On aperiodic and star-free formal power series in partially commuting variables

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Abstract. Formal power series over non-commuting variables have been investigated as representations of the behavior of automata with multiplicities. Here we introduce and investigate the concepts of aperiodic and of star-free formal power series over semirings and partially commuting variables. We prove that if the semiring K is idempotent and commutative, or if K is idempotent and the variables are non-commuting, then the product of any two aperiodic series is again aperiodic. We also show that if K is idempotent and the matrix monoids over K have a Burnside property (satisfied, e.g. by the tropical semiring), then the aperiodic and the star-free series coincide. This generalizes a classical result of Schützenberger (1961) for aperiodic regular languages and contains a result of Guaiana, Restivo and Salemi (1992) on aperiodic trace languages.

1 Introduction

In the theory of automata, Kleene's fundamental theorem on the coincidence of regular and rational languages in free monoids has been extended in many ways. Schützenberger [18] investigated formal power series over arbitrary semirings (e.g., like the natural numbers) and with non-commuting variables and showed that the recognizable formal power series, which represent precisely the behavior of automata with multiplicities (cf. Eilenberg [5]), coincide with the rational series. This was the starting point for a large amount of work on formal power series, cf. [17, 10, 1, 9] for surveys. Special cases of automata with multiplicities are networks with capacities (costs) and have been also investigated in operations research for algebraic optimization problems, cf. [22] and in the 'max-plus-community' [6].

Schützenberger [19] also showed that in free monoids the aperiodic regular languages coincide with the star-free languages. Such languages are important for and arise from counter-free automata, and have been also investigated intensively due to characterizations using first order logic (McNaughton and Papert [14]) or temporal logic (Kamp [8]).

An algebraic characterization of the sub-algebra of series over commutative fields generated by letters and geometric series was announced by Reutenauer [16] as an analogue of Schützenberger's theorem. However, in the case of the boolean semiring, his class of series restricts to a proper subclass of the aperiodic languages (dot-depth 3/2).

It is the aim of this paper to introduce and investigate the concepts of aperiodicity and star-freeness for formal power series over arbitrary semirings. In fact, we will allow the variables to be partially commutative. A recognizable series is called aperiodic if, when iterating any complex task in a representing automaton with multiplicities, there is a fixed bound of iterations after which the computed value remains stable. This is an assumption often made in optimization problems, cf. [22]. A series is star-free, if it can be constructed from finitely many polynomials using the operations sum, product and complement, with the latter being applied only to characteristic series. This generalizes the concepts of aperiodic and of star-free languages, respectively.

Whereas formal power series over non-commuting variables represent the behavior of sequential systems with costs, series over partially commuting variables can be viewed as the behavior of concurrent systems with multiplicities ('costs' for the actions). For an investigation of recognizable and rational formal power series over trace monoids, we refer the reader to [4].

Let us now give a summary of our main results. They all require that the semiring K of coefficients be idempotent. In applications, this is satisfied when 'addition' means the operation of taking minimum or maximum. We first show that the product of any two aperiodic series with non-commuting variables is again aperiodic. This means that the sequentialization of two such aperiodic systems stays aperiodic. We then show that this remains true even with partially commuting variables, but under the additional assumption that K be commutative. By example, we show that the idempotence of K is necessary.

For our further results we need that the matrix monoids $K^{n\times n}$ of $(n \times n)$-matrices over K have a Burnside property: each finitely generated torsion submonoid of $K^{n\times n}$ be finite. We note that since Burnside's question in 1902, this property has been deeply investigated in group and semigroup theory, cf. the surveys of Simon [21] and Pin [15] for its relevance to automata theory. By a deep result of Simon [20], the important tropical semiring $(\mathbb{N}\cup\{\infty\}, \min, +)$ (and several others) satisfies this property. We can then show that any series over K and partially commuting variables is aperiodic if and only if it is star-free. If here $K = \mathbb{B}$, the Boolean semiring, we obtain as a consequence that in trace monoids the aperiodic languages are precisely the star-free ones, a result of Guaiana, Restivo and Salemi [7] which in turn contains Schützenberger's classical result for aperiodic languages of words.

Note that this provides a syntactic construction of how to obtain the behavior of a given aperiodic concurrent system by combining singleton automata using the operations parallel sum, sequentialization, and complementation.

Some results presented here can be extended to more general monoids than trace monoids. Due to space limitations, we do not include these generalizations here.

