) > x\} > 0, \quad \forall x \in \mathbb{R}_+$.

Consider the continuous time and continuous state space Markov process $(X_t)_t$ formed by the marking and the residual firing times of the ongoing firings at instant t. Let $(T_n)_n$ be the instants when the marking changes and let $Y_n = X_{T_n^-}$. Then $(Y_n)_n$ is a Markov chain in discrete time. Under the above assumptions, it is not difficult to prove that $\{(M_b, 0)\}$ is a regeneration point for $(Y_n)_n$. It follows using standard arguments that $(Y_n)_n$ and $(X_t)_t$ have a unique stationary regime.

This result calls for some comments.

- Assumption (i) is always satisfied if transition b is recycled (*i.e.* $\{b^\bullet\} \cap {}^\bullet\{b\} = \{p_b\}$ where place p_b has an initial marking equal to 1). This is equivalent to the assumption that transition b operates like a single server queue.

- Closed Jackson networks are a subclass of live and bounded Free Choice nets (in which assumption (i) is always satisfied). Cyclic networks are a subclass of closed Jackson networks. In [15, 27, 22], second order results for closed Jackson netwoks are proved. The proofs are basically the same as the one sketched above. In the specific case of cyclic networks, the second order results hold true under much weaker assumptions [11, 23, 24]. This shows that conditions such as (i) and (ii) are only sufficient conditions for the existence and uniqueness of stationary regimes.

- When removing assumption (i), it becomes much more intricate to get second order results under reasonable sufficient conditions. For instance, second order results can be obtained if the firing time of b is exponentially distributed.

6 Some Extensions

6.1 Extended Free Choice nets

It is common in the literature to consider *Extended Free Choice nets* (EFCN) defined as follows: $\forall q_1, q_2 \in \mathcal{T},\ p \in {}^\bullet q_1 \cap {}^\bullet q_2 \Rightarrow {}^\bullet q_1 = {}^\bullet q_2$ (this is even the definition of *Free Choice nets* in [18]). The results in Theorem 3.1 hold for EFCN. Indeed, given an EFCN, one can apply Theorem 3.1 to the Free Choice net obtained from the EFCN by applying the local transformation illustrated on Figure 6.

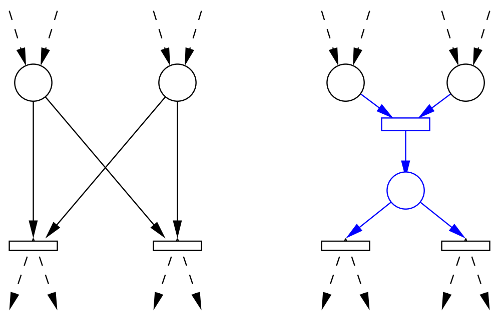


Figure 6: Transformation of an Extended Free Choice net into a Free Choice net.

On the other hand, the results from Sections 4 and 5 do not apply for EFCN. In fact, the routed version of a live and bounded EFCN is in general not live.

6.2 Petri nets with a live and bounded Free Choice expansion

In this section, we consider the class of Petri nets having a live and bounded *Free Choice expansion*. This class is strictly larger than the one of live and bounded Free Choice nets (and strictly smaller than the one of live and bounded Petri nets). The results related to routed nets in Sections 4 and 5 extend to this class. On the other hand, it is easily checked that Theorem 3.1 does not hold for this class.

Definition 6.1. *Given a Petri net* $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, M)$*, we define its* Free Choice expansion $\varphi(\mathcal{N}) = (\varphi(\mathcal{P}), \varphi(\mathcal{T}), \varphi(\mathcal{F}), \varphi(M))$ *as follows:*

- $\varphi(\mathcal{P}) = \mathcal{P} \cup \{s_{pq} \ : \ p \in \mathcal{P}, q \in p^\bullet\}$;
- $\varphi(\mathcal{T}) = \mathcal{T} \cup \{t_{pq} \ : \ p \in \mathcal{P}, q \in p^\bullet\}$;
- $\varphi(\mathcal{F}) = \mathcal{F} \cup \{(p, t_{pq}), (t_{pq}, s_{pq}), (s_{pq}, q) \ : \ p \in \mathcal{P}, q \in p^\bullet\}$;
- $\varphi(M) : \ \forall p \in \mathcal{P}, \varphi(M)_p = M_p, \ \forall p \notin \mathcal{P}, \varphi(M)_p = 0.$

Note that φ acts in a functional way (its components mapping sets to sets), which justifies our notation. Obviously, the resulting net $\varphi(\mathcal{N})$ is Free Choice. An example of this transformation is displayed in Figure 7.

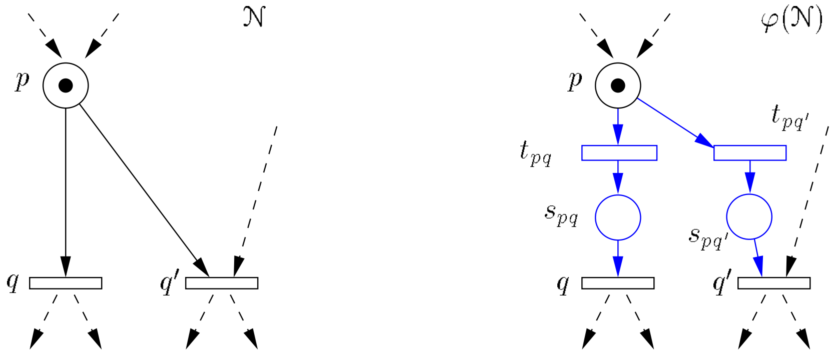


Figure 7: Free Choice expansion of a Petri net.

It is easy to see that $\varphi(\mathcal{N})$ is bounded if and only if $\mathcal{N}$ is bounded. Liveness is more subtle. If $\varphi(\mathcal{N})$ is live then clearly $\mathcal{N}$ is also live. On the other hand, it is possible that $\mathcal{N}$ be live, but not $\varphi(\mathcal{N})$. This is the case for the net on the left of Figure 4 (the net on the right of the same figure is 'almost' its Free Choice expansion). For a detailed comparison of the behaviors of $\mathcal{N}$ and $\varphi(\mathcal{N})$, see [20].

