A logic with relative knowledge operators

Stéphane Demri
Laboratoire LEIBNIZ - C.N.R.S.,
46 Avenue Félix Viallet,
38031 Grenoble, France
e-mail: demri@imag.fr

January 21st, 1998

Abstract. We study a knowledge logic that assumes that to each set of agents, an indiscernibility relation is associated and the agents decide the membership of objects or states up to this indiscernibility relation. Its language contains a family of relative knowledge operators. We prove the decidability of the satisfiability problem, we show its **EXPTIME**-completeness and as a side-effect we define a complete Hilbert-style axiomatization.

Key-words: modal logic, relative knowledge operator, decidability, complexity, Hilbert-style axiomatization.

1. Introduction

In several logical analyses of reasoning about knowledge, it is assumed that the knowledge of the agents depends on a degree of certainty with which they grasp objects from a given domain (see e.g. (Orło89)). Several knowledge logics based on this assumption use classes of semantical structures inherited from Pawlak's rough set theory (Paw81) and some of them are investigated in (Orło89). Some of these structures have features from Kripke-style semantics for logics of knowledge (see e.g. (HM92)), but also from *information systems* in (Paw81) and from Aumann's structures (Aum76). For instance, in (Orło89), semantical structures of the form $(OB, AGT, (ind_A)_{A \subseteq AGT})$ are introduced where OB is a set of *objects*, AGT is a set of *agents* and $(ind_A)_{A \subseteq AGT}$ is a family of equivalence relations over OB such that $(o, o') \in ind_A$ iff the objects o and o' cannot be distinguished by the set A of agents. It is also required that $ind_\emptyset = OB \times OB$ and for any $A, A' \subseteq AGT$, $ind_{A \cup A'} = ind_A \cap ind_{A'}$. These conditions come from the way the family $(ind_A)_{A \subseteq AGT}$ can be constructed from information systems (Paw81). Each relation ind_A is called an *indiscernibility relation.* Unlike Kripke-style semantics for logics of knowledge, OB is not a set of *knowledge states.* In (Orło89), for each set $A \subseteq AGT$ of agents, a knowledge operator $\mathcal{K}(A) : \mathcal{P}(OB) \rightarrow \mathcal{P}(OB)$ is defined such that for any $X \subseteq OB$,

$$(\star)\ \mathcal{K}(A)(X) \stackrel{\text{def}}{=} \{o \in OB : ind_A(o) \subseteq X\} \cup \{o \in OB : ind_A(o) \subseteq (OB \setminus X)\}$$

$\mathcal{K}(A)(X)$ represents the set of objects in OB for which the set A of agents can decide whether they belong to X. Since the structure $(OB, AGT, (ind_A)_{A \subseteq AGT})$ is obviously a Kripke-style structure and since it can be easily related to Aumann's structures (Aum76), it is not surprising that the knowledge operator $\mathcal{K}(A)$ is similar to those introduced in (Aum76) in the event-based approach of knowledge (see also (FHMV95)). Indeed in $(\star)$, X can be seen as an *event* (Aum76) whereas Aumann's knowledge operator is defined by $\mathcal{K}_a(A)(X) \stackrel{\text{def}}{=} \{o \in OB : ind_A(o) \subseteq X\}$. Moreover, each operator $\mathcal{K}(A)$ also corresponds to the operator $\triangle$ in logics of non-contingency[1] (see e.g. (MR66; Hum95; Kuh95)). The family $(\mathcal{K}(A))_{A \subseteq AGT}$ of knowledge operators associated to the structure $(OB, AGT, (ind_A)_{A \subseteq AGT})$ will be central in the rest of the paper.

The logic defined by Ewa Orłowska in (Orło89) (section 6) admits in its language a family of knowledge operators $K(\mathtt{A})$ where $\mathtt{A}$ is a Boolean expression (interpreted as a set of agents). As expected, the knowledge operator $K(\mathtt{A})$ is interpreted by $\mathcal{K}(A)$ where A is the interpretation of $\mathtt{A}$. In the rest of the paper, this logic is called LKO. The main goal of the present work is to answer the following open questions:

1. What is the computational complexity of the satisfiability problem for LKO?
2. How to define an adequate Hilbert-style system (called herein $\vdash_{lko}$) for LKO?

Actually, we show that the satisfiability problem is **EXPTIME**-complete and the definition of $\vdash_{lko}$ is inspired by the completeness proof of the system for $\mathrm{S5}_n^D$ given in (Bal97). In order to prove these results, we will define faithful translations between LKO and $S5_\omega^{\cap,\mathtt{U}}$, an auxiliary logic extending the knowledge logic $\mathrm{S5}_n^D$ (HM92). Although the $\mathrm{S5}_n^D$-satisfiability problem ($n \geq 2$) is known to be **PSPACE**-complete, we show that the satisfiability problem for $S5_\omega^{\cap,\mathtt{U}}$ is **EXPTIME**-complete. The lower bound is a mere consequence of more general results from (Hem96) and the introduction of the universal operator $[\mathtt{U}]$ -see e.g. (GP92)- can be seen as the main cause for this jump in the complexity classes. The upper bound is obtained by adapting arguments from (Pra79) where the satisfiability problem for the Propositional Dynamic Logic PDL is shown to be in **EXPTIME**. A complete Hilbert-style system[2] $\vdash_{aux}$ is also defined for $S5_\omega^{\cap,\mathtt{U}}$ by extending the system for the knowledge logic $\mathrm{S5}_n^D$ and by slightly adapting the copying construction used in the completeness proof from (Bal97) (see also (Vak91)). The

[1] The author thanks Lloyd Humberstone for pointing him to these works.
[2] 'aux' is sometimes used instead of $S5_\omega^{\cap,\mathtt{U}}$.

translation from LKO into $S5_{\omega}^{\cap,\mathtt{U}}$ takes advantage of the existence of normal forms for Boolean terms following developments from (Lem65; Kon97). This is sufficient[3] to define $\vdash_{lko}$ of which the axioms and inference rules mimic those of $\vdash_{aux}$. Axiomatization of LKO has been open until now although some fragments have already been axiomatized (see e.g. (MR68; Val88; Dem97)). Although the translation may exponentially increase the length of formulae, we will show that the satisfiability problem for LKO is in **EXPTIME** by observing that the translation increases linearly the number of subformulae -the full arguments are given in the paper. In order to show the lower complexity bound of LKO-satisfiability, we define a polynomial-time transformation from $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability into LKO-satisfiability. Some of the translations involved in the paper are instances of those defined in (DG98) and apart from the results proved herein, we wish to point out the close relationships between LKO and $S5_{\omega}^{\cap,\mathtt{U}}$.

The rest of the paper is structured as follows. In Section 2, we recall the definition of the logic with knowledge operators LKO. In Section 3, we introduce the auxiliary logic $S5_{\omega}^{\cap,\mathtt{U}}$, we prove the **EXPTIME**-completeness of the satisfiability problem and provide a complete Hilbert-style system $\vdash_{aux}$. In Section 4, we define faithful translations between LKO and $S5_{\omega}^{\cap,\mathtt{U}}$ and we show why the satisfiability problem for LKO is **EXPTIME**-complete. In Section 5, we define a Hilbert-style system $\vdash_{lko}$ for LKO and we show its completeness with respect to the LKO-validity by translating deductions in $\vdash_{aux}$ into deductions in $\vdash_{lko}$.

2. A logic with knowledge operators: LKO

The set of primitive symbols of the *modal language* L_{lko} is composed of a set $\mathtt{For}_0 = \{\mathtt{p}, \mathtt{q}, \ldots\}$ of propositional variables, the classical connectives $\neg, \wedge$ (negation and conjunction), and the countably infinite set $\{K(\mathtt{A}) : \mathtt{A} \in \mathtt{P}\}$ of unary modal operators where the set of *agent expressions* $\mathtt{P}$ is the smallest set containing a countably infinite set $\mathtt{P}_0 = \{\mathtt{C}_1, \mathtt{C}_2, \ldots\}$ of constants and it is closed under the Boolean operators $\cap, \cup, -$. The formation rules of the set $\mathtt{For}_{lko}$ of lko-formulae are those of the classical propositional calculus plus the rule: if $\mathtt{F} \in \mathtt{For}_{lko}$ and $\mathtt{A} \in \mathtt{P}$, then $K(\mathtt{A})\mathtt{F} \in \mathtt{For}_{lko}$. We use the connectives $\vee, \Rightarrow, \Leftrightarrow$ as abbreviations with their standard meaning. For any syntactic category $\mathtt{X}$ and any syntactic object $\mathtt{O}$, we write $\mathtt{X}(\mathtt{O})$ to denote the set composed of elements of $\mathtt{X}$ occurring in $\mathtt{O}$. Moreover for any syntactic object $\mathtt{O}$, we write $|\mathtt{O}|$ to

[3] Philippe Balbiani has communicated to the author the possibility of proving the soundness and completeness of $\vdash_{lko}$ by adapting some arguments from (Bal97) without translation.

denote its *length* (or *size*), i.e., the number of symbol occurrences in $\mathtt{O}$. For instance $\mathtt{For}_0(\mathtt{p} \vee \neg \mathtt{q}) = \{\mathtt{p}, \mathtt{q}\}$ and $|\mathtt{p} \vee \neg \mathtt{q}| = 4$. A *P-valuation* m is a map $m : \mathtt{P} \rightarrow \mathcal{P}(Ag)$ such that Ag is a non-empty set and for any $\mathtt{A}_1, \mathtt{A}_2 \in \mathtt{P}$,

- $m(\mathtt{A}_1 \cap \mathtt{A}_2) = m(\mathtt{A}_1) \cap m(\mathtt{A}_2)$;
- $m(\mathtt{A}_1 \cup \mathtt{A}_2) = m(\mathtt{A}_1) \cup m(\mathtt{A}_2)$;
- $m(-\mathtt{A}_1) = Ag \setminus m(\mathtt{A}_1)$.

For any $\mathtt{A}, \mathtt{B} \in \mathtt{P}$ we write $\mathtt{A} \equiv \perp$ (resp. $\mathtt{A} \sqsubseteq \mathtt{B}$, $\mathtt{A} \equiv \mathtt{B}$) when for any $\mathtt{P}$-valuation m, $m(\mathtt{A}) = \emptyset$ (resp. $m(\mathtt{A}) \subseteq m(\mathtt{B})$, $m(\mathtt{A}) = m(\mathtt{B})$). Both relations $\equiv$ and $\sqsubseteq$ are known to be decidable.