2 Preliminaries

Here we recall the necessary notation and background for formal power series and for trace theory. For more details, we refer the reader to [17, 1–3].

Let M be any monoid and $K = (K, +, \cdot, 0, 1)$ any semiring, i.e., $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot x = x \cdot 0 = 0$ for each $x \in K$. If multiplication is commutative, we say that K is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. For instance, the Boolean semiring $\mathbb{B} = (\{0, 1\}, +, \cdot, 0, 1)$ is both commutative and idempotent. The semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ is commutative but not idempotent. The semiring $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{1\})$ of languages over some alphabet Σ is idempotent but not commutative. The semiring $\mathbb{N}^{n \times n}$ of $(n \times n)$-matrices is neither commutative nor idempotent. Other semirings useful in computer science (and also in optimization problems of operations research [22]) are the min-plus (or max-plus or min-max) semirings over the integers or the reals. For instance the min-plus semiring over the reals $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ is both commutative and idempotent.

A formal power series is a mapping $S : M \to K$ defined by $m \mapsto (S, m)$. It is usually denoted as a formal sum $S = \sum_{m \in M} (S, m).m$. The set $\mathrm{Im}(S) = \{(S, m) \mid m \in M\}$ is called the *image* of S. The set $\mathrm{Supp}(S) = \{m \in M \mid (S, m) \neq 0\}$ is called the *support* of S. If $\mathrm{Supp}(S)$ is finite, then S is called a *polynomial*. The collection of all formal power series is denoted by $K\langle\langle M\rangle\rangle$, and its subset of all polynomials by $K\langle M\rangle$. We consider elements of K also as polynomials in the natural way, having a non-zero entry only at $1 \in M$. If $L \subseteq M$, we define the *characteristic series* of L by $1_L = \sum_{m \in L} 1 \cdot m$.

Let $n \geq 1$ and $[n] = \{1, \ldots, n\}$. We let $K^{n \times n}$ be the monoid of all $(n \times n)$-matrices over K (with matrix multiplication as usual). A series $S \in K\langle\langle M\rangle\rangle$ is called *recognizable*, if there exists an integer $n \geq 1$, a monoid morphism $\mu : M \to K^{n \times n}$ and vectors $\lambda \in K^{1 \times n}, \gamma \in K^{n \times 1}$ such that

$$(S, m) = \lambda \cdot \mu(m) \cdot \gamma = \sum_{i,j \in [n]} \lambda_i \mu(m)_{ij} \gamma_j$$

for each $m \in M$. In this case, the triple (λ, μ, γ) is called a *representation* of dimension n of the series S, and we often shortly write $S = (\lambda, \mu, \gamma)$ to denote this. We let $K^{rec}\langle\langle M\rangle\rangle$ denote the set of all recognizable formal power series.

With componentwise addition, $K\langle\langle M\rangle\rangle$ becomes a commutative monoid: the sum of two series $S, S' \in K\langle\langle M\rangle\rangle$ is defined for $m \in M$ by $(S + S', m) = (S, m) + (S', m)$. In order to define the *(Cauchy) product* of two series, we assume that M is a finitely generated monoid which carries a length function, that is a morphism ℓ from M to $(\mathbb{N}, +)$ such that $\ell(m) = 0$ iff $m = 1$. This ensures in particular that each element $m \in M$ has only finitely many factorizations $m = m_1 \cdot m_2$. Then, the *(Cauchy) product* of two series S, S' in $K\langle\langle M\rangle\rangle$ is the series defined for $m \in M$ by $(S \cdot S', m) = \sum_{m = m_1 \cdot m_2} (S, m_1) \cdot (S', m_2)$. This operation can also be defined in more general settings but such generalizations are not needed in the present paper. With sum and product, $K\langle\langle M\rangle\rangle$ is a semiring.

Next we recall basic notions from trace theory. Trace monoids were introduced by Mazurkiewicz as an important mathematical model for the behaviour of concurrent systems, see also [13, 2, 3] for their well-developed theory. A pair (Σ, I) is called a *trace alphabet*, if Σ is a finite set and I is an irreflexive symmetric binary *independence* relation on Σ. Let $\sim$ denote the smallest congruence on Σ^* containing $\{(ab, ba) \mid a \; I \; b\}$. The quotient monoid $\mathbb{M} = \mathbb{M}(\Sigma, I) := \Sigma^*/\sim$ is called the *trace monoid* (or free partially commutative monoid) over (Σ, I) and its elements are called traces. Note that a trace monoid is finitely generated and carries a length function.

