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Computational Complexity of Multimodal Logics Based on Rough Sets

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Abstract. We characterize the computational complexity of a family of approximation multimodal logics in which interdependent modal connectives are part of the language. Those logics have been designed to reason in presence of incomplete information in the sense of rough set theory. More precisely, we show that all the logics have a **PSPACE**-complete satisfiability problem and we define a family of tolerance approximation multimodal logics whose satisfiability is **EXPTIME**-complete. This illustrates that the **PSPACE** upper bound for this kind of multimodal logics is a very special feature of such logics. The **PSPACE** upper bounds are established by adequately designing Ladner-style tableaux-based procedures whereas the **EXPTIME** lower bound is established by reduction from the global satisfiability problem for the standard modal logic B.

Keywords: rough sets, tolerance rough sets, multimodal logics, computational complexity, Ladner-style algorithm.

1. Introduction

Rough sets and logics. Since the introduction of *rough sets* in [18], rough set theory has been an ever growing field on its own right and many directions have been already explored (see e.g. valuable surveys in [19, 15, 21, 22]). As rough sets can be viewed as manifestation of incomplete information, many logics based on rough set theory have been designed for mechanizing reasoning in the presence of incomplete information, leading to interesting notions of approximation (see e.g. [16, 5, 26, 31, 12, 28]). Many logics extend the current state of the art in modal logic theory and many new problems arise, challenging the existing proof techniques for modal logics. The introduction of the *copying* model-theoretic construction in [31] is certainly one of the best illustration of this phenomenon. In this work, we focus our attention on the computational complexity of the satisfiability problems for multimodal logics where interdependent modal connectives are part of the language. Actually, no general technique exists and we propose Ladner-style decision procedures for the family of approximation multimodal logics $\mathrm{AML}(\tau_m)$, $m \geq 1$ introduced and investigated in [26, 28]. Although it is known that each logic of the family is **PSPACE**-hard, the **PSPACE** upper bound has been an open problem up to now. Besides, those logics admit Hilbert-style proof systems using only Sahlqvist formulae [23] as extra modal axioms but this is unfortunately of no help to completely characterize the computational complexity of such logics.

Our contribution. The main contribution of the paper is to fully characterize the computational complexity of approximation multimodal logics. Actually, we show that all the approximation multimodal logics $\mathrm{AML}(\tau_m)$, $m \geq 1$, have a **PSPACE**-complete satisfiability problem (see e.g. [17] for a thorough introduction to complexity theory). So, each logic $\mathrm{AML}(\tau_m)$ captures the difficulty of the whole complexity class **PSPACE**, that is the class of (decision) problems that can be solved by a deterministic Turing machine in polynomial space in the length of the input string. **PSPACE**-hardness with respect to logarithmic space transformations is shown to be an easy consequence of **PSPACE**-hardness of the well-known modal logic K [13]. The main difficulty is to show that $\mathrm{AML}(\tau_m)$ satisfiability is in **PSPACE**. To do so we present an original construction that extends various previous works in [13, 11]. Furthermore, we define tolerance approximation multimodal logics $\mathrm{TAML}(\tau_m)$ and we show that these logics are **EXPTIME**-complete leading to the conclusion that the **PSPACE** upper bound for $\mathrm{AML}(\tau_m)$ is a very remarkable feature.

Related work. The procedure designed in this paper has a direct filiation with the works of Ladner [13] and Halpern and Moses [11]. Indeed, we shall use a tableau-based procedure to show that we do not need more than polynomial space to check satisfiability. We cannot take advantage of [10] where complexity of join modal logics is characterized ($\mathrm{AML}(\tau_m)$ contains interdependent modal connectives). Other proof-theoretical analysis about complexity issues for modal logics can be found in [14, 32]. The techniques we use in this paper have been successfully applied to another modal logic with interdependent modal connectives [3].

Organization of the paper. In the next section we recall definitions of different rough set models. In Section 3 we discuss basic properties of Approximation Multimodal Logics. In Section 4 we investigate properties of a closure operator for sets of formulae. In Section 5 we prove that the satisfiability problem of the Approximation Multimodal Logics is in **PSPACE**. In Section 6 we prove that the satisfiability problem of the Tolerance Approximation Multimodal Logics is **EXPTIME**-complete.

2. Selected Rough Set Models

Rough set approach has been used in a lot of applications aimed at description of concepts. In most cases only approximate descriptions of concepts can be constructed because of incomplete information about them. Let us consider a typical example for classical rough set approach when concepts are described by positive and negative examples. In such situations it is not always possible to describe concepts exactly, since some positive and negative examples of the concepts being described inherently cannot be distinguished one from another. Rough set theory was proposed [19] as a new approach to vague concept description from incomplete data. The rough set approach to processing of incomplete data is based on the lower and upper approximations. A rough set is defined as a pair of two crisp sets corresponding to approximations. If both approximations of a given subset of the universe are exactly the same, then one can say that the subset mentioned above is definable with respect to available information. Otherwise, one can consider it as roughly definable. Suppose we are given a finite non-empty set U of objects, called the universe. Each object of U is characterized by a description constructed, for example, from a set of attribute values. In standard rough set approach [19] introduced by Pawlak, an equivalence relation (reflexive, symmetric and transitive relation) on the universe of objects is defined from equivalence relations on the attribute values. In particular, this equivalence relation is constructed assuming the existence of the equality relation on attribute values. Two different objects are indiscernible in view of available information, if the same information can be associated with these objects. Thus, information associated with objects from the universe generates an indiscernibility relation in this universe. In the standard rough set model the lower approximation of any subset $X \subseteq U$ is defined as the union of all equivalence classes fully included in X. On the other hand the upper approximation of X is defined as the union of all equivalence classes with a non-empty intersection with X. In modal logic, those approximation operators correspond to necessity and possibility, respectively.

In real data sets usually there is some noise, caused for example from imprecise measurements or mistakes made during collecting data. In such situations the notions of "full inclusion" and "non-empty intersection" used in approximations definition are too restrictive. Some extensions in this direction have been proposed in the variable precision rough set model.

The indiscernibility relation can be also employed in order to define not only approximations of sets but also approximations of relations. Investigations on relation approximation are well motivated both from theoretical and practical points of view. Let us bring two examples. The equality approximation is fundamental for a generalization of the rough set approach based on a similarity relation approximating the equality relation in the value sets of attributes. Rough set methods in control processes require function approximation [21].

However, the classical rough set approach is based on the indiscernibility relation defined by means of the equality relations in different sets of attribute values. In many applications instead of these equalities some similarity (tolerance) relations are given only. This observation has stimulated some researchers to generalize the rough set approach to deal with such cases, i.e., to consider similarity (tolerance) classes instead of the equivalence classes as elementary definable sets. There is one more basic notion to be considered, namely the rough inclusion of concepts. This kind of inclusion should be considered instead of the exact set equality because of incomplete information about the concepts. The two notions mentioned above, namely the generalization of equivalence classes to similarity classes (or in more general cases to some neighborhoods) and the equality to rough inclusion have lead to a generalization of classical approximation spaces defined by the universe of objects together with the indiscernibility relation being an equivalence relation.

One of the problems we are interested in is the following: given a subset $X \subseteq U$ or a relation $R \subseteq U \times U$, define X or R in terms of the available information. Using an approach based on generalized approximation spaces introduced and investigated in [27] we can combine in one model not only some extension of the concept of indiscernibility relation but also some extension of the concept of standard inclusion used in definitions of approximations in the standard rough set model.

We recall general definition of an approximation space [27], [29] which can be used for example for introducing the tolerance based rough set model and the variable precision rough set model.

For every non-empty set U, let $\mathcal{P}(U)$ denote the set of all subsets of U.

Definition 2.1. A *parameterized approximation space* is a system $AS_{\#,\$} = (U, I_{\#}, \nu_{\$})$, where

- U is a non-empty set of objects,
- $I_{\#} : U \to \mathcal{P}(U)$ is an uncertainty function,
- $\nu_{\$} : \mathcal{P}(U) \times \mathcal{P}(U) \to [0,1]$ is a rough inclusion function.

and $\#, \$$ are denoting vectors of parameters.

The uncertainty function defines for every object x a set of similarly described objects. A constructive definition of uncertainty function can be based on the assumption that some metrics (distances) are given on attribute values. For example, if for some attribute $a \in A$, a metric $\delta_a : V_a \times V_a \longrightarrow [0,\infty)$ is given, where V_a is the set of all values of attribute a, then one can define the following uncertainty function:

$$y \in I_a^{f_a}(x) \text{ if and only if } \delta_a(a(x), a(y)) \leq f_a(a(x), a(y)),$$

where $f_a : V_a \times V_a \to [0,\infty)$ is a given threshold function.

A set $X \subseteq U$ is *definable in* the parametrized approximation space $AS_{\#,\$}$ if and only if it is a union of some values of the uncertainty function.

The rough inclusion function defines the degree of inclusion between two subsets of U [27], [29].

We will consider the standard rough inclusion (we assume that U is finite):

$$\nu_{SRI}(X,Y) = \begin{cases} \frac{card(X \cap Y)}{card(X)} & \text{if } X \neq \emptyset \\ 1 & \text{if } X = \emptyset \end{cases}.$$

$\nu_{SRI}(X,Y)$ coincides with the conditional probability $Pr(Y \mid X)$.

The lower and the upper approximations of subsets of U are defined as follows.

Definition 2.2. For an approximation space $AS_{\#,\$} = (U, I_{\#}, \nu_{\$})$ and any subset $X \subseteq U$ the lower and the upper approximations of X are defined by

$LOW\,(AS_{\#,\$}, X) = \{x \in U : \nu_{\$}\,(I_{\#}\,(x)\,, X) = 1\}\,,$

$UPP\,(AS_{\#,\$}, X) = \{x \in U : \nu_{\$}\,(I_{\#}\,(x)\,, X) > 0\}$, respectively.