DEFINITION 2.1. An *lko-frame* $\mathcal{F}$ is a structure $\mathcal{F} = (OB, Ag, (R_Q)_{Q \subseteq Ag})$ such that OB is a non-empty set of *objects*, Ag is a non-empty set of *agents* and $(R_Q)_{Q \subseteq Ag}$ is a family of equivalence relations over OB such that for any $Q, Q' \subseteq Ag$, $R_{Q \cup Q'} = R_Q \cap R_{Q'}$ and $R_\emptyset = OB \times OB$. An *lko-model* $\mathcal{M}$ is a structure $\mathcal{M} = (OB, Ag, (R_Q)_{Q \subseteq Ag}, m)$ such that $\mathcal{F} = (OB, Ag, (R_Q)_{Q \subseteq Ag})$ is an lko-frame and m is a mapping $m : \mathtt{For}_0 \cup \mathtt{P} \rightarrow \mathcal{P}(OB) \cup \mathcal{P}(Ag)$ such that for any $\mathtt{p} \in \mathtt{For}_0$, $m(\mathtt{p}) \subseteq OB$ and the restriction of m to $\mathtt{P}$ is a $\mathtt{P}$-valuation. We say that the lko-model $\mathcal{M}$ is *based on* $\mathcal{F}$. ∇

Intuitively, in an lko-frame $(o, o') \in R_Q$ means that the objects o and o' cannot be distinguished by the set Q of agents. Let $\mathcal{M} = (OB, Ag, (R_Q)_{Q \subseteq Ag}, m)$ be an lko-model. As usual, we say that a formula $\mathtt{F}$ is *satisfied by the object* $o \in OB$ *in* $\mathcal{M}$ (written $\mathcal{M}, o \models \mathtt{F}$) when the following conditions are satisfied.

- $\mathcal{M}, o \models \mathtt{p} \overset{\text{def}}{\Leftrightarrow} o \in m(\mathtt{p})$, for any $\mathtt{p} \in \mathtt{For}_0$;
- $\mathcal{M}, o \models \neg \mathtt{F} \overset{\text{def}}{\Leftrightarrow}$ not $\mathcal{M}, o \models \mathtt{F}$;
- $\mathcal{M}, o \models \mathtt{F} \wedge \mathtt{G} \overset{\text{def}}{\Leftrightarrow} \mathcal{M}, o \models \mathtt{F}$ and $\mathcal{M}, o \models \mathtt{G}$;
- $\mathcal{M}, o \models K(\mathtt{A})\mathtt{F} \overset{\text{def}}{\Leftrightarrow}$ either for any $o' \in R_{m(\mathtt{A})}(o)$, $\mathcal{M}, o' \models \mathtt{F}$ or for any $o' \in R_{m(\mathtt{A})}(o)$, $\mathcal{M}, o' \models \neg \mathtt{F}$.

We omit the standard definitions of satisfiability for the other logical operators. The *standard* necessity operator $[\mathtt{A}]$ can be defined by $[\mathtt{A}]\mathtt{F} \overset{\text{def}}{=} \mathtt{F} \wedge K(\mathtt{A})\mathtt{F}$. Observe that,

$$\{o \in OB : \mathcal{M}, o \models K(\mathtt{A})\mathtt{F}\} = \mathcal{K}(m(\mathtt{A}))(\{o \in OB : \mathcal{M}, o \models \mathtt{F}\})$$

A formula $\mathtt{F}$ is *true* in an lko-model $\mathcal{M}$ (written $\mathcal{M} \models \mathtt{F}$) $\overset{\text{def}}{\Leftrightarrow}$ for any $o \in OB$, $\mathcal{M}, o \models \mathtt{F}$. An lko-formula $\mathtt{F}$ is *true* in an lko-frame $\mathcal{F}$

(written $\mathcal{F} \models \texttt{F}$) $\stackrel{\text{def}}{\Leftrightarrow}$ F is true in every lko-model based on $\mathcal{F}$. An lko-formula F is said to be *LKO-valid* $\stackrel{\text{def}}{\Leftrightarrow}$ F is true in any lko-model. An lko-formula F is said to be *LKO-satisfiable* $\stackrel{\text{def}}{\Leftrightarrow}$ there exist an lko-model $\mathcal{M} = (OB, Ag, (R_Q)_{Q \subseteq Ag}, m)$ and $o \in OB$ such that $\mathcal{M}, o \models \texttt{F}$. Observe that $K(\texttt{A})\texttt{p} \Leftrightarrow [\texttt{A}]\texttt{p} \vee [\texttt{A}]\neg\texttt{p}$ is LKO-valid and it is therefore irrelevant[4] whether in the language the primitive knowledge operators are of the form $[\texttt{A}]$ or $K(\texttt{A})$ (see e.g. (Cre88)). In the sequel, we prefer to keep the operators $K(\texttt{A})$ since they are faithful to the original formulation in (Orło89) and they correspond to the knowledge operators in (Hin62) where $K(\texttt{A})\texttt{p}$ is interpreted by "the agents A know whether the proposition p holds".

3. The logic $S5_{\omega}^{\cap,\mathtt{U}}$

In this section, we shall define the logic $S5_{\omega}^{\cap,\mathtt{U}}$, we propose a complete Hilbert-style axiomatization and we show the decidability of the satisfiability problem and its **EXPTIME**-completeness. In Section 4, we shall define a transformation from the LKO-satisfiability into the $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability and a *polynomial-time* transformation from the $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability into the LKO-satisfiability. It is worth observing that although $S5_{\omega}^{\cap,\mathtt{U}}$ has the status of an *auxiliary* logic (since our initial motivation consists in better understanding the knowledge logic LKO defined in (Orło89)), the results for $S5_{\omega}^{\cap,\mathtt{U}}$ are crucial for understanding LKO. In this paper, we defend the thesis that $S5_{\omega}^{\cap,\mathtt{U}}$ and LKO are very similar.

3.1. Definition

The set of primitive symbols of the *modal language* L_{aux} is composed of the set $\texttt{For}_0$, the classical connectives $\neg$, $\wedge$, and the family $\{[\texttt{a}] : \texttt{a} \in \texttt{M} \cup \{\texttt{U}\}\}$ of unary modal operators where M is the smallest set containing a countably infinite set $\texttt{M}_0 = \{\texttt{c}_0, \texttt{c}_1, \ldots\}$ of modal constants and M is closed under the binary intersection operator $\cap$. The formation rules of the set $\texttt{For}_{aux}$ of $S5_{\omega}^{\cap,\mathtt{U}}$-formulae are those of the classical propositional calculus plus the rule: if $\texttt{F} \in \texttt{For}_{aux}$ and $\texttt{a} \in \texttt{M} \cup \{\texttt{U}\}$, then $[\texttt{a}]\texttt{F} \in \texttt{For}_{aux}$. As usual, by definition $\langle \texttt{a} \rangle \texttt{F} \stackrel{\text{def}}{=} \neg[\texttt{a}]\neg\texttt{F}$. $[\texttt{U}]$ will be interpreted as the *universal operator* (see e.g (GP92)).

[4] It is true for the expressive power but also for the complexity class of the satisfiability problems since the relations of the models are reflexive and the universal operator is in the language.

DEFINITION 3.1.A *standard* (resp. *non-standard*) $S5_{\omega}^{\cap,U}$-*frame* is a structure $(W, (R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}\cup\{\mathtt{U}\}})$ such that W is a non-empty set of *states* and $(R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}\cup\{\mathtt{U}\}}$ is a family of equivalence relations over W such that

- $R_{\mathtt{U}} = W \times W$ (resp. for any $\mathtt{a} \in \mathtt{M}$, $R_{\mathtt{a}} \subseteq R_{\mathtt{U}}$);
- for any $\mathtt{a}, \mathtt{b} \in \mathtt{M}$, $R_{\mathtt{a}\cap\mathtt{b}} = R_{\mathtt{a}} \cap R_{\mathtt{b}}$ (resp. $R_{\mathtt{a}\cap\mathtt{b}} \subseteq R_{\mathtt{a}} \cap R_{\mathtt{b}}$);
- for any $\mathtt{a}, \mathtt{b} \in \mathtt{M}$ such that $\mathtt{M}_0(\mathtt{a}) = \mathtt{M}_0(\mathtt{b})$, $R_{\mathtt{a}} = R_{\mathtt{b}}$.

A *standard* (resp. *non-standard*) $S5_{\omega}^{\cap,U}$-*model* $\mathcal{M} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}\cup\{\mathtt{U}\}}, m)$ is a structure such that $(W, (R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}})$ is a standard (resp. non-standard) $S5_{\omega}^{\cap,\mathtt{U}}$-frame and m is a mapping $m : \mathtt{For}_0 \rightarrow \mathcal{P}(W)$. We say that the standard (resp. the non-standard) $S5_{\omega}^{\cap,\mathtt{U}}$-model $\mathcal{M}$ is *based on* the standard (resp. non-standard) $S5_{\omega}^{\cap,\mathtt{U}}$-frame $(W, (R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}\cup\{\mathtt{U}\}})$. ∇

When interpreted by non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-models, a modal expression $\mathtt{a} \in \mathtt{M}$ can be equivalently represented by $\mathtt{M}_0(\mathtt{a})$ (this is due to the third point of Definition 3.1). Let $\mathcal{M} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in \mathtt{M}\cup\{\mathtt{U}\}}, m)$ be a non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model, $w \in W$ and $\mathtt{F}$ be an $S5_{\omega}^{\cap,\mathtt{U}}$-formula. The satisfiability relation $\models$ is defined as for the lko-models except for the following condition (which is standard in the Kripke-style semantics): $\mathcal{M}, w \models [\mathtt{a}]\mathtt{F} \overset{\text{def}}{\Leftrightarrow}$ for any $w' \in R_{\mathtt{a}}(w)$, $\mathcal{M}, w' \models \mathtt{F}$. The notions of $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability, $S5_{\omega}^{\cap,\mathtt{U}}$-validity, ... are defined similarly to those for LKO. $S5_{\omega}^{\cap,\mathtt{U}}$-validity and $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability are defined with the class of *standard* $S5_{\omega}^{\cap,\mathtt{U}}$-models although we shall see that it is *equivalent* to the notion defined with the class of non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-models. In Definition 3.2 below, a notion of a partial $S5_{\omega}^{\cap,\mathtt{U}}$-model is introduced in order to simplify further developments about the complexity of the $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability problem.

DEFINITION 3.2.Let $X \subseteq \mathtt{M} \cup \{\mathtt{U}\}$ and $\emptyset \neq Y \subseteq \mathtt{For}_0$. A $\langle X, Y\rangle$-*partial* $S5_{\omega}^{\cap,U}$-*model* is a structure $\mathcal{M} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in X\cup\{\mathtt{U}\}}, m)$ such that W is a non-empty set, m is a map $m : Y \rightarrow \mathcal{P}(W)$ and $(R_{\mathtt{a}})_{\mathtt{a}\in X\cup\{\mathtt{U}\}}$ is a family of equivalence relations on W such that for any $\mathtt{a} \in X$, $R_{\mathtt{a}} \subseteq R_{\mathtt{U}}$ and for any $\mathtt{a}, \mathtt{b} \in X$ if $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$, then $R_{\mathtt{a}} \subseteq R_{\mathtt{b}}$. ∇

The relation $\models$ can be easily extended to the class of partial $S5_{\omega}^{\cap,\mathtt{U}}$-models provided the $S5_{\omega}^{\cap,\mathtt{U}}$-formulae contain the adequate modal expressions and propositional variables.

PROPOSITION 3.3.Let $\mathtt{F}$ be an $S5_{\omega}^{\cap,\mathtt{U}}$-formula. $\mathtt{F}$ is satisfiable in a non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model iff $\mathtt{F}$ is satisfiable in a $\langle \mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5_{\omega}^{\cap,\mathtt{U}}$-model.

PROOF: Let $\mathcal{M} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m)$ be a non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model, $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$. Let $\mathcal{M}' = (W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}(\mathtt{F})\cup\{\mathtt{U}\}}, m')$ be the $\langle\mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5_{\omega}^{\cap,\mathtt{U}}$-model such that m' is the restriction of m to $\mathtt{For}_0(\mathtt{F})$. It is easy to show that $\mathcal{M}', w \models \mathtt{F}$.

