Tractable Transformations from Modal Provability Logics into First-Order Logic

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Abstract. We define a class of modal logics LF by uniformly extending a class of modal logics L. Each logic L is characterised by a class of first-order definable frames, but the corresponding logic LF is sometimes characterised by classes of modal frames that are not first-order definable. The class LF includes provability logics with deep arithmetical interpretations. Using Belnap's proof-theoretical framework Display Logic we characterise the "pseudo-displayable" subclass of LF and show how to define polynomial-time transformations from each such LF into the corresponding L, and hence into first-order classical logic. Theorem provers for classical first-order logic can then be used to mechanise deduction in these "psuedo-displayable second order" modal logics.

1 Introduction

Background. There are two main approaches to modal theorem proving in the literature. The *direct approach* consists in defining calculi dedicated to modal logics at the cost of modifying existing theorem provers (see e.g. [Fit83, AEH90, Mas94]). The *translational approach* consists in translating modal logics into logics for which theorem provers already exist, typically classical first-order logic (FOL). The relational translation into FOL (see e.g. [Mor76, Sch99, GHM98]) is the most common such translation although not the only one (see e.g. [Mor76, Ohl88, Her89, dMP95, Ohl98]). These two approaches cannot always be applied with equal success (see e.g. [HS97]). For instance, for the provability logic G which is characterized by a second-order class of modal frames (see e.g. [Boo93]), the relational translation is not possible unless FOL is augmented with fixed-point operators (see e.g. [NS98]). However, dedicated sequent-style calculi do exist for provability logics such as G or Grz (for Grzegorczyk); see e.g. [SV80, Lei81, Fit83, Val83, Avr84, Boo93, Gor99].

Display Logic. Display Logic (**DL**) [Bel82] is a proof-theoretical framework that generalises the structural language of Gentzen's sequents by using multiple complex structural connectives instead of Gentzen's comma. The term "display"

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comes from the nice property that any occurrence of a *structure* in a sequent can be displayed either as the entire antecedent or as the entire succedent of some sequent which is *structurally equivalent* to the initial sequent (see e.g. [Bel82]). Any display calculus satisfying the conditions (C1)-(C8) [Bel82] (see the appendix) enjoys cut-elimination. Kracht's characterisation of *properly displayable* modal logics [Kra96] means that any extension of the (poly)modal logic K obtained by the addition of the so-called *primitive axioms* admits a display calculus that obeys conditions (C1)-(C8) [Kra96], and therefore enjoys cut-elimination. Since every primitive axiom is a Sahlqvist formula, any such extension is first-order definable [Sah75].

Our contribution. Let ϕ be a modal formula and $\mathtt{F}(\phi)$ be a formula built from $\{\phi\}$ using $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Box$ and $\Diamond$ such that any subformula of the form $\Box\psi$ in $\mathtt{F}(\phi)$ occurs positively (when every $\phi_1 \Rightarrow \phi_2$ is written as $\neg\phi_1 \vee \phi_2$). Let $\mathtt{L}$ be a properly displayable modal logic and $\mathtt{LF}$ be the logic obtained from $\mathtt{L}$ by adding the axiom scheme $\Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi$. Here a *logic* is understood as a set of formulae and therefore is exactly a *(decision) problem* in the usual sense in complexity theory. That is, as a language viewed as a set of strings built upon a given alphabet.

By generalising results from [DG99a], we establish conditions permitting an $\mathcal{O}(n^3.log\ n)$-time transformation (also called a "many-one reduction" [Pap94]) g from $\mathtt{LF}$ into $\mathtt{L}$. If K4 $\subseteq \mathtt{L}$, then g can be in $\mathcal{O}(n.log\ n)$-time. Now, every primitive modal logic can be translated into FOL in linear-time (using a smart recycling of the variables). So in the general case, we define an $\mathcal{O}(n^3.log\ n)$-time transformation from $\mathtt{LF}$ into (possibly known) fragments of FOL even though a formula of second-order logic may be essential to describe the class of modal frames characterising $\mathtt{LF}$. This provides an alternative for mechanizing modal provability logics.

Our uniform definition of such mappings shows that **DL** is ideal for proof-theoretical analyses of calculi for $\mathtt{LF}$ and $\mathtt{L}$. In fact, the theoremhood preserving nature of our transformations are a characterisation of (weak) cut-elimination for many of these logics.

Plan of the paper. In Section 2, we define the class of modal logics $\mathtt{LF}$ studied in the paper. In Section 3, we define display calculi $\delta\mathtt{LF}$ for these logics and show these calculi to be sound and complete. In Section 4, we give necessary and sufficient conditions to establish that the display calculi $\delta\mathtt{LF}$ admit a (weak) cut-elimination theorem, and provide the promised transformations. Section 5 contains a similar analysis for traditional sequent-style calculi. Proofs are omitted because of lack of space and they can be found in [DG99b].

2 Provability Logics

Given a set $\mathtt{PRP} = \{\mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of *atomic formulae*, the formulae $\phi \in \mathtt{FML}$ are inductively defined as follows for $\mathtt{p}_i \in \mathtt{PRP}$:

$$\phi \ ::= \bot \mid \top \mid \mathtt{p}_i \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi \mid \phi_1 \Rightarrow \phi_2 \mid \Box\phi$$

Standard abbreviations include $\Leftrightarrow$, $\Diamond$: for instance, $\Diamond\phi \stackrel{\text{def}}{=} \neg\Box\neg\phi$. An occurrence of the subformula ψ in ϕ is *positive* [resp. *negative*] iff it is in the scope of an even [resp. odd] number of negations, where as usual, every occurrence of $\phi_1 \Rightarrow \phi_2$ is treated as an occurence of $\neg\phi_1 \vee \phi_2$. The standard axiomatic Hilbert system K consists of the axiom schemes

1. the tautologies of the Propositional Calculus (PC)
2. $\Box(\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow (\Box\mathtt{p} \Rightarrow \Box\mathtt{q})$

and the inference rules: *modus ponens* (from ϕ and $\phi \Rightarrow \psi$ infer ψ) and *necessitation* (from ϕ infer $\Box\phi$). When $\mathtt{L}$ is an extension of K (including K), we write $\phi \in \mathtt{L}$ to denote that ϕ is a theorem of $\mathtt{L}$. In the paper, we refer to the following well-known extensions $\mathtt{L}$ of K:

$\mathrm{T} \stackrel{\text{def}}{=} \mathrm{K} + \Box\mathtt{p} \Rightarrow \mathtt{p}$ $\qquad$ $\mathrm{K4} \stackrel{\text{def}}{=} \mathrm{K} + \Box\mathtt{p} \Rightarrow \Box\Box\mathtt{p}$
$\mathrm{S4} \stackrel{\text{def}}{=} \mathrm{K4} + \Box\mathtt{p} \Rightarrow \mathtt{p}$ $\qquad$ $\mathrm{G} \stackrel{\text{def}}{=} \mathrm{K4} + \Box(\Box\mathtt{p} \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p}$
$\mathrm{Grz} \stackrel{\text{def}}{=} \mathrm{S4} + \Box(\Box(\mathtt{p} \Rightarrow \Box\mathtt{p}) \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p}$

Following [Kra96], a formula is *primitive* iff it is of the form $\phi \Rightarrow \psi$ where both ϕ and ψ are built from $\mathtt{PRP} \cup \{\top\}$ with the help of $\wedge$, $\vee$, $\Diamond$, and where ϕ contains each atomic proposition at most once.