Approximations of concepts (sets or relations) are constructed on the basis of background knowledge. Hence it is very useful to define parameterized approximations with parameters tuned in the searching process for approximations of concepts. This idea is crucial for construction of concept approximations using rough set methods. In our notation $\#, \$$ are denoting vectors of parameters which can be tuned in the process of concept approximation.

In this paper we discuss multimodal logics based on standard and tolerance rough set models. In the tolerance rough set model we consider approximation spaces AS of the form $AS = (U, I, \nu_{SRI})$ with two conditions for an uncertainty function I :

- For every $x \in U$, we have $x \in I\,(x)$ (called reflexivity).
- For every $x, y \in U$, if $y \in I\,(x)$, then $x \in I\,(y)$ (called symmetry).

In the standard rough set model there is one additional condition called transitivity:

- For every $x, y, z \in U$, if $y \in I\,(x)$ and $z \in I\,(y)$, then $z \in I\,(x)$.

Example 2.1. Approximations of relations in the standard rough set model.
Let the universe $U = \{x_i : i = 1, \ldots, 11\}$ (see Table 1) and let R be a binary relation such that $R = \{(x_2, x_6), (x_2, x_7), (x_3, x_8), (x_3, x_9), (x_4, x_8), (x_4, x_9)\}$.

Assume that objects are described by two attributes a_1 and a_2. For attribute a_1 we consider three intervals: $\leq 160, [161, 180], \geq 181$. We define an uncertainty function I_{ind} by

$y \in I_{ind}(x)$ if and only if $a_1(x), a_1(y)$ are from the same interval and $a_2(x) = a_2(y)$.

The data table and the uncertainty function I_{ind} are described in Table 1.

One can obtain an equivalence (indiscernibility) relation R_{ind} by

$(x, y) \in R_{ind}$ if and only if $y \in I_{ind}(x)$.

The lower approximation of R is equal to $\{(x_3, x_8), (x_3, x_9), (x_4, x_8), (x_4, x_9)\}$. The upper approximation of R is equal to $\{(x_3, x_8), (x_3, x_9), (x_4, x_8), (x_4, x_9), (x_1, x_6), (x_1, x_7), (x_2, x_6), (x_2, x_7)\}$. The approximations of R are depicted in Figure 1.

Example 2.2. Approximations of sets in the tolerance rough set model.
We consider an uncertainty function I_{sim} defined by
$y \in I_{sim}(x)$ if and only if $|a_1(x) - a_1(y)| \leq 5$ (see also Table 1). We consider an approximation space $AS_{sim} = (U, I_{sim}, \nu_{SRI})$.

Let $X = \{x_2, x_6, x_8\}$. The lower approximation $LOW(AS_{sim}, X)$ of X is equal to $\{x_2\}$ and the upper approximation $UPP(AS_{sim}, X)$ of X is equal to $\{x_1, x_2, x_3, x_6, x_8\}$.

	a_1	a_2	I_{ind}	I_{sim}
x_1	155	f	$\{x_1, x_2\}$	$\{x_1, x_2, x_7\}$
x_2	158	f	$\{x_1, x_2\}$	$\{x_1, x_2, x_6\}$
x_3	168	f	$\{x_3, x_4\}$	$\{x_3, x_4, x_8\}$
x_4	172	f	$\{x_3, x_4\}$	$\{x_3, x_4, x_9\}$
x_5	184	f	$\{x_5\}$	$\{x_5, x_{10}, x_{11}\}$
x_6	161	m	$\{x_6, x_7\}$	$\{x_2, x_6, x_8\}$
x_7	152	m	$\{x_6, x_7\}$	$\{x_1, x_7\}$
x_8	164	m	$\{x_8, x_9\}$	$\{x_3, x_6, x_8\}$
x_9	176	m	$\{x_8, x_9\}$	$\{x_4, x_9, x_{10}\}$
x_{10}	181	m	$\{x_{10}, x_{11}\}$	$\{x_5, x_9, x_{10}\}$
x_{11}	187	m	$\{x_{10}, x_{11}\}$	$\{x_5, x_{11}\}$

Table 1. Data Table and Uncertainty Functions

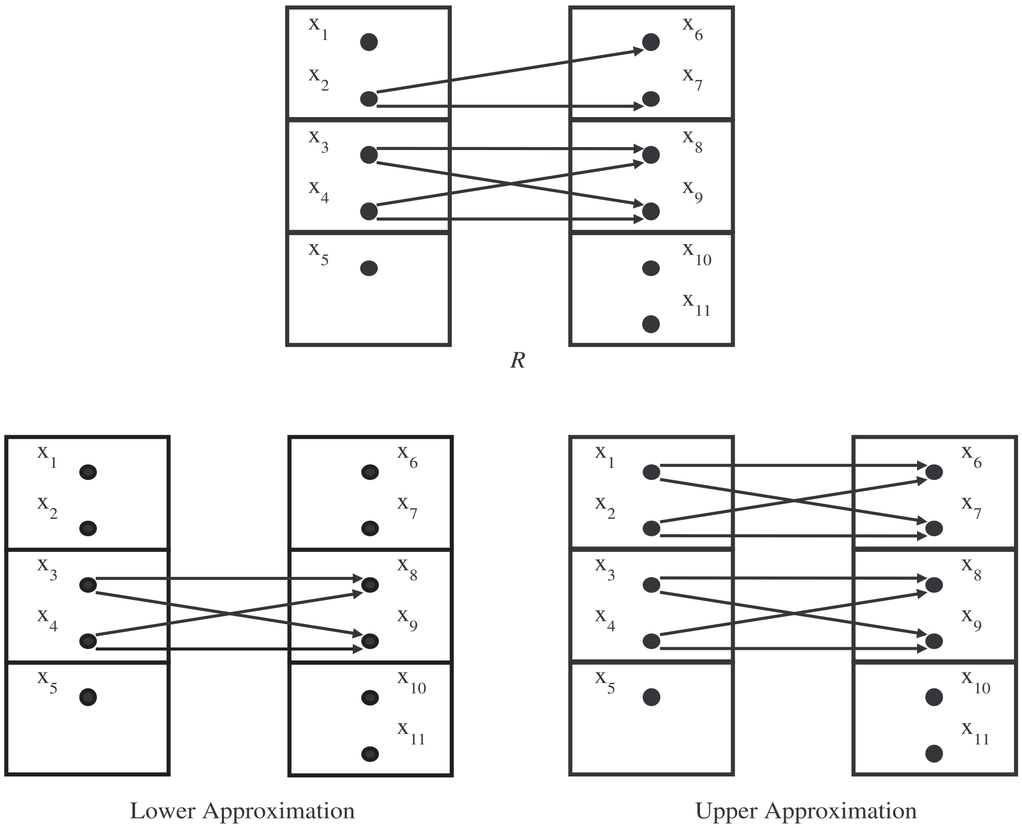

Figure 1. Approximations of Relation R

3. Approximation Multimodal Logics

For any set X, we write X^* to denote the set of finite strings built from elements of X. λ denotes the empty string. For any finite string s, we write $|s|$ [resp. $last(s)$] to denote its length [resp. the last element of s, if any]. For $s \in X^*$, for $j \in \{1, \ldots, |s|\}$, we write $s(j)$ [resp. $s[j]$] to denote the jth element of s [resp. to denote the initial substring of s of length j]. By convention $s[0] = \lambda$. For any $s \in X^*$, we write s^k to denote the string composed of k copies of s. For instance, $(ind \cdot r_1)^2 = ind \cdot r_1 \cdot ind \cdot r_1$ and $|(ind \cdot r_1)^2| = 4$.

Given a countably infinite set $\mathtt{For}_0 = \{\mathtt{p}_0, \mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of *propositional variables*, for $m \in \omega \cup \{\omega\}$ the $\mathtt{L}_m$-formulae ϕ are inductively defined as follows:

$$\phi ::= \mathtt{p}_k \mid \phi_1 \wedge \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi \mid [ind]\phi \mid [L(R_i)]\phi \mid [R_i]\phi$$

for $k \in \omega$, $i < m$. Each modal connective $[R_i]$ refers to a relation R_i in the semantical structures whereas $[L(R_i)]$ is the modal connective associated to the lower approximation $L(R_i)$ of R_i. The operator $[ind]$ is intended to allow us to reason about an indiscernibility relation from the underlying semantical structures (see below). We write $|\phi|$ to denote the *length* of the formula ϕ, that is the length of the string ϕ. We write $md(\phi)$ to denote the modal degree of ϕ, that is the modal depth of ϕ. md is naturally extended to finite sets of formulae, understood as conjunctions and by convention $md(\emptyset) = 0$.

For $m \in \omega \cup \{\omega\}$, we define an Hilbert-style proof system $\mathtt{H}_m$ whose axioms are:

1. the tautologies of the propositional calculus;
2. for $\mathtt{a} \in \{ind\} \cup \{R_i, L(R_i) : i < m\}$, $([\mathtt{a}]\phi \wedge [\mathtt{a}](\phi \Rightarrow \psi)) \Rightarrow [\mathtt{a}]\psi$;
3. $[ind]\phi \Rightarrow \phi$;
4. $\phi \Rightarrow [ind]\neg[ind]\neg\phi$;
5. $[ind]\phi \Rightarrow [ind][ind]\phi$;
6. for $i < m$, $[R_i]\phi \Rightarrow [L(R_i)]\phi$;
7. for $i < m$, $[L(R_i)]\phi \Rightarrow [ind][L(R_i)][ind]\phi$;

and the inference rules are:

1. from ϕ and $\phi \Rightarrow \psi$ infer ψ;
2. from ϕ infer $[ind]\phi$;
3. from ϕ infer $[L(R_i)]\phi$ for $i < m$;
4. from ϕ infer $[R_i]\phi$ for $i < m$.