Now assume that $\mathcal{M}' = (W, (R'_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}(\mathtt{F})\cup\{\mathtt{U}\}}, m')$ is a $\langle\mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5_{\omega}^{\cap,\mathtt{U}}$-model such that for some $w \in W$, $\mathcal{M}', w \models \mathtt{F}$. Let $\mathcal{M}$ be the non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model $(W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m)$ built as follows: for any $\mathtt{a} \in \mathtt{M}(\mathtt{F}) \cup \{\mathtt{U}\}$, $R_{\mathtt{a}} \stackrel{\text{def}}{=} R'_{\mathtt{a}}$ and for any $\mathtt{p} \in \mathtt{For}_0(\mathtt{F})$, $m(\mathtt{p}) \stackrel{\text{def}}{=} m'(\mathtt{p})$. So obviously $\mathcal{M}, w \models \mathtt{F}$ is guaranteed. Let us fix the other values of the structure $\mathcal{M}$ so that $\mathcal{M}$ is a non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model.

- for any $\mathtt{p} \in \mathtt{For}_0 \setminus \mathtt{For}_0(\mathtt{F})$, $m(\mathtt{p}) \stackrel{\text{def}}{=} W$ (arbitrary value);
- for any $\mathtt{a} \in \mathtt{M}$ such that $\mathtt{M}_0(\mathtt{a}) \cap \mathtt{M}_0(\mathtt{F}) = \emptyset$, $R_{\mathtt{a}} \stackrel{\text{def}}{=} R'_{\mathtt{U}}$;
- for any $\mathtt{a} \in \mathtt{M}$ such that $\mathtt{M}_0(\mathtt{a}) \subseteq \mathtt{M}_0(\mathtt{F})$ and none of the previous cases hold,
$$R_{\mathtt{a}} \stackrel{\text{def}}{=} R'_{\mathtt{U}} \cap \bigcap\{R'_{\mathtt{b}} : \mathtt{b} \in \mathtt{M}(\mathtt{F}),\ \mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})\}$$
- for any $\mathtt{a} \in \mathtt{M}$ such that $\mathtt{M}_0(\mathtt{a}) \cap \mathtt{M}_0(\mathtt{F}) = \{\mathtt{c}_1, \ldots, \mathtt{c}_n\}$ $(n \geq 1)$ and $\mathtt{M}_0(\mathtt{a}) \not\subseteq \mathtt{M}_0(\mathtt{F})$, $R_{\mathtt{a}} \stackrel{\text{def}}{=} R_{\mathtt{c}_1\cap\ldots\cap\mathtt{c}_n}$.

For instance with the definition above, for any $\mathtt{a} \in \mathtt{M}$ such that there is $\mathtt{b} \in \mathtt{M}(\mathtt{F})$ with $\mathtt{M}_0(\mathtt{a}) = \mathtt{M}_0(\mathtt{b})$, we have $R_{\mathtt{a}} = R'_{\mathtt{b}}$. It is a routine task to check that for any $\mathtt{a} \in \mathtt{M}$, $R_{\mathtt{a}} \subseteq R_{\mathtt{U}}$ and for any $\mathtt{a}, \mathtt{b} \in \mathtt{M}$, $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$ implies $R_{\mathtt{a}} \subseteq R_{\mathtt{b}}$ -different cases have to be distinguished according to the definition of $(R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}$. This leads to the fact that $\mathcal{M}$ is a non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-model. Q.E.D.

In the forthcoming constructions, partial $S5_{\omega}^{\cap,\mathtt{U}}$-models will be built instead of non-standard $S5_{\omega}^{\cap,\mathtt{U}}$-models.

3.2. A complete Hilbert-style system

By $\vdash_{aux}$ we understand the smallest set of $S5_{\omega}^{\cap,\mathtt{U}}$-formulae that satisfies the following conditions:

(PC) $\vdash_{aux}$ contains every tautology of the classical propositional calculus;

(MP) $\vdash_{aux}$ is closed under *modus ponens*, i.e. $\{\mathtt{F}, \mathtt{F} \Rightarrow \mathtt{G}\} \subseteq\vdash_{aux}$ implies $\mathtt{G} \in\vdash_{aux}$;

(SUB) $\vdash_{aux}$ is closed under *uniform substitution*, i.e $\mathtt{F} \in\vdash_{aux}$ implies $\mathtt{F}[\mathtt{p} \leftarrow \mathtt{G}] \in\vdash_{aux}$ where $\mathtt{F}[\mathtt{p} \leftarrow \mathtt{G}]$ is obtained from $\mathtt{F}$ by simultaneously substituting every occurrence of the propositional variable $\mathtt{p}$ by $\mathtt{G}$;

(NEC) $\vdash_{aux}$ is closed under *necessitation*, i.e $\mathtt{F} \in \vdash_{aux}$ implies $[\mathtt{a}]\mathtt{F} \in \vdash_{aux}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$;

$\vdash_{aux}$ contains every formula of the form

(K) $[\mathtt{a}](\mathtt{p} \Rightarrow \mathtt{q}) \wedge [\mathtt{a}]\mathtt{p} \Rightarrow [\mathtt{a}]\mathtt{q}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$;

(T) $[\mathtt{a}]\mathtt{p} \Rightarrow \mathtt{p}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$;

(5) $\langle\mathtt{a}\rangle\mathtt{p} \Rightarrow [\mathtt{a}]\langle\mathtt{a}\rangle\mathtt{p}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$;

($\cap$) $[\mathtt{a}]\mathtt{p} \vee [\mathtt{b}]\mathtt{p} \Rightarrow [\mathtt{a} \cap \mathtt{b}]\mathtt{p}$ for any $\mathtt{a}, \mathtt{b} \in \mathtt{M}$;

($\mathtt{U}$) $[\mathtt{U}]\mathtt{p} \Rightarrow [\mathtt{a}]\mathtt{p}$ for any $\mathtt{a} \in \mathtt{M}$;

(AC) $([\mathtt{a} \cap (\mathtt{a}' \cap \mathtt{a}'')]\mathtt{p} \Leftrightarrow [(\mathtt{a} \cap \mathtt{a}') \cap \mathtt{a}'']\mathtt{p}) \wedge ([\mathtt{a} \cap \mathtt{a}']\mathtt{p} \Leftrightarrow [\mathtt{a}' \cap \mathtt{a}]\mathtt{p})$ for any $\mathtt{a}, \mathtt{a}', \mathtt{a}'' \in \mathtt{M}$.

$\vdash_{aux}$ is a slight extension of the Hilbert-style system for $S5_n^D$ (see e.g. (Bal97)) where some axioms have been added in order to take into account the universal operator $[\mathtt{U}]$ and a countably infinite set $\mathtt{M}$ of modal constants. In a standard way, we define the notions of theoremhood (written $\vdash_{aux}$) and deducibility in $\vdash_{aux}$. Observe that ($\cap$) and (AC) can be replaced by $[\mathtt{a}]\mathtt{p} \Rightarrow [\mathtt{b}]\mathtt{p}$ when $\mathtt{M}_0(\mathtt{a}) \subseteq \mathtt{M}_0(\mathtt{b})$ while preserving the set of theorems. We indifferently write $\vdash_{aux} \mathtt{F}$ or $\mathtt{F} \in \vdash_{aux}$ and we shall also consider $\vdash_{aux}$ as an Hilbert-style system in some obvious way from the definition of the set $\vdash_{aux}$ of $S5_\omega^{\cap,\mathtt{U}}$-formulae. This will be also true for the other forthcoming systems. It is easy to show that if $\vdash_{aux} \mathtt{F}$, then $\mathtt{F}$ is valid in all non-standard $S5_\omega^{\cap,\mathtt{U}}$-models. Using the copying technique from (Vak91; Bal97) we can show the following result:

PROPOSITION 3.4.For any $S5_\omega^{\cap,\mathtt{U}}$-formula $\mathtt{F}$, $\mathtt{F}$ is $S5_\omega^{\cap,\mathtt{U}}$-valid iff $\vdash_{aux} \mathtt{F}$.

Observe that the use of the copying technique is crucial in the proof of Proposition 3.4 since the canonical model of $\vdash_{aux}$ is not a standard $S5_\omega^{\cap,\mathtt{U}}$-model. Using arguments similar to those used in (Bal97), one can show that $\vdash_{aux} \mathtt{F}$ iff $\mathtt{F}$ is $S5_\omega^{\cap,\mathtt{U}}$-valid iff $\mathtt{F}$ is valid in all non-standard $S5_\omega^{\cap,\mathtt{U}}$-models.

3.3. Decidability of the logic $S5_\omega^{\cap,\mathtt{U}}$

Let us show that the satisfiability problem for $S5_\omega^{\cap,\mathtt{U}}$ is decidable by using a filtration-like construction. Let $\mathtt{F}$ be an $S5_\omega^{\cap,\mathtt{U}}$-formula. We write $\Gamma_\mathtt{F}$ to denote the set $\Gamma_\mathtt{F} = sub(\mathtt{F})$, where $sub(\mathtt{F})$ is the set of subformulae of $\mathtt{F}$. Let $\Gamma_\mathtt{F}^+$ be the set $\Gamma_\mathtt{F} \cup \{\neg\mathtt{G} : \mathtt{G} \in \Gamma_\mathtt{F}\}$. The cardinality of $\Gamma_\mathtt{F}^+$ is at most $2 \times card(sub(\mathtt{F})) \leq 2 \times |\mathtt{F}|$. Let $\mathcal{M}^c = (W^c, (R^c_\mathtt{a})_{\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}}, m^c)$ be the canonical model for $\vdash_{aux}$ (see e.g. (Mak66)). As usual, for any subset X of $\mathtt{For}$ and any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$, we write $[\mathtt{a}]X$ to denote the set $\{\mathtt{G} : [\mathtt{a}]\mathtt{G} \in X\}$. W^c is the set of all *maximally* $\vdash_{aux}$*-consistent* sets, for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$ and any $X, Y \in W^c$, $X R^c_{\mathtt{a}} Y \stackrel{\mathrm{def}}{\Leftrightarrow} [\mathtt{a}]X \subseteq Y$. Moreover for any $\mathtt{p} \in \mathtt{For}_0$, $X \in m^c(\mathtt{p}) \stackrel{\mathrm{def}}{\Leftrightarrow} \mathtt{p} \in X$. It is a routine task to check that $\mathcal{M}^c$ is a non-standard $S5^{\cap,\mathtt{U}}_{\omega}$-model. As a consequence,

1. for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$ and any $X, Y \in W^c$, $X R^c_{\mathtt{a}} Y$ implies $[\mathtt{a}]X = [\mathtt{a}]Y$;
2. for any $\mathtt{a} \in \mathtt{M}$, and any $X, Y \in W^c$, $X R^c_{\mathtt{a}} Y$ implies for any $\mathtt{b} \in \mathtt{M} \cup \{\mathtt{U}\}$, $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$ implies $X R^c_{\mathtt{b}} Y$.