Example 1. Neither of the formulae $\Box\mathtt{p} \Rightarrow \mathtt{p}$ and $\Box\mathtt{p} \Rightarrow \Box\Box\mathtt{p}$ are primitive, but their logically equivalent (in K) forms $\mathtt{p} \Rightarrow \Diamond\mathtt{p}$ and $\Diamond\Diamond\mathtt{p} \Rightarrow \Diamond\mathtt{p}$ are both primitive.

Definition 1. [Kra96] A logic (defined via Hilbert system) $\mathtt{L}$ is *properly displayable* iff $\mathtt{L}$ is obtained from K by adding primitive formulae as axioms.

By Example 1, the logics T, K4 and S4 are properly displayable. In general, many of the traditional axioms for the most well-known modal logics are not primitive, but most of them have a primitive equivalent [Kra96]. In [Kra96], it is shown that every properly displayable logic admits a display calculus satisfying the conditions (C1)-(C8) [Bel82] and therefore enjoys cut-elimination. In what follows, we write $\delta\mathtt{L}$ to denote the display calculus for $\mathtt{L}$ defined in [Kra96].

A *formula generation map* $\mathtt{F} : \mathtt{FML} \rightarrow \mathtt{FML}$ is an application such that there is a formula $\psi_{\mathtt{F}}$ containing only one atomic proposition, say $\mathtt{p}$, and no logical constants, such that for $\phi \in \mathtt{FML}$, $\mathtt{F}(\phi)$ is obtained from $\psi_{\mathtt{F}}$ by replacing every occurrence of $\mathtt{p}$ by ϕ. Moreover, we assume that no subformula of the form $\Box\varphi$ occurs negatively in $\psi_{\mathtt{F}}$. $\mathtt{F}$ is also written $\lambda\mathtt{p}.\psi_{\mathtt{F}}$. For any properly displayable logic $\mathtt{L}$ and any formula generation map $\mathtt{F}$, we write $\mathtt{LF}$ to denote the logic obtained from $\mathtt{L}$ by addition of the scheme

$$\Box(\mathtt{F}(\mathtt{p}) \Rightarrow \mathtt{p}) \Rightarrow \Box\mathtt{p} \qquad (1)$$

Observe that $\Box(\mathtt{F}(\mathtt{q}) \Rightarrow \mathtt{q}) \Rightarrow \Box\mathtt{q}$ is not a Sahlqvist formula. This does not exclude the possibility to find a Sahlqvist formula logically equivalent (in the basic modal logic K) to it. For instance, this is the case when $\mathtt{F}(\mathtt{q}) = \neg\mathtt{q}$ since then, $\Box(\mathtt{F}(\mathtt{q}) \Rightarrow \mathtt{q})) \Rightarrow \Box\mathtt{q}$ is just $\Box\mathtt{q} \Rightarrow \Box\mathtt{q}$, and this has an equivalent primitive form $\top \Rightarrow \top$. Moreover, in numerous cases $\mathtt{LF}$ is not properly displayable. For instance, let $\mathtt{F}_G$ and $\mathtt{F}_{Grz}$ be $\lambda\mathtt{p}.\Box\mathtt{p}$ and $\lambda\mathtt{p}.\Box(\mathtt{p} \Rightarrow \Box\mathtt{p})$ respectively. Then, by definition $\mathrm{G} = \mathrm{K4}\mathtt{F}_G$ and $\mathrm{Grz} = \mathrm{S4}\mathtt{F}_{Grz}$. Since $\mathtt{F}_G$ and $\mathtt{F}_{Grz}$ are modal axioms that correspond to essentially second-order conditions on frames, the logics G and Grz are not properly displayable in the sense of Definition 1.

3 Display calculi

In this section, $\mathtt{L}$ is a properly displayable logic, $\mathtt{F}$ is a formula generation map and $\mathtt{LF}$ is the corresponding extension of $\mathtt{L}$ by the axiom scheme (1).

We briefly recall the main features of the calculus $\delta\mathtt{L}$ from [Wan94, Kra96]. On the structural side, we have structural connectives $*$ (unary), $\circ$ (binary), I (nullary) and $\bullet$ (unary). A *structure* $\mathtt{X} \in \mathtt{struc}(\delta\mathtt{L})$ is inductively defined as follows for $\phi \in \mathtt{FML}$:

$$\mathtt{X} ::= \phi \mid *\mathtt{X} \mid \mathtt{X}_1 \circ \mathtt{X}_2 \mid I \mid \bullet\mathtt{X}$$

We assume that the unary connectives bind tighter than the binary ones. We use *formula variables* like ϕ, ψ, φ to stand for formulae, and use *structure variables* like $\mathtt{X}$, $\mathtt{Y}$ and $\mathtt{Z}$ to stand for arbitrary structures from $\mathtt{struc}(\delta\mathtt{L})$.

A *sequent* is a pair of structures of the form $\mathtt{X} \vdash \mathtt{Y}$ with $\mathtt{X}$ the *antecedent* and $\mathtt{Y}$ the *succedent*. The rules of $\delta\mathtt{L}$ are presented in Figures 1-4. Additional structural rules satisfying conditions (C1)-(C8) are also needed but are omitted here since they depend on the primitive axioms defining $\mathtt{L}$ (see [Kra96] for details).

$$(\mathrm{Id})\ \ \mathtt{p} \vdash \mathtt{p} \qquad\qquad (\mathrm{cut})\ \ \frac{\mathtt{X} \vdash \phi \quad \phi \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}$$

Fig. 1. Fundamental logical axioms and cut rule

The *display postulates* (reversible rules) in Figure 2 deal with the manipulation of structural connectives.

In any structure $\mathtt{Z}$, the structure $\mathtt{X}$ occurs *negatively* [resp. *positively*] iff $\mathtt{X}$ occurs in the scope of an odd number [resp. an even number] of occurrences of $*$ [Bel82]. In a sequent $\mathtt{Z} \vdash \mathtt{Z}'$, an occurrence of $\mathtt{X}$ is an *antecedent part* [resp. *succedent part*] iff it occurs positively in $\mathtt{Z}$ [resp. negatively in $\mathtt{Z}'$] or it occurs negatively in $\mathtt{Z}'$ [resp. positively in $\mathtt{Z}$] [Bel82]. Two sequents $\mathtt{X} \vdash \mathtt{Y}$ and $\mathtt{X}' \vdash \mathtt{Y}'$ are *structurally equivalent* iff there is a derivation of the first sequent from the second (and vice-versa) only using display postulates from Figure 2.