An example of a non-Free Choice Petri net such that $\varphi(\mathcal{N})$ is live and bounded is proposed in Figure 8.

Lemma 4.1 and 4.2 undergo the following modifications.

Lemma 6.2. *Let $\mathcal{N}$ be a Petri net with Free Choice expansion $\varphi(\mathcal{N})$. We have the following implications:*

$$\text{1. } \mathcal{N} \text{ is bounded} \iff \text{2. } \varphi(\mathcal{N}) \text{ is bounded} \iff \text{3. } (\mathcal{N}, u) \text{ is bounded for any } u;$$
$$\text{a. } \mathcal{N} \text{ is live} \Longleftarrow \text{ b. } \varphi(\mathcal{N}) \text{ is live} \iff \text{c. } (\mathcal{N}, u) \text{ is live for any equitable } u.$$

The equivalence between $a.$ and $c.$ which was proved in Lemma 4.2 for Free Choice nets is not true in general.

Proof. We have just seen that 3. implies 1. and that 2. and 3. are equivalent. The proof of the equivalence between 1. and 3. was done in Lemma 4.1.
Now let us prove the equivalence between $b.$ and $c.$ Assume there exists an equitable routing u such that $(\mathcal{N}, u)$ is not live. Construct the set X of nodes of $\mathcal{N}$ as in the proof of Lemma 4.2 (the construction there does not require the Free Choice assumption). In $\varphi(\mathcal{N})$, the set $\varphi(X) \cap \varphi(\mathcal{P})$ is a siphon which can be emptied using the same firing sequence as for X. We deduce that $\varphi(X) \cap \varphi(\mathcal{P})$ cannot contain an initially marked trap, hence $\varphi(\mathcal{N})$ cannot be live by Commoner's Theorem A.9. $\square$

Lemma 6.2 shows that the liveness and boundedness of a routed Petri net is directly linked to the one of its unrouted Free Choice expansion.

Theorem 4.4, Lemma 4.5, Theorem 5.1, Lemma 5.2 and Proposition 5.4 still hold when replacing the assumption *live and bounded Free Choice net* by the assumption *Petri net with a live and bounded Free Choice expansion.* In Appendix A.4, the proof of Theorem 4.4 is actually carried out under the general assumption. As for the other results, it is not difficult to extend them by first considering the Free Choice expansion and then showing that the results still hold for the original Petri net.

Example 6.3. Consider the live and bounded Petri net of Figure 8. Clearly, it is not a Free Choice net, but its Free Choice expansion is live and bounded. Consider a stochastic routed

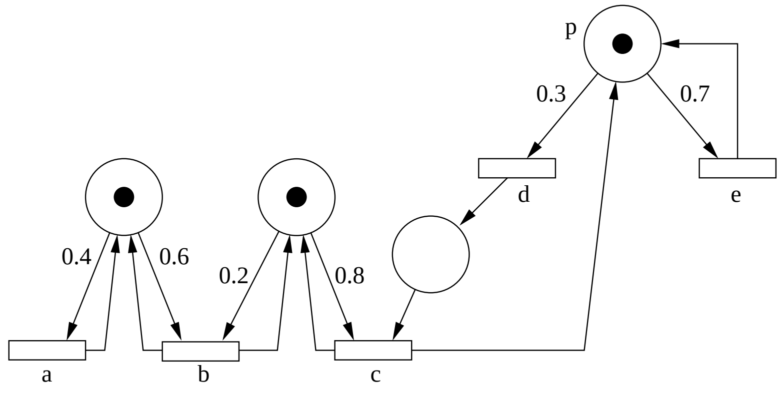


Figure 8: The values on the arcs are the routing probabilities.

version of the Petri net. As detailed above, the results of Theorem 5.1 and Proposition 5.4 apply. In particular, let R be defined as in Proposition 5.4 and let $\lambda = (\lambda_t, t \in \mathcal{T})$ be the vector of throughputs (the transitions being listed in alphabetical order). We have

$$R = \begin{pmatrix} 0.4 & 0.3 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.1 & 0.4 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}, \quad \lambda = c \begin{pmatrix} 0.04 & 0.05 & 0.21 & 0.21 & 0.49 \end{pmatrix} .$$

If we assume for instance that the routing probabilities of place p are $P\{u_p(1) = d\} = x,\ P\{u_p(1) = e\} = 1 - x$, then we obtain $\lambda = c\,(2x, 3x, 12x, 12x, 12 - 12x)\,/(12 + 17x)$.

Acknowledgments

We would like to thank Javier Esparza whose suggestions greatly helped us when we were blocked in our attempts to block transitions.

A Appendix

A.1 Reverse firings

Lemma A.1. *Let $(\mathcal{N}, M)$ be a T-net. We have $\forall q, r \in \mathcal{T}, q \neq r,\ M \xrightarrow{q} M', M \xrightarrow{r} M'' \Rightarrow M' \xrightarrow{r}, M'' \xrightarrow{q}$. Moreover, we have $\forall q, r \in \mathcal{T},\ M \xrightarrow{qr}, q^\bullet \cap {}^\bullet r = \emptyset \Rightarrow M \xrightarrow{r}$.*

Proof. Firing a transition in a T-net does not disable any other transition. $\square$

Let $\mathcal{N}$ be a Petri net. For a transition q and two markings M_1 and M_2, we write

$$M_2 \xrightarrow{q^-} M_1 \ \text{ if } \ M_1 \xrightarrow{q} M_2 \,.$$

Given $u = u_1 \cdots u_n, u_i \in \mathcal{T}$, we set $u^- = u_n^- \cdots u_1^-$. We write $M_2 \xrightarrow{u^-} M_1$ if $M_1 \xrightarrow{u} M_2$. We say that the *firing* of u^-, or the *reverse firing* of u, transforms the marking M_2 into M_1. Let us define $\mathcal{T}^- = \{q^- :\ q \in \mathcal{T}\}$, the set of *reverse transitions.* Given $u \in (\mathcal{T} \cup \mathcal{T}^-)^*$, its *Parikh vector* is $\vec{u} = (|u|_a - |u|_{a^-})_{a \in \mathcal{T}}$. A *generalized firing sequence* of $(\mathcal{N}, M)$ is a word $u \in (\mathcal{T} \cup \mathcal{T}^-)^*$ such that for all $k \leqslant |u|$, $M + N \cdot \vec{u}_{[k]} \geqslant (0, \ldots, 0)$.