For each $X \in W^c$, we write $|X|$ to denote the set $X \cap \Gamma^+_{\mathtt{F}}$ (X is a *maximally* $\vdash_{aux}$*-consistent* set of $S5^{\cap,\mathtt{U}}_{\omega}$-formulae). Let $\mathcal{M}^c_{\mathtt{F}}$ be the structure $(W^{fc}, (R^{fc}_{\mathtt{a}})_{\mathtt{a} \in \mathtt{M}(\mathtt{F}) \cup \{\mathtt{U}\}}, m^{fc})$ such that

- $W^{fc} \stackrel{\mathrm{def}}{=} \{|X| : X \in W^c\}$ $(card(W^{fc}) \leq 2^{card(\Gamma^+_{\mathtt{F}})})$;
- for any $|X|, |Y| \in W^{fc}$, $|X| R^{fc}_{\mathtt{U}} |Y| \stackrel{\mathrm{def}}{\Leftrightarrow}$ for any $[\mathtt{U}]\mathtt{G} \in \Gamma_{\mathtt{F}}$, $\mathcal{M}^c, X \models [\mathtt{U}]\mathtt{G}$ iff $\mathcal{M}^c, Y \models [\mathtt{U}]\mathtt{G}$ (or, equivalently, $[\mathtt{U}]|X| = [\mathtt{U}]|Y|$);
- for any $|X|, |Y| \in W^{fc}$, for any $\mathtt{a} \in \mathtt{M}(\mathtt{F})$ $|X| R^{fc}_{\mathtt{a}} |Y| \stackrel{\mathrm{def}}{\Leftrightarrow} |X| R^{fc}_{\mathtt{U}} |Y|$ and for any $\mathtt{b} \in \mathtt{M}(\Gamma_{\mathtt{F}})$ such that $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$ and any $[\mathtt{b}]\mathtt{G} \in \Gamma_{\mathtt{F}}$, $\mathcal{M}^c, X \models [\mathtt{b}]\mathtt{G}$ iff $\mathcal{M}^c, Y \models [\mathtt{b}]\mathtt{G}$ (or, equivalently, $[\mathtt{b}]|X| = [\mathtt{b}]|Y|$);
- $m^{fc} : \mathtt{For}_0(\mathtt{F}) \to \mathcal{P}(W^{fc})$ and for any $\mathtt{p} \in \mathtt{For}_0(\mathtt{F})$, $m^{fc}(\mathtt{p}) \stackrel{\mathrm{def}}{=} \{|X| : X \in m^c(\mathtt{p}), \mathtt{p} \in \Gamma_{\mathtt{F}}\}$ (or equivalently $m^{fc}(\mathtt{p}) = \{|X| : \mathtt{p} \in |X|\}$).

It is a routine task to check that $\mathcal{M}^{fc}_{\mathtt{F}}$ is a $\langle \mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5^{\cap,\mathtt{U}}_{\omega}$-model and for any $\mathtt{G} \in \Gamma^+_{\mathtt{F}}$, and any $X \in W^c$, $\mathtt{G} \in X$ iff $\mathcal{M}^c, X \models \mathtt{G}$ (*Truth Lemma*) iff $\mathcal{M}^c_{\mathtt{F}}, |X| \models \mathtt{G}$ iff $\mathtt{G} \in |X|$ ($\mathcal{M}^c_{\mathtt{F}}$ is actually a $\Gamma_{\mathtt{F}}$-filtration of $\mathcal{M}^c$). Indeed one can show,

1. for any $\mathtt{a} \in \mathtt{M}(\mathtt{F}) \cup \{\mathtt{U}\}$, $(X, Y) \in R^c_{\mathtt{a}}$ implies $(|X|, |Y|) \in R^{fc}_{\mathtt{a}}$ (by easy verification);
2. for any $|X|, |Y| \in W^{fc}$, if $(|X|, |Y|) \in R^{fc}_{\mathtt{a}}$ and $\mathcal{M}^c, X \models [\mathtt{a}]\mathtt{G}$ for some $[\mathtt{a}]\mathtt{G} \in \Gamma_{\mathtt{F}}$, then $\mathcal{M}^c, Y \models \mathtt{G}$ (by easy verification);
3. and, finally, for any $\mathtt{G} \in \Gamma^+_{\mathtt{F}}$ and any $X \in W^c$, $\mathcal{M}^c, X \models \mathtt{G}$ iff $\mathcal{M}^{fc}_{\mathtt{F}}, |X| \models \mathtt{G}$ (by induction on the size of $\mathtt{G}$).

So for any $S5^{\cap,\mathtt{U}}_{\omega}$-formulae $\mathtt{F}$, $\mathtt{F}$ is $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiable iff $\mathtt{F}$ is satisfiable in a $\langle \mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5^{\cap,\mathtt{U}}_{\omega}$-model (by Proposition 3.3) iff $\mathtt{F}$ is satisfiable in $\mathcal{M}^c_{\mathtt{F}}$. Since W^{fc} is finite and $card(W^{fc}) \leq 2^{2 \times card(sub(\mathtt{F}))}$), the validity problem (and the satisfiability problem) of the logic $S5^{\cap,\mathtt{U}}_{\omega}$ is decidable.

3.4. EXPTIME-complete satisfiability problem

By applying Theorem 5.1 in (Hem96), the $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability problem can be shown to be **EXPTIME**-hard: it is sufficient to consider the sublanguage of $S5_{\omega}^{\cap,\mathtt{U}}$ with the three modal operators $[\mathtt{U}]$, $[\mathtt{c}_0]$ and $[\mathtt{c}_1]$. Let us show that the $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability is in **EXPTIME**, that is, it can be solved in deterministic exponential time in the length of the tested formula. To do so, we shall apply a technique used in (HM92) to show that the satisfiability problem for the knowledge logic $\mathrm{S5}_n^C$ is in **EXPTIME** (originally due to Vaughan Pratt (Pra79)).

Let $\mathtt{F}$ be an $S5_{\omega}^{\cap,\mathtt{U}}$-formula and define $\Gamma_{\mathtt{F}}$ and $\Gamma_{\mathtt{F}}^{+}$ as in Section 3.3. The set W^{fc} of the non-standard model $\mathcal{M}_{\mathtt{F}}^{c}$ is actually the greatest set of maximal $\vdash_{aux}$-consistent subsets of $\Gamma_{\mathtt{F}}^{+}$, that is for any $X \in W^{fc}$,

1. $X \subset \Gamma_{\mathtt{F}}^{+}$ (strict inclusion);
2. for any $\mathtt{G} \in \Gamma_{\mathtt{F}}$, $\{\mathtt{G}, \neg\mathtt{G}\} \not\subseteq X$;
3. for any $\mathtt{G} \in \Gamma_{\mathtt{F}}$, $\{\mathtt{G}, \neg\mathtt{G}\} \cap X \neq \emptyset$;
4. if $\neg\neg\mathtt{G} \in X$, then $\mathtt{G} \in X$;
5. if $\mathtt{G}_1 \wedge \mathtt{G}_2 \in X$, then $\{\mathtt{G}_1, \mathtt{G}_2\} \subseteq X$;
6. if $\neg(\mathtt{G}_1 \wedge \mathtt{G}_2) \in X$, then $\{\neg\mathtt{G}_1, \neg\mathtt{G}_2\} \cap X \neq \emptyset$;
7. if $\neg[\mathtt{a}]\mathtt{G} \in X$, then there is $Y \in W^{fc}$ such that $\neg\mathtt{G} \in Y$, $[\mathtt{U}]X = [\mathtt{U}]Y$ and for any $\mathtt{b} \in \mathtt{M}(\mathtt{F})$ such that $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$, $[\mathtt{b}]X = [\mathtt{b}]Y$;
8. if $[\mathtt{a}]\mathtt{G} \in X$ then $\mathtt{G} \in X$.

A set X of $S5_{\omega}^{\cap,\mathtt{U}}$-formulae is said to be *maximally propositionally consistent* with respect to $\Gamma_{\mathtt{F}}^{+}$ if X satisfies conditions (1)-(6) above. So $\mathtt{F}$ is $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiable iff $\mathtt{F}$ is satisfiable in $\mathcal{M}_{\mathtt{F}}^{c}$ iff there is a maximal $\vdash_{aux}$-consistent subset X of $\Gamma_{\mathtt{F}}^{+}$ such that $\mathtt{F} \in X$. Hence for any set X that are maximally propositionally consistent with respect to $\Gamma_{\mathtt{F}}^{+}$, if X does not satisfy one of conditions (7)-(8) then the $S5_{\omega}^{\cap,\mathtt{U}}$-formula $\bigwedge_{\mathtt{G} \in X} \mathtt{G}$ is not $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiable.

Proposition 3.5. The satisfiability problem for $S5_{\omega}^{\cap,\mathtt{U}}$ is in **EXPTIME**, i.e. there is deterministic Turing machine dtm and there is a polynomial $p(n)$ such that for any $S5_{\omega}^{\cap,\mathtt{U}}$-formulae $\mathtt{F}$, dtm(F) is "yes" if $\mathtt{F}$ is $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiable, otherwise dtm(F) is "no" and dtm(F) is computed in deterministic time $2^{p(|\mathtt{F}|)}$.

Observe that the complexity class **EXPTIME** is independent of machine model and **EXPTIME** is closed under polynomial transformations ("many-one" reductions) -see e.g. (Joh90). The proof of

Proposition 3.5 follows the main lines of Theorem 6.20 in (HM92) with adequate modifications in order to deal with $S5_\omega^{\cap,\mathtt{U}}$. It can be viewed as a partial construction of W^{fc} ($\mathtt{F}$ being the tested formula) that runs in exponential-time in $|\mathtt{F}|$. Actually it even runs in time $p_1(|\mathtt{F}|)+2^{p_2(card(sub(\mathtt{F})))}$ for some polynomials $p_1(n)$ and $p_2(n)$. This feature will be of great importance to show that LKO-satisfiability is in **EXPTIME**.

PROOF: Let $\mathtt{F}$ be an $S5_\omega^{\cap,\mathtt{U}}$-formula. We shall either construct a $\langle \mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F})\rangle$-partial $S5_\omega^{\cap,\mathtt{U}}$-model satisfying $\mathtt{F}$ or prove that none exists.

Let $\mathtt{M}(\mathtt{F}) = \{\mathtt{a}_1, \ldots, \mathtt{a}_n\}$ (we arbitrarily order the modal expressions occurring in $\mathtt{F}$) and let $M = (M_{i,j})_{i,j \in \{1,\ldots,n\}\times\{1,\ldots,n\}}$ be the matrix such that for any $i, j \in \{1, \ldots, n\}$, $M_{i,j} \stackrel{\text{def}}{=} \top$ if $\mathtt{M}_0(\mathtt{a}_i) \subseteq \mathtt{M}_0(\mathtt{a}_j)$ otherwise $M_{i,j} \stackrel{\text{def}}{=} \perp$. M can be built in deterministic polynomial-time in $|\mathtt{F}|$ and there is a simple representation of M such that for any $i, j \in \{1, \ldots, n\}$, the value $M_{i,j}$ can be found in the matrix in deterministic polynomial-time in $card(sub(\mathtt{F}))$ ($n < card(sub(\mathtt{F}))$). When dealing with $\mathtt{F}$, we replace in the computations $\mathtt{a}_i$ by i and we consult M when necessary.