Theorem 2. *([Bel82]) For every sequent $\mathtt{X} \vdash \mathtt{Y}$ and every antecedent [resp. succedent] part $\mathtt{Z}$ of $\mathtt{X} \vdash \mathtt{Y}$, there is a structurally equivalent sequent $\mathtt{Z} \vdash \mathtt{Y}'$ [resp. $\mathtt{X}' \vdash \mathtt{Z}$] that has $\mathtt{Z}$ (alone) as its antecedent [resp. succedent]. $\mathtt{Z}$ is said to be* displayed *in $\mathtt{Z} \vdash \mathtt{Y}'$ [resp. $\mathtt{X}' \vdash \mathtt{Z}$].*

$$\frac{X \circ Y \vdash Z}{X \vdash Z \circ *Y} \qquad \frac{X \circ Y \vdash Z}{Y \vdash *X \circ Z} \qquad \frac{X \vdash Y \circ Z}{X \circ *Z \vdash Y} \qquad \frac{X \vdash Y \circ Z}{*Y \circ X \vdash Z}$$

$$\frac{*X \vdash Y}{*Y \vdash X} \qquad \frac{X \vdash *Y}{Y \vdash *X} \qquad \frac{**X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash **Y}{X \vdash Y} \qquad \frac{X \vdash \bullet Y}{\bullet X \vdash Y}$$

Fig. 2. Display postulates

$$\frac{}{\bot \vdash I}\ (\bot\vdash) \qquad \frac{X \vdash I}{X \vdash \bot}\ (\vdash\bot) \qquad \frac{I \vdash X}{\top \vdash X}\ (\top\vdash) \qquad \frac{}{I \vdash \top}\ (\vdash\top)$$

$$\frac{X \vdash \phi \quad \psi \vdash Y}{\phi \Rightarrow \psi \vdash *X \circ Y}\ (\Rightarrow\vdash) \qquad \frac{X \circ \phi \vdash \psi}{X \vdash \phi \Rightarrow \psi}\ (\vdash\Rightarrow)$$

$$\frac{\phi \circ \psi \vdash X}{\phi \wedge \psi \vdash X}\ (\wedge\vdash) \qquad \frac{X \vdash \phi \quad Y \vdash \psi}{X \circ Y \vdash \phi \wedge \psi}\ (\vdash\wedge) \qquad \frac{*\phi \vdash X}{\neg\phi \vdash X}\ (\neg\vdash) \qquad \frac{X \vdash *\phi}{X \vdash \neg\phi}\ (\vdash\neg)$$

$$\frac{X \vdash \bullet\phi}{X \vdash \Box\phi}\ (\vdash\Box_{L}) \qquad \frac{\phi \vdash X}{\Box\phi \vdash \bullet X}\ (\Box\vdash) \qquad \frac{\phi \vdash X \quad \psi \vdash Y}{\phi \vee \psi \vdash X \circ Y}\ (\vee\vdash) \qquad \frac{X \vdash \phi \circ \psi}{X \vdash \phi \vee \psi}\ (\vdash\vee)$$

Fig. 3. Operational rules

Theorem 3. *[Kra96] For all $\phi \in \mathtt{FML}$, $I \vdash \phi$ has a cut-free proof in $\delta\mathtt{L}$ iff $\phi \in \mathtt{L}$.*

To define the calculus $\delta\mathtt{LF}$ we need one additional notion. Let m be a map $m : \mathtt{FML} \times \{0,1\} \to \mathtt{struc}(\delta\mathtt{L})$ that transforms certain logical connectives into structural connectives, inductively defined as follows ($i \in \{0,1\}$):

$$m(\mathtt{p}, i) \stackrel{\text{def}}{=} \mathtt{p} \text{ for any } \mathtt{p} \in \mathtt{PRP}$$

$m(\top, i) \stackrel{\text{def}}{=} \top$	$m(\bot, i) \stackrel{\text{def}}{=} \bot$
$m(\phi_1 \vee \phi_2, 0) \stackrel{\text{def}}{=} \phi_1 \vee \phi_2$	$m(\phi_1 \vee \phi_2, 1) \stackrel{\text{def}}{=} m(\phi_1, 1) \circ m(\phi_2, 1)$
$m(\phi_1 \wedge \phi_2, 0) \stackrel{\text{def}}{=} m(\phi_1, 0) \circ m(\phi_2, 0)$	$m(\phi_1 \wedge \phi_2, 1) \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2$
$m(\phi_1 \Rightarrow \phi_2, 0) \stackrel{\text{def}}{=} \phi_1 \Rightarrow \phi_2$	$m(\phi_1 \Rightarrow \phi_2, 1) \stackrel{\text{def}}{=} *m(\phi_1, 0) \circ m(\phi_2, 1)$
$m(\Box\phi, i) \stackrel{\text{def}}{=} \Box\phi$	$m(\neg\phi, i) \stackrel{\text{def}}{=} *m(\phi, 1-i)$.

The second argument of m indicates when the first argument is read as an antecedent part ($i = 0$) or as a succedent part ($i = 1$). The calculus $\delta\mathtt{LF}$ has the same structures as $\delta\mathtt{L}$, and is obtained from $\delta\mathtt{L}$ by replacing the $(\vdash \Box_{L})$-rule from Figure 3 by the $(\vdash \Box_{LF})$ rule:

$$\frac{X \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}{X \vdash \Box\phi}\ (\vdash \Box_{LF})$$

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{I \circ \mathtt{X} \vdash \mathtt{Z}}\ (I_l) \qquad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \vdash I \circ \mathtt{Z}}\ (I_r) \qquad \frac{I \vdash \mathtt{Y}}{*I \vdash \mathtt{Y}}\ (Q_l) \qquad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash *I}\ (Q_r)$$

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}\ (weak_l) \quad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}\ (weak_r) \quad \frac{I \vdash \mathtt{X}}{\bullet I \vdash \mathtt{X}}\ (nec^l) \quad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash \bullet I}\ (nec^r)$$

$$\frac{\mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3) \vdash \mathtt{Z}}{(\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3 \vdash \mathtt{Z}}\ (assoc_l) \qquad \frac{\mathtt{Z} \vdash \mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3)}{\mathtt{Z} \vdash (\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3}\ (assoc_r)$$

$$\frac{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}\ (com_l) \quad \frac{\mathtt{Z} \vdash \mathtt{Y} \circ \mathtt{X}}{\mathtt{Z} \vdash \mathtt{X} \circ \mathtt{Y}}\ (com_r) \quad \frac{\mathtt{X} \circ \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (contr_l) \quad \frac{\mathtt{Y} \vdash \mathtt{X} \circ \mathtt{X}}{\mathtt{Y} \vdash \mathtt{X}}\ (contr_r)$$

Fig. 4. Other basic structural rules

The $(\vdash \Box_{\mathtt{LF}})$-rules for δGrz and δG are respectively:

$$\frac{\mathtt{X} \vdash \bullet(*\Box(\phi \Rightarrow \Box\phi) \circ \phi)}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_{Grz}) \qquad \frac{\mathtt{X} \vdash \bullet(*\Box\phi \circ \phi)}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_G)$$

The calculus $\delta\mathtt{LF}$ satisfies conditions (C2)-(C7). In particular, δG satisfies the conditions (C1)-(C7). The $(\vdash \Box_G)$-rule in δG is similar to the GLR rule in [SV82] (see also [Avr84]). Analogously, the $(\vdash \Box_{Grz})$-rule in δGrz is similar to the (GRZc) rule in [BG86] or to the $(\Rightarrow \Box)$ rule in [Avr84]. An intuitively obvious way to understand the $(\vdash \Box_{\mathtt{LF}})$-rule is to recall the *double nature* of the $\Box$-formulae in $\mathtt{LF}$ as illustrated by the $\mathtt{LF}$-theorem $\Box\phi \Leftrightarrow \Box(\mathtt{F}(\phi) \Rightarrow \phi)$.