The axiom schema 2. allows us to perform propositional reasoning in the scope of modal connectives, thanks also to the inference rule 1. (Modus Ponens) and to the so-called Necessitation rules 2.-4. Those are standard ingredients to axiomatize normal modal logics. The axiom schemes 3.-5. correspond to the fact that the indiscernibility relation associated to $[ind]$ is an equivalence relation whereas the axiom schema 6. is an inclusion axiom that corresponds to the property that a lower approximation relation $L(R_i)$ is included in R_i.

The system $\mathtt{H}_0$ is the Hilbert-style proof system for the standard modal logic S5 [9] whereas for $1 \leq m < \omega$, $\mathtt{H}_m$ is the proof system defined in [28, Section 4.2] for the Approximation Multimodal Logic AML(τ_m). The system $\mathtt{H}_\omega$ can be viewed as a limit for the systems $\mathtt{H}_m$, $m < \omega$.

For $m \in \omega \cup \{\omega\}$, an m-frame is defined as a structure $\langle W, R_{ind}, (R_i)_{i<m}, (LR_i)_{i<m}\rangle$ such that W is a nonempty set and R_{ind}, the LR_is and the R_is are binary relations on W. An AML$_m$*-frame* is an m-frame such that

1. R_{ind} is an equivalence relation on W;
2. for $i < m$, $LR_i \subseteq R_i$;
3. for $i < m$, $R_{ind} \circ LR_i \circ R_{ind} \subseteq LR_i$.

Because of the condition 1., the condition 3. above can be replaced by for $i < m$, $R_{ind} \circ LR_i \circ R_{ind} = LR_i$. Each relation LR_i is viewed as the lower approximation of the relation R_i with respect to the *indiscerniblity relation* R_{ind}. An [resp. AML$_m$-model] m-model is a structure of the form $\langle W, R_{ind}, (R_i)_{i<m}, (LR_i)_{i<m}, V\rangle$ such that $\langle W, R_{ind}, (R_i)_{i<m}, (LR_i)_{i<m}\rangle$ is an [resp. AML$_m$-frame] m-frame and $V : \mathtt{For}_0 \rightarrow \mathcal{P}(W)$ is a meaning function (valuation).

As is usual for modal logics, the $\mathtt{L}_m$-formula ϕ is *satisfied by* $w \in W$ *in* $\mathcal{M}$ $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathcal{M}, w \models \phi$ where the satisfaction relation $\models$ is inductively defined as follows:

- $\mathcal{M}, w \models \mathtt{p} \stackrel{\text{def}}{\Leftrightarrow} w \in V(\mathtt{p})$, for every propositional variable $\mathtt{p}$;
- $\mathcal{M}, w \models [ind]\phi \stackrel{\text{def}}{\Leftrightarrow}$ for every $w' \in R_{ind}(w)$, $\mathcal{M}, w' \models \phi$;
- $\mathcal{M}, w \models [L(R_i)]\phi \stackrel{\text{def}}{\Leftrightarrow}$ for every $w' \in LR_i(w)$, $\mathcal{M}, w' \models \phi$;
- $\mathcal{M}, w \models [R_i]\phi \stackrel{\text{def}}{\Leftrightarrow}$ for every $w' \in R_i(w)$, $\mathcal{M}, w' \models \phi$.

We omit the standard conditions for the propositional connectives. An $\mathtt{L}_m$-formula ϕ is said to be AML$_m$*-satisfiable* [resp. AML$_m$*-valid*] $\stackrel{\text{def}}{\Leftrightarrow}$ there exist an AML$_m$-model $\mathcal{M}$ and $w \in W$ such that $\mathcal{M}, w \models \phi$ [resp. for all the AML$_m$-models $\mathcal{M}$, for $w \in W$, $\mathcal{M}, w \models \phi$].

A standard result in modal logic allows to state that:

Theorem 3.1. *[23] For $m \in \omega \cup \{\omega\}$, for any $\mathtt{L}_m$-formula ϕ, ϕ is a theorem of $\mathtt{H}_m$ iff ϕ is* AML$_m$*-valid.*

Consequently, since for $m < \omega$, the system $\mathtt{H}_m$ axiomatizes the logic AML(τ_m) in [28], the computational complexity of AML$_m$-satisfiability (or AML$_m$-validity) is identical to the problems for AML(τ_m), respectively.

Lemma 3.1. AML$_0$*-satisfiability is* **NP***-complete and for $1 \leq m \leq \omega$,* AML$_m$*-satisfiability is* **PSPACE***-hard.*

Proof:
AML_0-satisfiability is **NP**-complete since AML_0-satisfiability is equivalent to the satisfiability problem for the standard modal logic S5 that is known to be **NP**-complete [13]. AML_m-satisfiability is **PSPACE**-hard since one can easily show that AML_m-satisfiability restricted to formulae containing only the modal connective $[R_1]$ is equivalent to the satisfiability problem for the standard modal logic K that is known to be **PSPACE**-hard [13]. $\Box$

Lemma 3.2. *If* AML_ω*-satisfiability is in* **PSPACE**, *then for* $1 \leq m < \omega$, AML_m*-satisfiability is* **PSPACE**-*complete.*

Proof:
For $1 \leq m < \omega$, one can check in **PSPACE** that $\phi \in \mathtt{L}_\omega$ belongs to $\mathtt{L}_m$. Moreover, for $\phi \in \mathtt{L}_m$, ϕ is AML_m-satisfiable iff ϕ is AML_ω-satisfiable, which guarantees the desired result. $\Box$

In the sequel, we shall therefore focus on AML_ω-satisfiability.

4. Preliminary Results

In Definition 4.1 below, we introduce a closure operator for sets of $\mathtt{L}_\omega$-formulae as it is done for Propositional Dynamic Logic (PDL, for short) in [7].

Definition 4.1. Let X be a set of $\mathtt{L}_\omega$-formulae. Let $\mathtt{cl}(X)$ be the smallest set of formulae such that:

- $X \subseteq \mathtt{cl}(X)$;
- if $\neg\phi \in \mathtt{cl}(X)$, then $\phi \in \mathtt{cl}(X)$;
- if $\phi_1 \wedge \phi_2 \in \mathtt{cl}(X)$, then $\phi_1, \phi_2 \in \mathtt{cl}(X)$;
- if $[\mathtt{a}]\phi \in \mathtt{cl}(X)$, then $\phi \in \mathtt{cl}(X)$ for $\mathtt{a} \in \{ind\} \cup \{L(R_i), R_i : i \in \omega\}$;
- if $[R_i]\phi \in \mathtt{cl}(X)$, then $[L(R_i)]\phi \in \mathtt{cl}(X)$;
- if $[ind][L(R_i)][ind]\phi \in \mathtt{cl}(X)$, then $[L(R_i)]\phi, [ind][L(R_i)]\phi \in \mathtt{cl}(X)$;
- if $[L(R_i)]\phi \in \mathtt{cl}(X)$ and $\phi \neq [ind]\phi'$, then $[ind][L(R_i)][ind]\phi \in \mathtt{cl}(X)$;
- if $[L(R_i)][ind]\phi \in \mathtt{cl}(X)$, then $[ind][L(R_i)][ind]\phi \in \mathtt{cl}(X)$.

A set X of formulae is said to be *closed* $\overset{\text{def}}{\Leftrightarrow}$ $\mathtt{cl}(X) = X$. Observe that for any finite set X of formulae, $md(\mathtt{cl}(X)) \leq md(X) + 2$.

Lemma 4.1. *Let* ϕ *be a formula. Then,* $card(\mathtt{cl}(\{\phi\})) < 6 \times |\phi|$.

Proof:
Let sub_ϕ be the set of subformulae of the formula ϕ. Obviously, $sub_\phi \subseteq \mathtt{cl}(\{\phi\})$. Moreover, $\mathtt{cl}(\{\phi\})$ is the union of the following sets:

- sub_ϕ;

- $\{[L(R_i)]\psi, [ind][L(R_i)]\psi : [ind][L(R_i)][ind]\psi \in sub_\phi\}$;
- $\{[ind][L(R_i)][ind]\psi, [L(R_i)][ind]\psi, [ind]\psi, [ind][L(R_i)]\psi : [L(R_i)]\psi \in sub_\phi,\ \psi \neq [ind]\psi'\}$;
- $\{[ind][L(R_i)][ind]\psi, [L(R_i)]\psi, [ind][L(R_i)]\psi : [L(R_i)][ind]\psi \in sub_\phi\}$
- $\{[L(R_i)]\psi, [ind][L(R_i)][ind]\psi, [L(R_i)][ind]\psi, [ind]\psi, [ind][L(R_i)]\psi :$ $[R_i]\psi \in sub_\phi,\ \psi \neq [ind]\psi'\}$;
- $\{[L(R_i)][ind]\psi, [ind][L(R_i)][ind]\psi, [L(R_i)]\psi, [ind][L(R_i)]\psi : [R_i][ind]\psi \in sub_\phi\}$.

Each set above is of the cardinality at most $5 \times card(sub_\phi)$. So $card(\mathtt{cl}(\{\phi\})) < 6 \times |\phi|$, since $card(sub_\phi) < |\phi|$. □

What is important to establish the **PSPACE** upper bound is that $card(\mathtt{cl}(\{\phi\}))$ is bounded by a polynomial in $|\phi|$.

In Definition 4.2 below, we introduce a family $(\mathtt{cl}(s,\phi))_{s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*}$ of finite subsets of $\mathtt{cl}(\phi)$ with the following intention. For $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$, for any AML$_\omega$-model and for $w \in W$, in order to check whether $\mathcal{M}, w \models \psi$ for some $\psi \in \mathtt{cl}(s,\phi)$, one should only need to check whether $\mathcal{M}, w' \models \varphi$ for all $\varphi \in \mathtt{cl}(s',\phi)$ and $w' \in W$ where s' extends s by one letter and w' is an immediate neighbor of w.