If $w \in \Sigma^*$, we let $[w]$ denote the equivalence class of w in $\mathbb{M}$. Also, let $\alpha(w)$ be the set of all letters of Σ occurring in w, called the *alphabet* of w. Since equivalent words have the same alphabet, we may put $\alpha([w]) = \alpha(w)$.

We say that two subsets A, B of Σ are independent and we write $A \; I \; B$ if $A \times B \subseteq I$. We also say that two words w, w' of Σ^* are independent, denoted by $w \; I \; w'$, if $\alpha(w) \; I \; \alpha(w')$. Similarly, we define $[w] \; I \; [w']$, $w \; I \; A$, etc.

The following generalized Levi's factorization is a very useful and classical result in trace theory.

Lemma 1. *Let $u, v, w_1, \ldots, w_n \in \mathbb{M}$. Then, $uv = w_1 \cdots w_n$ if and only if there are $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ such that $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n$, $w_i = u_i v_i$ for all $1 \leq i \leq n$ and $v_i \; I \; u_j$ for all $1 \leq i < j \leq n$. Moreover, the traces $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ with this property are unique.*

A monoid N is said to be *aperiodic* if there exists some integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$. Clearly, this condition implies that any group contained in N is trivial. The converse is true when the monoid N is finite [5].

A language $L \subseteq M$ is *aperiodic* if there exists a morphism $\varphi : M \to N$ into some finite aperiodic monoid such that $L = \varphi^{-1}(\varphi(L))$. When this holds, we say that the morphism φ recognizes L. The collection of all *star-free* languages in M is defined as the smallest set of languages of M containing all finite languages and which is closed under the operations union, product and complement.

3 Aperiodic series

In this section we introduce *aperiodic* and *weakly aperiodic* series and we study their closure properties.

Definition 1. *A recognizable series $S \in K\langle\langle M \rangle\rangle$ is* aperiodic *if there exists a representation $S = (\lambda, \mu, \gamma)$ with $\mu(M)$ aperiodic, i.e. there is some integer $m \geq 0$ such that $\mu(u^m) = \mu(u^{m+1})$ for all $u \in M$. In this case, we say that the morphism μ is aperiodic.*

There are several ways to weaken this definition. First, we say that a recognizable series is *weakly aperiodic* if there exists some integer $m \geq 0$ such that $(S, uv^m w) = (S, uv^{m+1} w)$ for all $u, v, w \in M$. Clearly, all aperiodic series are also weakly aperiodic. With the techniques of Section 4, one can show that the converse is true when the semiring K is locally finite.

Another possibility would be to require that the submonoid $\mu(M)$ is group-free, i.e. $\mu(M)$ does not contain any non-trivial group as a subsemigroup. Clearly the condition of Definition 1 is stronger. Also with the techniques of Section 4, one can show that the two definitions are equivalent in the case of locally finite semirings.

It is well-known [17] that if a language $L \subseteq M$ is recognizable then so is its characteristic series $1_L \in K\langle\langle M\rangle\rangle$. The converse does not hold for arbitrary semirings but it holds for a wide class of semirings such as positive semirings ($\mathbb{B}$,$\mathbb{N}$, ...) or locally finite semirings (see next section). We show now that the same is also true for aperiodic languages and aperiodic characteristic series.

Proposition 1. *Let K be an arbitrary semiring, let M be an arbitrary monoid and let $L \subseteq M$ be recognizable. Then L is aperiodic iff its characteristic series $1_L \in K\langle\langle M\rangle\rangle$ is aperiodic.*

Proof. First, let N be a finite aperiodic monoid and let $\varphi : M \to N$ be a morphism recognizing L, i.e. $L = \varphi^{-1}(\varphi(L))$. Let $n = |N|$. We identify N with $[n] = \{1, \dots, n\}$, 1 being indeed the neutral element of N. We define $\mu : M \to K^{n\times n}$, $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ by

$$\mu(u)_{i,j} = \begin{cases} 1 & \text{if } i \cdot \varphi(u) = j \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_j = \begin{cases} 1 & \text{if } j \in \varphi(L) \\ 0 & \text{otherwise} \end{cases}$$

One can check that (λ, μ, γ) is an aperiodic representation of 1_L.

Conversely, assume that 1_L is aperiodic, then it is also weakly aperiodic which implies the existence of an integer $m \geq 0$ such that $uv^m w \in L$ iff $uv^{m+1}w \in L$ for all $u, v, w \in M$. Therefore, the syntactic monoid of L is aperiodic. Since we assumed L recognizable, the syntactic monoid of L is also finite and we obtain that L is aperiodic.