The following set of rewriting rules are fundamental in what follows.

$$\forall a \in \mathcal{T},\ aa^- \leadsto e, a^-a \leadsto e, \quad \forall a, b \in \mathcal{T}, a \neq b,\ ab^- \leadsto b^-a, b^-a \leadsto ab^- \,. \tag{A.1}$$

For two words $u, v \in (\mathcal{T} \cup \mathcal{T}^-)^*$, we write $u \overset{*}{\leadsto} v$ if we can obtain v from u by successive application of a finite number of rewritings.

Lemma A.2. *Let $u, v \in (\mathcal{T} \cup \mathcal{T}^-)^*$ be such that $u \overset{*}{\leadsto} v$. If u is a generalized firing sequence, then v is also a generalized firing sequence.*

Proof. The proof follows easily from the fact that, for two distinct transitions a and b, we have $a^\bullet \cap b^\bullet = \emptyset$. $\square$

A.2 Results on Free Choice nets

We list here some results used in the paper, in particular in the proof of Theorem 3.1. All of them are proved in [18]; for the original references, see the bibliographic notes of [18].

Theorem A.3 ([18], Theorem 2.25). *A live and bounded connected Petri net is strongly connected.*

A vector $X \in \mathbb{N}^{\mathcal{T}}$ is a *T-invariant* if $N \cdot X = (0, \ldots, 0)$. If u is a firing sequence such that $M \xrightarrow{u} M$ then $\vec{u}$ is a T-invariant.

Proposition A.4 ([18], Prop. 3.16). *In a connected T-net, the T-invariants are the vectors* $(x, \ldots, x)$ *for* $x \in \mathbb{N}$.

Proposition A.5 ([25], Theorem 19). *In a live T-net* $(\mathcal{N}, M)$ *with incidence matrix* N*, if a vector* $x \in \mathbb{N}^{\mathcal{T}}$ *is such that* $M + N \cdot x \geqslant (0, \ldots, 0)$*, then there exists a firing sequence* u *such that* $\vec{u} = x$.

A Petri net is *k-bounded* if for every reachable marking M and for every place p, we have $M_p \leqslant k$.

Proposition A.6 ([18], Theorem 3.18). *A live T-net* $(\mathcal{N}, M)$ *is* k*-bounded if and only if, for every place* p*, there exists a circuit which contains* p *and holds at most* k *tokens under* M.

A subnet $\mathcal{N}' = (\mathcal{P}', \mathcal{T}', \mathcal{F}', M')$ of $\mathcal{N}$ is a *T-component* (resp. *S-component*) if $\mathcal{N}'$ is a strongly connected T-net (resp. S-net) and satisfies: $\forall q \in \mathcal{T}'$, ${}^\bullet q, q^\bullet \in \mathcal{P}'$ (resp. $\forall p \in \mathcal{P}'$, ${}^\bullet p, p^\bullet \in \mathcal{T}'$). A set of subnets of $\mathcal{N}$ forms a *covering* of $\mathcal{N}$ if each node and arc belongs to at least one of the subnets.

Theorem A.7 ([18], Theorems 5.6 and 5.18). *Live and bounded Free Choice nets are covered by S-components and by T-components.*

The *cluster* $[x]$ of a node x in $\mathcal{N}$ is the smallest subset of $\mathcal{P} \cup \mathcal{T}$ such that

(i) $x \in [x]$; (ii) $p \in \mathcal{P} \cap [x] \;\Rightarrow\; p^\bullet \in \mathcal{T} \cap [x]$; (iii) $q \in \mathcal{T} \cap [x] \;\Rightarrow\; {}^\bullet q \in \mathcal{P} \cap [x]$.

If $\mathcal{G}$ is a subnet of $\mathcal{N}$, then the *cluster* $[\mathcal{G}]$ of $\mathcal{G}$ is the union of the clusters of all the nodes in $\mathcal{G}$.

Theorem A.8 ([18], Theorem 5.20). *Let* $\mathcal{N}'$ *be a T-component of a live and bounded Free Choice net* $(\mathcal{N}, M_0)$*. There exists a firing sequence* σ *containing no transition from* $[\mathcal{N}']$ *and such that* $M_0 \xrightarrow{\sigma} M$ *and* $(\mathcal{N}', M|_{\mathcal{N}'})$ *is live.*

Actually, Theorem 5.20 in [18] states that the sequence σ does not contain any transitions from $\mathcal{N}'$; however, the proof given in [18] also provides the result stated above (and this strong version is the one we need).

A *siphon* is a set of places S such that ${}^\bullet S \subset S^\bullet$. A *trap* is a set of places S such that $S^\bullet \subset {}^\bullet S$. In particular, if a siphon (resp. a trap) is empty (resp. non-empty) under marking M, then it remains empty (resp. non-empty) under all markings in $R(M)$. The following theorem, known as Commoner's Theorem, gives a necessary and sufficient condition of liveness in Free Choice nets.

Theorem A.9 ([18], Theorems 4.21 and 4.27). *A Free Choice net is live if and only if every siphon contains an initially marked trap.*

A subnet $\mathcal{N}' = (\mathcal{P}', \mathcal{T}', F', M')$ of $\mathcal{N}$ is a *CP-subnet* if (i) $\mathcal{N}'$ is a non-empty and connected T-net; (ii) $\forall p \in \mathcal{P}'$, ${}^\bullet p, p^\bullet \subseteq \mathcal{T}'$; (iii) the subnet generated by $(\mathcal{P} - \mathcal{P}') \cup (\mathcal{T} - \mathcal{T}')$ is strongly connected. A *way-in* (resp. *way-out*) transition of a Petri net is a transition a such that ${}^\bullet a = \emptyset$ (resp $a^\bullet = \emptyset$).

