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ABOUT THE DECISION OF REACHABILITY FOR REGISTER MACHINES

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Abstract. We study the decidability of the following problem: given p affine functions $f_1, \ldots, f_p$ over $\mathbb{N}^k$ and two vectors $v_1, v_2 \in \mathbb{N}^k$, is v_2 reachable from v_1 by successive iterations of $f_1, \ldots, f_p$ (in this given order)?
We show that this question is decidable for $p = 1, 2$ and undecidable for some fixed p.

Résumé. Nous étudions le problème d'accessibilité suivant: étant données p fonctions affines $f_1, \ldots, f_p$ sur $\mathbb{N}^k$ et deux vecteurs d'entiers v_1, v_2, est-ce que v_2 est accessible depuis v_1 par l'application successive de l'itération de f_1, puis de f_2 et ainsi de suite jusque f_p?
Nous montrons que cette question est décidable pour $p = 1, 2$ et indécidable pour un certain p_0 fixé (et donc pour tout $p \geq p_0$).

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Introduction

Reachability is a fundamental question for computation models: a typical safety property of a reactive system is the unreachability of some catastrophic state. Reachability is straightforwardly decidable (in a time linear in the number of states) for finite-state systems. For other (infinite-state) computation models, it is most of the time undecidable.

In this paper, we study the border between decidability and undecidability for a particular computation model: configurations are vectors of non-negative integers. Each move from a configuration to its successor is given by an affine function $f(X) = AX + B$ where A is a matrix of non-negative integers and B

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is a vector of integers. Such affine functions are used to model the evolution of dynamical systems like the age repartition of trees of a forestry development or the human population growth (see [9]): the initial vector represents the initial repartition and the affine function describe the evolution of this repartition during a year. They can also be used to compute limit trajectories (see [1]).

Petri nets with transfer are a particular case of this model (components of A are 0 or 1), hence reachability is in general undecidable, see [6]. Many register machines can be also modeled using such a computation model.

On the decidability side, in [2] B. Boigelot shows that

$$\{f_1^{k_1} \cdots f_p^{k_p}(X) \mid X \in \mathbb{N}^m, k_1, \ldots, k_p \in \mathbb{N}\}$$

is definable in WS_1S (weak monadic second order logic with one successor), hence reachability is decidable, when $f_1, \ldots, f_k$ are affine functions such that the matrix $A_1, \ldots, A_n$ are diagonalizing and their eigenvalues satisfy some conditions.

Instead of restricting the operations on the vectors, we consider here some restriction on the control. For instance, it has been shown in [3] that reachability for extended counter machines becomes decidable when the control is *flat*. We consider here the iteration of some affine functions with such a flat control. More precisely, given arbitrary affine functions $f_1, \ldots, f_p$, we assume that $f_1, \ldots, f_p$ are applied in a fixed order: first f_1 is applied an arbitrary number of times and then f_1 is not used again, then f_2 is applied an arbitrary number of times and then f_2 is not used again, etc ... Under these conditions, we prove that reachability is decidable for $p = 1$ (Section 2.1), for $p = 2$ (Section 3) and undecidable for some p (Section 4).

1. Preliminaries

1.1. Introduction

Notation: $\mathcal{A}_k(\mathbb{N})$ is the set of affine functions $f : \mathbb{N}^k \to \mathbb{N}^k$ such that $f(X) = AX + B$ where A is a matrix with nonnegative integer components and B is a vector in $\mathbb{Z}^k$.
Notation: $M_k(\mathbb{N})$ is the set of matrices of size $k \times k$ with nonnegative integer components.
Notation: E_i^k denotes the vector in $\mathbb{N}^k$ such that the j^{th} coordinate is 1 if $j = i$, 0 otherwise.

We consider the following decision problem: given $f_1, \ldots, f_p \in \mathcal{A}_k(\mathbb{N})$, given $U, V \in \mathbb{N}^k$, does V belong to $\{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \ldots, n_p \in \mathbb{N}\}$?

Example 1.1. We consider

$$f(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} X + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Petri net extension	affine function
Petri net	$A = Id$
Double Petri net	$Id \leq A \leq 2Id$
Generalized Transfer Petri net	$0 \leq A \quad \forall j A_j \neq 0$
Reset Petri net	$0 \leq A \leq Id$

FIGURE 1. Example of transition classes which can be modeled by $f(X) = AX + B$, cf [7]

Then $f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-1 \\ b \\ c+b \end{pmatrix}$ if $a \geq 1$, $\quad f^n\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-n \\ b \\ c+nb \end{pmatrix}$ if $a \geq n$.

If $a = 0$, $f\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is not defined because $f(X)$ is not allowed to take negative values.

Then, given $U = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$, where $a, b.c \in \mathbb{N}$, deciding if V belongs to $\{f^n(U) \mid n \in \mathbb{N}\}$ is equivalent to decide if $c = ab$.

This type of transition functions ($f(X) = AX + B$) is more general than many other transitions which are considered in the literature (Figure 1). For example, if $A = Id$, we obtain a Petri net. But, on the other hand, there is a strong restriction on the control: $f_1, \ldots, f_p$ have to be iterated in a fixed order, which is not the case in Petri nets. Such a control corresponds to the notion of "flat automata" in [3].

1.2. Useful Properties

We consider a partial order on vectors of integers in the following way:

Definition 1.2. Let $U, V \in \mathbb{N}^k$, $U \leq V$ if and only if U's coordinates are all smaller than those of V.

The relation $\leq$ is a well quasi-order([5]). Moreover, it is easy to verify that for every $f \in \mathcal{A}_k(\mathbb{N})$ ($f(X) = AX + B$), f is "increasing" for $\leq$: if $U_1 \leq U_2$ then $f(U_1) \leq f(U_2)$. This last property uses that A has only non-negative components but does not require anything on B's components. That is why we can allow B to have negative components.

NOTATION: We write $V \tilde{>} U$ if at least one of V's coordinates is greater than the corresponding one of U, i.e., if $U \not\geq V$.
Clearly, if $V_1 \geq V_2$ and $V_2 \tilde{>} U$ then $V_1 \tilde{>} U$.