We use the label (dp) as shown below left to denote that the sequent s' is obtained from the sequent s by an unspecified finite number (possibly zero) of display postulate applications from Figure 2. The $(\vdash \Box_{\mathtt{L}})$-rule from $\delta\mathtt{L}$ is derivable in $\delta\mathtt{LF}$ as shown below right:

$$\frac{s}{s'}\ (dp) \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathtt{X} \vdash \bullet\phi}{\bullet\mathtt{X} \vdash \phi}\ (dp)}{m(\mathtt{F}(\phi), 0) \circ \bullet\mathtt{X} \vdash \phi}\ (weak_l)}{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}\ (dp)}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}})$$

To prove soundness of $\delta\mathtt{LF}$ with respect to $\mathtt{LF}$-theoremhood, we use the mappings $a : \mathtt{struc}(\delta\mathrm{L}) \to \mathtt{FML}$ and $s : \mathtt{struc}(\delta\mathrm{L}) \to \mathtt{FML}$ recalled below:

$$a(\phi) \stackrel{\text{def}}{=} s(\phi) \stackrel{\text{def}}{=} \phi \text{ for any } \phi \in \mathtt{FML}$$

$$\begin{array}{ll@{\qquad\qquad}ll}
a(I) & \stackrel{\text{def}}{=} \top & s(I) & \stackrel{\text{def}}{=} \bot \\
a(*\mathtt{X}) & \stackrel{\text{def}}{=} \neg s(\mathtt{X}) & s(*\mathtt{X}) & \stackrel{\text{def}}{=} \neg a(\mathtt{X}) \\
a(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} a(\mathtt{X}) \wedge a(\mathtt{Y}) & s(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} s(\mathtt{X}) \vee s(\mathtt{Y}) \\
a(\bullet\mathtt{X}) & \stackrel{\text{def}}{=} \Diamond^- a(\mathtt{X}) & s(\bullet\mathtt{X}) & \stackrel{\text{def}}{=} \Box s(\mathtt{X})
\end{array}$$

The modality $\Diamond^-$ is the backward existential modality associated with $\Box$. That is, as is usual with **DL**, we extend the language by adding the unary modal operator $\Box^-$. We write $\mathtt{L}^+$ [resp. $\mathtt{L}^+\mathtt{F}$] to denote the extension of $\mathtt{L}$ [resp. $\mathtt{LF}$] obtained by adding the axiom schemes $\Box^-\mathtt{p} \Rightarrow (\Box^-(\mathtt{p} \Rightarrow \mathtt{q}) \Rightarrow \Box^-\mathtt{q})$, $\mathtt{q} \Rightarrow \Box\Diamond^-\mathtt{q}$, $\mathtt{q} \Rightarrow \Box^-\Diamond\mathtt{q}$ and the necessitation rule: from ϕ infer $\Box^-\phi$. The language is extended appropriately by adding $\Box^-$, and $\Diamond^-\phi$ is defined as $\neg\Box^-\neg\phi$.

Theorem 4. *If* $\mathtt{X} \vdash \mathtt{Y}$ *is derivable in* $\delta\mathtt{LF}$*, then* $a(\mathtt{X}) \Rightarrow s(\mathtt{Y}) \in \mathtt{L}^+\mathtt{F}$*.*

The maps a and s can be found for instance in [Kra96] where they are called τ_1 and τ_2 respectively. The maps a and s give an intuitive way to interpret the meaning of the structural connectives depending on the polarity of their occurrence (either as antecedent part or as succedent part).

Lemma 5. *The following rules are admissible in* $\delta\mathtt{LF}$*:*

$$\frac{\phi_1 \wedge \phi_2 \vdash \mathtt{X}}{\phi_1 \circ \phi_2 \vdash \mathtt{X}}\ (\circ\vdash) \qquad \frac{\mathtt{X} \vdash \phi_1 \Rightarrow \phi_2}{\mathtt{X} \vdash *\phi_1 \circ \phi_2}\ (adm1) \qquad \frac{\mathtt{X} \vdash \phi_1 \vee \phi_2}{\mathtt{X} \vdash \phi_1 \circ \phi_2}\ (\vdash\circ)$$

$$\frac{\neg\phi \vdash \mathtt{X}}{*\phi \vdash \mathtt{X}}\ (*\vdash) \qquad \frac{\mathtt{F}(\phi) \vdash \mathtt{X}}{m(\mathtt{F}(\phi), 0) \vdash \mathtt{X}}\ (adm2) \qquad \frac{\mathtt{X} \vdash \neg\phi}{\mathtt{X} \vdash *\phi}\ (\vdash *)$$

Moreover, for each of these rules, if the premiss has a cut-free proof in $\delta\mathtt{LF}$*, then the conclusion also has a cut-free proof in* $\delta\mathtt{LF}$*.*

The proof of admissibility of the rules $(\vdash\circ)$, $(adm1)$, $(\circ\vdash)$, $(*\vdash)$ and $(\vdash *)$ is similar to [Kra96, Lemma 9]. Admissibility of $(adm2)$ is a mere consequence of the admissibility of the above rules.

Lemma 6. $\phi \vdash \phi$ *is cut-free derivable in* $\delta\mathtt{LF}$ *for any formula* ϕ*.*

Lemma 6 requires induction on the formation of ϕ. Theorem 7 is the **DL** version of Theorem 1 in [Avr84] for Gentzen-style calculi.

Theorem 7. *A formula* $\phi \in \mathtt{LF}$ *iff* $I \vdash \phi$ *is derivable in* $\delta\mathtt{LF}$*.*

Proof. The right to left direction is just an instance of Theorem 4. The left to right direction *requires* uses of the cut rule (to simulate the application of the *modus ponens* rule) and proceeds by induction on the length of the derivation in $\mathtt{LF}$ (viewed as an Hilbert-style system). Most of the cases can be found in [Wan94, Kra96, Wan98]. It remains to show that $I \vdash \Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi$ has a proof in $\delta\mathtt{LF}$ which is done below using the fact that $\mathtt{F}(\phi) \vdash \mathtt{F}(\phi)$ and $\phi \vdash \phi$ are derivable in $\delta\mathtt{LF}$ by Lemma 6:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \vdash \phi \qquad \dfrac{\mathtt{F}(\phi) \vdash \mathtt{F}(\phi)}{m(\mathtt{F}(\phi), 0) \vdash \mathtt{F}(\phi)}\ (adm2)}{\mathtt{F}(\phi) \Rightarrow \phi \vdash *m(\mathtt{F}(\phi), 0) \circ \phi}\ (\Rightarrow\vdash)}{\Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \bullet(*m(\mathtt{F}(\phi), 0) \circ \phi)}\ (\Box\vdash)}{\Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}})}{I \circ \Box(\mathtt{F}(\phi) \Rightarrow \phi) \vdash \Box\phi}\ (I_l)}{I \vdash \Box(\mathtt{F}(\phi) \Rightarrow \phi) \Rightarrow \Box\phi}\ (\vdash\Rightarrow)$$

As stated previously, any display calculus satisfying the conditions (C2)-(C8) from [Bel82] admits cut-elimination. Unfortunately $\delta\mathtt{LF}$ does not satisfy (C8) (see the appendix).