Let $W^1(\mathtt{F})$ be the set of all the sets that are maximally propositionally consistent with respect to $\Gamma^+_{\mathtt{F}}$. The set $W^1(\mathtt{F})$ contains less than $2^{card(\Gamma^+_{\mathtt{F}})}$ sets and $W^{fc} \subseteq W^1(\mathtt{F})$. $W^1(\mathtt{F})$ can be built in deterministic exponential-time in $card(sub(\mathtt{F}))$. We define a sequence of structures $\mathcal{M}^j = (W^j, (R^j_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}(\mathtt{F})\cup\{\mathtt{U}\}}, m^j)$, for $j \in \omega$ with $W^1(\mathtt{F}) = W^1 \supset W^2 \supset W^3 \supset \ldots$ ($\supset$ is the strict set inclusion). Suppose we have defined W^j. Then define $(R^j_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}(\mathtt{F})\cup\{\mathtt{U}\}}$ and m^j as follows:

- for any $\mathtt{p} \in \mathtt{For}_0(\mathtt{F})$, $X \in m^j(\mathtt{p}) \stackrel{\text{def}}{\Leftrightarrow} \mathtt{p} \in X$;
- $X R^j_{\mathtt{U}} Y \stackrel{\text{def}}{\Leftrightarrow} [\mathtt{U}]X = [\mathtt{U}]Y$;
- for any $\mathtt{a} \in \mathtt{M}(\mathtt{F})$ $X R^j_{\mathtt{a}} Y \stackrel{\text{def}}{\Leftrightarrow} [\mathtt{U}]X = [\mathtt{U}]Y$ and for any $\mathtt{b} \in \mathtt{M}(\Gamma_{\mathtt{F}})$ such that $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$, $[\mathtt{b}]X = [\mathtt{b}]Y$.

A set $X \in W^j$ is *consistent* $\stackrel{\text{def}}{\Leftrightarrow}$ (if $\neg[\mathtt{a}]\mathtt{G} \in X$ then then there is $Y \in W^j$ such that $\neg\mathtt{G} \in Y$, $[\mathtt{U}]X = [\mathtt{U}]Y$ and for any $\mathtt{b} \in \mathtt{M}(\mathtt{F})$ such that $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$, $[\mathtt{b}]X = [\mathtt{b}]Y$) and X satisfies (8).

If every set in W^j is consistent and $\mathtt{F} \in X$ for some $X \in W^j$ then return "yes" ($\mathtt{F}$ is $S5_\omega^{\cap,\mathtt{U}}$-satisfiable). If there is no consistent state $X \in W^j$ such that $\mathtt{F} \in X$ then return "no" ($\mathtt{F}$ is not $S5_\omega^{\cap,\mathtt{U}}$-satisfiable). Otherwise let W^{j+1} be the set of all consistent sets in W^j and continue the construction.

Since $W^{j+1} \subset W^j$ (strict inclusion) and W^1 has at most $2^{card(\Gamma^+_{\mathtt{F}})}$ elements, this construction terminates after at most exponentially many stages. Computing which sets of W^j are consistent can be done in deterministic polynomial-time in $card(W^j)$ which is at most exponential in $card(sub(\mathtt{F}))$. Observe that at each stage we only need to compute the relations $R^j_{\mathtt{a}}$ for $\mathtt{a} \in \mathtt{M}(\Gamma_{\mathtt{F}}) \cup \{\mathtt{U}\}$ in order to determine which are the consistent sets of W^j and the cardinality of $\mathtt{M}(\Gamma_{\mathtt{F}}) \cup \{\mathtt{U}\}$ is linear in $card(sub(\mathtt{F}))$. Thus the whole construction can be done in deterministic exponential-time in $card(sub(\mathtt{F}))$ plus the time to compute the matrix $(M_{i,j})_{i,j \in \{1,\ldots,n\} \times \{1,\ldots,n\}}$ that is polynomial in $|\mathtt{F}|$.

One can show that $\mathcal{M}^j$ is a $\langle \mathtt{M}(\mathtt{F}), \mathtt{For}_0(\mathtt{F}) \rangle$-partial $S5^{\cap,\mathtt{U}}_{\omega}$-model and by induction on the structure of the formulae that if all the sets in W^j are consistent then $\mathcal{M}^j, X \models \mathtt{G}$ iff $\mathtt{G} \in X$ for any $X \in W^j$ and any $\mathtt{G} \in \Gamma_{\mathtt{F}}$. By way of example in the induction step assume $\mathcal{M}^j, X \not\models [\mathtt{a}]\mathtt{G}$. So there is $Y \in W^j$ such that $(X, Y) \in R^j_{\mathtt{a}}$ and $\mathcal{M}^j, Y \models \neg\mathtt{G}$. By induction hypothesis, $\neg\mathtt{G} \in Y$ (since $\neg\mathtt{G} \in \Gamma^+_{\mathtt{F}}$) and by definition of $R^j_{\mathtt{a}}$, $[\mathtt{a}]X = [\mathtt{a}]Y$. Suppose $[\mathtt{a}]\mathtt{G} \in X$. Hence $[\mathtt{a}]\mathtt{G} \in Y$, by (8) $\mathtt{G} \in Y$ which leads to a contradiction. Similarly assume $\mathcal{M}^j, X \models [\mathtt{a}]\mathtt{G}$ and suppose $[\mathtt{a}]\mathtt{G} \not\in X$. By (2)-(3), $\neg[\mathtt{a}]\mathtt{G} \in X$ and since X is consistent in W^j, there is $Y \in W^j$ such that $\neg\mathtt{G} \in Y$, $[\mathtt{U}]X = [\mathtt{U}]Y$ and for any $\mathtt{b} \in \mathtt{M}(\mathtt{F})$ such that $\mathtt{M}_0(\mathtt{b}) \subseteq \mathtt{M}_0(\mathtt{a})$, $[\mathtt{b}]X = [\mathtt{b}]Y$. So $(X, Y) \in R^c_{\mathtt{a}}$, $\mathcal{M}^j, Y \models \mathtt{G}$ and by induction hypothesis $\mathtt{G} \in Y$ which leads to a contradiction. It follows that $\mathtt{F}$ is $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiable when there is X in such a set W^j (all its elements are consistent) such that $\mathtt{F} \in X$. Furthermore, if there are no consistent sets $X \in W^j$ such that $\mathtt{F} \in X$ then $\mathtt{F}$ is not $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiable. Indeed $W^{fc} \subseteq W^j$ and $\mathtt{F}$ is not $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiable iff there is no $X \in W^{fc}$ such that $\mathtt{F} \in X$. This concludes the correctness of the algorithm. Q.E.D.

4. Decidability and complexity for LKO-satisfiability

In Section 3, we have studied the computational complexity of the $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiability problem because the knowledge logic LKO happens to be similar to $S5^{\cap,\mathtt{U}}_{\omega}$. Indeed in this section we shall define a transformation from LKO-satisfiability into $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiability that entails that LKO-satisfiability is in **EXPTIME**.

Let $\mathtt{F}$ be an lko-formula. Without any loss of generality, one can assume that if $card(\mathtt{P}_0(\mathtt{F})) = n$ then $\mathtt{P}_0(\mathtt{F}) = \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$, $\mathtt{P}_0(\mathtt{F})$ contains the n first constants in the enumeration of the countably infinite set $\mathtt{P}_0$. Indeed LKO-satisfiability and LKO-validity are not sensitive to the renaming of constants. When $\mathtt{F}$ does not contain any constants from $\mathtt{P}_0$, $\mathtt{F}$ is LKO-valid iff $\mathtt{F}$ is $S5^{\cap,\mathtt{U}}_{\omega}$-valid iff $\mathtt{F}$ is valid in the Propositional Calculus. So assume in the sequel $n \geq 1$, that is $\mathtt{P}_0(\mathtt{F}) \neq \emptyset$.

For any integer $k \in \{0, \ldots, 2^n-1\}$ we write $\mathtt{A}^*_k$ to denote the Boolean expression (also called *component*) $\mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$ where for any $i \in \{1, \ldots, n\}$, $\mathtt{A}_i \stackrel{\text{def}}{=} \mathtt{C}_i$ if $bit_i(k) = 0$ ($bit_i(k)$ denotes the ith bit in the binary representation of k) otherwise $\mathtt{A}_i \stackrel{\text{def}}{=} -\mathtt{C}_i$. It is standard to prove that for any $\mathtt{P}$-valuation $m : \mathtt{P} \rightarrow \mathcal{P}(Ag)$, for any $k \neq k' \in \{0, \ldots, 2^n - 1\}$, $m(\mathtt{A}^*_k) \cap m(\mathtt{A}^*_{k'}) = \emptyset$ and, $\bigcup\{m(\mathtt{A}^*_k) : k \in \{0, \ldots, 2^n-1\}\} = Ag$. In the realm of information logics derived from rough set theory, components have been also used in (Kon97) to define various proof systems.

For any agent expression $\mathtt{A} \in \mathtt{P}$ such that $\mathtt{P}_0(\mathtt{A}) \subseteq \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$ either $\mathtt{A} \equiv \perp$ or there is a unique non-empty set $Y = \{\mathtt{A}^*_{i_1}, \ldots, \mathtt{A}^*_{i_l}\}$ such that $\mathtt{A} \equiv \mathtt{A}^*_{i_1} \cup \ldots \cup \mathtt{A}^*_{i_l}$. Let $\mathtt{A} \in \mathtt{P}$ occurring in $\mathtt{F}$ such that $\mathtt{A} \not\equiv \perp$ and $\mathtt{A} \equiv \mathtt{A}^*_{i_1} \cup \ldots \cup \mathtt{A}^*_{i_l}$. The *normal form*[5] of $\mathtt{A}$, written $N(\mathtt{A})$, is merely the expression $\mathtt{A}^*_{i_1} \cup \ldots \cup \mathtt{A}^*_{i_l}$. If $\mathtt{A} \equiv \perp$ then $N(\mathtt{A}) \stackrel{\text{def}}{=} \mathtt{C}_1 \cap -\mathtt{C}_1$. We write $N(\mathtt{F})$ to denote the formula obtained from $\mathtt{F}$ by substituting each occurrence of $\mathtt{A}$ by $N(\mathtt{A})$. $N(\mathtt{F})$ is unique modulo the associativity and commutativity of $\cup$ and $\cap$ and there exists an effective procedure to compute $N(\mathtt{F})$. It is a routine task to check that $\mathtt{F} \Leftrightarrow N(\mathtt{F})$ is LKO-valid. Without any loss of generality, we can assume that $\mathtt{M}_0$ contains the stock $\{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$ of distinct modal constants. We define a mapping g from the set of lko-formulae into the set of $S5^{\cap,\mathtt{U}}_{\omega}$-formulae such that $g(\mathtt{F}) = g'_{\mathtt{F}}(N(\mathtt{F}))$ where $g'_{\mathtt{F}}$ is defined as follows:

- for any $\mathtt{p} \in \mathtt{For}_0$, $g'_{\mathtt{F}}(\mathtt{p}) = \mathtt{p}$; $g'_{\mathtt{F}}(\neg \mathtt{G}) = \neg g'_{\mathtt{F}}(\mathtt{G})$;
- $g'_{\mathtt{F}}(\mathtt{F}_1 \wedge \mathtt{F}_2) = g'_{\mathtt{F}}(\mathtt{F}_1) \wedge g'_{\mathtt{F}}(\mathtt{F}_2)$;
- $g'_{\mathtt{F}}(K(\mathtt{A}^*_{i_1} \cup \ldots \cup \mathtt{A}^*_{i_l})\mathtt{G}) = [\mathtt{c}_{i_1} \cap \ldots \cap \mathtt{c}_{i_l}]g'_{\mathtt{F}}(\mathtt{G}) \vee [\mathtt{c}_{i_1} \cap \ldots \cap \mathtt{c}_{i_l}]\neg g'_{\mathtt{F}}(\mathtt{G})$;
- $g'_{\mathtt{F}}(K(\mathtt{C}_1 \cap -\mathtt{C}_1)\mathtt{G}) = [\mathtt{U}]g'_{\mathtt{F}}(\mathtt{G}) \vee [\mathtt{U}]\neg g'_{\mathtt{F}}(\mathtt{G})$.