NOTATION: The *size* of V, written $|V|$, is the sum of the absolute values of its coordinates.

2. Decidability Results

2.1. Decidability for one function

Since $\leq$ is a well quasi-order and affine functions are increasing for $\leq$, the problem stated in introduction is decidable when $p = 1$.

Theorem 2.1. *Given $f \in \mathcal{A}_k(\mathbb{N})$, $U, V \in \mathbb{N}^k$, $V \in \{f^n(U) \mid n \in \mathbb{N}\}$ is decidable.*

Proof: We consider the following sequence: $U, f(U), f^2(U), \ldots, f^n(U), \ldots$.
If there exists N such that $f^N(U) \not\geq 0$, then the sequence is finite and $V \in \{f^n(U) \mid n \in \mathbb{N}\}$ is decidable.
Otherwise there exist $N_1 < N_2$ such that $f^{N_1}(U) \leq f^{N_2}(U)$ because $\leq$ is a well quasi-order. Let $l = N_2 - N_1$. For all $0 \leq i \leq l-1$,

$$f^{N_1}(U) \leq f^{N_2}(U) \Rightarrow f^i(f^{N_1}(U)) \leq f^i(f^{N_2}(U))$$

since f is increasing, thus $f^{N_1+i}(U) \leq f^{N_2+i}(U)$.

So we have:

$$\begin{cases} f^{N_1}(U) & \leq & f^{N_1+l}(U) & \leq & \cdots & \leq & f^{N_1+kl}(U) & \leq & \cdots \\ f^{N_1+1}(U) & \leq & f^{N_1+1+l}(U) & \leq & \cdots & \leq & f^{N_1+1+kl}(U) & \leq & \cdots \\ & & & & \vdots & & & & \\ f^{N_1+(l-1)}(U) & \leq & f^{N_1+(l-1)+l}(U) & \leq & \cdots & \leq & f^{N_1+(l-1)+kl}(U) & \leq & \cdots \end{cases}$$

- Either for one of these sequences two consecutive terms are equal, then this sequence becomes constant.
- Or all these sequences are strictly increasing.

In concrete terms, we proceed on the following way: we compute successively $U, f(U), f^2(U), \ldots, f^n(U), \ldots$ (and at each step we check the equality to V) until we find $f^{N_1}(U) \leq f^{N_2}(U)$, unless the sequence is finite.
In this case, the algorithm stops, there is no n such that $V = f^n(U)$.
In the other case (if we find $f^{N_1}(U) \leq f^{N_2}(U)$), we compute successively

$$f^{N_1}(U) \leq f^{N_1+l}(U) \leq f^{N_1+2l}(U) \leq \cdots \leq f^{N_1+kl}(U) \leq \cdots$$

until either one of the coordinates is greater than the corresponding one of V, or the sequence becomes constant (if two consecutive terms are equal). In the second case, $\{f^n(U) \mid n \in \mathbb{N}\}$ is a finite set: we check if $V \in \{f^n(U) \mid n \in \mathbb{N}\}$. Otherwise (if one of the coordinates is greater than the corresponding one of V), we try again with

$$f^{N_1+1}(U) \leq f^{N_1+1+l}(U) \leq f^{N_1+1+2l}(U) \leq \cdots \leq f^{N_1+1+kl}(U) \leq \cdots$$

and so on with the l sequences. $\square$

Nevertheless, even if the algorithm is effective, we have no bound regarding its complexity. There is another algorithm which is more complicated, yielding however an explicit upper bound which is a tower of exponentials whose height depends linearly on k, the number of registers, see [4]. The main idea of this algorithm is to associate to the matrix A its dependence graph G_A (there is an edge between i and j in G_A, labelled by $A_{i,j}$ if and only if $A_{i,j} \neq 0$). Then, we break down the graph G_A into strongly connected components and study precisely the behavior of each component when the affine function is iterated.

3. Decidability for two functions

A result similar to Theorem 2.1 also holds for the composition of two functions in a given order.

Theorem 3.1. *Given $f, g \in \mathcal{A}_k(\mathbb{N})$, $U, V \in \mathbb{N}^k$, then $V \in \{g^n f^m(U) \mid n, m \in \mathbb{N}\}$ is decidable.*

Proof: we proceed with three steps:

(1) For m fixed, we compute n_m such that either $(g^n f^m(U))_{n \geq n_m}$ is not defined or $(g^n f^m(U))_{n \geq n_m}$ is periodic or $\forall n \geq n_m, g^n f^m(U) \tilde{>} V$.
(2) For n fixed, we show that $V \in \{g^n f^m(U) \mid m \in \mathbb{N}\}$ is decidable.
(3) We combine the first two steps with a kind of cross-ruling.

3.1. Behavior of $(g^n f^m(U))_n$, m fixed

We just give here a refinement of Theorem 2.1.

Theorem 3.2. *Given $f, g \in \mathcal{A}_k(\mathbb{N})$, $U, V \in \mathbb{N}^k$, given $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ (computable) such that:*

(1) *either $\forall n \geq n_m \quad g^n f^m(U) \tilde{>} V$,*
(2) *or there exist N_m, l_m (computable) such that*
$\forall n \geq n_m \quad \forall i < l_m \quad g^{N_m+i+nl_m} f^m(U) = g^{N_m+i+n_m l_m} f^m(U)$,
i.e., there exist N'_m, l_m such that $\forall n \geq N'_m \quad \exists i < l_m \quad g^n f^m(U) = g^{N'_n+i} f^m(U)$,
(3) *or $g^{n_m} f^m(U) \not\geq 0$, thus for all $n \geq n_m$, $g^n f^m(U)$ is not defined.*

Proof: We just refine the proof of Theorem 2.1:
Let $U' = f^m(U)$, we consider the following sequence: $U', f(U'), f^2(U'), \ldots, f^n(U'), \ldots$
Either there exists n_m such that $g^{n_m} f^m(U') \not\geq 0$ (case 3).
Or there exist $N_1 < N_2$ such that $g^{N_1}(U') \leq g^{N_2}(U')$. In this case, we consider the following sequences:

$$\begin{cases} g^{N_1}(U') \leq g^{N_1+l}(U') \leq \cdots \leq g^{N_1+kl}(U') \leq \cdots \\ g^{N_1+1}(U') \leq g^{N_1+1+l}(U') \leq \cdots \leq g^{N_1+1+kl}(U') \leq \cdots \\ \vdots \\ g^{N_1+(l-1)}(U') \leq g^{N_1+(l-1)+l}(U') \leq \cdots \leq g^{N_1+(l-1)+kl}(U') \leq \cdots \end{cases}$$

- Either for one of these sequences, two consecutive terms are equal, thus this sequence is stabilized which implies that all these sequences are stabilized (case 2).
- Or all these sequences are increasing and we compute each of them until one of the coordinates of a term of the sequence is greater than the corresponding one of V (case 1).