Specifically, the cut instance below breaks (C8):

$$\dfrac{\dfrac{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0) \circ \phi)}{\mathtt{X} \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}}) \qquad \dfrac{\phi \vdash \mathtt{Y}}{\Box\phi \vdash \bullet\mathtt{Y}}\ (\Box \vdash)}{\mathtt{X} \vdash \bullet\mathtt{Y}}\ (cut)$$

where there is a formula ψ in $m(\mathtt{F}(\phi),0)$ that is not a subformula of ϕ. For instance, such cases are easy to find with the display calculi δG and δGrz. Furthermore, in order to infer $\mathtt{X} \vdash \bullet\mathtt{Y}$ from $\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0) \circ \phi)$ and $\phi \vdash \mathtt{Y}$, no cut can be used on ψ if (C8) has to be satisfied. In the display calculus $\delta\mathtt{LF}$, for all the derivations of the sequent $\mathtt{X}'' \vdash \mathtt{Y}''$ from $\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\phi),0) \circ \phi)$, if a cut with a cut-formula that is not a subformula of ϕ is forbidden, then either $\mathtt{X}'' \vdash \mathtt{Y}''$ contains ψ as the subformula of some formula/structure (see the introduction rules different from $(\vdash \Box_{\mathtt{LF}})$), or $\mathtt{X}'' \vdash \mathtt{Y}''$ contains $\Box\phi$ as the subformula of some formula/structure (see the $(\vdash \Box_{\mathtt{LF}})$-rule). So, *in the general case*, there is no way to derive $\mathtt{X} \vdash \bullet\mathtt{Y}$ since neither ψ nor $\Box\phi$ are guaranteed to occur in it.

We say that $\mathtt{LF}$ is *pseudo displayable* iff for any $\phi \in \mathtt{FML}$, $I \vdash \phi$ has a proof in $\delta\mathtt{LF}$ iff $I \vdash \phi$ has a cut-free proof in $\delta\mathtt{LF}$. "Pseudo" because strong cut-elimination is couched in terms of arbitrary sequents $\mathtt{X} \vdash \mathtt{Y}$ rather than sequents of the form $I \vdash \phi$. For mechanisation "pseudo" is sufficient for our needs since we want to check whether $\phi \in \mathtt{LF}$. We now provide a characterisation of the class of pseudo displayable logics and show that both G and Grz are pseudo displayable.

4 Transformations from LF into L

In this section, $\mathtt{L}$ is a properly displayable logic and $\mathtt{F}$ is a formula generation map. Let $f : \mathtt{FML} \times \{0,1\} \rightarrow \mathtt{FML}$ be the following map for $i \in \{0,1\}$:

for any $\mathtt{p} \in \mathtt{PRP}$, $f(\mathtt{p},0) \stackrel{\text{def}}{=} f(\mathtt{p},1) \stackrel{\text{def}}{=} \mathtt{p}$ $\quad f(\top,i) \stackrel{\text{def}}{=} \top$ $\quad f(\bot,i) \stackrel{\text{def}}{=} \bot$

$f(\phi_1 \oplus \phi_2, i) \stackrel{\text{def}}{=} f(\phi_1,i) \oplus f(\phi_2,i)$ for $\oplus \in \{\wedge,\vee\}$

$f(\neg\phi,0) \stackrel{\text{def}}{=} \neg f(\phi,1)$ $\quad f(\neg\phi,1) \stackrel{\text{def}}{=} \neg f(\phi,0)$

$f(\phi_1 \Rightarrow \phi_2,1) \stackrel{\text{def}}{=} f(\phi_1,0) \Rightarrow f(\phi_2,1)$ $\quad f(\phi_1 \Rightarrow \phi_2,0) \stackrel{\text{def}}{=} f(\phi_1,1) \Rightarrow f(\phi_2,0)$

$f(\Box\phi,1) \stackrel{\text{def}}{=} \Box(f(\mathtt{F}(\phi),0) \Rightarrow f(\phi,1))$ $\quad f(\Box\phi,0) \stackrel{\text{def}}{=} \Box f(\phi,0)$

In $f(\phi,i)$, the index i carries information about the *polarity* of ϕ in the translation process as in [BH94]. The map f also generalises one of the maps from G into K4 defined in [BH94]. By simultaneous induction one can show that for any $\phi \in \mathtt{FML}$ and for any $i \in \{0,1\}$, $\phi \Leftrightarrow f(\phi,i) \in \mathtt{LF}$. Moreover, for any $\phi \in \mathtt{FML}$: $f(\phi,0) \Rightarrow \phi \in \mathtt{L}$, $\phi \Rightarrow f(\phi,1) \in \mathtt{L}$ and therefore $f(\phi,0) \Rightarrow f(\phi,1) \in \mathtt{L}$.

Lemma 8. *Every positive [resp. negative] occurrence of*

1. *$\Box\psi$ in $f(\phi,1)$ is of the form $\Box(f(\mathtt{F}(\varphi),0) \Rightarrow f(\varphi,1))$ [resp. $\Box f(\varphi,0)$] for some subformula φ of ϕ;*

2. *$\neg\psi$ in $f(\phi,1)$ is of the form $\neg f(\varphi,0)$ [resp. $\neg f(\varphi,1)$] for some subformula φ of ϕ;*
3. *$\psi_1 \Rightarrow \psi_2$ in $f(\phi,1)$ is of the form $f(\varphi_1,0) \Rightarrow f(\varphi_2,1)$ [resp. $f(\varphi_1,1) \Rightarrow f(\varphi_2,0)$] for some subformulae φ_1, φ_2 of ϕ;*
4. *$\psi_1 \oplus \psi_2$ ($\oplus \in \{\wedge,\vee\}$) in $f(\phi,1)$ is of the form $f(\varphi_1,1) \oplus f(\varphi_2,1)$ [resp. $f(\varphi_1,0) \oplus f(\varphi_2,0)$] for some subformulae φ_1, φ_2 of ϕ;*

The proof of Lemma 8 is by an easy verification. Lemma 8 is used in the proof of Theorem 9 below. We extend the map f to structures in the following way ($i \in \{0,1\}$):

$$f(I,i) \stackrel{\text{def}}{=} I \qquad f(*\mathtt{X},i) \stackrel{\text{def}}{=} *f(\mathtt{X},1-i)$$
$$f(\mathtt{X}\circ\mathtt{Y},i) \stackrel{\text{def}}{=} f(\mathtt{X},i)\circ f(\mathtt{Y},i) \qquad f(\bullet\mathtt{X},i) \stackrel{\text{def}}{=} \bullet f(\mathtt{X},i)$$

By induction on the structure of ϕ, the rule below is admissible in $\delta\mathtt{L}$:

$$\frac{f(m(\phi,0),0) \vdash \mathtt{X}}{f(\phi,0) \vdash \mathtt{X}}\ (adm3)$$

Theorem 9 below is the main result of the paper.