From the above result, we deduce that a semiring for which L is recognizable whenever 1_L is recognizable satisfies also that L is aperiodic whenever 1_L is aperiodic.

Proposition 2. *Let K be an arbitrary semiring and let M be an arbitrary monoid. Aperiodic series in $K\langle\langle M\rangle\rangle$ are closed under sum and external product.*

Proof. Easy.

Recall that polynomials are the series with finite supports and they correspond to finite languages when the semiring is $\mathbb{B}$. Finite languages are aperiodic in trace monoids. Therefore, from Propositions 1 and 2 we get

Corollary 1. *Let K be an arbitrary semiring. Then, all polynomials in $K\langle\langle \mathbb{M}\rangle\rangle$ are aperiodic.*

Our aim is now to show that aperiodic series over free monoids are closed under Cauchy product when the semiring K is idempotent. For this, we will use special representations whose existence are proved in the next two lemmas.

When (λ, μ, γ) is a representation of dimension n, we denote by $I = \{i \in [n] \mid \lambda_i \neq 0\}$ the set of so-called initial states and by $F = \{j \in [n] \mid \gamma_j \neq 0\}$ the set of so-called final states.

Lemma 2. *Let K be an arbitrary semiring, M be an arbitrary monoid and $S \in K\langle\langle M \rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ such that j is final iff $\gamma_j = 1$, and for all $w \in M \setminus \{1\}$, $\mu(w)_{ij} = 0$ whenever i is final. Moreover, we may also require that μ is aperiodic if S is an aperiodic series and that there is exactly one final state.*

Proof. Let (λ, μ, γ) be a representation of S of dimension n. We define a representation $(\lambda', \mu', \gamma')$ of dimension $n+1$ by $\lambda'_i = \lambda_i$ if $1 \leq i \leq n$, $\lambda'_{n+1} = \sum_{1 \leq j \leq n} \lambda_j \gamma_j$, $\gamma' = (0, \ldots, 0, 1)^t$ and for $w \in M \setminus \{1\}$,

$$\mu'(w)_{ij} = \begin{cases} \mu(w)_{ij} & \text{if } 1 \leq i, j \leq n \\ \sum_{1 \leq q \leq n} \mu(w)_{iq} \gamma_q & \text{if } 1 \leq i \leq n, j = n+1 \\ 0 & \text{otherwise (i.e. if } i = n+1). \end{cases}$$

One can check that $(\lambda', \mu', \gamma')$ is an aperiodic representation of S which fulfills the requirements of the lemma.

Using a similar proof, we also obtain

Lemma 3. *Let K be an arbitrary semiring, M be an arbitrary monoid and $S \in K\langle\langle M \rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ such that i is initial iff $\lambda_i = 1$ and for all $w \in M \setminus \{1\}$, $\mu(w)_{ij} = 0$ whenever j is initial. Moreover, we may also require that μ is aperiodic if S is an aperiodic series and that there is exactly one initial state.*

Note that it is also possible to find a representation which satisfies both requirements of Lemmas 2 and 3 if the series S is proper, i.e. if $(S, 1) = 0$.

We are now ready to prove that aperiodic series over the free monoid Σ^* are closed under Cauchy product when the semiring K is idempotent.

Theorem 1. *Assume that the semiring K is idempotent. Let $S_1, S_2 \in K\langle\langle \Sigma^* \rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension n_1 which satisfies the requirements of Lemma 2 and let $(\lambda^2, \mu^2, \gamma^2)$ be an aperiodic representation of S_2 of dimension n_2 which satisfies the requirements of Lemma 3. Let $n = n_1 \cdot n_2$ and identify $[n]$ with $[n_1] \times [n_2]$. We define the morphism $\mu : \Sigma^* \to K^{n \times n}$ by giving its value over letters $a \in \Sigma$:

$$\mu(a)_{(j_1,j_2)(k_1,k_2)} = \delta_{j_2,k_2} \mu^1(a)_{j_1,k_1} + \delta_{j_1,k_1} \mu^2(a)_{j_2,k_2} f(a, j_1)$$

where

$$\delta_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(v,j) = \begin{cases} 1 & \text{if } v = 1 \text{ or } j \text{ is final} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(a, j_1)\mu^1(a)_{j_1,k_1} = 0$, hence at most one of the two terms in the definition of $\mu(a)_{(j_1,j_2)(k_1,k_2)}$ is non-zero.