□

3.2. Behavior of $(g^n f^m(U))_m$, n fixed

To control $(g^n f^m(U))_m$, we first establish a very useful lemma.

Lemma 3.3. *Let $f, g \in \mathcal{A}_k(\mathbb{N})$, $U \in \mathbb{N}^k$, let $l \in \mathbb{N}, l > 0$. We consider the following sequence: $g^n(U), g^n f^l(U), g^n f^{2l}(U), \ldots, g^n f^{ml}(U), \ldots$ (n fixed), then either the sequence is eventually stabilized or it is never constant more than $k+1$ steps. More formally, this sequence has the following property:*

$$\forall m_0 \quad (\, g^n f^{m_0 l}(U) = g^n f^{(m_0+1)l}(U) = \cdots = g^n f^{(m_0+k+1)l}(U) \\ \Longrightarrow \forall m \geq m_0 \quad g^n f^{ml}(U) = g^n f^{m_0 l}(U)\,).$$

Proof: The proof of this lemma uses elementary properties of algebra.
Assume $g^n f^{m_0 l}(U) = g^n f^{(m_0+1)l}(U) = \cdots = g^n f^{(m_0+k+1)l}(U)$. Let us show that $g^n f^{(m_0+k+2)l}(U) = g^n f^{m_0 l}(U)$, which proves by induction that

$$\forall m \geq m_0 \quad g^n f^{ml}(U) = g^n f^{m_0 l}(U).$$

$f^{(m_0+1)l}(U) - f^{m_0 l}(U), \ldots, f^{(m_0+k+1)l}(U) - f^{m_0 l}(U)$ are $k+1$ vectors of the k-dimensional vector space $\mathbb{Q}^k$, so they are linearly dependent in $\mathbb{Q}$, thus they are linearly dependent in $\mathbb{Z}$ (by multiplying by an appropriate integer). Thus

$$\exists q_1, \ldots, q_{k+1} \in \mathbb{Z} \quad \sum_{i=1}^{k+1} q_i (f^{(m_0+i)l}(U) - f^{m_0 l}(U)) = 0.$$

Let N denote the greatest i such that q_i is different from 0 $(1 \leq N \leq k+1)$.

$$\text{Then} \quad q_N (f^{(m_0+N)l}(U) - f^{m_0 l}(U)) + \sum_{i=1}^{N-1} q_i (f^{(m_0+i)l}(U) - f^{m_0 l}(U)) = 0.$$

Applying f^{k+2-N} yields:

$$q_N (f^{(m_0+k+2)l}(U) - f^{(m_0+k+2-N)l}(U)) \\ + \sum_{i=1}^{N-1} q_i (f^{(m_0+i+k+2-N)l}(U) - f^{(m_0+k+2-N)l}(U)) = 0,$$

so,

$$q_N g^n(f^{(m_0+k+2)l}(U) - f^{(m_0+k+2-N)l}(U)) \\ + \underbrace{\sum_{i=1}^{N-1} q_i g^n(f^{(m_0+i+k+2-N)l}(U) - f^{(m_0+k+2-N)l}(U))}_{=0} = 0.$$

since (by hypothesis)

$$\forall 1 \le i \le N-1 \quad g^n f^{(m_0+\overbrace{i+k+2-N}^{1\le \quad \le k+1})l}(U) = g^n f^{(m_0+\overbrace{k+2-N}^{1\le \quad \le k+1})l}(U).$$

Conclusion: $g^n f^{(m_0+k+2)l}(U) = g^n f^{m_0 l}(U)$. □

Definition 3.4. A sequence is k-*almost increasing* if this sequence is non-decreasing and if it is never constant more than k steps.

Lemma 3.5. *For n fixed, $V \in \{g^n f^m(U) \mid m \in \mathbb{N}\}$ is decidable.*

Proof: We compute the sequence $(f^m(U))_m$ until:
either there exists N such that $f^N(U) \not\geq 0$, the sequence stops and we test if $V \in \{g^n f^m(U) \mid m < N\}$,
or there exist N, l such that $f^N(U) \le f^{N+l}(U)$, then

$$\begin{cases} f^N(U) \le f^{N+l}(U) \le \cdots \le f^{N+kl}(U) \le \cdots \\ f^{N+1}(U) \le f^{N+1+l}(U) \le \cdots \le f^{N+1+kl}(U) \le \cdots \\ \vdots \\ f^{N+(l-1)}(U) \le f^{N+(l-1)+l}(U) \le \cdots \le f^{N+(l-1)+kl}(U) \le \cdots \end{cases}$$

which implies $\begin{cases} g^n f^N(U) \le g^n f^{N+l}(U) \le \cdots \le g^n f^{N+kl}(U) \le \cdots \\ g^n f^{N+1}(U) \le g^n f^{N+1+l}(U) \le \cdots \le g^n f^{N+1+kl}(U) \le \cdots \\ \vdots \\ g^n f^{N+(l-1)}(U) \le g^n f^{N+(l-1)+l}(U) \le \cdots \le g^n f^{N+(l-1)+kl}(U) \le \cdots \end{cases}$

Using Lemma 3.3 (with $U' = f^N(U)$), each sequence is:

- either $(k+1)$-almost increasing,
- or eventually stabilized.