Theorem 9. *The statements below are equivalent:*

1. *For all formulae ϕ, $\phi \in \mathtt{LF}$ iff $f(\phi,1) \in \mathtt{L}$.*
2. *$\mathtt{LF}$ is pseudo displayable.*

That is, (weak) cut-elimination of $\delta\mathtt{LF}$ is equivalent to the theoremhood preserving nature of f from $\mathtt{LF}$ into $\mathtt{L}$. Its proof is purely syntactic and therefore it does *not depend on the class of modal frames that possibly characterises* $\mathtt{LF}$. Moreover, Theorem 9 goes beyond the mechanisation aspect since it provides a characterisation of the class of pseudo displayable logics which is not directly based on a cut-elimination procedure.

The proof of Theorem 9 is long and tedious. For instance, when (2) holds and $\phi \in \mathtt{LF}$, to show that $f(\phi,1) \in \mathtt{L}$, we show that in any cut-free proof Π of $I \vdash \phi$ in $\delta\mathtt{LF}$, for any sequent $\mathtt{X} \vdash \mathtt{Y}$ in Π, $f(\mathtt{X},0) \vdash f(\mathtt{Y},1)$ has a cut-free proof in $\delta\mathtt{L}$. By way of example, the proof step (in $\delta\mathtt{LF}$) shown below left is transformed into the proof steps (in $\delta\mathtt{L}$) shown below right:

$$\frac{\mathtt{X} \vdash \bullet(*m(\mathtt{F}(\psi),0)\circ\psi)}{\mathtt{X} \vdash \Box\psi}\ (\vdash \Box_{\mathtt{LF}})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f(\mathtt{X},0) \vdash \bullet(*f(m(\mathtt{F}(\psi),0),0)\circ f(\psi,1))}{\bullet f(\mathtt{X},0) \vdash *f(m(\mathtt{F}(\psi),0),0)\circ f(\psi,1)}\ (dp)}{\bullet f(\mathtt{X},0) \vdash f(\psi,1)\circ *f(m(\mathtt{F}(\psi),0),0)}\ (com_r)}{f(m(\mathtt{F}(\psi),0),0) \vdash *\bullet f(\mathtt{X},0)\circ f(\psi,1)}\ (dp)}{f(\mathtt{F}(\psi),0) \vdash *\bullet f(\mathtt{X},0)\circ f(\psi,1)}\ (adm3)}{\bullet f(\mathtt{X},0)\circ f(\mathtt{F}(\psi),0) \vdash f(\psi,1)}\ (dp)}{\bullet f(\mathtt{X},0) \vdash f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1)}\ (\vdash\Rightarrow)}{f(\mathtt{X},0) \vdash \bullet(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\ (dp)}{f(\mathtt{X},0) \vdash \Box(f(\mathtt{F}(\psi),0) \Rightarrow f(\psi,1))}\ (\vdash \Box_{\mathtt{L}})$$

$f^{-1}(I,i) \stackrel{\text{def}}{=} I$ $\quad f^{-1}(\top,i) \stackrel{\text{def}}{=} \top$ $\quad f^{-1}(\bot,i) \stackrel{\text{def}}{=} \bot$

$f^{-1}(\mathtt{X} \circ \mathtt{Y},i) \stackrel{\text{def}}{=} f^{-1}(\mathtt{X},i) \circ f^{-1}(\mathtt{Y},i)$ or undefined

$f^{-1}(\bullet\mathtt{X},i) \stackrel{\text{def}}{=} \bullet f^{-1}(\mathtt{X},i)$ or undefined

$f^{-1}(*\mathtt{X},1-i) \stackrel{\text{def}}{=} *f^{-1}(\mathtt{X},i)$ or undefined

$f^{-1}(\mathrm{p},i) \stackrel{\text{def}}{=} \mathrm{p}$ for any $\mathrm{p} \in \mathtt{PRP}$

for $\oplus \in \{\wedge,\vee\}$, $f^{-1}(\phi \oplus \psi,i) \stackrel{\text{def}}{=} f^{-1}(\phi,i) \oplus f^{-1}(\psi,i)$ or undefined

$f^{-1}(\phi \Rightarrow \psi,1) \stackrel{\text{def}}{=} f^{-1}(\phi,0) \Rightarrow f^{-1}(\psi,1)$ or undefined

$f^{-1}(\phi \Rightarrow \psi,0) \stackrel{\text{def}}{=} f^{-1}(\phi,1) \Rightarrow f^{-1}(\psi,0)$ or undefined

$f^{-1}(\neg\phi,1-i) \stackrel{\text{def}}{=} \neg f^{-1}(\phi,i)$ or undefined

$f^{-1}(\Box\phi,0) \stackrel{\text{def}}{=} \Box f^{-1}(\phi,0)$ or undefined

$$f^{-1}(\Box\phi,1) \stackrel{\text{def}}{=} \begin{cases} \Box f^{-1}(\phi_2,1) \text{ if } \phi = (\phi_1 \Rightarrow \phi_2) \text{ and } f^{-1}(\phi_2,1) \text{ is defined} \\ \text{undefined otherwise} \end{cases}$$

where "$x \stackrel{\text{def}}{=} y$ or undefined" means: $x \stackrel{\text{def}}{=} \begin{cases} y \text{ if all components of } y \text{ are defined} \\ \text{undefined otherwise} \end{cases}$

Fig. 5. Definition of $f^{-1}(.)$ for $i \in \{0,1\}$.

When (1) holds, $f(\phi,1) \in \mathtt{L}$ implies $I \vdash f(\phi,1)$ has a cut-free proof Π in $\delta\mathtt{L}$, and $\phi \in \mathtt{LF}$ implies $I \vdash \phi$ is provable in $\delta\mathtt{LF}$. To show that $I \vdash \phi$ has a *cut-free* proof in $\delta\mathtt{LF}$, we prove that for every sequent $\mathtt{X} \vdash \mathtt{Y}$ in Π, the sequent $f^{-1}(\mathtt{X},0) \vdash f^{-1}(\mathtt{Y},1)$ admits a cut-free proof in $\delta\mathtt{LF}$ with the partial function f^{-1} defined in Figure 5. Since $\delta\mathtt{L}$ satisfies (C1)-(C8) (see the appendix), for every sequent $\mathtt{X} \vdash \mathtt{Y}$ in Π, both $f^{-1}(\mathtt{X},0)$ and $f^{-1}(\mathtt{Y},1)$ are defined (Lemma 8 is also used to show this property). Because the map $f^{-1} : \mathtt{struc}(\delta\mathtt{L}) \times \{0,1\} \to \mathtt{struc}(\delta\mathtt{L})$ satisfies $f^{-1}(f(\phi,1),1) = \phi$ and $f^{-1}(f(\phi,0),0) = \phi$, the end-sequent $I \vdash f(\phi,1)$ of Π becomes $I \vdash \phi$ in this procedure, as desired.

The proof of Theorem 9 actually shows that **DL** is particularly well-designed to reason about polarity, succedent and antecedent parts. One of the translations from G into K4 in [BH94] is exactly the map f when $\mathtt{L}$ is K4 and $\mathtt{F}$ is $\mathtt{F}_G$.