Proposition A.10 ([18], Prop. 7.8). *Let $(\mathcal{N}, M_0)$ be a live and bounded Free Choice net, let $\hat{\mathcal{N}}$ be a CP-subnet of $\mathcal{N}$ and let $\hat{\mathcal{T}}$ be the set of transitions of $\hat{\mathcal{N}}$ and $\hat{\mathcal{T}}_{in}$ the set of way-in transitions of $\hat{\mathcal{N}}$. Then there exists a marking M and a firing sequence $\sigma \in (\hat{\mathcal{T}} - \hat{\mathcal{T}}_{in})^*$ such that $M_0 \xrightarrow{\sigma} M$ and M enables no transition of $\hat{\mathcal{T}} - \hat{\mathcal{T}}_{in}$. Furthermore, the subnet of $(\mathcal{N}, M)$ generated by $(\mathcal{T} - \hat{\mathcal{T}}) \cup (\mathcal{P} - \hat{\mathcal{P}})$ is live and bounded.*

Proposition A.11 ([18], Prop. 7.10). *Let $\hat{\mathcal{N}}$ be a CP-subnet of a live and bounded Free Choice net and let $\hat{\mathcal{T}}_{in}$ be the set of way-in transitions of $\hat{\mathcal{N}}$. We have $|\hat{\mathcal{T}}_{in}| = 1$.*

A.3 Proof of Theorem 3.1

Let us recall the statement of Theorem 3.1.

Let $(\mathcal{N}, M_0)$ be a live and bounded Free Choice net. If b is a non-conflicting transition, then there exists a unique reachable marking M_b in which the only enabled transition is b. Furthermore, M_b can be reached from any reachable marking and without firing transition b.

We recall that M_b is the *blocking marking* associated with b.

Proof. It follows from the definition that we have

$$\forall M \in R(M_0), \quad R'_b(M) \subset R_b(M) \subset R_b(M_0)\,. \tag{A.2}$$

According to Theorem A.7, there exists a covering of $\mathcal{N}$ by T-components that we denote by $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$. The proof will proceed by induction on n.

We assume first that $n = 1$, that is, $\mathcal{N}$ is a T-net. Note that all the transitions are non-conflicting. The proof has four parts, each showing one of the following auxiliary results. Given a transition b, one has for all $M \in R(M_0)$:

$$1.\ R'_b(M) \neq \emptyset\,; \quad 2.\ |R'_b(M)| = 1\,; \quad 3.\ R'_b(M) = R'_b(M_0)\,; \quad 4.\ R_b(M) = R'_b(M)\,.$$

1. The T-net $\mathcal{N}$ is covered by circuits with a bounded number of tokens, say K (Proposition A.6). We block transition b in the marking $M \in R(M_0)$. If γ is a circuit of the covering containing b, it prevents any transition in γ from firing strictly more than K times. Now, let q be a transition such that there exist circuits $\gamma_1, \ldots, \gamma_l$ from the covering such that b belongs to γ_1, q belongs to γ_l, and γ_i and γ_{i+1} have a common transition for $i = 1, \ldots, l-1$. Then q can fire at most $l \cdot K$ times. Since $\mathcal{N}$ is strongly connected, any transition can fire at most $n \cdot K$ times, where n is the number of circuits in the covering.

2. The proof is almost the same as for Lemma 4.5. Let us consider $M_1, M_2 \in R'_b(M)$ with $M \xrightarrow{\sigma_1} M_1$ and $M \xrightarrow{\sigma_2} M_2$ and $|\sigma_1|_b = |\sigma_2|_b = 0$. We want to prove that $M_1 = M_2$. There exist possibly several firing sequences with Parikh vectors $\vec{\sigma}_1$ and $\vec{\sigma}_2$. Among these firing sequences, we choose the two with the longest common prefix, and we denote them by $u_1 = xv_1$ and $u_2 = xv_2$ (recall that $\vec{u}_1 = \vec{\sigma}_1$ and $\vec{u}_2 = \vec{\sigma}_2$). Let $\tilde{M}$ be such that $M \xrightarrow{x} \tilde{M}$. If $v_1 = v_2 = e$, then $M_1 = M_2 = \tilde{M}$. Assume that $v_1 \neq e$ and let a be the first letter of v_1. Since $|u_1|_a > 0$, we deduce that $a \neq b$. The transition a is enabled in $\tilde{M}$. Furthermore, by

definition, a is not enabled in M_2. This implies that the firing sequence v_2 must contain a; thus, we can set $v_2 = yaz$ with $|y|_a = 0$. Since a is enabled in $\tilde{M}$, it follows that ayz is a firing sequence and $\tilde{M} \xrightarrow{ayz} M_2$. To summarize, we have found two firing sequences u_1 and $u_2' = xayz$ with respective Parikh vectors $\vec{\sigma}_1$ and $\vec{\sigma}_2$ and with a common prefix at least equal to xa. This is a contradiction.

3. Let σ be such that $M_0 \xrightarrow{\sigma} M$. If $|\sigma|_b = 0$, it follows from the previous point that $R_b'(M) = R_b'(M_0)$. Let us assume that $|\sigma|_b > 0$. Let $\sigma = q_1 \cdots q_n$ with $q_i \in \mathcal{T}$ and $M_0 \xrightarrow{q_1} M_1 \xrightarrow{q_2} M_2 \cdots M_{n-1} \xrightarrow{q_n} M_n = M$. Let k be any index such that $q_k = b$, that is $M_{k-1} \xrightarrow{b} M_k$. Using Propositions A.4 and A.5, there exists a firing sequence θ with Parikh vector $\vec{\theta} = (1, \ldots, 1) - \vec{b}$ and such that $M_k \xrightarrow{\theta} M_{k-1}$, that is $M_{k-1} \xrightarrow{\theta^-} M_k$ (see Section A.1). By replacing every b by θ^- in σ, we get a generalized firing sequence $\sigma' \in ((\mathcal{T} - \{b\}) \cup (\mathcal{T}^- - \{b^-\}))^*$ such that $M_0 \xrightarrow{\sigma'} M$. Using the rewriting rules in (A.1)

$$\begin{tikzcd}
M_0 \arrow[r, dashed] \arrow[drrr, "u"'] & M_j \arrow[r, "b"] & M_{j+1} \arrow[l, bend right, "\theta"'] \arrow[rr, dashed] & & M_k \arrow[r, "b"] & M_{k+1} \arrow[l, bend right, "\theta"'] \arrow[r, dashed] & M \arrow[dlll, "v"] \\
 & & & \tilde{M} \arrow[d, "w"] & & & \\
 & & & M' & & &
\end{tikzcd}$$