Hence, for each of these sequences, there exists m_i (computable) such that $\forall m \ge m_i$:

- either $g^n f^{N+i+ml}(U) \widetilde{>} V$, thus each term of the sequence $(g^n f^{N+i+ml}(U))_{m \ge m_i}$ is different to V,
- or $g^n f^{N+i+ml}(U) = g^n f^{N+i+m_i l}(U)$.

In concrete terms, we compute each term W of $(g^n f^{N_1+i+ml}(U))_m$ until:

- either $W \widetilde{>} V$ which implies $V \notin \{g^n f^m(U) \mid m \in \mathbb{N}\}$,

- or $k+2$ consecutive terms have the same value, it means that the sequence is stabilized, which implies $V \notin \{g^n f^m(U) \mid m \in \mathbb{N}\}$,
- $W = V$ which implies $V \in \{g^n f^m(U) \mid m \in \mathbb{N}\}$.

One of these 3 cases is bound to happen. $\square$

3.3. Proof of Theorem 3.1

We need to control $g^n f^m(U)$ when n and m vary at the same time. We first establish a technical lemma used to initialize the proof of Theorem 3.1.

Lemma 3.6. *Let $f, g \in \mathcal{A}_k(\mathbb{N})$, $U \in \mathbb{N}^k$, let $N_1, l \in \mathbb{N}^k$, $N_1, l > 0$.*
If $f^{N_1}(U) \leq f^{N_1+l}(U) \leq f^{N_1+2l}(U) \leq \cdots \leq f^{N_1+ml}(U) \leq \cdots$

$$\text{and if } \begin{cases} \exists n_0 & g^{n_0} f^{N_1}(U) \ngeq 0 \\ \exists n_1 & g^{n_1} f^{N_1+l}(U) \ngeq 0 \\ & \vdots \\ \exists n_k & g^{n_k} f^{N_1+kl}(U) \ngeq 0 \end{cases}$$

where n_i is the smallest n such that $g^n f^{N_1+il}(U) \ngeq 0$, then
(1) $n_0 \leq n_1 \leq \cdots \leq n_k$,
(2) $\forall m \geq k \quad g^{n_k} f^{N_1+ml}(U) \ngeq 0$

Proof: The proof of (1) is easy: $g^{n_0-1} f^{N_1+l}(U) \geq g^{n_0-1} f^{N_1}(U) \geq 0$ thus $n_1 \geq n_0$ and so on.
The proof of (2) uses again elementary results of algebra.
$f^{N_1}(U), \ldots, f^{N_1+kl}(U)$ are $k+1$ vectors linearly dependent in $\mathbb{Q}$ thus in $\mathbb{Z}$, thus:

$$\exists p_0, \ldots, p_k \in \mathbb{Z} \quad \sum_{i=0}^{k} p_i f^{N_1+il}(U) = 0.$$

Let N denote the greatest i such that p_i is not equal to 0. Assume $p_N > 0$ (if it is not the case, multiply the equation by -1).
Let $I = \{i \mid p_i > 0, i \neq N\}$, $J = \{j \mid p_j < 0\}$. For $j \in J$, let $q_j = -p_j > 0$.

$$p_N f^{N_1+Nl}(U) + \sum_{i \in I} p_i f^{N_1+il}(U) = \sum_{j \in J} q_j f^{N_1+jl}(U) \tag{1}$$

which implies

$$p_N f^{N_1+Nl}(U) \leq \sum_{j \in J} q_j f^{N_1+jl}(U)$$

applying $f^{(k+1-N)l}$ yields:

$$p_N f^{N_1+(k+1)l}(U) \leq \sum_{j \in J} q_j f^{N_1+\overbrace{(k+1-N+j)}^{\leq k} l}(U)$$

$$\Rightarrow p_N g^{n_k} f^{N_1+(k+1)l}(U) \leq \sum_{j\in J} q_j \underbrace{g^{n_k} f^{N_1+(k+1-N+j)l}(U)}_{i_0^{\text{th}} \text{ coordonate } <0} \tag{2}$$

Applying g^{n_k} to $f^{N_1}(U) \leq \cdots \leq f^{N_1+kl}(U)$ yields

$$g^{n_k} f^{N_1}(U) \leq \cdots \leq g^{n_k} f^{N_1+kl}(U).$$

Since $g^{n_k} f^{N_1+kl}(U) \ngeq 0$, there exists i_0 such that the i_0^{th} coordinate of $g^{n_k} f^{N_1+kl}(U)$ is negative, so for all $1 \leq i \leq k$, the i_0^{th} coordinate of $g^{n_k} f^{N_1+il}(U)$ is negative. Thus (using 2), the i_0^{th} coordinate of $g^{n_k} f^{N_1+(k+1)l}(U)$ is negative, so:

$$g^{n_k} f^{N_1+(k+1)l}(U) \ngeq 0.$$

Hence $\forall m \geq k \quad g^{n_k} f^{N_1+ml} \ngeq 0$ by induction. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: We consider the sequence $(f^n(U))_n$.

- Either there exists N such that $f^N(U) \ngeq 0$. It is the easy case:

  $$\{g^n f^m(U) \mid n, m \in \mathbb{N}\} = \{g^n f^m(U) \mid n \in \mathbb{N}, m < N\}$$

  For all $m < N$, we test if $V \in \{g^n f^m(U) \mid n \in \mathbb{N}\}$.

- Or there exist N, l such that $f^N(U) \leq f^{N+l}(U) \leq \cdots \leq f^{N+kl}(U) \leq \cdots$. Let us show that $V \in \{g^n f^{N+ml}(U) \mid n, m \in \mathbb{N}\}$ is decidable, which proves that $V \in \{g^n f^{N+i+ml}(U) \mid n, m \in \mathbb{N}\}$ is decidable for all $1 \leq i \leq l$ (take $U' = f^i(U)$). Thus it implies $V \in \{g^n f^m(U) \mid n, m \in \mathbb{N}\}$ is decidable.
  We first consider the sequence $(g^n f^N(U))_n$. Either there exists n_0 such that $g^{n_0} f^N(U) \ngeq 0$, then we consider the sequence $(g^n f^{N+l}(U))_n$, or there exist n_0, l_0 such that $g^{n_0} f^N(U) \leq g^{n_0+l} f^N(U) \leq \cdots$, thus $\forall n \quad g^n f^N(U) \geq 0$. We repeat this (at most k times) until we obtain Case 1 or Case 2.