We can prove by induction that for all $w \in \Sigma^*$ we have

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} \mu^1(u)_{i_1 j_1} \mu^2(v)_{i_2 j_2} f(v, j_1).$$

Now, we define $\lambda \in K^{1\times n}$ by $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$ and $\gamma \in K^{n\times 1}$ by $\gamma_{(j_1,j_2)} = \gamma^1_{j_1}\gamma^2_{j_2}$. We can show that $S = S_1 \cdot S_2 = (\lambda, \mu, \gamma)$. Finally, it remains to show that the morphism μ is aperiodic. Let $q > 1$ be such that $\mu^i(w^q) = \mu^i(w^{q-1})$ for all $w \in \Sigma^*$ and $i = 1, 2$. We claim that for $m > 2q$ we have $\mu(w^m) = \mu(w^{m+1})$ for all $w \in \Sigma^*$.

Fix $(i_1, i_2), (j_1, j_2) \in [n]$ and let $\varphi(u, v) = \mu^1(u)_{i_1 j_1}\mu^2(v)_{i_2 j_2} f(v, j_1)$ for all $u, v \in \Sigma^*$. If $m > 2q$ and $w^m = uv$ then we have either $u = w^q u'$ or $v = v' w^q$. In the first case, using the fact that $\mu^1(w^q) = \mu^1(w^{q-1}) = \mu^1(w^{q+1})$ we deduce that $\varphi(u, v) = \varphi(w^{q-1}u', v) = \varphi(w^{q+1}u', v)$. We proceed similarly in the second case and we obtain $\{\varphi(u, v) \mid w^m = uv\} = \{\varphi(u, v) \mid w^{m+1} = uv\}$.

From this, using the fact that sum is idempotent, we deduce

$$\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{w^m=uv} \varphi(u, v) = \sum_{w^{m+1}=uv} \varphi(u, v) = \mu(w^{m+1})_{(i_1,i_2)(j_1,j_2)}$$

which completes the proof of the theorem.

The idempotence of sum is needed to obtain this result, even for the free monoid. Indeed, let $K = (\mathbb{N}, +, \cdot)$ and $\Sigma = \{a\}$. The language Σ^* is clearly aperiodic and therefore its characteristic series $S = 1_{\Sigma^*}$ is aperiodic as well (Proposition 1). Now, for all $m \geq 0$, $(S \cdot S, a^m) = \sum_{a^m=uv}(S, u)(S, v) = m + 1$. Therefore, the series S^2 is not weakly aperiodic, whence is not aperiodic.

We aim now at establishing a similar result for arbitrary trace monoids. Alphabetic representations have been introduced in [4] in order to study the closure of recognizable series over trace monoids under product and star. We will use a simpler form of alphabetic representations which is sufficient to show that aperiodic series are also closed under product.

We say that a representation (λ, μ, γ) of dimension n is *(past-)alphabetic*, if there exists a function $\overleftarrow{\alpha} : [n] \to \mathcal{P}(\Sigma)$ such that for all $u \in \mathbb{M}$, the following two conditions are satisfied:
(1) Whenever $\mu(u)_{ij} \neq 0$, then $\overleftarrow{\alpha}(j) = \overleftarrow{\alpha}(i) \cup \alpha(u)$
(2) whenever $\lambda_i \neq 0$, then $\overleftarrow{\alpha}(i) = \emptyset$.
We call $(\lambda, \mu, \gamma; \overleftarrow{\alpha})$ an alphabetic representation of S.

Proposition 3. *Let K be an arbitrary semiring and $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be an aperiodic series over traces. Then there exists an alphabetic representation $S = (\lambda, \mu, \gamma; \overleftarrow{\alpha})$ with μ aperiodic.*

Proof. The construction of the representation follows the same line as in [4, Proposition 6]. One can check in addition that this representation is aperiodic.

We will now prove that aperiodic series over trace monoids are closed under Cauchy product when the semiring K is idempotent and commutative.