Figure 9: Using reverse firings to avoid b.

and applying Lemma A.2, we find a generalized firing sequence σ'' such that $\sigma' \overset{*}{\rightsquigarrow} \sigma''$ and such that $\sigma'' = uv^-, u \in (\mathcal{T} - \{b\})^*, v^- \in (\mathcal{T}^- - \{b^-\})^*$. Let $\tilde{M}$ be the marking such that $M_0 \xrightarrow{u} \tilde{M} \xleftarrow{v} M$. Let M' be the unique element of $R_b'(\tilde{M})$. Since we have $M \xrightarrow{v} \tilde{M}$ with $|v|_b = 0$, we obtain that $R_b'(M) = \{M'\}$. By definition there exists a firing sequence $w \in (\mathcal{T} - \{b\})^*$ such that $\tilde{M} \xrightarrow{w} M'$. We deduce that we have $M_0 \xrightarrow{uw} M'$ with $uw \in (\mathcal{T} - \{b\})^*$. This implies that $R_b'(M_0) = R_b'(M)$. The whole argument is illustrated in Figure 9.

4. Clearly we have $R_b'(M) \subset R_b(M)$. For the converse, consider $\tilde{M} \in R_b(M)$ and $u \in \mathcal{T}^*$ such that $M \xrightarrow{u} \tilde{M}$. If $|u|_b = 0$ then $\tilde{M} \in R_b'(M)$; so assume $|u|_b > 0$ and set $u = vbw$ with $|w|_b = 0$. Let $\hat{M}$ be the marking such that $M \xrightarrow{vb} \hat{M}$. By construction, we have $\tilde{M} \in R_b'(\hat{M})$. Now, by point 3. above, this implies that $\tilde{M} \in R_b'(M)$.

Assume now that $\mathcal{N}$ is covered by the T-components $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$, with $n \geqslant 2$, and let b be a non-conflicting transition. We also assume the covering to be minimal, *i.e.* such that no T-component can be removed from it. Let $\mathcal{P}_i$ and $\mathcal{T}_i$ be the places and transitions of $\mathfrak{T}_i$. Set $\mathcal{N}_+ = \mathcal{N}[\bigcup_{j=1}^{n-1} \mathcal{P}_i \cup \mathcal{T}_i]$ and $\mathcal{N}_- = \mathcal{N}[(\mathcal{P} - \mathcal{P}_+) \cup (\mathcal{T} - \mathcal{T}_+)]$, where $\mathcal{P}_+$ and $\mathcal{T}_+$ are the places and transitions of $\mathcal{N}_+$. Since the covering is minimal, the subnet $\mathcal{N}_-$ is non-empty.

Now, it is always possible to re-number the $\mathfrak{T}_i$'s such that $b \in \mathcal{N}_+$ and $\mathcal{N}_+$ is strongly connected. This is shown in the first part of the proof of Proposition 7.11 in [18] (see also Proposition 4.5 in [17]).

On the other hand, the net $\mathcal{N}_-$ has no reason to be connected. Let us denote by $\kappa_1, \dots, \kappa_m$, the connected components of $\mathcal{N}_-$. According to Propositions 4.4. and 4.5 in [17], the nets κ_j are CP-subnets of $\mathcal{N}$ (see Appendix A.2). This result is also demonstrated in the second part of the proof of Proposition 7.11 in [18].

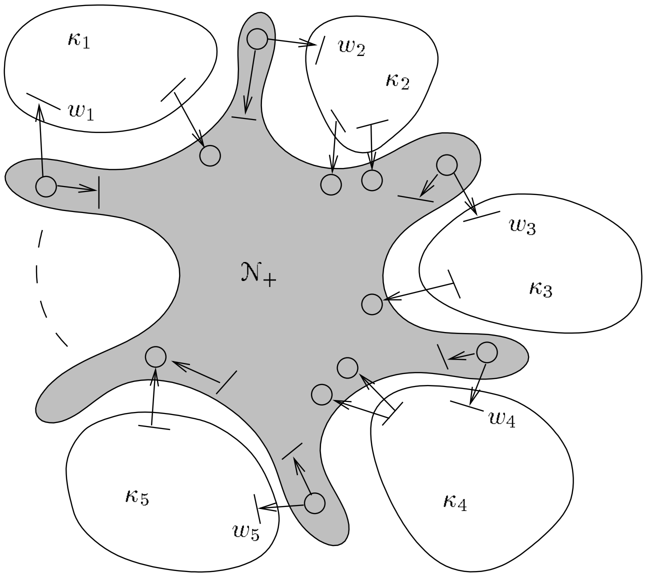


Figure 10: The net $\mathcal{N}$ decomposed into $\mathcal{N}_+$ and the CP-subnets $\kappa_1, \dots, \kappa_m$.

The decomposition of $\mathcal{N}$ into $\mathcal{N}_+$ and $\kappa_1, \dots, \kappa_m$, is illustrated in Figure A.3. By Proposition A.11, each κ_i has a single way-in transition denoted w_i. Furthermore, w_i has a unique input place that we denote p_i. Indeed, let us consider $p \in {}^\bullet w_i$. We have $p \in \mathcal{N}_+$. Since $\mathcal{N}_+$ is strongly connected, the set of successors of p in $\mathcal{N}_+$ is non-empty, and we conclude that $|p^\bullet| > 1$. Now by the Free Choice property, p must be the only predecessor of w_i.