  Case 1:

  $$\begin{array}{c} \exists n_0 \quad g^{n_0} f^{N_1}(U) \ngeq 0 \\ \exists n_1 \quad g^{n_1} f^{N_1+l}(U) \ngeq 0 \\ \vdots \\ \exists n_k \quad g^{n_k} f^{N_1+kl}(U) \ngeq 0 \end{array}$$

  Applying Lemma 3.6 yields $\forall m \geq k \quad g^{n_k} f^{N_1+ml}(U) \ngeq 0$. Thus, it is enough to test if $V \in \{g^n f^{N+ml}(U) \mid m \in \mathbb{N}\}$ for all $n < n_k$ (see figure 2).

  Case 2: There exists $i \leq k$ such that $\forall n \quad g^n f^{N+il}(U) \geq 0$. We rename $N := N + il$. So,

  $$\forall n \, \forall m \quad g^n f^{N+ml}(U) \geq g^n f^N(U) \geq 0. \tag{3}$$

  Let $K = (k+1)|V|$. We consider $(g^n f^{N+Kl}(U))_n$. Applying Theorem 3.2 yields 3 cases:

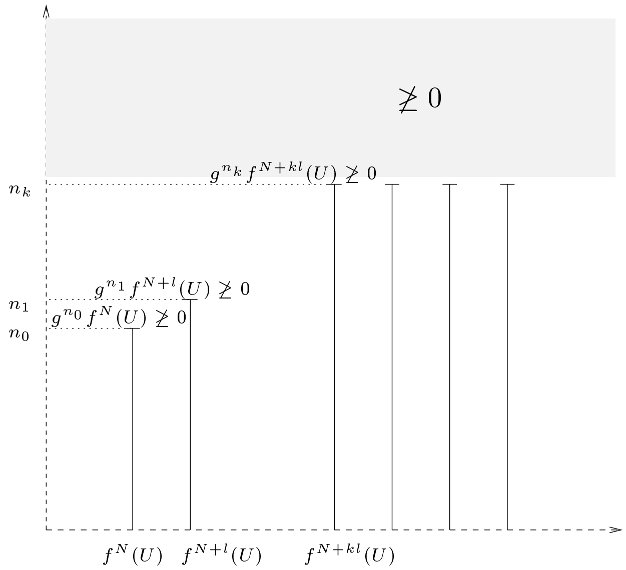

FIGURE 2. Case 1

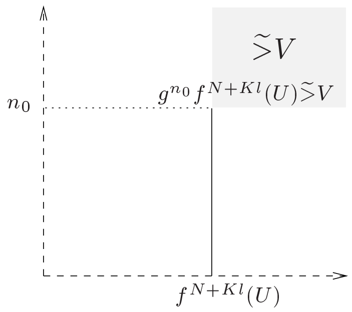

FIGURE 3. Case 3

(1) There exists n_0 such that $g^{n_0} f^{N+Kl}(U) \not\gtrsim 0$ which is inconsistent with Equation 3.
(2) There exists n_0 such that $\forall n \geq n_0 \quad g^n f^{N+Kl}(U)\widetilde{>}V$. In this case, (see Figure 3)

$$\forall n \geq n_0 \, \forall m \geq K \quad g^n f^{N+ml}(U) \geq g^n f^{N+Kl}(U)\widetilde{>}V. \qquad (4)$$

Thus we test if $V \in \{g^n f^{N+ml}(U) \mid n \in \mathbb{N}\}$ for all $m < K$ and we test if $V \in \{g^n f^{N+ml}(U) \mid m \in \mathbb{N}\}$ for all $n < n_0$ (which is decidable by Lemma 3.5 with $U' = f^N(U)$ and $f' = f^l$).

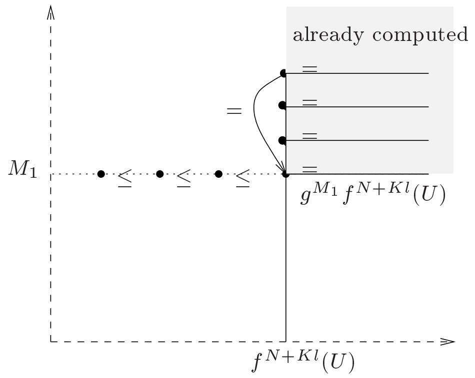

FIGURE 4. Case 3

(3) There exist M_1, l_1, such that (see Figure 4)

$$\forall n \geq M_1 \quad \exists i < l_1 \quad g^n f^{N+Kl}(U) = g^{M_1+i} f^{N+Kl}(U) \leq V. \tag{5}$$

In this last case, we consider then:

$$g^{M_1} f^{N+Kl}(U) \geq g^{M_1} f^{N+(K-1)l}(U) \geq \cdots \geq g^{M_1} f^{N}(U).$$

Applying Lemma 3.3 yields:

- either this sequence is already stabilized, so
$$\forall m \geq K \quad g^{M_1} f^{N+ml}(U) = g^{M_1} f^{N+Kl}(U), \tag{6}$$
- or this sequence is $(k+1)$-almost increasing (at least until the K^{th} term of the sequence), so:
$$|g^{M_1} f^{N+Kl}(U)| \geq \frac{K}{k+1} + |g^{M_1} f^{N}(U)| = |V| + |g^{M_1} f^{N}(U)|$$
$$\Rightarrow \quad g^{M_1} f^{N+Kl}(U) \widetilde{>} V$$
which is inconsistent with Equation 5.