Definition 4.2. Let ϕ be a formula. For $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$, let $\mathtt{cl}(s,\phi)$ be the smallest set such that:

1. $\mathtt{cl}(\lambda,\phi) = \mathtt{cl}(\{\phi\})$;
2. $\mathtt{cl}(s,\phi)$ is closed;
3. if $[ind]\psi \in \mathtt{cl}(s,\phi)$, then $[ind]\psi \in \mathtt{cl}(s \cdot ind, \phi)$;
4. if $[R_i]\psi \in \mathtt{cl}(s,\phi)$, then $\psi \in \mathtt{cl}(s \cdot r_i, \phi)$;
5. if $[L(R_i)]\psi \in \mathtt{cl}(s,\phi)$, then $\psi \in \mathtt{cl}(s \cdot lr_i, \phi)$.

Definition 4.2 is not a typical inductive definition. Actually, the clauses 1.-5. below define a certain operator which least fixed point allows to define a function $s \mapsto \mathtt{cl}(s,\phi)$ by Knaster-Tarski fixed-point Theorem.

Example 4.1. Let ϕ be the formula $[R_1]\mathtt{p}_0$. The set $\mathtt{cl}(\lambda,\phi) = \mathtt{cl}(\{\phi\})$ is equal to

$$\{[R_1]\mathtt{p}_0, \mathtt{p}_0, [L(R_1)]\mathtt{p}_0, [ind][L(R_1)][ind]\mathtt{p}_0, [L(R_1)][ind]\mathtt{p}_0, [ind]\mathtt{p}_0, [ind][L(R_1)]\mathtt{p}_0\}$$

Below are some examples of sets of the form $\mathtt{cl}(s,\phi)$:

- $\mathtt{cl}(ind,\phi) = \mathtt{cl}(\{\phi\}) \setminus \{[R_1]\mathtt{p}_0\}$;
- $\mathtt{cl}(ind \cdot lr_1,\phi) = \{\mathtt{p}_0, [ind]\mathtt{p}_0\}$;
- $\mathtt{cl}(ind \cdot lr_1 \cdot ind,\phi) = \{\mathtt{p}_0, [ind]\mathtt{p}_0\}$;
- $\mathtt{cl}(ind \cdot lr_1 \cdot ind \cdot lr_1,\phi) = \emptyset$.

One can check that for any $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$ such that lr_1 or r_1 occurs more than twice, $\mathtt{cl}(s, \phi) = \emptyset$.

Lemma 4.2 contains some basic properties about the sets $\mathtt{cl}(s, \phi)$.

Lemma 4.2. *Let ϕ be a formula and $s, s' \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$ such that s is a prefix s'. Then,*

(I) $\mathtt{cl}(s', \phi) \subseteq \mathtt{cl}(s, \phi)$;

(II) if $\psi \in \mathtt{cl}(s, \phi)$, $md(\psi) = md(\mathtt{cl}(s, \phi))$ *and* $md(\psi) \neq 0$, *then* ψ *is of the form* $[ind]\psi'$;

(III) if $md(\mathtt{cl}(s, \phi)) = 0$, *then* $\mathtt{cl}(s \cdot ind, \phi) = \mathtt{cl}(s \cdot lr_i, \phi) = \mathtt{cl}(s \cdot r_i, \phi) = \emptyset$;

(IV) for $k \geq 1$, $\mathtt{cl}(s \cdot ind, \phi) = \mathtt{cl}(s \cdot ind^k, \phi)$;

(V) $md(\mathtt{cl}(s \cdot lr_i, \phi)) \leq max(0, md(\mathtt{cl}(s, \phi)) - 1)$;

(VI) $md(\mathtt{cl}(s \cdot r_i, \phi)) \leq max(0, md(\mathtt{cl}(s, \phi)) - 1)$;

(VII) If the lr_is and r_is occur more than $md(\phi) + 3$ times in s, then $\mathtt{cl}(s, \phi) = \emptyset$.

Proof:
(I) This is immediate since both sets $\mathtt{cl}(s', \phi)$ and $\mathtt{cl}(s, \phi)$ are closed.
(II) $\mathtt{cl}(s, \phi)$ is closed and by a simple inspection of Definition 4.1, this is immediate.
(III) This is immediate by Definition 4.2 since only the formulae prefixed by a modal connective can be propagated from s into $s \cdot ind$, $s \cdot r_i$ and $s \cdot lr_i$, respectively.
(IV) The proof is by an easy verification by induction on k.
(V) Suppose that $md(\mathtt{cl}(s, \phi)) \geq 1$, otherwise the proof is immediate by (III). Let $\psi \in \mathtt{cl}(s, \phi)$ be such that $md(\psi) = md(\mathtt{cl}(s, \phi))$. Let us show that $\psi \notin \mathtt{cl}(s \cdot lr_i, \phi)$. By (I) we are then done. By (II), ψ is of the form $[ind]\psi'$. Suppose that $\psi \in \mathtt{cl}(s \cdot lr_i, \phi)$. So, there is $[L(R_i)]\varphi \in \mathtt{cl}(s, \phi)$ such that $\psi \in \mathtt{cl}(\varphi)$.
Case V.1: $\varphi \neq [ind]\varphi'$.
So, $[ind][L(R_i)][ind]\varphi \in \mathtt{cl}(s, \phi)$. Since $\psi \in \mathtt{cl}(\varphi)$, $md(\psi) \leq md(\varphi) + 2$. However, $md(\varphi) + 2 < md([ind][L(R_i)][ind]\varphi)$ which is in contradiction with $md(\psi) = md(\mathtt{cl}(s, \phi))$.
Case V.2: $\varphi = [ind]\varphi'$.
So, $[ind][L(R_i)][ind]\varphi' \in \mathtt{cl}(s, \phi)$. One can easily show that for any formula of the form $[ind]\varphi''$, $md(\mathtt{cl}([ind]\varphi'')) \leq md([ind]\varphi'') + 1$. Indeed, the occurrences of $[ind]$ are not the cause for increasing the modal degree. Since $\psi \in \mathtt{cl}(\varphi)$, $md(\psi) \leq md(\varphi) + 1$. However, $md(\varphi) + 1 < md([ind][L(R_i)][ind]\varphi')$ which is in contradiction with $md(\psi) = md(\mathtt{cl}(s, \phi))$.
(VI) The proof is similar to (V).
(VII) This is a direct consequence of (V) and (VI) since $md(\mathtt{cl}(\lambda, \phi)) \leq md(\phi) + 2$. □

Definition 4.3. Let X, Y be sets of $\mathtt{L}_\omega$-formulae. The binary relation $\approx_{ind}$ is defined as follows: $X \approx_{ind} Y \stackrel{\text{def}}{\Leftrightarrow}$

1. for all $[ind]\psi \in X$, $[ind]\psi \in Y$;
2. for all $[ind]\psi \in Y$, $[ind]\psi \in X$.

The binary relation $\approx_{r_i}$ is defined as follows: $X \approx_{r_i} Y \stackrel{\text{def}}{\Leftrightarrow}$ for all $[R_i]\psi \in X$, $\psi \in Y$. The binary relation $\approx_{lr_i}$ is defined as follows: $X \approx_{lr_i} Y \stackrel{\text{def}}{\Leftrightarrow}$

1. for all $[R_i]\psi \in X$, $\psi \in Y$;
2. for all $[L(R_i)]\psi \in X$, $\psi \in Y$;
3. for all $[ind][L(R_i)][ind]\psi \in X$, $\psi \in Y$.

In the definition of $\approx_{lr_i}$, condition 2. faithfully reflects the standard semantics of modal necessity. Condition 1. encodes syntactically the fact that $LR_i \subseteq R_i$ holds true in the AML_ω-models. Condition 3. is maybe the most surprising unless one observes that $[ind][L(R_i)][ind]\psi \Leftrightarrow [L(R_i)]\varphi$ is AML_ω-valid and therefore condition 3. is a variant of condition 1.

Let $\texttt{clos}$ be the set of subsets Y of $\texttt{cl}(\{\phi\})$ such that $[ind]\psi \in Y$ implies $\psi \in Y$. Observe that $\approx_{ind}$ is an equivalence relation on $\texttt{clos}$ and $\approx_{lr_i} \subseteq \approx_{r_i}$.

Definition 4.4. Let X be a subset of $\texttt{cl}(s, \phi)$ for some $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$ and for some formula ϕ. The set X is said to be *s-consistent* $\stackrel{\text{def}}{\Leftrightarrow}$ for $\psi \in \texttt{cl}(s, \phi)$:

1. if $\psi = \neg\varphi$, then $\varphi \in X$ iff not $\psi \in X$;
2. if $\psi = \varphi_1 \wedge \varphi_2$, then $\{\varphi_1, \varphi_2\} \subseteq X$ iff $\psi \in X$;
3. if $\psi = [ind]\varphi$ and $\psi \in X$, then $\varphi \in X$;
4. if $\psi = [ind][L(R_i)][ind]\varphi$ and $\psi \in X$, then $[L(R_i)]\varphi \in X$;
5. if $\psi = [L(R_i)][ind]\varphi$ and $\psi \in X$, then $[ind][L(R_i)][ind]\varphi \in X$;
6. if $\psi = [L(R_i)]\varphi$, $\varphi \neq [ind]\varphi'$ and $\psi \in X$, then $[ind][L(R_i)][ind]\varphi \in X$;
7. if $\psi = [R_i]\varphi$, and $\psi \in X$, then $[L(R_i)]\varphi \in X$.

Roughly speaking, the s-consistency entails the maximal propositional consistency with respect to the set $\texttt{cl}(s, \phi)$ of formulae. Furthermore, the modal conditions 3.-7. in Definition 4.4 are added in order to take into account the reflexivity of R_{ind}, the inclusion $LR_i \subseteq R_i$ and the equality $LR_i = R_{ind} \circ LR_i \circ R_{ind}$.