We first show that $R'_b(M_0)$ is non-empty. We proceed as follows.

a. Using Proposition A.10, for all $i = 1, \dots, m$, there exists a firing sequence $\sigma_{\kappa_i} \in (\mathcal{T}_{\kappa_i} - \{w_i\})^*$ such that no transitions in $\mathcal{T}_{\kappa_i} - \{w_i\}$ is enabled after firing σ_{κ_i}. Let M'_0 be the marking obtained from M_0 after firing the sequence $\sigma = \sigma_{\kappa_1} \cdots \sigma_{\kappa_m}$. No transition from $\mathcal{N}_-$ is enabled in M'_0 except possibly the way-in transitions.

b. Consider the subnet $(\mathcal{N}_+, M'_0|_{\mathcal{N}_+})$. We first prove that it is live and bounded. By Proposition A.10, under the marking M'_0, the net $\mathcal{N} - \kappa_m$ is a live and bounded Free Choice net. Now, we can prove that κ_{m-1} is a CP-subnet of $\mathcal{N} - \kappa_m$ by the same arguments as the ones used to prove that κ_{m-1} is a CP-subnet of $\mathcal{N}$. Again by Proposition A.10, the net $\mathcal{N} - (\kappa_m \cup \kappa_{m-1})$ is a live and bounded Free Choice net. By removing in the same way all the CP-subnets, we finally conclude that $(\mathcal{N}_+, M'_0|_{\mathcal{N}_+})$ is a live and bounded Free Choice net. Furthermore, $\mathcal{N}_+$ admits a covering by T-components of cardinality $n-1$. By the induction hypothesis, there exists a firing sequence x avoiding b and which disables all the transitions in $\mathcal{T}_+$ except b. Let M_b be the marking of $\mathcal{N}$ obtained from M'_0 after firing x (now viewed as a firing sequence of $\mathcal{N}$).

c. By construction, no transition from $\mathcal{T}_+$ except b is enabled in $(\mathcal{N}, M_b)$. Let us prove that the transitions w_i are also disabled in M_b. The transition w_i is enabled if its input place p_i is marked. Let a be an output transition of p_i belonging to $\mathcal{N}_+$. By the free choice property, we have $\{p_i\} = {}^\bullet a = {}^\bullet w_i$. Since a is conflicting and b is non-conflicting, we have $a \neq b$, which implies that a is not enabled and that p_i is not marked.

Clearly, the above proof also works for $(\mathcal{N}, M)$ where $M \in R(M_0)$. Hence we have

$$\forall M \in R(M_0), \quad R'_b(M) \neq \emptyset \,. \tag{A.3}$$

We have thus completed the first step of the proof. We now prove the following assertion.

Assertion (A_0): *The T-net κ_i has a unique* reference *marking in which the only enabled transition is w_i. Furthermore, starting from the reference marking, if w_i is fired h_i times, then the other transitions can fire at most h_i times. If all the transitions in κ_i are fired h_i times, then the net goes back to the reference marking.*

Proof of (A_0): First, according to Proposition 5.1 in [17], there is a reachable marking M_R where no transition is enabled except w_i. Now using the same argument as in point 2 above (or as in the proof of Lemma 4.5), we obtain that M_R is the only such marking. According to Proposition 5.2 in [17], a property of M_R is: for all transition $q \neq w_i$, there is an unmarked path from w_i to q. The rest of assertion (A_0) follows easily.

By assertion (A_0), the markings M'_0, M_b, and M'_b coincide on all the subnets κ_i. We turn our attention to the following assertion.

Assertion (A_1): *If M' is a marking reachable from M'_0 which coincides with M'_0 on all the places of $\kappa_1, \cdots, \kappa_m$, then the marking M' is reachable from M'_0 by firing and reverse firing of transitions from $\mathcal{N}_+$ only.*

We first show how to complete the proof assuming (A_1). Consider $M'_b \in R_b(M_0)$. We want to show that $M'_b = M_b$. Apply (A_1) to the marking M'_b: it is reachable from M'_0 by firing and reverse firing of transitions from $\mathcal{N}_+$ only. We have seen above that $(\mathcal{N}_+, M'_0|_{\mathcal{N}_+})$ is a live and bounded Free Choice net. It follows readily that $(\mathcal{N}_+, M'_b|_{\mathcal{N}_+})$ is also live and bounded. Since $\mathcal{N}_+$ admits a covering by T-components of cardinality $n-1$, we can apply the induction hypothesis to $\mathcal{N}_+$: if M and M' are two markings of $\mathcal{N}_+$ such that $M \xrightarrow{q} M'$ or $M \xrightarrow{q^-} M'$ for some q in $\mathcal{T}_+$, then the blocking markings reached from M and M' are the same. By repeating the argument for all transitions (which are fired or reverse fired) on the path from $M'_0|_{\mathcal{N}_+}$ to $M'_b|_{\mathcal{N}_+}$, we get that $M'_b|_{\mathcal{N}_+} = M_b|_{\mathcal{N}_+}$. It follows that $M'_b = M_b$, i.e. $R_b(M_0) = \{M_b\}$. Coupled with the results in (A.2) and (A.3), it implies that $R_b(M) = R'_b(M) = \{M_b\}$ for any reachable marking M. The only remaining point consists in proving assertion (A_1).

Proof of (A_1): Let τ be a firing sequence leading from M'_0 to M' and let $h_i = |\tau|_{w_i}$ for $i = 1, \ldots, m$. The proof proceeds by induction on $h = h_1 + \cdots + h_m$. The case $h = 0$ is trivial, since, under M'_0, no transition in $\kappa_1, \ldots, \kappa_m$, can fire without firing the way-in transitions first.

Now let us consider the case where $h_1 + \ldots + h_m > 0$. Since M_0' and M' coincide on $\kappa_1, \ldots, \kappa_m$, it follows from (A_0) that all the transitions in κ_i have fired h_i times in the sequence τ. Without loss of generality (by re-numbering the κ_i's) we can assume that the last way-in transition fired in the sequence τ is w_1. By commuting the last occurrence of w_1 with the transitions in τ which can fire independently of it, we can assume that all the transitions in κ_i for $i = 2, \cdots, m$, have fired h_i times and all the transitions in κ_1 have fired $h_1 - 1$ times before w_1 is fired for the last time. This means that the marking M_1 reached just before w_1 is fired for the last time coincides with M_0' on all the $\kappa_i's$.