Thus we claim that

$$\forall n \geq M_1 \, \forall m \geq K \quad g^n f^{N+ml}(U) = g^n f^{N+Kl}(U). \tag{7}$$

Indeed, let $n \geq M_1$, $m \geq K$,

$$g^n f^{N+ml}(U) = g^{n'}(g^{M_1} f^{N+ml}(U))$$
$$\overset{\text{(Equation 6)}}{=} g^{n'}(g^{M_1} f^{N+Kl}(U)) = g^n f^{N+Kl}(U).$$

$$\text{Thus } \{g^n f^{N+ml}(U) \mid n,m \in \mathbb{N}\} = \bigcup_{n \le M_1} \{g^n f^{N+ml}(U) \mid m \in \mathbb{N}\} \cup \bigcup_{m \le K} \{g^n f^{N+ml}(U) \mid n \in \mathbb{N}\}.$$

Thus $V \in \{g^n f^{N+ml}(U) \mid n,m \in \mathbb{N}\}$ is decidable which completes the proof. $\square$

4. Undecidability in the general case

In the general case: if an arbitrary number p of functions are iterated in a fixed order, $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \dots, n_p \in \mathbb{N}\}$ becomes undecidable.

Theorem 4.1. *Given $f_1, \dots, f_p \in \mathcal{A}_k(\mathbb{N})$, $U, V \in \mathbb{N}^k$, then $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \dots, n_p \in \mathbb{N}\}$ is undecidable.*

We present here the sketch of the proof, lemmas needed for the proof are developed in sections 4.1, 4.2 and 4.3, we conclude with the proof in section 4.4. Section 4.5 presents a refinement of Theorem 4.1.

Sketch proof: We start from Theorem 3.10 of [8] which is a stronger form of *Hilbert's Tenth Problem*, shown to be equivalent:

Theorem 4.2. *[8] There is a polynomial $\mathcal{P}(x, y_1, \dots, y_{N_0})$ with integer coefficients such that no algorithm exists for deciding whether or not an arbitrary equation on the form*

$$\mathcal{P}(x_0, y_1, \dots, y_{N_0}) = 0$$

where x_0 is a positive integer, has a solution in nonnegative integers $y_1, \dots, y_{N_0}$.

From now on, $\mathcal{P}$ is reserved for the polynomial mentioned in Theorem 4.2 and N_0 is reserved for the degree of $\mathcal{P}$ minus 1.

A straightforward corollary is the following:

Corollary 4.3. *There is no algorithm for deciding whether or not an arbitrary equation on the form $Q(x_1, \dots, x_{N_0}) = 0$ where Q is a polynomial with integer coefficients and N_0 variables, has a solution in nonnegative integers $x_1, \dots, x_{N_0}$.*

The idea of the proof is to establish a correspondence between each polynomial P of N_0 variables and a system of affine functions (computable from P) which simulates the computation of $P(x_1, \dots, x_{N_0})$ for each tuple of N_0 integers.

We establish the correspondence in the following way:

(1) The first functions C_{m_j,x_i} create a tuple of integers, (see Lemma 4.4)
(2) The functions $f_{m_j,i}$ compute the monomials of P, (Lemma 4.5)
(3) The functions $A^+_{m_i}, A^-_{m_i}, D$ compute P, keeping positiveness at each step, (Lemma 4.7)

Then we conclude with Corollary 4.3.

Notation: If m is the monomial $a_{\alpha_1,\dots,\alpha_{N_0}} x_1^{\alpha_1} \dots x_{N_0}^{\alpha_{N_0}}$, let N_m denote $2(\alpha_1 + \dots + \alpha_{N_0}) - 1$ and $K_m = \frac{N_m - 1}{2}$.

4.1. PREPARATION STEP: GENERATION OF TUPLE OF INTEGERS

Lemma 4.4. *For every monomial $m = a_{\alpha_1,\dots,\alpha_{N_0}} x_1^{\alpha_1} \dots x_{N_0}^{\alpha_{N_0}}$, there exist N_0 affine functions $C_{m,x_1},\dots,C_{m,x_{N_0}} \in \mathcal{A}_{N_m}(\mathbb{N})$ such that $\forall\, a_1,\dots,a_{N_0} \in \mathbb{N}$,*

$$C^{a_{N_0}}_{m,x_{N_0}} \cdots C^{a_1}_{m,x_1} O_{N_m} = \left.\begin{pmatrix} \left.\begin{matrix} a_1 \\ \vdots \\ a_1 \end{matrix}\right\} \alpha_1 \\ \vdots \\ \left.\begin{matrix} a_{N_0} \\ \vdots \\ a_{N_0} \end{matrix}\right\} \alpha_{N_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} N_m$$

where O_{N_m} is the null vector of size N_m. More formally, if $\alpha_1 + \dots + \alpha_j \leq i < \alpha_1 + \dots + \alpha_j + \alpha_{j+1}$, the i^{th} coordinate of $C^{a_{N_0}}_{m,x_{N_0}} \dots C^{a_1}_{m.x_1} O_{N_m}$ is equal to a_j.

Proof of Lemma 4.4:
We build N_0 functions $C_{m,x_j} = Id + V_{m,x_j}$ where the i^{th} coordinate of V_{m,x_j} is 1 if $\alpha_1 + \dots + \alpha_j \leq i < \alpha_1 + \dots + \alpha_j + \alpha_{j+1}$, 0 otherwise. □

4.2. MULTIPLICATIONS FUNCTIONS

4.2.1. *Definitions*

We generalize Example 1.1 : if i, j, l, N are distinct integers ($N \geq i, j, l$), there exists a function $f_{i.j\to l} : \mathbb{N}^N \to \mathbb{N}^N$ such that $f_{i.j\to l}$ computes the product of the i^{th} coordinate by the j^{th} and writes the result on the l^{th} coordinate :

$$\exists n \quad f^n_{i.j\to l}(x_i E_i^N + x_j E_j^N) = x_j E_j^N + d E_l^N \text{ iff } d = x_i x_j.$$