Theorem 2. *Assume that the semiring K is idempotent and commutative. Let $S_1, S_2 \in K\langle\langle\mathbb{M}\rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Again, we use the construction of [4, Theorem 7]. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension n_1 and let $(\lambda^2, \mu^2, \gamma^2; \overleftarrow{\alpha})$ be an alphabetic representation of S_2 of dimension n_2 with μ_2 aperiodic (Proposition 3). Let $n = n_1 \cdot n_2$ and identify $[n]$ with $[n_1] \times [n_2]$. Next, we define $\mu : \mathbb{M} \to K^{n \times n}$ by

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} I(u, i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$$

where

$$I(u, i) = \begin{cases} 1 & \text{if } u \; I \; \overleftarrow{\alpha}(i) \\ 0 & \text{otherwise.} \end{cases}$$

It was shown in [4, Theorem 7] that μ is a morphism and $S = (\lambda, \mu, \gamma)$ where $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ are defined by $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$ and $\gamma_{(k_1,k_2)} = \gamma^1_{k_1}\gamma^2_{k_2}$.

We have assumed μ^1 and μ^2 aperiodic so let $q > 1$ be such that $\mu^1(w^q) = \mu^1(w^{q-1})$ and $\mu^2(w^q) = \mu^2(w^{q-1})$ for all $w \in \mathbb{M}$. We will show that μ is aperiodic. Throughout the rest of the proof, we fix $(i_1, i_2), (j_1, j_2) \in [n]$. For $w \in \mathbb{M}$ and $m \geq 0$, we define

$$X_m(w) = \{(u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{M}^{2m} \mid \forall i \in [m], w = u_i v_i \\ \text{and } \forall 1 \leq i < j \leq m, v_i \; I \; u_j\}.$$

Also, for $x = (u_1, \ldots, u_m, v_1, \ldots, v_m)$, we set $\varphi(x) = I(u, i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$ where $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_m$ (recall that $(i_1, i_2), (j_1, j_2)$ have been fixed). Using the generalized Levi's factorization (Lemma 1) and the definition of μ we deduce immediately that $\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{x\in X_m(w)} \varphi(x)$. Note that, thanks to the unicity of the factorization in Levi's Lemma, we do not use that sum is idempotent for this result.

Claim. If $m > q(|\Sigma|+1)$ then $\varphi(X_m(w)) = \varphi(X_{m+1}(w))$. Actually, we will prove that for all $x \in X_m(w)$, there exist $y \in X_{m-1}(w)$ and $z \in X_{m+1}(w)$ such that $\varphi(x) = \varphi(y) = \varphi(z)$.

Let $x = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in X_m(w)$. For $1 \leq i < j \leq m$ we have $v_i \; I \; u_j$, which implies that $\alpha(v_i) \subseteq \alpha(w) \setminus \alpha(u_j) \subseteq \alpha(v_j)$. Therefore we have $\alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$ and from the hypothesis on m we deduce that there exists $1 \leq p \leq m - q$ such that $\alpha(v_p) = \cdots = \alpha(v_{p+q})$.

Let $p < k \leq p + q$, we have $\alpha(w) = \alpha(u_k) \cup \alpha(v_k)$ and $\alpha(u_k) \; I \; \alpha(v_{k-1}) = \alpha(v_k)$. Hence we obtain $\alpha(u_k) = \alpha(w) \setminus \alpha(v_k)$. Since there is at most one way to split the trace w into two independent traces whose alphabet are fixed, we deduce that $u_{p+1} = \cdots = u_{p+q} = \bar{u}$ and $v_{p+1} = \cdots = v_{p+q} = \bar{v}$.

Let $u' = u_1 \cdots u_p$, $u'' = u_{p+q+1} \cdots u_m$, $v' = v_1 \cdots v_p$ and $v'' = v_{p+q+1} \cdots v_m$. Then we have

$$\begin{aligned}
\mu^1(u) &= \mu^1(u'\bar{u}^q u'') = \mu^1(u'\bar{u}^{q+1} u'') = \mu^1(u'\bar{u}^{q-1} u'') \\
\mu^2(v) &= \mu^2(v'\bar{v}^q v'') = \mu^2(v'\bar{v}^{q+1} v'') = \mu^2(v'\bar{v}^{q-1} v'') \\
\alpha(u) &= \alpha(u'\bar{u}^q u'') = \alpha(u'\bar{u}^{q+1} u'') = \alpha(u'\bar{u}^{q-1} u'')
\end{aligned}$$

The claim follows since this implies that $\varphi(x) = \varphi(y) = \varphi(z)$ for

$$\begin{aligned}
y &= (u_1, \ldots, u_{p+q-1}, u_{p+q+1}, \ldots, u_m, v_1, \ldots, v_{p+q-1}, v_{p+q+1}, \ldots, v_m) \in X_{m-1}(w) \\
z &= (u_1, \ldots, u_{p+q}, u_{p+q}, \ldots, u_m, v_1, \ldots, v_{p+q}, v_{p+q}, \ldots, v_m) \in X_{m+1}(w).
\end{aligned}$$

Using the claim, we deduce immediately that for $m > q(|\Sigma| + 1)$ it holds

$$\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{x \in X_m(w)} \varphi(x) = \sum_{x \in X_{m+1}(w)} \varphi(x) = \mu(w^{m+1})_{(i_1,i_2)(j_1,j_2)}.$$

Here we have used that sum is idempotent in K since $|X_m(w)| < |X_{m+1}(w)|$ as soon as $w \neq 1$.