Let τ_{κ_i} be a firing sequence of κ_i leading from the reference marking of κ_i to itself (see (A_0)). We have $|\tau_{\kappa_i}|_t = 1$ for $t \in \kappa_i$, and $|\tau_{\kappa_i}|_t = 0$ otherwise (see (A_0)). By further commutation of transitions which can fire independently, the sequence τ can be rearranged and decomposed as displayed in (A.4), where arrows $\rightharpoondown$ mean "*only transitions in $\kappa_1, \ldots, \kappa_m$ are fired*"; arrows $\rightharpoonup$ mean "*only transitions in $\mathcal{N}_+$ are fired*"; and arrows $\longleftrightarrow$ mean "*only transitions and reverse transitions from $\mathcal{N}_+$ are fired*":

$$M_0 \overset{\sigma}{\rightharpoondown} M_0' \overset{v}{\longleftrightarrow} M_1 \overset{\tau_{\kappa_1}}{\rightharpoondown} M_2 \overset{u}{\rightharpoonup} M'. \tag{A.4}$$

The firing sequence $M_0' \overset{v}{\longleftrightarrow} M_1$, with v being a generalized firing sequence containing only (reverse) transitions from $\mathcal{N}_+$, exists by the induction hypothesis on (A_1). In the subnet κ_1, the firing sequence τ_{κ_1} leads from the reference marking to itself. However the sequence has some side effects in the net $\mathcal{N}_+$, since a token has been removed from the place p_1 and one token has been added in each output place of a way-out transition of κ_1. The challenge is now to "erase" this change in $\mathcal{N}_+$ while using only transitions from $\mathcal{N}_+$.

To do this, consider the subnet $\mathcal{G} = \mathcal{N}_+ \cup \kappa_1$. We have proved in point b. above that the net $(\mathcal{N}_+, M_0'|_{\mathcal{N}_+})$ is a live and bounded Free Choice net. It follows clearly that $\mathcal{G}$ is live and bounded under the marking $M_0'|_{\mathcal{G}}$. This implies that $\mathcal{G}$ is also live and bounded under the marking $M_1|_{\mathcal{G}}$ (since, in $\mathcal{N}$, the marking M_1 is obtained from M_0' by firing and reverse firing of transitions from $\mathcal{G}$). By Theorem A.7, the net $(\mathcal{G}, M_1|_{\mathcal{G}})$ can be covered by T-components. Let $\mathcal{Z}$ be a T-component of the covering which contains w_1. By definition, $\mathcal{Z}$ must also contain all the places in $w_1^\bullet$. Since $\mathcal{Z}$ is strongly connected, it must contain the unique output transition of each place in $w_1^\bullet$. By repeating the argument, we get that the whole subnet κ_1 is included in $\mathcal{Z}$.

In the following, we play with the three nets $\mathcal{N}$, $\mathcal{G}$ and $\mathcal{Z}$ (with $\mathcal{Z} \subset \mathcal{G} \subset \mathcal{N}$). To avoid very heavy notations, we use the same symbol for the marking in one of the three nets and its restrictions/expansions to the other two. For instance we use M_1 for $M_1, M_1|_{\mathcal{G}}$ or $M_1|_{\mathcal{Z}}$. We hope this is done without ambiguity.

Applying Theorem A.8 to $(\mathcal{G}, M_1)$, there exists a marking M_3 and a firing sequence x such that $M_1 \xrightarrow{x} M_3$, the subnet $(\mathcal{Z}, M_3)$ is live and x contains no transition from $[\mathcal{Z}]$. Recall that $[\mathcal{Z}]$ is the cluster of $\mathcal{Z}$. By construction, x contains only transitions from $\mathcal{N}_+$. In particular, the markings M_1 and M_3 coincide on the subnet κ_1; moreover, no transition of κ_1 except possibly w_1 is enabled in M_3. Now we claim that w_1 is enabled in M_3. By definition of a cluster, the input place p of w_1 belongs to $[\mathcal{Z}]$, as well as all the output transitions of p. We deduce that x does not contain the output transitions of p, and w_1 is enabled in M_3 since it was enabled in M_1.

Consequently, the sequence τ_{κ_1} is a firing sequence in $(\mathcal{Z}, M_3)$. Let M_4 be the marking defined by $M_3 \xrightarrow{\tau_{\kappa_1}} M_4$. Let $\mathcal{T}_{\mathcal{Z}}$ be the set of places of $\mathcal{Z}$. We consider the vector $X \in \mathbb{N}^{\mathcal{T}_{\mathcal{Z}}}$ defined by

$X_t = 0$ if t belongs to κ_1 and $X_t = 1$ otherwise. By construction and Assertion $(A0)$, we have $X + \vec{\tau}_{\kappa_1} = (1, \ldots, 1)$. According to Proposition A.4, this implies that $M_4 + N_{\mathcal{Z}} \cdot X = M_3$, where $N_{\mathcal{Z}}$ is the incidence matrix of $\mathcal{Z}$. According to Proposition A.5, there exists a firing sequence θ of $(\mathcal{Z}, M_4)$ such that $\vec{\theta} = X$. This implies that θ^- is a generalized firing sequence leading from M_3 to M_4.

Now we want to prove that x is a firing sequence of $(\mathcal{N}, M_2)$. The firing of τ_1 involves only places from $\mathcal{Z}$ (the places from κ_1, the input place of the way-in transition, and the output places of the way-out transitions). This implies that M_1 and M_2 coincide on the places which do not belong to $[\mathcal{Z}]$. Now x contains only transitions outside of $[\mathcal{Z}]$, and if t is a transition outside of $[\mathcal{Z}]$ then the input places of t do not belong to $[\mathcal{Z}]$ either. Since x is a firing sequence of $(\mathcal{N}, M_1)$, we deduce that it is also a firing sequence of $(\mathcal{N}, M_2)$. We have

$$M_2 + N \cdot \vec{x} = M_1 + N \cdot (\vec{\tau}_{\kappa_1} + \vec{x}) = M_3 + N \cdot \vec{\tau}_{\kappa_1} = M_4 \,.$$

Hence we obtain $M_2 \xrightarrow{x} M_4$ and $M_4 \xrightarrow{x^-} M_2$. Summarizing the above steps, we have obtained that $\varpi = vx\theta^- x^- u$ is a generalized firing sequence leading from M'_0 to M' and involving only transitions and reverse transitions from $\mathcal{N}_+$. This concludes the proof of (A_1). The various steps are illustrated in Figure 11, with the shaded area highlighting ϖ.