4.2.2. *Computation of the monomials*

Lemma 4.5. *For every monomial $m = a_{\alpha_1,\dots,\alpha_{N_0}} x_1^{\alpha_1} \dots x_{N_0}^{\alpha_{N_0}}$, there exist K_m affine functions $f_{m,1},\dots,f_{m,K_m} \in \mathcal{A}_{N_m}(\mathbb{N})$ such that $\forall\, a_1,\dots,a_{K_m+1} \in \mathbb{N}$,*

$$\begin{matrix} \exists k_1,\dots,k_{K_m} \\ \exists c_1,\dots,c_{K_m} \end{matrix} \qquad f^{k_{K_m}}_{m,K_m} \cdots f^{k_1}_{m,1} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_{K_m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ 0 \\ \vdots \\ 0 \\ c_2 \\ \vdots \\ c_{K_m} \\ b \end{pmatrix} \textit{ iff } \quad b = a_1 \cdots a_{K_m+1}.$$

Proof of Lemma 4.5 Let $f_{m,1} = f_{1,2\to K_m+2}$, $f_{m,i} = f_{i+1,K_m+i\to K_m+i+1}$ for $2 \leq i \leq K_m$. □

Together with Lemma 4.4, we obtain a straightforward corollary:

Corollary 4.6. *For every monomial* $m = a_{\alpha_1,\ldots,\alpha_{N_0}} x_1^{\alpha_1} \ldots x_{N_0}^{\alpha_{N_0}}$*, there are affine functions* $f_{m,1},\ldots,f_{m,K_m}, C_{m,x_1},\ldots,C_{m,x_{N_0}} \in \mathcal{A}_{N_m}(\mathbb{N})$ *such that*

$$\left\{ b \in \mathbb{N} \;\middle|\; \begin{array}{c} \exists k_1,\cdots,k_{K_m} \in \mathbb{N}, \\ \exists c_1,\ldots,c_{K_m} \in \mathbb{N}, \\ \exists a_1,\ldots,a_{N_0} \in \mathbb{N}, \end{array} \quad f_{m,K_m}^{k_{K_m}} \cdots f_{m,1}^{k_1} C_{m,x_{N_0}}^{a_{N_0}} \cdots C_{m,x_1}^{a_1} O_{N_m} = \begin{pmatrix} 0 \\ c_1 \\ 0 \\ \vdots \\ 0 \\ c_2 \\ \vdots \\ c_{K_m} \\ b \end{pmatrix} \right\}$$
$$= \left\{ n_1^{\alpha_1} \cdots n_{N_0}^{\alpha_{N_0}} \mid n_1,\ldots,n_{N_0} \in \mathbb{N} \right\}$$

4.3. Computation of P

Lemma 4.7. *Let P be a polynomial of N_0 variables, whose constant term is 0. There exists k, there exist affine functions $f_1,\ldots,f_n \in \mathcal{A}_k(\mathbb{N})$ such that*

$$\{b \in \mathbb{N} \mid \exists k_1,\cdots k_n \quad f_1^{k_1}\ldots f_n^{k_n} 0_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix}\} = \{P(n_1,\cdots,n_{N_0}) \mid n_1,\ldots,n_{N_0} \in \mathbb{N}\}$$

Proof of Lemma 4.7: Let P be a polynomial of N_0 variables, whose constant term is 0.

For each monomial m of P, we construct affine functions

$$f_{m,1},\ldots,f_{m,K_m}, C_{m,x_1},\ldots,C_{m,x_{N_0}} \in \mathcal{A}_{N_m}(\mathbb{N})$$

as described in Corollary 4.6.

We establish a one-to-one correspondence between each monomial m of P and a "block" $\mathcal{B}_m$ whose size is $N_m \times N_m$. From now on, we will only consider block matrix and block vectors, of the form:

$$\left(\begin{array}{cccc|c} \mathcal{B}_{m_1} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{m_2} & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \mathcal{B}_{m_L} & 0 \\ \hline 0 & \cdots & \cdots & 0 & 1 \end{array}\right), \quad \begin{pmatrix} \mathcal{V}_{m_1} \\ \mathcal{V}_{m_2} \\ \vdots \\ \mathcal{V}_{m_L} \\ a \end{pmatrix}$$

where L is the number of monomials.

We transform the functions $f_{m,i}$ and C_{m,x_i} into functions $\tilde{f}_{m,i}$ and $\tilde{C}_{m,x_i}$ in $\mathcal{A}_{N+1}(\mathbb{N})$, where $N = \sum_{i=1}^{K} N_{m_i}$, in the following way:

If $C_{m,x_j} = Id + V_{m,x_j}$ then $\tilde{C}_{m,x_j} = Id + \begin{pmatrix} \mathcal{V}_{m_1} \\ \mathcal{V}_{m_2} \\ \vdots \\ \mathcal{V}_{m_K} \\ 0 \end{pmatrix}$ where $\mathcal{V}_{m_j} = \begin{cases} V_{m,x_j} & \text{if } m_j = m, \\ 0 & \text{otherwise.} \end{cases}$

If $f_{m,i}(X) = A_{m,i}X + V_{m,i}$, then $\tilde{f}_{m,i}(X) = \left(\begin{array}{cccc|c} \mathcal{B}_{m_1} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{m_2} & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \mathcal{B}_{m_K} & 0 \\ \hline 0 & \cdots & \cdots & 0 & 1 \end{array}\right) X + \begin{pmatrix} \mathcal{V}_{m_1} \\ \mathcal{V}_{m_2} \\ \vdots \\ \mathcal{V}_{m_K} \\ 0 \end{pmatrix}$

where $\begin{cases} \mathcal{B}_{m_j} = A_{m,i} \quad \text{and} \quad \mathcal{V}_{m_j} = V_{m,i} & \text{if } m_j = m, \\ \mathcal{B}_{m_j} = Id \quad \text{and} \quad \mathcal{V}_{m_j} = 0 & \text{otherwise.} \end{cases}$

Besides, for each m, let

$$I_m = \{i_m, i_m + 2, i_m + 3, \ldots, i_m + K_m - 1, i_m + K_m, i_m + N_m - 1\}$$

where i_m is the coordinate of the first line of block $\mathcal{B}_m$. This set corresponds to the coordinates we have to test: after the iterations of $f_{m,1}$, then $f_{m,2}$, then, ..., then f_{m,K_m}, we obtain the product of the $K_m + 1$ first coordinates if and only if the coordinates whose numbers are in $I_m - \{i_m + N_m - 1\}$ are equal to zero.