Lemma 4.3. *Let $\mathcal{M}$ be an AML_ω-model, $w \in W$, $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$, ϕ be a $\mathtt{L}_\omega$-formula. Then, $\{\psi \in \texttt{cl}(s, \phi) : \mathcal{M}, w \models \psi\}$ is s-consistent.*

The proof of the above lemma is by an easy verification using the fact that $\texttt{cl}(s, \phi)$ is closed. In a sense, Lemma 4.3 states the correctness of the notion of s-consistency.

Lemma 4.4. *Let X_i be an s_i-consistent set, $i = 1, \ldots, 4$.*

(I) If $X_1 \approx_{ind} X_2 \approx_{lr_i} X_3 \approx_{ind} X_4$, then $X_1 \approx_{lr_i} X_4$.

(II) Reciprocally, if $X_1 \approx_{lr_i} X_4$, then $X_1 (\approx_{ind} \circ \approx_{lr_i} \circ \approx_{ind}) X_4$.

Proof:
(I) This property can be shown mainly because $[ind][L(R_i)][ind]\psi \in X_1$ iff either $[L(R_i)][ind]\psi \in X_1$ or $[L(R_i)]\psi \in X_1$, and if $[ind][L(R_i)][ind]\psi \in X_1$, then $[ind]\psi, \psi \in X_4$.
(II) Assume $X_1 \approx_{lr_i} X_4$. Since $X_1 \approx_{ind} X_1$ and $X_4 \approx_{ind} X_4$, $X_1(\approx_{ind} \circ \approx_{lr_i} \circ \approx_{ind})X_4$. □

Lemma 4.5. *Let $\mathcal{M}$ be an AML_ω-model, $w, w' \in W$, $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$, $s', s'' \in \{\lambda, ind, lr_i, r_i\}$ and ϕ be a $\mathtt{L}_\omega$-formula. Let*

$$X_w \stackrel{def}{=} \{\psi \in \mathtt{cl}(s \cdot s', \phi) : \mathcal{M}, w \models \psi\} \quad X_{w'} \stackrel{def}{=} \{\psi \in \mathtt{cl}(s \cdot s'', \phi) : \mathcal{M}, w' \models \psi\}$$

Then,

(I) X_w is $s.s'$-consistent and $X_{w'}$ is $s.s''$-consistent;

(II) if $\langle s', s''\rangle = \langle \lambda, lr_i\rangle$ and $\langle w, w'\rangle \in LR_i$, then $X_w \approx_{lr_i} X_{w'}$;

(III) if $\langle s', s''\rangle = \langle \lambda, r_i\rangle$ and $\langle w, w'\rangle \in R_i$, then $X_w \approx_{r_i} X_{w'}$;

(IV) if $\langle s', s''\rangle \in \{\langle \lambda, ind\rangle, \langle ind, \lambda\rangle, \langle \lambda, \lambda\rangle\}$ and $\langle w, w'\rangle \in R_{ind}$, then $X_w \approx_{ind} X_{w'}$.

The proof is by an easy verification using the previous lemmas.

5. AML_ω-satisfiability is in PSPACE

In this section, we present the algorithm `WORLD` and prove its termination and correctness. We also estimate the computational complexity of `WORLD`.

5.1. The algorithm

In Figure 2, the function $\mathtt{WORLD}(\Sigma, s, \phi)$ returning a Boolean is defined. Σ is a non-empty finite sequence of subsets of $\mathtt{cl}(\{\phi\})$ and $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$. The function `WORLD` is actually defined on the model of the function `K-WORLD` in [13] (see also [25, 4, 14, 32]). The results given in Section 4 are crucial to guarantee that `WORLD` is *correct* and *terminates*. By a successful call of $\mathtt{WORLD}(\Sigma, s, \phi)$ we mean that it returns true.

Most of the ingenuity to guarantee that the algorithms terminate are in the definition of $\mathtt{cl}(s, \phi)$, s-consistency and the syntactic relations of the form $\approx$. Indeed, $\mathtt{cl}(s \cdot \mathtt{a}, \phi)$ contains the formulae that can be possibly propagated from $\mathtt{cl}(s, \phi)$. In the easiest case, $\mathtt{cl}(s \cdot \mathtt{a}, \phi) \subset \mathtt{cl}(s, \phi)$ (strict inclusion) but this is not the general case here. Then the syntactic relation of the form $\approx$ and s-consistency further restrict the formulae that can be propagated. Still, we may be in trouble to guarantee termination. That is why the detection of cycles is introduced (see e.g. [13]). It is precisely, the appropriate combination of all these ingredients that guarantees termination and the **PSPACE** upper bound. What we present is a formalization of Ladner-like algorithms based on [25] and we believe it is the proper framework to allow further extensions (see e.g. [2, 3]).

`function` $\texttt{WORLD}(\Sigma, s, \phi)$

if $last(\Sigma)$ is not s-consistent, then return false;

% 'ind' segment
for $[ind]\psi \in \texttt{cl}(s, \phi) \setminus last(\Sigma)$ do

if there is no $X \in \Sigma$ such that $\Sigma = \Sigma_1 X \Sigma_2$, s is of the form $s_1.s_2$ with $|s_2| = |\Sigma_2|$ and $s_2 \in \{ind\}^*$, $\psi \notin X$, $last(\Sigma) \approx_{ind} X$, then
for each $X_\psi \subseteq \texttt{cl}(s \cdot ind, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \approx_{ind} X_\psi$, call $\texttt{WORLD}(\Sigma \cdot X_\psi, s \cdot ind, \phi)$. If all these calls return false, then return false;

% 'r_i' segment
for $[R_i]\psi \in \texttt{cl}(s, \phi) \setminus last(\Sigma)$ do

for each $X_\psi \subseteq \texttt{cl}(s \cdot r_i, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \approx_{r_i} X_\psi$, call $\texttt{WORLD}(\Sigma \cdot X_\psi, s \cdot r_i, \phi)$. If all these calls return false, then return false;

% 'lr_i' segment
for $[L(R_i)]\psi \in \texttt{cl}(s, \phi) \setminus last(\Sigma)$ do

for each $X_\psi \subseteq \texttt{cl}(s \cdot lr_i, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \approx_{lr_i} X_\psi$, call $\texttt{WORLD}(\Sigma \cdot X_\psi, s \cdot lr_i, \phi)$. If all these calls return false, then return false;

Return true.

Figure 2. Algorithm WORLD

We prove that for any set $X \subseteq \mathtt{cl}(\{\phi\})$, $\mathtt{WORLD}(X, \lambda, \phi)$ always terminates and requires polynomial space in $|\phi|$. To do so, we shall take advantage of the fact that if $\mathtt{WORLD}(\Sigma, s, \phi)$ calls $\mathtt{WORLD}(\Sigma', s', \phi)$ (at any recursion depth), then $|s'| > |s|$.

Each subset $X \subseteq \mathtt{cl}(\{\phi\})$ can be represented as a bit string of length $6 \times |\phi|$ (see Lemma 4.1). By implementing Σ as a global stack, each level of the recursion uses space in $\mathcal{O}(|\phi|)$. For instance, in the parts of WORLD of the form

> "for each $X_\psi \subseteq \mathtt{cl}(s \cdot \mathtt{a}, \phi) \setminus \{\psi\}$ such that $last(\Sigma) \approx_{\mathtt{a}} X_\psi$, call $\mathtt{WORLD}(\Sigma \cdot X_\psi, s \cdot \mathtt{a}, \phi)$.
> If all these calls return false, then return false"

the implementation uses a bit string of length $6 \times |\phi|$ to encode each X_ψ (this value is incremented for each new X_ψ) and a Boolean indicating whether there were a call returning true.

5.2. Termination and space upper bounds

Lemma 5.1. *Let $X \subseteq \mathtt{cl}(\{\phi\})$ and ϕ be an $\mathtt{L}_\omega$-formula.*
If $\mathtt{WORLD}(\Sigma, s, \phi)$ is called in $\mathtt{WORLD}(X, \lambda, \phi)$ (at any recursion depth $|s|$) and s is of the form $s'.s''$ with $s'' \in \{ind\}^*$ and $|s''| \geq 6 \times |\phi| + 1$, then in the "ind" segment of $\mathtt{WORLD}(\Sigma, s, \phi)$, no recursive call to WORLD is executed.

Proof:
So Σ is of the form $\Sigma_1.\Sigma_2$ with $\Sigma_2 = X_0 \ldots X_k$ and $|s''| = k \geq 6 \times |\phi| + 1$. By definition of WORLD, for $i \in \{0, \ldots, k\}$, $X_i \approx_{ind} X_{i+1}$. If X_k is not s-consistent, then we are obviously done. Otherwise, assume that $[ind]\psi \in \mathtt{cl}(s, \phi) \setminus X_k$ and suppose that there is no $X \in \Sigma$ such that $\Sigma = \Sigma'_1 X \Sigma'_2$, s is of the form $s_1 \cdot s_2$ with $|s_2| = |\Sigma'_2|$ and $s_2 \in \{ind\}^*$, $\psi \notin X$, $X_k \approx_{ind} X$. For $0 \leq i \leq j \leq k$, $X_i \approx_{ind} X_j$. That is, for $0 \leq i \leq j \leq k$,

$$\{[ind]\varphi : [ind]\varphi \in X_i\} = \{[ind]\varphi : [ind]\varphi \in X_j\}.$$

Since for $\alpha \in \{0, \ldots, k-1\}$,

$$\mathtt{WORLD}(\Sigma_1 \cdot X_1 \cdot X_2 \ldots X_{\alpha-1}, s[|\Sigma_1 \cdot X_1 \cdot X_2 \ldots X_{\alpha-1}| - 1], \phi)$$

calls

$$\mathtt{WORLD}(\Sigma_1 \cdot X_1 \cdot X_2 \ldots X_\alpha, s[|\Sigma_1 \cdot X_1 \cdot X_2 \ldots X_\alpha| - 1], \phi)$$

and there are formulae $\psi_1, \ldots, \psi_{k'}$ in $\mathtt{cl}(\{\phi\})$ such that $\psi_\alpha \notin X_\alpha$ and for $\alpha' \in \{1, \ldots, \alpha - 1\}$, $\psi_\alpha \in X_{\alpha'}$. Hence $\psi_1, \ldots, \psi_{k'}$ are k' *different* formulae in $\mathtt{cl}(\{\phi\})$. Hence k' is in $\mathcal{O}(|\phi|)$. More precisely, $k' < 6 \times |\phi|$. So the maximal length of Σ_2 is in $\mathcal{O}(|\phi|)$. More precisely, $|\Sigma_2| \leq 6 \times |\phi|$, a contradiction.