In the definition of $g'_{\mathtt{F}}$, each component $\mathtt{A}^*_k$ is associated to the modal constant $\mathtt{c}_k$ and therefore the main idea of the translation consists in viewing the components as constants. That is why the subscript $\mathtt{F}$ in $g'_{\mathtt{F}}$ is used. However, observe that in order to define $g'_{\mathtt{F}}$ we only need to know that $card(\mathtt{P}_0(\mathtt{F})) = n$. For instance,

$$g(K(\mathtt{C}_2 \cap \mathtt{C}_1)\mathtt{p} \wedge K(\mathtt{C}_2 \cap -\mathtt{C}_2)\mathtt{q}) = ([\mathtt{c}_0]\mathtt{p} \vee [\mathtt{c}_0]\neg\mathtt{p}) \wedge ([\mathtt{U}]\mathtt{q} \vee [\mathtt{U}]\neg\mathtt{q})$$

PROPOSITION 4.1.$\mathtt{F}$ is LKO-satisfiable iff $g(\mathtt{F})$ is $S5^{\cap,\mathtt{U}}_{\omega}$-satisfiable.

Since $g(\neg\mathtt{F}) \Leftrightarrow \neg g(\mathtt{F})$ is $S5^{\cap,\mathtt{U}}_{\omega}$-valid, Proposition 4.1 also entails that $\mathtt{F}$ is LKO-valid iff $g(\mathtt{F})$ is $S5^{\cap,\mathtt{U}}_{\omega}$-valid.

PROOF: First assume that $\mathtt{F}$ is LKO-satisfiable. So there exist an lko-model $\mathcal{M} = (OB, Ag, (R_Q)_{Q \subseteq Ag}, m)$ and $o \in OB$ such that $\mathcal{M}, o \models$

[5] N computes precisely the canonical disjunctive normal form for the Propositional Calculus (see e.g. (Lem65))

$N(\mathtt{F})$. Let $\mathcal{M}'$ be the standard $S5_{\omega}^{\cap,\mathtt{U}}$-model $(OB, (R'_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m')$ such that,

- for any $k \in \{0, \ldots, 2^n - 1\}$, $R'_{\mathtt{C}_k} \stackrel{\text{def}}{=} R_{m(\mathtt{A}_k^*)}$;
- for any $\mathtt{c} \in \mathtt{M}_0 \setminus \{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$, $R'_{\mathtt{C}} \stackrel{\text{def}}{=} R'_{\mathtt{C}_0}$ (arbitrary value);
- for any $\mathtt{a} \in \mathtt{M}$, $R'_{\mathtt{a}} \stackrel{\text{def}}{=} \bigcap_{\mathtt{C}\in\mathtt{M}_0(\mathtt{a})} R'_{\mathtt{C}}$ and $R'_{\mathtt{U}} \stackrel{\text{def}}{=} OB \times OB$;
- for any $\mathtt{p} \in \mathtt{For}_0$, $m'(\mathtt{p}) \stackrel{\text{def}}{=} m(\mathtt{p})$.

It is a routine task to check that $\mathcal{M}', o \models g(\mathtt{F})$ holds.

Now assume that $g(\mathtt{F})$ is $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiable. So there exist a standard $S5_{\omega}^{\cap,\mathtt{U}}$-model $\mathcal{M}' = (W', (R'_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m')$ and $w \in W'$ such that $\mathcal{M}', w \models g(\mathtt{F})$. Let $X_1, \ldots, X_n$ be *finite* sets and $\star \not\in X_1 \cup \ldots \cup X_n$ with for any $k \in \{0, \ldots, 2^n - 1\}$, $Y_1 \cap \ldots \cap Y_n \neq \emptyset$ where $Y_i \stackrel{\text{def}}{=} X_i$ if $bit_i(k) = 0$ otherwise $Y_i \stackrel{\text{def}}{=} (X_1 \cup \ldots \cup X_n \cup \{\star\}) \setminus X_i$. One can always build such sets satisfying the conditions below:

1. $card(X_1 \cup \ldots \cup X_n \cup \{\star\}) = 2^n$;
2. for any $i \in \{1, \ldots, n\}$, $card(X_i) = 2^{n-1}$
3. for any $k \in \{0, \ldots, 2^n - 1\}$, $Y_1 \cap \ldots \cap Y_n$ is a singleton.

For example, we can build the X_i's in the following way: $X_1 \cup \ldots \cup X_n \subseteq \{0, \ldots, 2^n - 2\}$ and for any $i \in \{1, \ldots, n\}$, for any $k \in \{0, \ldots, 2^n - 2\}$, $k \in X_i$ iff $bit_i(k) = 0$ ($2^n - 1$ can therefore play the rôle of the element $\star$). Let $\mathcal{M} = (W', Ag, (R_Q)_{Q\subseteq Ag}, m)$ be the lko-model such that

- $Ag \stackrel{\text{def}}{=} X_1 \cup \ldots \cup X_n \cup \{\star\}$;
- the restriction of m to $\mathtt{For}_0$ is m';
- for any $i \in \{1, \ldots, n\}$, $m(\mathtt{C}_i) \stackrel{\text{def}}{=} X_i$. The interpretation of the other constants is not constrained until the restriction of m to $\mathtt{P}$ is a $\mathtt{P}$-valuation, which is always possible;
- $R_{\emptyset} \stackrel{\text{def}}{=} W' \times W'$;
- for any $k \in \{0, \ldots, 2^n - 1\}$, for the *unique* $x \in Y_1 \cap \ldots \cap Y_n$, $R_{\{x\}} \stackrel{\text{def}}{=} R'_{\mathtt{C}_k}$ where $Y_i \stackrel{\text{def}}{=} X_i$ if $bit_i(k) = 0$ otherwise $Y_i \stackrel{\text{def}}{=} Ag \setminus X_i$;
- for any $Q \subseteq Ag$ such that $card(Q) \geq 2$, $R_Q \stackrel{\text{def}}{=} \bigcap_{x\in Q} R_{\{x\}}$ (finite intersection).

One can check that the definition of $\mathcal{M}$ is correct and $\mathcal{M}$ is an lko-model. In order to check whether $\mathcal{M}, w \models N(\mathtt{F})$ holds it is sufficient to satisfy that for any $k \in \{0, \ldots, 2^n - 1\}$, $R_{m(\mathtt{A}_k^*)} = R'_{\mathtt{C}_k}$.

$$
\begin{aligned}
R_{m(\mathtt{A}_k^*)} &= \bigcap\{R_{\{x\}} : x \in m(\mathtt{A}_k^*)\}\\
&= \bigcap\{R_{\{x\}} : x \in m(\mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n)\}\\
&(\mathtt{A}_i = \mathtt{C}_i \text{ if } bit_i(k) = 0 \text{ otherwise } \mathtt{A}_i = -\mathtt{C}_i)\\
&= \bigcap\{R_{\{x\}} : x \in Y_1 \cap \ldots \cap Y_n\}\\
&(Y_i = X_i \text{ if } bit_i(k) = 0 \text{ otherwise } Y_i = Ag \setminus X_i)\\
&= R'_{\mathtt{C}_k} \ (card(Y_1 \cap \ldots \cap Y_n) = 1)
\end{aligned}
$$

Q.E.D.

COROLLARY 4.2.(1) The validity problem for the logic LKO is decidable. (2) For any $\mathtt{F} \in \mathtt{For}_{lko}$, $\mathtt{F}$ is LKO-satisfiable iff $\mathtt{F}$ is satisfiable in some lko-model $(OB, Ag, \ldots)$ such that Ag is finite and $card(Ag) \leq 2^{card(\mathtt{P}_0(\mathtt{F}))}$.

Decidability of LKO-satisfiability has been open until now. As far as LKO-satisfiability is concerned, it is worth observing that we can restrict ourselves to the set of lko-models having a *finite* set of agents. For the sake of comparison, all the knowledge logics studied in (HM92) involve a finite set of agents *in the language*. In the rest of this section, we shall present another feature that is common to LKO and to the logics studied in (HM92), namely LKO-satisfiability is **EXPTIME**-complete.

In (DG98), various similar translations are defined between information logics and (combinatory) dynamic logics in order to establish decidability results for information logics. Observe that for any $S5_{\omega}^{\cap,\mathtt{U}}$-formulae $\mathtt{F}$ there is an lko-formula $\mathtt{G}$ such that $\mathtt{G}$ is LKO-valid iff $\mathtt{F}$ is $S5_{\omega}^{\cap,\mathtt{U}}$-valid. Actually $\mathtt{G}$ can be computed by an effective procedure called g_Y^{-1}. Indeed, let $\mathtt{F}$ be an $S5_{\omega}^{\cap,\mathtt{U}}$-formula such that $\mathtt{M}_0(\mathtt{F}) = \{\mathtt{c}_0, \ldots, \mathtt{c}_K\}$. Without any loss of generality one can assume that if $card(\mathtt{M}_0(\mathtt{F})) = K + 1$ then $\mathtt{M}_0(\mathtt{F}) = \{\mathtt{c}_0, \ldots, \mathtt{c}_K\}$, $\mathtt{M}_0(\mathtt{F})$ contains the $K + 1$ first constants in the enumeration of the countably infinite set $\mathtt{M}_0$. Indeed $S5_{\omega}^{\cap,\mathtt{U}}$-satisfiability and $S5_{\omega}^{\cap,\mathtt{U}}$-validity are not sensitive to the renaming of modal constants. Let n be the smallest natural number such that $2^n - 1 \geq K$. Take n agent constants from $\mathtt{P}_0$, say $Y = \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$. If $\mathtt{M}_0(\mathtt{F}) = \emptyset$ then we just consider $Y = \{\mathtt{C}_1\}$. For any $k \in \{0, \ldots, 2^n - 1\}$ we define the agent expression $\mathtt{A}_k^* = \mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$ where for any $s \in \{1, \ldots, n\}$, $\mathtt{A}_s \stackrel{\text{def}}{=} \mathtt{C}_s$ if $bit_s(k) = 0$ otherwise $\mathtt{A}_s \stackrel{\text{def}}{=} -\mathtt{C}_s$. We define the mapping $g_Y^{-1} : \mathtt{M} \cup \{\mathtt{U}\} \cup \mathtt{For}_{aux} \to \mathtt{P} \cup \mathtt{For}_{lko}$ in the following way:

- for any $k \in \{0, \ldots, 2^n - 1\}$, $g_Y^{-1}(\mathtt{c}_k) = \mathtt{A}_k^*$;
- for any $\mathtt{c} \in \mathtt{M}_0 \setminus \{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$, $g_Y^{-1}(\mathtt{c}) = \mathtt{C}_1$ (arbitrary value);
- $g_Y^{-1}(\mathtt{a} \cap \mathtt{b}) = g_Y^{-1}(\mathtt{a}) \cup g_Y^{-1}(\mathtt{b})$; $g_Y^{-1}(\mathtt{U}) = \mathtt{C}_1 \cap -\mathtt{C}_1$;

– for any $\mathtt{p} \in \mathtt{For}_0$, $g_Y^{-1}(\mathtt{p}) = \mathtt{p}$;

– $g_Y^{-1}(\mathtt{G} \wedge \mathtt{H}) = g_Y^{-1}(\mathtt{G}) \wedge g_Y^{-1}(\mathtt{H})$; $g_Y^{-1}(\neg\mathtt{G}) = \neg g_Y^{-1}(\mathtt{G})$;

– $g_Y^{-1}([\mathtt{a}]\mathtt{G}) = g_Y^{-1}(\mathtt{G}) \wedge K(g_Y^{-1}(\mathtt{a}))g_Y^{-1}(\mathtt{G})$.