Commutativity is needed because we are dealing with trace monoids. An example was given in [4, Section 8] showing that recognizable series over trace monoids are not closed under product in general.

As usual, in the special case of the boolean semiring we obtain the result on trace languages as a corollary.

Corollary 2 ([7]). *The product of two aperiodic trace languages is again aperiodic.*

4 Aperiodic series over semirings with Burnside matrix monoids.

In this section we will derive a representation of aperiodic formal power series over semirings with local finiteness conditions which will be crucial later on.

Recall that a semiring K is locally finite, if each finitely generated subsemiring is finite. A monoid is called locally finite, if each finitely generated submonoid is finite. Clearly, any commutative idempotent monoid is locally finite. Also, a semiring $(K, +, \cdot, 0, 1)$ is locally finite iff both monoids $(K, +, 0)$ and $(K, \cdot, 1)$ are locally finite.

For example, the max-min semiring $\mathcal{R}_{\max,\min} = (\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ of positive reals, used in operations research for maximum capacity problems of networks, is locally finite. In fact, any distributive lattice $(L, \vee, \wedge, 0, 1)$ with smallest element 0 and largest element 1 is a locally finite semiring.

Note that if K is a locally finite semiring, then the matrix monoids $K^{n\times n}$ are locally finite for all n. The converse is also true.

For applications on aperiodic series, we use weaker assumptions on K. A monoid is called *torsion* (or *periodic*), if each of its cyclic submonoids is finite. Each locally finite monoid is torsion. The converse, which was posed as a problem for groups by Burnside in 1902 (and answered negatively by Golod in 1964), leads to deep problems in semigroup theory, see the surveys by Simon [21] and Pin [15] for its relevance to automata theory. We will say that a semiring K has *Burnside matrix monoids*, if in the monoids $K^{n\times n} (n \in \mathbb{N})$ each finitely generated torsion submonoid is finite. Before continuing, let us point out examples of such semirings. Clearly, if K is locally finite then K has Burnside matrix monoids. But also all of the following important (not locally finite) semirings have been shown to have Burnside matrix monoids:

Theorem 3. *Each of the following semirings has Burnside matrix monoids:*

- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ *and its completion* $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ *(Mandel and Simon [11]),*
- $\mathcal{M} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ *(Simon [20]),*
- $\mathcal{P} = (\mathbb{N} \cup \{-\infty, \infty\}, \max, +, -\infty, 0)$ *(Mascle [12]),*
- $\mathcal{R}at = (Rat(a^*), \cup, \cdot, \emptyset, a^*)$ *(Mascle [12]).*

Now let $S \in K\langle\langle M\rangle\rangle$. If $x \in K$, we let $S^{-1}(x) = \{m \in M \mid (S, m) = x\}$. Now we obtain the following finitary description of aperiodic series:

Proposition 4. *Let M be a finitely generated monoid, let K be a semiring with Burnside matrix monoids and $S \in K\langle\langle M\rangle\rangle$ be aperiodic. Then,* $\mathrm{Im}(S)$ *is finite and* $S = \sum_{x\in\mathrm{Im}(S)} x \cdot 1_{S^{-1}(x)}$ *is a finite linear combination of characteristic series with each $S^{-1}(x)$ aperiodic. In particular,* $\mathrm{Supp}(S)$ *is aperiodic.*

Proof. Let (λ, μ, γ) be an aperiodic representation of S. Since μ is aperiodic and M is finitely generated, $\mu(M)$ is a finitely generated torsion submonoid of $K^{n\times n}$ and hence finite by assumption on K. Then, $\mathrm{Im}(S) = \{\lambda \cdot A \cdot \gamma \mid A \in \mu(M)\}$ is also finite and clearly $S = \sum_{x\in\mathrm{Im}(S)} x \cdot 1_{S^{-1}(x)}$. It is easy to show that $\mu : M \to \mu(M)$ recognizes each $S^{-1}(x)$. Therefore, each $S^{-1}(x)$ is aperiodic in M.