So necessarily, there is $X \in \Sigma$ such that $\Sigma = \Sigma'_1 X \Sigma'_2$, s is of the form $s_1.s_2$ with $|s_2| = |\Sigma'_2|$ and $s_2 \in \{ind\}^*$, $\psi \notin X$, $last(\Sigma) \approx_{ind} X$. Hence, in the "ind" segment of $\mathtt{WORLD}(\Sigma, s, \phi)$, no recursive call to WORLD is executed. □

The proof of Lemma 5.1 contains a technique used to show that S5 satisfiability is in **NP** [13].

Lemma 5.2. *Let $X \subseteq \mathtt{cl}(\{\phi\})$ and ϕ be a $\mathtt{L}_\omega$-formula.*
If $\mathtt{WORLD}(\Sigma, s, \phi)$ is called in $\mathtt{WORLD}(X, \lambda, \phi)$ (at any recursion depth $|s|$) and s contains more than $k \geq md(\phi)+3$ occurrences of either the r_is or the lr_is, then in the "lr_i" segment of $\mathtt{WORLD}(\Sigma, s, \phi)$ and in the "r_i" segment of $\mathtt{WORLD}(\Sigma, s, \phi)$, no recursive call to $\mathtt{WORLD}$ is executed.

Proof:
Since $\mathtt{cl}(s, \phi) = \emptyset$ (by Lemma 4.2(VII)), this is immediate. □

Theorem 5.1. *Let $X \subseteq \mathtt{cl}(\{\phi\})$.*

(I) $\mathtt{WORLD}(X, \lambda, \phi)$ *terminates and requires at most space in $\mathcal{O}(|\phi|^3)$;*

(II) Let $\mathtt{WORLD}(\Sigma, s, \phi)$ be a call in the computation of $\mathtt{WORLD}(X, \lambda, \phi)$. Then, $|\Sigma| \leq \alpha$ and $|s| \leq \alpha$ with $\alpha = (6 \times |\phi| + 1) \times (md(\phi) + 3)$.

Any call $\mathtt{WORLD}(Y, s, \phi)$ from $\mathtt{WORLD}(X, \lambda, \phi)$ (at any recursion depth) satisfies:

- $ind^{6|\phi|+1}$ is not a substring of s (by Lemma 5.1);
- the number of occurrences of the r_is and lr_is are less than $md(\phi) + 3$ (by Lemma 5.2).

Consequently, the length of s is at most $(6 \times |\phi| + 1) \times (md(\phi) + 3)$. So the recursion depth is in $\mathcal{O}(|\phi|^2)$. Each level of the recursion requires space in $\mathcal{O}(|\phi|)$. Hence $\mathtt{WORLD}(X, \lambda, \phi)$ requires space at most in $\mathcal{O}(|\phi|^3)$.

Theorem 5.1 is certainly an important step to prove that satisfiability is in **PSPACE** but this is not sufficient. Indeed, until now we have no guarantee that $\mathtt{WORLD}$ is actually correct.

5.3. Correctness

Correctness shall be shown in the next two lemmas.

Lemma 5.3. *Let ϕ be an $\mathtt{L}_\omega$-formula and $Y \subseteq \mathtt{cl}(\{\phi\})$ such that $\phi \in Y$. If $\mathtt{WORLD}(Y, \lambda, \phi)$ returns true, then ϕ is AML_ω-satisfiable.*

Proof:
Assume that $\mathtt{WORLD}(Y, \lambda, \phi)$ returns true. Let us build an AML_ω-model $\mathcal{M} = \langle W, R_{ind}, (R_i)_{i\in\omega}, (LR_i)_{i\in\omega}, V\rangle$ for which there is $w \in W$ such that for all $\psi \in \mathtt{cl}(\{\phi\})$, $\mathcal{M}, w \models \psi$ iff $\psi \in Y$.

Let S be the set of strings s over $(\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$ such that $|s| \leq (6 \times |\phi| + 1) \times (md(\phi) + 3)$. We define W as the set of pairs $\langle X, s\rangle$ for which there is a finite sequence $\langle \Sigma_1, s_1\rangle, \ldots, \langle \Sigma_k, s_k\rangle$ $(k \geq 1)$ such that

1. for $i \in \{1, \ldots, k\}$, $\mathtt{WORLD}(\Sigma_i, s_i, \phi)$ is called successfully in $\mathtt{WORLD}(X, s, \phi)$ (at any depth of the recursion);
2. $\Sigma_1 = Y$; $s_1 = \lambda$; $last(\Sigma_k) = X$; $s_k = s$;
3. for $i \in \{1, \ldots, k\}$, $\mathtt{WORLD}(\Sigma_i, s_i, \phi)$ returns true;
4. for $i \in \{1, \ldots, k-1\}$, $\mathtt{WORLD}(\Sigma_i, s_i, \phi)$ calls directly $\mathtt{WORLD}(\Sigma_{i+1}, s_{i+1}, \phi)$.

The conditions 3. and 4. state that we only record the pairs $\langle X, s\rangle \in \mathtt{clos} \times S$ that contribute to make $\mathtt{WORLD}(Y, \lambda, \phi)$ true. $\langle Y, \lambda\rangle \in W$ by definition. Furthermore, for all $\langle X, s\rangle \in W$, $X \subseteq \mathtt{cl}(s, \phi)$ and X is s-consistent.

Let us define the auxiliary binary relation R'_{ind} on W as follows: $\langle X, s\rangle R'_{ind} \langle X', s'\rangle \overset{\text{def}}{\Leftrightarrow}$ there is a successful call $\mathtt{WORLD}(\Sigma, s, \phi)$ in $\mathtt{WORLD}(Y, \lambda, \phi)$ (at any depth of the recursion) such that

1. either
   (a) $last(\Sigma) = X$;
   (b) $\mathtt{WORLD}(\Sigma, s, \phi)$ calls successfully $\mathtt{WORLD}(\Sigma', s', \phi)$ in the "ind" segment of $\mathtt{WORLD}(\Sigma, s, \phi)$ $last(\Sigma') = X'$;
2. or there is a finite sequence $\langle \Sigma_1, s_1\rangle, \ldots, \langle \Sigma_k, s_k\rangle$ such that:
   (a) $last(\Sigma_k) = X$; $last(\Sigma_1) = X'$; $\Sigma_k = \Sigma$; $s_k = s$; $s_1 = s'$;
   (b) for $i \in \{1, \ldots, k\}$, $\langle last(\Sigma_i), s_i\rangle \in W$;
   (c) for $i \in \{1, \ldots, k-1\}$, $\mathtt{WORLD}(\Sigma_i, s_i, \phi)$ calls successfully $\mathtt{WORLD}(\Sigma_{i+1}, s_{i+1}, \phi)$ in the "ind" segment of $\mathtt{WORLD}$;
   (d) the call $\mathtt{WORLD}(\Sigma_k, s_k, \phi)$ enters in the "ind" segment of $\mathtt{WORLD}$ and for some formula $[ind]\psi \in \mathtt{cl}(s, \phi) \setminus X$, no recursive call to $\mathtt{WORLD}$ is necessary thanks to Σ_1, $\psi \notin X'$, $X \approx_{ind} X'$.

For $i \in \omega$ such that either $[R_i]$ or $[L(R_i)]$ occurs in ϕ, let us define the auxiliary binary relation R'_i [resp. LR'_i] on W as follows: $\langle X, s\rangle R'_i \langle X', s'\rangle$ [resp. $\langle X, s\rangle LR'_i \langle X', s'\rangle$] $\overset{\text{def}}{\Leftrightarrow}$ there is a successful call $\mathtt{WORLD}(\Sigma, s, \phi)$ in $\mathtt{WORLD}(Y, \lambda, \phi)$ (at any depth of the recursion) such that

1. $last(\Sigma) = X$;
2. $\mathtt{WORLD}(\Sigma, s, \phi)$ calls $\mathtt{WORLD}(\Sigma', s \cdot r_i, \phi)$ [resp. $\mathtt{WORLD}(\Sigma', s \cdot lr_i, \phi)$] in the "$r_i$" [resp. "$lr_i$"] segment of $\mathtt{WORLD}(\Sigma, s, \phi)$; $last(\Sigma') = X'$.

For $i \in \omega$ such that neither $[R_i]$ nor $[L(R_i)]$ occurs in ϕ, $R'_i = LR'_i = \emptyset$ (dummy values). The definition of $\mathcal{M}$ can be now completed:

- $R_{ind} \overset{\text{def}}{=} (R'_{ind} \cup R'^{-1}_{ind})^*$;
- For $i \in \omega$, $LR_i \overset{\text{def}}{=} R_{ind} \circ LR'_i \circ R_{ind}$;
- For $i \in \omega$, $R_i \overset{\text{def}}{=} R'_i \cup LR_i$;
- For $\mathtt{p} \in \mathtt{For}_0$, $V(\mathtt{p}) \overset{\text{def}}{=} \{\langle X, s\rangle \in W : \mathtt{p} \in X\}$.