It is easy to see that $N(g_Y^{-1}(\mathtt{F})) = g_Y^{-1}(\mathtt{F})$ and $g(g_Y^{-1}(\mathtt{F}))$ is $S5_\omega^{\cap,\mathtt{U}}$-valid iff $\mathtt{F}$ is $S5_\omega^{\cap,\mathtt{U}}$-valid since $\mathtt{p} \wedge ([\mathtt{a}]\mathtt{p} \vee [\mathtt{a}]\neg\mathtt{p}) \Leftrightarrow [\mathtt{a}]\mathtt{p}$ is $S5_\omega^{\cap,\mathtt{U}}$-valid. Hence $\mathtt{F}$ is $S5_\omega^{\cap,\mathtt{U}}$-valid iff $g(g_Y^{-1}(\mathtt{F}))$ is $S5_\omega^{\cap,\mathtt{U}}$-valid iff $g_Y^{-1}(\mathtt{F})$ is LKO-valid (by Proposition 4.1).

PROPOSITION 4.3.The LKO-satisfiability problem is **EXPTIME**-complete.

PROOF: Since $g_Y^{-1}(\mathtt{F})$ ($\mathtt{F}$ is an $S5_\omega^{\cap,\mathtt{U}}$-formula) cannot be computed in polynomial-time in $|\mathtt{F}|$ -two occurrences of $g_Y^{-1}(\mathtt{G})$ in the recursive definition of $g_Y^{-1}([\mathtt{a}]\mathtt{G})$-, we shall define a variant of $g_Y^{-1}(\mathtt{F})$ that uses the renaming of subformulae so that we could conclude that the satisfiability problem for LKO is **EXPTIME**-hard. Let us consider the map g_{Ybis}^{-1} defined as g_Y^{-1} except for the following condition: $g_{Ybis}^{-1}([\mathtt{a}]\mathtt{G}) = \mathtt{p_G} \wedge K(g_{Ybis}^{-1}(\mathtt{a}))\mathtt{p_G}$ where $\mathtt{p_G}$ is a *new* propositional variable and add $\langle \mathtt{p_G}, g_{Ybis}^{-1}(\mathtt{G})\rangle$ to the set `Renaming`. Let us compute $\mathtt{F}'$ such that $\mathtt{F}'$ is LKO-satisfiable iff $g_Y^{-1}(\mathtt{F})$ is LKO-satisfiable and $\mathtt{F}'$ is computed in deterministic polynomial-time. First initialize `Renaming` to the empty set. For any $\mathtt{G} \in sub(\mathtt{F})$, compute $g_{Ybis}^{-1}(\mathtt{G})$ (possibly by updating `Renaming`). Observe that for any $\mathtt{G} \in sub(\mathtt{F})$, $g_{Ybis}^{-1}(\mathtt{G})$ can be computed in polynomial-time in $|\mathtt{G}|$ and `Renaming` is also of polynomial size in $|\mathtt{F}|$ at the end of the procedure above. The whole procedure can be computed in deterministic polynomial-time in $|\mathtt{F}|$. Using standard arguments for the renaming of subformulae one can show that $g_Y^{-1}(\mathtt{F})$ is LKO-satisfiable iff $\mathtt{F}' \stackrel{\text{def}}{=}$

$$g_{Ybis}^{-1}(\mathtt{F}) \bigwedge_{\langle \mathtt{p_G}, g_{Ybis}^{-1}(\mathtt{G})\rangle \in \mathtt{Renaming}} (\mathtt{p_G} \Leftrightarrow g_{Ybis}^{-1}(\mathtt{G})) \wedge K(\mathtt{C}_1 \cap -\mathtt{C}_1)(\mathtt{p_G} \Leftrightarrow g_{Ybis}^{-1}(\mathtt{G}))$$

is LKO-satisfiable, $\mathtt{F}'$ being computed in deterministic polynomial-time. So there is a polynomial-time transformation from $S5_\omega^{\cap,\mathtt{U}}$-satisfiability into LKO-satisfiability.

In order to prove the upper bound, take any lko-formula $\mathtt{F}$. $N(\mathtt{F})$ can be computed in deterministic time $2^{p(|\mathtt{F}|)}$ for some polynomial $p(n)$ (use of Boolean truth-tables to compute the normal forms of all $\mathtt{A} \in \mathtt{P}(\mathtt{F})$). Moreover $card(sub(\mathtt{F})) = card(sub(N(\mathtt{F})))$, $card(sub(g(\mathtt{F}))) \leq 3 \times card(sub(\mathtt{F}))$ and one can show that there is a polynomial $p'(n)$ such that $|g(\mathtt{F})| \leq 2^{p'(|\mathtt{F}|)}$.

Hence deciding whether $g(\mathtt{F})$ is $S5_\omega^{\cap,\mathtt{U}}$-satisfiable can be solved in deterministic time $2^{p(|\mathtt{F}|)} + p_1(2^{p'(|\mathtt{F}|)}) + 2^{p_2(3\times card(sub(\mathtt{F})))}$ which is bounded by $2^{q(|\mathtt{F}|)}$ for some polynomial $q(n)$ -see Section 3.4. Q.E.D.

For the sake of comparison, the satisfiability problems for the Propositional Dynamic Logic PDL and for the knowledge logic $S5_n^C$ $(n > 1)$ are also **EXPTIME**-complete.

5. Hilbert-style axiomatization for LKO

In this section, as a side-effect of the results from the previous sections we define the Hilbert-style calculus $\vdash_{lko}$ for LKO and we show its completeness with respect to LKO-validity by using g, g_Y^{-1} and the system $\vdash_{aux}$. By $\vdash_{lko}$ we understand the smallest set of lko-formulae that satisfies the following conditions:

1. $\vdash_{lko}$ is closed under *modus ponens*, *uniform substitution* and $\vdash_{lko}$ contains every tautology of the classical propositional calculus;
2. $\vdash_{lko}$ is closed under *necessitation*, i.e $\mathtt{F} \in \vdash_{lko}$ implies $K(\mathtt{A})\mathtt{F} \in \vdash_{lko}$ for any $\mathtt{A} \in \mathtt{P}$;
3. `lko` contains every formula of the form:

($\cap'$) $K(\mathtt{A})\mathtt{p} \Rightarrow K(\mathtt{A}')\mathtt{p}$ when $\mathtt{A} \sqsubseteq \mathtt{A}'$

(K$'$) $\mathtt{F} \wedge K(\mathtt{A})\mathtt{p} \wedge K(\mathtt{A})(\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow K(\mathtt{A})\mathtt{q}$;

(T5$'$) $K(\mathtt{A})(K(\mathtt{A})\mathtt{p} \Rightarrow \mathtt{p})$;

(EqK) $K(\mathtt{A})\mathtt{p} \Leftrightarrow K(\mathtt{A})\neg\mathtt{p}$;

(U$'$) $\mathtt{p} \wedge K(\mathtt{A}_1)\mathtt{p} \Rightarrow K(\mathtt{A}_2)\mathtt{p}$ when $\mathtt{A}_1 \equiv \perp$.

Although the Boolean Modal Logic BML defined in (GP90) admits Boolean terms as modal indices in its language, in BML these terms are interpreted by binary relations unlike LKO. However, in the definition of $\vdash_{lko}$, ($\cap'$) also corresponds to an axiom schema from the axiomatization of BML defined in (GP90). Moreover, by deleting (U') from the definition of $\vdash_{lko}$ the set of theorems remains unchanged but it is introduced to emphasize the correspondence with $\vdash_{aux}$.

Proposition 5.1.If $\vdash_{lko}$ $\mathtt{F}$ then $\mathtt{F}$ is LKO-valid.

Proof: By an easy verification. Q.E.D.

The proof of Proposition 5.2 below can be easily obtained from similar statements in (MR68; Dem97).

Proposition 5.2. 1. If $\vdash_{lko} \mathtt{F} \Leftrightarrow \mathtt{F}'$ then $\vdash_{lko} \mathtt{G} \Leftrightarrow \mathtt{G}'$ where $\mathtt{G}'$ is obtained from $\mathtt{G}$ by simultaneously replacing some occurrences of $\mathtt{F}$ by $\mathtt{F}'$.

2. For any $\mathtt{A} \in \mathtt{P}$, $\vdash_{lko} K(\mathtt{A})(\mathtt{p} \wedge K(\mathtt{A})\mathtt{p})$.

Another useful result is stated in Proposition 5.3 below.

PROPOSITION 5.3. Let $\mathtt{F}$ be an lko-formula (see the notations of Section 4) and g_Y^{-1} be the mapping with $Y = \mathtt{P}_0(\mathtt{F})$.

1. $\vdash_{lko} g_Y^{-1}(g(\mathtt{F})) \Leftrightarrow \mathtt{F}$;
2. For any $S5_\omega^{\cap,\mathsf{U}}$-formulae $\mathtt{G},\mathtt{H}$, $g_Y^{-1}(\mathtt{G}[\mathtt{p} \leftarrow \mathtt{H}]) = g_Y^{-1}(\mathtt{G})[\mathtt{p} \leftarrow g_Y^{-1}(\mathtt{H})]$.

PROOF: (1) By Proposition 5.2(1) and the axiom schema $(\cap')$, $\vdash_{lko}$ $\mathtt{F} \Leftrightarrow N(\mathtt{F})$. However, $g_Y^{-1}(g(\mathtt{F})) = g_Y^{-1}(g'_{\mathtt{F}}(N(\mathtt{F}))) = \alpha(N(\mathtt{F}))$ where $\alpha : \mathtt{For}_{lko} \rightarrow \mathtt{For}_{lko}$ is defined as follows:

- for any $\mathtt{p} \in \mathtt{For}_0$, $\alpha(\mathtt{p}) = \mathtt{p}$;
- $\alpha(\neg\mathtt{G}) = \neg\alpha(\mathtt{G})$; $\alpha(\mathtt{G}_1 \wedge \mathtt{G}_2) = \alpha(\mathtt{G}_1) \wedge \alpha(\mathtt{G}_2)$;
- $\alpha(K(\mathtt{A})\mathtt{G}) = (K(\mathtt{A})\neg\alpha(\mathtt{G}) \wedge \neg\alpha(\mathtt{G})) \vee (K(\mathtt{A})\alpha(\mathtt{G}) \wedge \alpha(\mathtt{G}))$.