Corollary 10. *G is pseudo displayable.*

Let $\delta^- G$ be the calculus δG minus the cut-rule. Thus $\delta^- G$ satisfies *all* the conditions (C1)-(C8) and for any $\phi \in \mathtt{FML}$, $\phi \in G$ iff $I \vdash \phi$ has a proof in $\delta^- G$. At first glance, this seems to contradict the fact that G is not properly displayable in the sense of [Kra96]. However, in [Kra96], all the axioms added to K are transformed into structural rules. By contrast, in $\delta^- G$, one of the axioms is encoded in the $(\vdash \Box_G)$-rule. This opens an avenue to define display calculi satisfying (C1)-(C8) for modal logics that are not properly displayable.

Theorem 11. *For every pseudo displayable logic* $\mathtt{LF}$*, there is an* $\mathcal{O}(n^3.log\ n)$*-time transformation* g *such that any* $\phi \in \mathtt{LF}$ *iff* $g(\phi) \in \mathtt{L}$.

Form of ϕ_i	ψ_i
$\top$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow \top) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow \top))$
$\perp$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow \perp) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow \perp))$
p	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,0} \Leftrightarrow \mathrm{p}_{i,1}))$
$\neg\phi_j$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow \neg\mathrm{p}_{j,0}) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow \neg\mathrm{p}_{j,1}))$
$\phi_{i_1} \wedge \phi_{i_2}$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow (\mathrm{p}_{i_1,1} \wedge \mathrm{p}_{i_2,1})) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow (\mathrm{p}_{i_1,0} \wedge \mathrm{p}_{i_2,0})))$
$\phi_{i_1} \vee \phi_{i_2}$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow (\mathrm{p}_{i_1,1} \vee \mathrm{p}_{i_2,1})) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow (\mathrm{p}_{i_1,0} \vee \mathrm{p}_{i_2,0})))$
$\phi_{i_1} \Rightarrow \phi_{i_2}$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow (\mathrm{p}_{i_1,0} \Rightarrow \mathrm{p}_{i_2,1})) \wedge \Box(\mathrm{p}_{i,0} \Leftrightarrow (\mathrm{p}_{i_1,1} \Rightarrow \mathrm{p}_{i_2,0})))$
$\Box\phi_j$	$(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{i,1} \Leftrightarrow \Box(\mathtt{F}'(\mathrm{p}_{j,0},\mathrm{p}_{j,1}) \Rightarrow \mathrm{p}_{j,1})) \wedge \Box^i(\mathrm{p}_{i,0} \Leftrightarrow \Box\mathrm{p}_{j,0}))$

Fig. 6. Definition of ψ_i

The right-hand side of the definition of $f(\Box\psi, 1)$ may require several calls to $f(\psi, 0)$ and $f(\psi, 1)$, so f is not necessarily computable in $\mathcal{O}(n^3.log\ n)$-time. However, we can use a variant of f using a standard renaming technique (see e.g. [Min88]). Let $md(\phi)$ denote the *modal depth* of ϕ, let $\Box^0\varphi \stackrel{\text{def}}{=} \varphi$ and $\Box^{i+1}\varphi \stackrel{\text{def}}{=} \Box\Box^i\varphi$. Then for any extension $\mathtt{L}$ of K, $\phi \in \mathtt{L}$ iff $(\bigwedge_{i=0}^{md(\phi)} \Box^i(\mathrm{p}_{new} \Leftrightarrow \psi)) \Rightarrow \phi' \in \mathtt{L}$ where ϕ' is obtained by replacing every occurrence of ψ in ϕ by p_{new}, a new propositional variable not occurring in ϕ. When $\mathtt{K4} \subseteq \mathtt{L}$, $\phi \in \mathtt{L}$ iff $(\Box(\mathrm{p}_{new} \Leftrightarrow \psi) \wedge (\mathrm{p}_{new} \Leftrightarrow \psi)) \Rightarrow \phi' \in \mathtt{L}$.

Proof. (Theorem 11) The key point to define g is to observe that there is a map $\mathtt{F}' : \mathtt{FML} \times \mathtt{FML} \rightarrow \mathtt{FML}$ and a formula $\psi_{\mathtt{F}'}$ containing at most *two* atomic propositions, say $\mathtt{p}$ and $\mathtt{q}$, such that

- $\mathtt{F}'(\varphi_1, \varphi_2)$ is obtained from $\psi_{\mathtt{F}'}$ by replacing simultaneously every occurrence of $\mathtt{p}$ [resp. $\mathtt{q}$] by φ_1 [resp. φ_2];
- for any $\varphi \in \mathtt{FML}$, $f(\mathtt{F}(\varphi), 0) = \mathtt{F}'(f(\varphi, 0), f(\varphi, 1))$.

For instance, if $\mathtt{F} = \lambda\mathtt{p}.\mathtt{p} \wedge \neg\mathtt{p}$ then $\mathtt{F}' = \lambda\mathtt{pq}.\mathtt{p} \wedge \neg\mathtt{q}$.

Let ϕ be a modal formula we wish to translate from $\mathtt{LF}$ into $\mathtt{L}$. Let $\phi_1, \ldots, \phi_m$ be an enumeration (without repetition) of all subformulae of ϕ, in increasing order of size. We shall build a formula $g(\phi)$ using the set $\{\mathrm{p}_{i,j} : 1 \leq i \leq m,\ j \in \{0,1\}\}$ of atomic propositions such that $g(\phi) \in \mathtt{L}$ iff $f(\phi, 1) \in \mathtt{L}$. We could also just consider the set $\{\mathrm{p}_i : i \in \omega\}$ of atomic propositions and use a 1-1 mapping from $\omega^2 \rightarrow \omega$, but for simplicity, the present option is the most convenient.

Moreover, $g(\phi)$ can be computed in time $\mathcal{O}(|\phi|^3.log\ |\phi|)$. For $i \in \{1, \ldots, m\}$, we create a formula ψ_i as shown in Figure 6 and let $g(\phi) \stackrel{\text{def}}{=} (\bigwedge_{i=1}^{m} \psi_i) \Rightarrow \mathrm{p}_{m,1}$. If K4 $\subseteq \mathtt{L}$, the generalised conjunction in Figure 6 is needed only for $i \in \{0, 1\}$.

Each $|\psi_i|$ is in $\mathcal{O}(|\phi|^2)$ since $\Sigma_{i=0}^{md(\phi)} i$ is in $\mathcal{O}(|\phi|^2)$. So $|g(\phi)|$ is in $\mathcal{O}(|\phi| \times (|\phi|^2 \times log\ |\phi|))$. As usual in complexity theory, the extra $log\ |\phi|$ factor in the size of ϕ is because we need an index of size $\mathcal{O}(log\ |\phi|)$ for these different atomic propositions. That is, these indices are represented in binary writing.

The map g is a generalisation of the map from Grz into S4 defined in [DG99a]. One of the maps from G into K4 in [BH94] is linear-time and does not use renaming (which allows us to treat the general case). We are currently investigating if their map can be generalised by considering the map $f' : \mathtt{FML} \times \{0,1\} \rightarrow \mathtt{FML}$ inductively defined as f except that $f'(\Box\phi, 1) \stackrel{\text{def}}{=} \Box(\mathtt{F}(\phi) \Rightarrow f'(\phi, 1))$. Another map in $\mathbf{P}$ from G into K4 is given in [Fit83, Chapter 5].