$\mathcal{M}$ is an AML$_\omega$-model and W is of cardinality $2^{\mathcal{O}(|\phi|)}$. One can show:

(i) $\langle X, s\rangle R'_{ind} \langle X', s'\rangle$ implies $X \approx_{ind} X'$;

(ii) $\langle X, s\rangle R'_i \langle X', s'\rangle$ implies $X \approx_{r_i} X'$;

(iii) $\langle X, s\rangle LR'_i \langle X', s'\rangle$ implies $X \approx_{lr_i} X'$;

So,

(iv) $\langle X, s\rangle R_{ind}\langle X', s'\rangle$ implies for all $[ind]\psi \in X$, $\psi \in X'$ ($\approx_{ind}$ is an equivalence relation on $\mathtt{clos}$);

(v) $\langle X, s\rangle R_i\langle X', s'\rangle$ implies for all $[R_i]\psi \in X$, $\psi \in X'$ (using $\approx_{lr_i} \subseteq \approx_{r_i}$);

(vi) $\langle X, s\rangle LR_i\langle X', s'\rangle$ implies for all $[L(R_i)]\psi \in X$, $\psi \in X'$ (by Lemma 4.4(I)).

By induction on the structure of ψ we show that for all $\langle X, s\rangle \in W$, for all $\psi \in \mathtt{cl}(s, \phi)$, $\psi \in X$ iff $\mathcal{M}, \langle X, s\rangle \models \psi$. The case when ψ is a propositional variable is by definition of V.
Induction Hypothesis: for all $\psi \in \mathtt{cl}(\{\phi\})$ such that $|\psi| \leq n$, for all $\langle X, s\rangle \in W$, if $\psi \in \mathtt{cl}(s, \phi)$, then $\psi \in X$ iff $\mathcal{M}, \langle X, s\rangle \models \psi$.
Let ψ be a formula in $\mathtt{cl}(\{\phi\})$ such that $|\psi| \leq n + 1$. The cases when the outermost connective of ψ is Boolean is a consequence of the s-consistency of X and of the induction hypothesis. Let us treat the other cases.
Case 1: $\psi = [ind]\psi'$. Let $\langle X, s\rangle \in W$ such that $\psi \in \mathtt{cl}(s, \phi)$. By definition of W, there is Σ such that $last(\Sigma) = X$ and $\mathtt{WORLD}(\Sigma, s, \phi)$ returns true. If $\psi \in X$, then by (iv), for all $\langle X', s'\rangle \in R_{ind}(\langle X, s\rangle)$, $\psi' \in X'$. One can show that $\psi' \in \mathtt{cl}(s', \phi)$ (by using Lemma 4.2(IV)). By the induction hypothesis, $\mathcal{M}, \langle X', s'\rangle \models \psi'$ and therefore $\mathcal{M}, \langle X, s\rangle \models \psi$. Now, if $\psi \notin X$, two cases are distinguished.
Case 1.1: there is X' in Σ such that $X \approx_{ind} X'$, $\psi' \notin X'$ and $\Sigma = \Sigma' X' \Sigma_2$, s is of the form $s' \cdot s_2$ with $|\Sigma_2| = |s_2|$ and $s_2 \in \{ind\}^*$.
By definition of W, $\mathtt{WORLD}(\Sigma' \cdot X', s', \phi)$ returns true (see the conditions 3. and 4. defining W). Hence, $\langle X, s\rangle R'_{ind}\langle X', s'\rangle$ by definition and therefore $\langle X, s\rangle R_{ind}\langle X', s'\rangle$. One can show that $\psi' \in \mathtt{cl}(s', \phi)$ since s is of the form $s' \cdot ind^k$ for some $k \geq 0$ (see Lemma 4.2(IV)). By the induction hypothesis, $\mathcal{M}, \langle X', s'\rangle \not\models \psi'$ and therefore $\mathcal{M}, \langle X, s\rangle \not\models \psi$.
Case 1.2: the condition in the Case 1.1 does not hold. So, $\mathtt{WORLD}(\Sigma, s, \phi)$ calls successfully $\mathtt{WORLD}(\Sigma', s \cdot ind, \phi)$ in the "ind" segment of $\mathtt{WORLD}$, $last(\Sigma') = X'$ and $\psi' \notin last(\Sigma')$, $X \approx_{ind} X'$, and $X' \subseteq \mathtt{cl}(s \cdot ind, \phi)$. This is so since $\mathtt{WORLD}(\Sigma, s, \phi)$ returns true. By definition of R'_{ind}, $\langle X, s\rangle R'_{ind}\langle X', s'\rangle$. Furthermore, one can easily show that $\psi' \in \mathtt{cl}(s \cdot ind, \phi)$. By the induction hypothesis, $\mathcal{M}, \langle X', s \cdot ind\rangle \not\models \psi'$ and therefore $\mathcal{M}, \langle X, s\rangle \not\models \psi$.
Case 2: $\psi = [R_i]\psi'$. This is analogous to the Case 1.1 using (v).
Case 3: $\psi = [L(R_i)]\psi'$. This is analogous to the Case 1.1 using (vi).
As a conclusion, since $\phi \in Y$ and $\mathtt{WORLD}(Y, \lambda, \phi)$ returns true, $\mathcal{M}, \langle Y, \lambda\rangle \models \phi$ and therefore ϕ is AML_ω-satisfiable. □

Lemma 5.4. *Let ϕ be an $\mathtt{L}_\omega$-formula. If ϕ is AML_ω-satisfiable, then there is $Y \subseteq \mathtt{cl}(\{\phi\})$ such that $\phi \in Y$ and $\mathtt{WORLD}(Y, \lambda, \phi)$ returns true.*

Proof:
Assume that ϕ is AML_ω-satisfiable. Hence, we obtain that there is an AML_ω-model $\mathcal{M}^0 = \langle W^0, R^0_{ind}, (R^0_i)_{i\in\omega}, (LR^0_i)_{i\in\omega}, V^0\rangle$ and $w^0 \in W^0$ such that $\mathcal{M}^0, w^0 \models \phi$. Actually we show that

(i) for any $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$, for any sequence $\Sigma = X_0 \ldots X_{|s|}$ such that for $j \in \{0, \ldots, |s|\}$,

1. $X_j \subseteq \mathtt{cl}(s[j], \phi)$ is $s[j]$-consistent;
2. if $j \neq |s|$, then $X_j \approx_{s(j)} X_{j+1}$;
3. if $j \neq 0$, then there is $[s(j)]\psi \in \mathtt{cl}(s[j-1], \phi) \setminus X_{j-1}$ such that
   - 3.1. $\psi \notin X_j$;
   - 3.2. if $s(j) = ind$, then there is no $k' \in \{0, \ldots, j-1\}$ such that $X_{j-1} \approx_{ind} X_{k'}$, $\psi \notin X_{k'}$ and $s[j-1] = s[k'] \cdot ind^{j-k'}$.

if there exist an AML_ω-model $\mathcal{M} = \langle W, R_{ind}, (R_i)_{i \in \omega}, (LR_i)_{i \in \omega}, V \rangle$ and $w \in W$ satisfying for all $\psi \in \mathtt{cl}(s, \phi)$, $\mathcal{M}, w \models \psi$ iff $\psi \in X_{|s|}$, then $\mathtt{WORLD}(\Sigma, s, \phi)$ returns true.

Consequently, by taking $s = \lambda$ and $X_0 = \{\psi \in \mathtt{cl}(\{\phi\}) : \mathcal{M}^0, w^0 \models \psi\}$, we obtain that $\mathtt{WORLD}(X_0, \lambda, \phi)$ returns true. The proof of (i) is by induction on the length of s.
Base case: $|s| > (6 \times |\phi| + 1) \times (md(\phi) + 3)$. By the proof of Theorem 5.1, no sequence Σ of length $|s|$ satisfies for $j \in \{0, \ldots, |s|\}$ the conditions 1.-3. So the property trivially holds.
Induction step: assume $s \in (\{ind\} \cup \{r_i, lr_i : i \in \omega\})^*$ is of length $n-1$. Let Σ be a sequence of length $|s|$ such that for $j \in \{0, \ldots, |s|\}$ the conditions 1.-3. hold true. Let $\mathcal{M} = \langle W, R_{ind}, (R_i)_{i \in \omega}, (LR_i)_{i \in \omega}, V \rangle$ be an AML_ω-model and $w \in W$ such that for all $\psi \in \mathtt{cl}(s, \phi)$, $\mathcal{M}, w \models \psi$ iff $\psi \in X_{|s|}$. Since $X_{|s|}$ is s-consistent, $\mathtt{WORLD}(\Sigma, s, \phi)$ returns false only if either the segment "ind" or the segment "r_i" or the segment "lr_i" returns false. Let $[ind]\psi \in \mathtt{cl}(s, \phi) \setminus X_{|s|}$. By hypothesis, $\mathcal{M}, w \not\models [ind]\psi$. So there is $w' \in W$ such that $\langle w, w' \rangle \in R_{ind}$ and $\mathcal{M}, w' \not\models \psi$. Let $Y \subseteq \mathtt{cl}(s \cdot ind, \phi)$ be such that for all $\varphi \in \mathtt{cl}(s \cdot ind, \phi)$, $\varphi \in Y \overset{\text{def}}{\Leftrightarrow} \mathcal{M}, w' \models \varphi$. So, $\psi \notin Y$ and $X_{|s|} \approx_{ind} Y$.