Since $\vdash_{lko} (K(\mathtt{A})\neg\mathtt{p} \wedge \neg\mathtt{p}) \vee (K(\mathtt{A})\mathtt{p} \wedge \mathtt{p}) \Leftrightarrow K(\mathtt{A})\mathtt{p}$, by induction and Proposition 5.2(1) $\vdash_{lko} \alpha(N(\mathtt{F})) \Leftrightarrow N(\mathtt{F})$. Hence $\vdash_{lko} g_Y^{-1}(g(\mathtt{F})) \Leftrightarrow \mathtt{F}$.
(2) By induction on the size of $\mathtt{G}$. Q.E.D.

We can now present the main result that entails the completeness of $\vdash_{lko}$ with respect to LKO-validity.

PROPOSITION 5.4. If $\vdash_{aux} g(\mathtt{F})$ then $\vdash_{lko} \mathtt{F}$.

PROOF: Let $\langle \mathtt{G}_1, \ldots, \mathtt{G}_K \rangle$ be a deduction of $g(\mathtt{F})$ in $\vdash_{aux}$ ($\mathtt{G}_K = g(\mathtt{F})$). We show that for any $i \in \{1, \ldots, K\}$, $\vdash_{lko} g_Y^{-1}(\mathtt{G}_i)$ with $Y = \mathtt{P}_0(\mathtt{F})$. By manipulation at the propositional level, and since $\vdash_{lko} g_Y^{-1}(g(\mathtt{F})) \Leftrightarrow \mathtt{F}$, we get $\vdash_{lko} \mathtt{F}$. The proof is by induction on K
Base case: $\mathtt{G}_1$ is an axiom of $\vdash_{aux}$

1. If $\mathtt{G}_1$ is a tautology of the propositional calculus then obviously $\vdash_{lko} \mathtt{G}_1$ and $g_Y^{-1}(\mathtt{G}_1) = \mathtt{G}_1$.
2. If $\mathtt{G}_1$ is of the form $([\mathtt{a}](\mathtt{p} \Rightarrow \mathtt{q}) \wedge [\mathtt{a}]\mathtt{p}) \Rightarrow [\mathtt{a}]\mathtt{q}$ then $g_Y^{-1}(\mathtt{G}_1)$ is equal to

$$(\mathtt{p} \Rightarrow \mathtt{q}) \wedge K(g_Y^{-1}(\mathtt{a}))(\mathtt{p} \Rightarrow \mathtt{q}) \wedge \mathtt{p} \wedge K(g_Y^{-1}(\mathtt{a}))(\mathtt{p}) \Rightarrow \mathtt{q} \wedge K(g_Y^{-1}(\mathtt{a}))(\mathtt{q})$$

   By (K$'$), $\vdash_{lko} K(g_Y^{-1}(\mathtt{a}))(\mathtt{p} \Rightarrow \mathtt{q}) \wedge \mathtt{p} \wedge K(g_Y^{-1}(\mathtt{a}))(\mathtt{p}) \Rightarrow K(g_Y^{-1}(\mathtt{a}))(\mathtt{q})$ and $\vdash_{lko} (\mathtt{p} \Rightarrow \mathtt{q}) \wedge \mathtt{p} \Rightarrow \mathtt{q}$. By manipulation at the propositional level we get $\vdash_{lko} g_Y^{-1}(\mathtt{G}_1)$.
3. If $\mathtt{G}_1$ is of the form $[\mathtt{a}]\mathtt{p} \Rightarrow \mathtt{p}$ then $g_Y^{-1}(\mathtt{G}_1) = K(g_Y^{-1}(\mathtt{a}))\mathtt{p} \wedge \mathtt{p} \Rightarrow \mathtt{p}$ which can be derived in $\vdash_{lko}$ by an easy manipulation at the propositional level.

4. If $\mathtt{G}_1$ is of the form $\langle \mathtt{a} \rangle \mathtt{p} \Rightarrow [\mathtt{a}]\langle \mathtt{a} \rangle \mathtt{p}$ then $g_Y^{-1}(\mathtt{G}_1) = \neg \mathtt{G} \Rightarrow \neg \mathtt{G} \wedge K(g_Y^{-1}(\mathtt{a}))(\neg \mathtt{G})$ where $\mathtt{G} = (\neg \mathtt{p} \wedge K(g_Y^{-1}(\mathtt{a}))\neg \mathtt{p})$. By Proposition 5.2(2) and by considering the substitution rule, $\vdash_{lko} K(g_Y^{-1}(\mathtt{a}))(\neg \mathtt{p} \wedge K(g_Y^{-1}(\mathtt{a}))\neg \mathtt{p})$. By (EqK) and Proposition 5.2(1), $\vdash_{lko} K(g_Y^{-1}(\mathtt{a}))(\neg \mathtt{G})$. By manipulation at the propositional level we get $\vdash_{lko} g_Y^{-1}(\mathtt{G}_1)$.
5. We omit the details of the cases when $\mathtt{G}_1$ is an instance of $(\cap)$, $(\mathtt{U})$ or (AC) since they present no extra difficulties.

Induction step: In case $\mathtt{G}_{i+1}$ is an axiom, see the treatment of the base case. Now assume $\mathtt{G}_{i+1}$ is computed from an inference rule of $\vdash_{aux}$.

1. $\mathtt{G}_{i+1}$ is obtained from $\mathtt{G}_j$ by the substitution rule. So $\mathtt{G}_{i+1} = \mathtt{G}_j[\mathtt{p} \leftarrow \mathtt{G}]$. Since $j \leq i$, by induction hypothesis, $\vdash_{lko} g_Y^{-1}(\mathtt{G}_j)$. By Proposition 5.3(2), $g_Y^{-1}(\mathtt{G}_{i+1}) = g_Y^{-1}(\mathtt{G}_j)[\mathtt{p} \leftarrow g_Y^{-1}(\mathtt{G})]$ and therefore $\vdash_{lko} g_Y^{-1}(\mathtt{G}_{i+1})$ by applying the substitution rule on $g_Y^{-1}(\mathtt{G}_j)$ with $\mathtt{p} \leftarrow g_Y^{-1}(\mathtt{G})$.
2. We omit the details of the cases when $\mathtt{G}_{i+1}$ is obtained from the necessitation rule or the modus ponens rule since they present no extra difficulties.

Q.E.D.

PROPOSITION 5.5.If $\mathtt{F}$ is LKO-valid then $\vdash_{lko} \mathtt{F}$.

PROOF: $\mathtt{F}$ is LKO-valid implies $g(\mathtt{F})$ is $S5_{\omega}^{\cap,\mathtt{U}}$-valid, which implies $\vdash_{aux} g(\mathtt{F})$. By Proposition 5.4, $\vdash_{lko} \mathtt{F}$. Q.E.D.

As a conclusion, for any lko-formula $\mathtt{F}$, $\vdash_{lko} \mathtt{F}$ iff $\mathtt{F}$ is LKO-valid.

6. Concluding remarks

In this paper we have shown that the satisfiability problem for LKO is **EXPTIME**-complete and we have defined a sound and complete Hilbert-style system for it. In (Orło89), the logic LKO′ has been also introduced where the lko′-models are like the lko-models except that the relations are only reflexive and symmetric. According to (Orło89) (p. 571) the axiomatization of LKO′ seems to be more complicated since the indiscernibility relations are not necessarily transitive. By using constructions similar to those in the present paper, the system $\vdash_{lko'}$ that consists in $\vdash_{lko}$ except that (T5′) is replaced by

(B′) $\mathtt{p} \Rightarrow K(\mathtt{A})(\mathtt{p} \vee \neg K(\mathtt{A})\mathtt{p})$ for any $\mathtt{A} \in \mathtt{P}$

(T5U) $K(\mathtt{A})(K(\mathtt{A})\mathtt{p} \Rightarrow \mathtt{p})$ for any $\mathtt{A} \equiv \perp$

is sound and complete for LKO$'$. This problem has been open until now (Orło89). Decidability of LKO$'$ can also be established as well as the **EXPTIME**-completeness of the satisfiability problem.

Ackowledgments

Thanks are due to Thierry Boy de la Tour, Dov Gabbay, Lloyd Humberstone, Beata Konikowska, Maarten Marx and Ewa Orłowska. Special thanks are due to Philippe Balbiani who initiates me to the copying constructions.

References

R. Aumann. Agreeing to disagree. *Annals of Statistics*, 4(3):1236–1239, 1976.

Philippe Balbiani. Axiomatization of logics based on Kripke models with relative accessibility relations. In Ewa Orłowska, editor, *Incomplete Information: Rough Set Analysis*, pages 553–578. Physica Verlag, Heidelberg, 1997.

Max Cresswell. Necessity and contingency. *Studia Logica*, 47(2):145–149, 1988.

Stéphane Demri. A completeness proof for a logic with an alternative necessity operator. *Studia Logica*, 58(1):99–112, 1997.

Stéphane Demri and Dov Gabbay. On modal logics characterized by models with relative accessibility relations, 1998. Submitted.

Ronald Fagin, Joseph Halpern, Yoram Moses, and Moshe Vardi. *Reasoning about Knowledge*. The MIT Press, 1995.

George Gargov and Solomon Passy. A note on boolean modal logic. In P. Petkov, editor, *Summer School and Conference on Mathematical Logic '88*, pages 299–309. Plenum Press, New-York, 1990.

Valentin Goranko and Solomon Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.

Edith Hemaspaandra. The price of universality. *Notre Dame Journal of Formal Logic*, 37(2):173–203, 1996.

Jaakko Hintikka. *Knowledge and Belief*. Cornell University Press, Ithaca, New-York, 1962.

Joseph Halpern and Yoram Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.

Lloyd Humberstone. The logic of non-contingency. *Notre Dame Journal of Formal Logic*, 36(2):214–229, 1995.

David Johnson. A catalog of complexity classes. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume A, Algorithms and Complexity*, pages 68–161. Elsevier, 1990.

Beata Konikowska. A logic for reasoning about relative similarity. *Studia Logica*, 58(1):185–226, 1997.

Steven Kuhn. Minimal logic of non-contingency. *Notre Dame Journal of Formal Logic*, 36(2):230–234, 1995.

E. Lemmon. *Beginning Logic*. Chapman and Hall, London, 1965.

David Makinson. On some completeness theorems in modal logic. *The Journal of Symbolic Logic*, 12:379–384, 1966.

H. Montgomery and R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 9:318–328, 1966.

H. Montgomery and R. Routley. Non-contingency axioms for S4 and S5. *Logique et Analyse*, 11:422–424, 1968.

Ewa Orłowska. Logic for reasoning about knowledge. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 35:559–572, 1989.

Zdzislaw Pawlak. Information systems theoretical foundations. *Information Systems*, 6(3):205–218, 1981.

Vaughan Pratt. Models of program logics. In *Proceedings of the 20th IEEE Symposium on Foundations of Computer Science*, pages 115–122, 1979.

Dimiter Vakarelov. Modal logics for knowledge representation systems. *Theoretical Computer Science*, 90:433–456, 1991.

Mars Valiev. Interpretation of modal logics as epistemic logics (In Russian). In *Proceedings of the Conference of Borzomi*, pages 76–77, 1988.