5 Another characterisation

Theorem 9 has a counterpart when $\mathtt{LF}$ has a traditional Gentzen system although additional conditions are required. In this section, $\mathtt{L}$ is a properly displayable logic and $\mathtt{F}$ is a formula generation map.

Definition 12. Assume $\mathtt{L}$, properly displayable by assumption, has a traditional Gentzen system $\mathtt{GL}$ in which

1. $\mathtt{GL}$ extends a standard Gentzen system for PC containing contraction, weakening, exchange and cut (shown below left), and $\mathtt{GL}$ satisfies cut-elimination;
2. any sequent $\Gamma \vdash \Delta$ is derivable in $\mathtt{GL}$ iff $(\bigwedge_{\phi \in \Gamma} \phi) \Rightarrow (\bigvee_{\phi \in \Delta} \phi) \in \mathtt{L}$;
3. the $(\Box \vdash)$-rule (if any) and the $(\vdash \Box)$-rule have the form

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \Box\phi \vdash \Delta}\ (\Box \vdash) \qquad\qquad \frac{\Sigma' \vdash \phi}{\Sigma \vdash \Box\phi}\ (\vdash \Box)$$

   where there is a map $h : \mathtt{FML} \rightarrow \mathtt{FML}$ such that $h(\Sigma) = \Sigma'$, $h(f(\Sigma, 0)) = f(\Sigma', 0)$ for any sequence Σ to which the $(\vdash \Box)$-rule is applicable. Both h and f (defined via $\mathtt{F}$) are naturally extended to sequences. Moreover, $f(\Sigma, 0)$ satisfies any further conditions on the $(\vdash \Box)$-rule iff Σ does too: for example, that Σ must contain only formulae beginning with $\Box$.

These sequents consist of comma-separated lists of formulae. Then, $\mathtt{LF}$ is *pseudo Gentzenisable* iff the Gentzen system $\mathtt{GLF}$ obtained from $\mathtt{GL}$ by replacing the $(\vdash \Box)$-rule by $(\vdash \Box_{\mathtt{LF}})$ shown below right enjoys cut-elimination.

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma', \phi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (cut) \qquad\qquad \frac{\mathtt{F}(\phi), \Sigma' \vdash \phi}{\Sigma \vdash \Box\phi}\ (\vdash \Box_{\mathtt{LF}})$$

Definition 12(3.) ensures that (C4) is satisfied when the *structures* are simply sequences of formulae. Following [Avr84, Theorem 1], we can show that $\mathtt{GLF}$ is sound and complete for $\mathtt{LF}$. Here is the Gentzen counterpart of Theorem 9.

Theorem 13. *If $\mathtt{L}$ satisfies assumptions (1)-(3) from Definition 12 then the statements below are equivalent:*

1. *For all $\phi \in \mathtt{FML}$, $\phi \in \mathtt{LF}$ iff $f(\phi, 1) \in \mathtt{L}$.*
2. *$\mathtt{LF}$ is pseudo Gentzenisable.*

By [Avr84, Corollary 3.1], Grz is pseudo Gentzenisable and therefore Grz is pseudo displayable. So by Theorem 13, for any $\phi \in \mathtt{FML}$, $\phi \in Grz$ iff $f(\phi, 1) \in S4$ where f is defined with $\mathtt{F} = \mathtt{F}_{Grz}$. See [DG99a] for a more detailed case-study showing how to translate Grz into a *decidable* subset of first-order logic.

6 Concluding remarks

We can summarise the general situation as follows:

Theorem 14. *Assume logic* $\mathtt{L}$ *is properly displayable and* $\mathtt{F}$ *is a formula generation map. Then,*

(I) (a) $\mathtt{LF}$ *is pseudo displayable iff*
(b) for all $\phi \in \mathtt{FML}$, $\phi \in \mathtt{LF}$ *iff* $f(\phi, 1) \in \mathtt{L}$ *(where f is defined via* $\mathtt{F}$*).*
(II) If $\mathtt{L}$ *satisfies the assumptions (1)-(3) from Definition 12, then (b) holds iff* $\mathtt{LF}$ *is pseudo Gentzenisable.*
(III) When (a) holds [resp. and when $K4 \subseteq \mathtt{L}$*], there is an* $\mathcal{O}(n^3.log\ n)$*-time [resp. an* $\mathcal{O}(n.log\ n)$*-time] transformation from* $\mathtt{LF}$ *into* $\mathtt{L}$*.*

Our contribution can be summarised as follows:

1. Automated Reasoning: We gave a uniform framework to translate every pseudo-displayable modal logic $\mathtt{LF}$, which may not be first-order definable, into the first-order definable primitive modal logic $\mathtt{L}$. The transformations are at most in $\mathcal{O}(n^3.log\ n)$-time. Since a linear-time transformation from each $\mathtt{L}$ into FOL is known, we can use classical theorem provers for any pseudo-displayable logic $\mathtt{LF}$.
2. Cut-elimination: (Often a desirable property for mechanisation) Although the calculi $\delta\mathtt{LF}$ do not satisfy condition (C8), we can nevertheless characterise weak cut-elimination by the theoremhood preserving nature of f.
3. Display Logic: We defined **DL** calculi for logics that are not properly displayable à la [Kra96].

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Appendix: Belnap's Conditions.

For every sequent rule Belnap [Bel82, page 388] first defines the following notions: in an application *Inf* of a sequent rule (ρ), "constituents occurring as part of occurrences of structures assigned to structure-variables are defined to be **parameters** of *Inf* ; all other constituents are defined as **nonparametric**, including those assigned to formula-variables. Constituents occupying similar positions in occurrences of structures assigned to the same structure-variable are defined as **congruent** in *Inf* ". The eight (actually seven) conditions shown below are from [Kra96] and [Wan98]:

(C1) Each formula which is a constituent of some premiss of a rule ρ is a subformula of some formula in the conclusion of ρ.
(C2) Congruent parameters are occurrences of the same structure.
(C3) Each parameter is congruent to at most one constituent in the conclusion. Equivalently, no two constituents of the conclusion are congruent to each other.
(C4) Congruent parameters are either all antecedent parts or all succedent parts of their respective sequent.
(C5) If a formula is non-parametric in the conclusion of a rule ρ, it is either the entire antecedent, or the entire succedent. Such a formula is called a **principal** formula.
(C6/7) Each rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.
(C8) If there are inferences $\mathcal{I}_1$ and $\mathcal{I}_2$ with respective conclusions $\mathtt{X} \vdash \varphi$ and $\varphi \vdash \mathtt{Y}$ with φ *principal* in both inferences, and if cut is applied to obtain $\mathtt{X} \vdash \mathtt{Y}$, then
 (i) either $\mathtt{X} \vdash \mathtt{Y}$ is identical to one of $\mathtt{X} \vdash \varphi$ and $\varphi \vdash \mathtt{Y}$;
 (ii) or there is derivation of $\mathtt{X} \vdash \mathtt{Y}$ from the premisses of $\mathcal{I}_1$ and $\mathcal{I}_2$ in which every cut-formula of any application of cut is a proper subformula of φ.