5 Star-free series

In a monoid M, the collection of star-free languages is defined as the smallest system of languages in M containing all finite languages and being closed under the operations union, complement and product. We define a corresponding notion for formal power series as follows. If K is a semiring and $L \subseteq M$, the *complement* of the characteristic series 1_L is simply the characteristic series $\overline{1_L} := 1_{\overline{L}}$ of the complement $\overline{L}$ of L. Hence complement is only a partially defined operation on $K\langle\langle M\rangle\rangle$, provided K has more than 2 elements.

Definition 2. *The collection of all star-free series in $K\langle\langle\mathbb{M}\rangle\rangle$ is the smallest collection of formal power series containing all polynomials and being closed under the operations union, product and the partial operation complement.*

Note that if S is star-free and $k \in K$, then the series $k \cdot S$ is star-free, since k is a polynomial.

Proposition 5. *Let K be idempotent and $L \subseteq \mathbb{M}$ star-free. Then 1_L is a star-free series.*

Proof. By structural induction on L. If L is finite, 1_L is a polynomial. If $L = L_1 \cup L_2$, we have $1_L = 1_{L_1} + 1_{L_2}$ since $(K,+)$ is idempotent. For the same reason, $L = L_1 \cdot L_2$ implies $1_L = 1_{L_1} \cdot 1_{L_2}$. Finally, $1_{\overline{L}} = \overline{1_L}$ by definition.

Schützenberger [19] showed that in the free monoid Σ^*, aperiodic languages coincide with star-free languages. This was generalized by Guaiana Restivo and Salemi [7] to trace monoids. Here we further extend this equivalence to formal power series over trace monoids and idempotent semirings with Burnside matrix monoids, in particular for the semirings $\mathcal{M}$, $\mathcal{P}$, $\mathcal{R}at$ (Theorem 3) and $\mathcal{R}_{\max,\min}$, or any distributive lattice with 0 and 1.

Theorem 4. *Let K be an idempotent semiring with Burnside matrix monoids. For a series $S \in K\langle\langle\mathbb{M}\rangle\rangle$, the following are equivalent:*

(1) S is aperiodic,
(2) S is star-free,
(3) S is a finite linear combination of characteristic series of aperiodic languages.

Proof. (1) $\Rightarrow$ (3) is Proposition 4.

(3) $\Rightarrow$ (2) follows from Proposition 5 since in trace monoids, aperiodic languages are star-free.

(2) $\Rightarrow$ (1): We proceed by induction. By Corollary 1 all polynomials are aperiodic. By Proposition 2 the sum of two aperiodic series is again aperiodic.

Now we show that the product of two aperiodic series S_1 and S_2 is again aperiodic. By Proposition 4, we have $S_1 = \sum_{x \in I_1} x \cdot 1_{S_1^{-1}(x)}$ and $S_2 = \sum_{y \in I_2} y \cdot 1_{S_2^{-1}(y)}$ with $I_i = \mathrm{Im}(S_i)$ finite $(i = 1,2)$ and each $S_1^{-1}(x), S_2^{-1}(y)$ aperiodic in $\mathbb{M}$. For all $a, b \in K$ and $A, B \subseteq \mathbb{M}$ we have $(a \cdot 1_A) \cdot (b \cdot 1_B) = ab \cdot (1_A \cdot 1_B)$ and since K is idempotent we also have $1_A \cdot 1_B = 1_{A \cdot B}$. Hence,

$$S_1 \cdot S_2 = \sum_{x \in I_1, y \in I_2} x \cdot y \cdot 1_{S_1^{-1}(x)} \cdot 1_{S_2^{-1}(y)} = \sum_{x \in I_1, y \in I_2} x \cdot y \cdot 1_{S_1^{-1}(x) \cdot S_2^{-1}(y)},$$

and each $S_1^{-1}(x) \cdot S_2^{-1}(y)$ is aperiodic in $\mathbb{M}$. Now by Propositions 1 and 2, $S_1 \cdot S_2$ is aperiodic.

Finally, let 1_L be aperiodic. Then $L = \mathrm{Supp}(1_L)$ is aperiodic by Proposition 4, hence $\overline{L}$ is aperiodic. Then $\overline{1_L} = 1_{\overline{L}}$ is aperiodic by Proposition 1.

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