In the case when there is no $X \in \Sigma$ such that $\Sigma = \Sigma_1 X \Sigma_2$, s is of the form $s_1.s_2$ with $|s_2| = |\Sigma_2|$ and $s_2 \in \{ind\}^*$, $\psi \notin X$, $last(\Sigma) \approx_{ind} X$, by the induction hypothesis, $\mathtt{WORLD}(\Sigma.Y, s \cdot ind, \phi)$ returns true. Therefore, $\mathtt{WORLD}(\Sigma, s, \phi)$ does not return false in the "ind" segment of $\mathtt{WORLD}$. Similarly, we can show that $\mathtt{WORLD}(\Sigma, s, \phi)$ does not return false neither in the "r_i" segment nor in the "lr_i" segment of $\mathtt{WORLD}$.
Consequently, $\mathtt{WORLD}(\Sigma, s, \phi)$ returns true. □

Since $\mathtt{WORLD}$ is correct, the proof of Lemma 5.3 provides the finite model property for AML_ω and an exponential bound for the size of the models exists. These results are obtained as a by-product of the complexity result.

Finally,

Theorem 5.2. AML_ω*-satisfiability is in* **PSPACE**.

Proof:
By Lemma 5.3 and Lemma 5.4, for any formula ϕ, ϕ is AML_ω-satisfiable iff there is $X \subseteq \mathtt{cl}(\{\phi\})$ such that $\mathtt{WORLD}(X, \lambda, \phi)$ returns true. By Theorem 5.1, $\mathtt{WORLD}(X, \lambda, \phi)$ requires space in $\mathcal{O}(|\phi|^3)$ and the bit string necessary to remember which sets $X \subseteq \mathtt{cl}(\phi)$ have been already dealt with requires space in $\mathcal{O}(|\phi|)$. So AML_ω-satisfiability is in **PSPACE**. □

6. Tolerance Approximation Multimodal Logics

The complexity of the approximation multimodal logics is now completely characterized. The reader familiar with complexity issues for modal logics may be a bit disappointed since after all, **PSPACE** is the complexity class for modal logics. In a sense, the sophisticated developments of the previous sections do not lead to very surprising results. However, this impression is erroneous as the rest of this section will show.

For $m \in \omega \cup \{\omega\}$, a *tolerance* AML_m*-frame* (or TAML_m-frame) is defined as an AML_m-frame except that R_{ind} is reflexive and symmetric instead of being an equivalence relation. In the sequel, we write R_{sim} instead of R_{ind} and $[sim]$ instead of $[ind]$. Such an alternative definition can be also justified by considering that a lower approximation of a relation is computed from a similarity relation (see e.g. [12]). All the other definitions can be easily adapted. We shall use the term of TAML_m-satisfiability for satisfiability with respect to the class of TAML_m-models. Observe that in the TAML_m-models, for $k \geq 1$, for $i < m$, $R^k_{sim} \circ LR_i \circ R^k_{sim} \subseteq LR_i$. Since R_{sim} is reflexive, it leads to the condition $R^*_{sim} \circ LR_i \circ R^*_{sim} \subseteq LR_i$, or equivalently $R^*_{sim} \circ LR_i \circ R^*_{sim} = LR_i$.

Lemma 6.1. *For $m \in \omega \cup \{\omega\}$, TAML_m-satisfiability is in* **EXPTIME**.

Proof:
Let $m \in \omega \cup \{\omega\}$. We define a logarithmic space transformation into satisfiability for converse-PDL that is known to be in **EXPTIME** (see e.g. [8]). The map f is inductively defined as follows ($\{\mathtt{c}_i : i \in \omega\}$ is a set of distinct program constants):

- $f(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{p}$ for $\mathtt{p} \in \mathtt{For}_0$ and f is homomorphic with respect to the Boolean connectives;
- $f([sim]\phi) \stackrel{\text{def}}{=} [(\mathtt{c}_1 \cup \mathtt{c}_1^{-1}) \cup \top?]f(\phi)$;
- $f([L(R_i)]\phi) \stackrel{\text{def}}{=} [((\mathtt{c}_1 \cup \mathtt{c}_1^{-1}) \cup \top?)^*; \mathtt{c}_{(2\times(i+1))+1}; ((\mathtt{c}_1 \cup \mathtt{c}_1^{-1}) \cup \top?)^*]f(\phi)$;
- $f([R_i]\phi) \stackrel{\text{def}}{=} [(((\mathtt{c}_1 \cup \mathtt{c}_1^{-1}) \cup \top?)^*; \mathtt{c}_{(2\times(i+1))+1}; ((\mathtt{c}_1 \cup \mathtt{c}_1^{-1}) \cup \top?)^*) \cup \mathtt{c}_{2\times(i+1)}]f(\phi)$.

One can show that ϕ is TAML_m-satisfiable iff $f(\phi)$ is satisfiable in converse-PDL. □

The logarithmic space transformation defined in the proof of Lemma 6.1 is a variant of standard mappings from modal logics into PDL (see e.g. [7, 6, 30]). The most surprising result is the **EXPTIME** complexity lower bound.

Lemma 6.2. TAML_1*-satisfiability is* **EXPTIME**-*hard.*

Proof:
Let $\mathtt{L}(\Box)$ be the standard modal language (extension of the language for the propositional calculus by adding the standard modal connective $\Box$). Let B-GSAT be the set of standard modal formulae ϕ such that there is a Kripke model $\mathcal{M} = \langle W, R, V\rangle$ satisfying for all $w \in W$, $\mathcal{M}, w \models \phi$ and R is reflexive and symmetric. 'B-GSAT' stands for the global satisfiability problem for the standard modal logic B that is known to be **EXPTIME**-hard [1, Theorem 1].

Let us define a logarithmic space transformation from B-GSAT into TAML$_1$-satisfiability. Let ϕ be a formula of $\mathtt{L}(\Box)$. Let $f(\phi)$ be the formula $\neg[L(R_1)]\neg\top \wedge [L(R_1)]\phi'$ where ϕ' is obtained from the formula ϕ by replacing every occurrence of $\Box$ by $[sim]$. Let us show that (i) ϕ belongs to B-GSAT iff (ii) $\neg[L(R_1)]\neg\top \wedge [L(R_1)]\phi'$ is TAML$_1$-satisfiable.
(i) $\rightarrow$ (ii) Assume that ϕ belongs to B-GSAT. Hence, there is a Kripke model $\mathcal{M} = \langle W, R, V\rangle$ such that for $w \in W$, $\mathcal{M}, w \models \phi$ and R is reflexive and symmetric. Let w_0 be an arbitrary element of W. Let $\mathcal{M}' = \langle W', R_{sim}, LR_1, R_1, V'\rangle$ be the TAML$_1$-model such that

- $W' \stackrel{\text{def}}{=} W$; $V' \stackrel{\text{def}}{=} V$;
- $R_{sim} = R$; $R_1 \stackrel{\text{def}}{=} LR_1 = R^*_{sim}$.

One can check that $R^*_{sim} \circ LR_1 \circ R^*_{sim} \subseteq LR_1$. Obviously $\mathcal{M}', w_0 \models \neg[L(R_1)]\neg\top$ since $\langle w_0, w_0\rangle \in LR_1$. For $w \in W$, $\mathcal{M}', w \models \phi'$ since ϕ is in B-GSAT. In particular, for $w \in R^*_{sim}(w_0)$, $\mathcal{M}', w \models \phi'$. So, $\mathcal{M}', w_0 \models \neg[L(R_1)]\neg\top \wedge [L(R_1)]\phi'$.
(ii) $\rightarrow$ (i) Assume that $\neg[L(R_1)]\neg\top \wedge [L(R_1)]\phi'$ is TAML$_1$-satisfiable. So, there is a TAML$_1$-model $\mathcal{M} = \langle W, R_{sim}, LR_1, R_1, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \models \neg[L(R_1)]\neg\top \wedge [L(R_1)]\phi'$. So, there is $w_0 \in LR_1(w)$. Since $R_{sim} \circ LR_1 \circ R_{sim} \subseteq LR_1$ and R_{sim} is reflexive, $LR_1 \circ R^*_{sim} \subseteq LR_1$. So, for all $w' \in R^*_{sim}(w_0)$, $\mathcal{M}, w' \models \phi'$ since for all $w' \in (LR_1 \circ R^*_{sim})(w)$, $\mathcal{M}, w' \models \phi'$. Let $\mathcal{M}' = \langle W', R', V'\rangle$ be the Kripke model such that

- $W' \stackrel{\text{def}}{=} R^*_{sim}(w_0)$;
- R' is the restriction of R_{sim} to W';
- V' is the restriction of V to W'.

So for all $w' \in W'$, $\mathcal{M}', w' \models \phi$. Hence, ϕ is in B-GSAT since R' is also reflexive and symmetric.
$\Box$

Corollary 6.1. TAML$_1$-*satisfiability restricted to formulae without the modal connective* $[R_1]$ *is* **EXPTIME**-*hard.*

The proof of Lemma 6.2 entails Corollary 6.1. Therefore, we have isolated another *simple* bimodal logic that is **EXPTIME**-hard although the independent fusion of the corresponding (mono)modal logics is in **PSPACE** (see e.g. [24]).

Theorem 6.1. TAML$_0$-*satisfiability is* **PSPACE**-*complete and for* $1 \leq m \leq \omega$, TAML$_m$-*satisfiability is* **EXPTIME**-*complete.*

7. Concluding Remarks

We have shown that all the approximation multimodal logics AML(τ_m), $1 \leq m < \omega$, introduced and investigated in [26], [28] have a **PSPACE**-complete satisfiability problem. This should not come as a real surprise since **PSPACE** is known to be the complexity class for modal logics. However, we have shown that a tolerance variant of the logics AML(τ_m) leads to **EXPTIME**-complete satisfiability problems. Hence, a further analysis about the **PSPACE**-completeness of the approximation multimodal logics shall certainly help understanding the complexity of other rough set theory based logics and in a more general setting the computational complexity of numerous polymodal logics with interdependent modal connectives. This is part of our future work.

In the literature, most of the logics derived from rough set theory do not satisfy the requirements proposed in [20] for the applied logics in approximation reasoning. Indeed, the features of the logics are not driven from the data. Investigations on algorithmic methods for extracting logical structures (like approximate schemes of reasoning) from data seems to be a very important research direction.

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