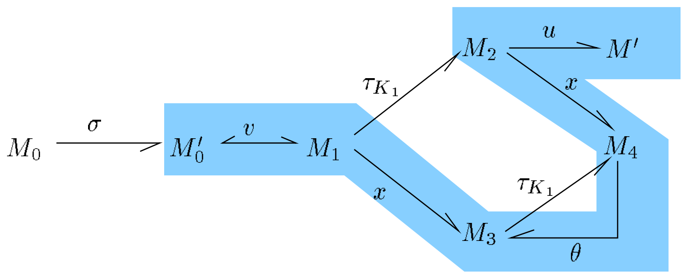


Figure 11: Proof of assertion (A_1).

$\square$

A.4 Proof of Theorem 4.4

We prove the result for Petri nets whose Free Choice expansion is live and bounded, along the lines of Section 6. Here is the precise statement proved below.

Let $(\mathcal{N}, M_0)$ be a Petri net whose Free Choice expansion $\varphi(\mathcal{N})$ is live and bounded. For any transition b, there exists a blocking marking M_b such that for every equitable routing u and all $M \in R(M_0, u)$, we have $R_b(M, u) = R'_b(M, u) = \{M_b\}$.

Proof. Consider $\varphi(\mathcal{N})$ and set $\mathcal{P}' = \varphi(\mathcal{P}) - \mathcal{P}$ and $\mathcal{T}' = \varphi(\mathcal{T}) - \mathcal{T}$, The function φ maps a marking M of $\mathcal{N}$ into a marking $\varphi(M)$ of $\varphi(\mathcal{N})$ as defined above. Now, we define a reverse transformation $\psi : \mathbb{N}^{\varphi(\mathcal{P})} \to \mathbb{N}^{\mathcal{P}}$ which transforms a marking $\tilde{M}$ of $\varphi(\mathcal{N})$ into a marking $\psi(\tilde{M})$ of $\mathcal{N}$:

$$\psi(\tilde{M}) = (\psi(\tilde{M})_p)_{p \in \mathcal{P}} \quad \text{and} \quad \psi(\tilde{M})_p = \tilde{M}_p + \sum_{(p,q) \in \mathcal{F}} \tilde{M}_{s_{pq}} \,.$$

Note that for any marking M in $\mathcal{N}$, we have $\psi \circ \varphi(M) = M$.

A *pointed marking* (M, f) of $\mathcal{N}$ is a pair formed by a marking M and an assignment f of each token of the marking to an output transition. Formally, f is an application from $\{(p,t), p \in \mathcal{P}, t \in p^\bullet\}$ to $\mathbb{N}$, satisfying $\sum_{t \in p^\bullet} f(p,t) = M_p$ for all place p. In $(\mathcal{N}, M_0, u)$, given $M_0 \xrightarrow{\sigma} M'$, we denote by (M', u, σ) the pointed marking formed by M' and the assignment induced by u and σ: the tokens in place p are assigned as in (4.1). To a pointed marking (M, f) of $\mathcal{N}$, we associate the marking $\varphi(M, f)$ in $\varphi(\mathcal{N})$ obtained from $\varphi(M)$ by firing all the transitions in $\mathcal{T}'$ which are compatible with the assignment. Note that we have $\psi \circ \varphi(M, f) = M$. We have illustrated this in Figure 12; small letters next to a token indicate the transition to which the token is routed.

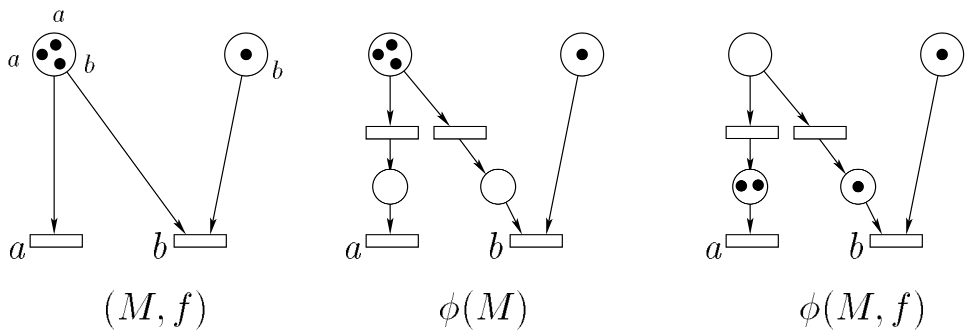


Figure 12: The original net with pointed marking (M, f) (left) and the effect of φ.

Consider the Free Choice net $\varphi(\mathcal{N})$. By construction, any transition b of $\mathcal{T}$ is a non-conflicting transition for $\varphi(\mathcal{N})$. Using Theorem 3.1, there exists a marking M'_b in $\varphi(\mathcal{N})$ such that for all $M \in R(\varphi(\mathcal{N}), \varphi(M_0))$, we have $R_b(\varphi(\mathcal{N}), M) = R'_b(\varphi(\mathcal{N}), M) = \{M'_b\}$. Let us set $M_b = \psi(M'_b)$. Consider now the routed Petri net $(\mathcal{N}, M_0, u)$. We want to prove first that M_b is such that $R_b(\mathcal{N}, M, u) = \{M_b\}$ for all $M \in R(\mathcal{N}, M_0, u)$. Assume that there exists $M' \in R_b(\mathcal{N}, M, u)$ and let σ, τ be such that $M_0 \xrightarrow{\sigma} M \xrightarrow{\tau} M'$. Let us consider the pointed marking $x = (M', u, \sigma\tau)$ and the marking $\varphi(x)$ of $\varphi(\mathcal{N})$. Assume that there is a transition $t \neq b$ of $\varphi(\mathcal{N})$ which is enabled in $\varphi(x)$. By construction, we have $t \in \mathcal{T}$, and t is also enabled in $\psi \circ \varphi(x) = M'$, which is a contradiction. We conclude that b is the only transition enabled in $\varphi(x)$, that is $\varphi(x) = M'_b$, which implies that $M' = M_b$.
Now we prove that $R'_b(\mathcal{N}, M, u)$ is non-empty for any reachable marking M. Starting from M, we build a firing sequence of the routed net by always firing an enabled transition different from b. By Lemma 4.3, it is impossible to build an infinite such sequence. Hence, we end up in a marking such that no transition is enabled except b, this marking belongs to $R'_b(\mathcal{N}, M, u)$. Since $R'_b(\mathcal{N}, M, u) \subset R_b(\mathcal{N}, M, u)$, this finishes the proof. □

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