We split the monomials of P into 2 categories:

$$I = \{m \mid a_{\alpha_1,\ldots,\alpha_{N_0}} \geq 0\}, \quad J = \{m \mid a_{\alpha_1,\ldots,\alpha_{N_0}} < 0\}$$

($a_{\alpha_1,\ldots,\alpha_{N_0}}$ is the coefficient of the monomial $m = a_{\alpha_1,\ldots,\alpha_{N_0}} x_1 \ldots x_{N_0}$.)

For every m in I (resp. in J), we define:

$$A_m^+ \text{ (resp. } A_m^-) = Id - E_{i_m+2K_m}^{N+1} + a_{\alpha_1,\ldots,\alpha_{N_0}} E_{N+1}^{N+1}.$$

$i_m + 2K_m$ is the number of the line where the multiplication's result of $x_1^{n_1} \cdots x_{N_0}^{n_{N_0}}$ is written.

The aim of this distinction between $m \in I$ and $m \in J$ is to iterate first all the A_m^+ (we first add the nonnegative terms), then the A_m^-. In this way, if $P(a_1, \ldots, a_{N_0}) \geq 0$, we make sure that during the computation of $P(a_1, \ldots, a_{N_0})$, the intermediate vectors remain nonnegative at each iteration.

Let D denote a diagonal matrix such that if $i \in \bigcup_m I_m$ then $D_{i,i} = 1$ else $D_{i,i} = 0$. Actually, D assigns irrelevant coordinates to 0.
Applying Corollary 4.6, we obtain the following equivalence:

$$\exists x_1 \ldots \exists x_{N_0} \quad P(x_1, \ldots, x_{N_0}) = a \quad a \geq 0$$

if and only if

$$\exists n_D \exists n^-_{A_{m_1}} \dots \exists n^-_{A_{m_M}} \exists n^+_{A_{m_1}} \dots \exists n^+_{A_{m_{M'}}}$$
$$\exists n_{m_1,1} \dots \exists n_{m_1,K_{m_1}} \dots \exists n_{m_L,1} \dots \exists n_{m_L,K_{m_L}} \exists n_1 \dots \exists n_{N_0}$$
$$D^{n_D}\left(\prod_{i=1}^{M} A^-_{m_1}{}^{n^-_{A_{m_1}}}\right)\left(\prod_{i=1}^{M'} A^+_{m_1}{}^{n^+_{A_{m_1}}}\right)\left(\prod_{i=1}^{L}\prod_{j=1}^{K_{m_i}} f_{m_i,j}^{n_{m_i,j}}\right)\left(\prod_{i=1}^{N_0} C_{x_i}^{n_i}\right)\begin{pmatrix}0\\ \vdots\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\\ a\end{pmatrix}$$

where L is the number of monomials, M the number of nonnegative monomials and M' the number of negative monomials. This completes the proof of Lemma 4.7. □

4.4. Conclusion: Reduction to Hilbert's Tenth Problem

We can now prove Theorem 4.1.
Proof: For each polynomial P of degree N_0, we associate:

$$\tilde{P} = \begin{cases} P - a_{0,\dots,0} & \text{if } a_{0,\dots,0} < 0 \\ -P + a_{0,\dots,0} & \text{if } a_{0,\dots,0} \geq 0 \end{cases}$$

Then $P(x_1, \dots, x_{N_0}) = 0$ iff $\tilde{P}(x_1, \dots, x_{N_0}) = |a_{0,\dots,0}|$.
The constant term of $\tilde{P}$ is 0, thus we build

$$D, A^-_{m_1}, \dots, A^-_{m_M} A^+_{m_1}, \cdots, A^+_{m_{M'}}, f_{m_1,1}, \dots, f_{m_1,N_{m_1}}, \dots, f_{m_L,1}, \dots, f_{m_L,N_{m_L}}$$

from $\tilde{P}$ as described in section 4.3.
Thus,

$$\exists x_1 \dots \exists x_{N_0} \quad P(x_1, \dots, x_{N_0}) = 0$$

if and only if

$$\exists x_1 \dots \exists x_{N_0} \quad \tilde{P}(x_1, \dots, x_{N_0}) = |a_{0,\dots,0}|$$

if and only if (see section 4.3)

$$\begin{pmatrix}0\\ \vdots\\ 0\\ |a_{0,\dots,0}|\end{pmatrix} \in \left\{ D^{n_D} \cdots C_{x_{N_0}}^{n_{N_0}} \begin{pmatrix}0\\ \vdots\\ 0\\ 0\end{pmatrix} \middle| \, n_D, \dots, n_{N_0} \in \mathbb{N} \right\}$$

If $V \in \{f_p^{n_p} \dots f_1^{n_1}(U) \mid n_1, \dots, n_p \in \mathbb{N}\}$ was decidable, then the Hilbert's tenth problem would be decidable too.
Conclusion: $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \dots, n_p \in \mathbb{N}\}$ is undecidable. □

4.5. Existence of a bound

We have shown that, given $p \in \mathbb{N}$, $f_1, \ldots, f_p \in \mathcal{A}_k(\mathbb{N})$ $U, V \in \mathbb{N}^k$, the problem $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \ldots, n_p \in \mathbb{N}\}$ is undecidable.

Actually, the number of functions f_i can be fixed in advance (provided the number of functions is large enough), it is not a parameter of the problem:

Theorem 4.8. *There are $f_1, \ldots, f_p$ affine functions in $\mathcal{A}_k(\mathbb{N})$ such that, given $U, V \in \mathbb{N}^k$, the problem $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \ldots, n_p \in \mathbb{N}\}$ is undecidable.*

It follows that:

Corollary 4.9. *There exists $K \in \mathbb{N}$, such that: for all "fixed" $p \geq K$, given $f_1, \ldots, f_p \in \mathcal{A}_k(\mathbb{N})$, $U, V \in \mathbb{N}^k$, the problem $V \in \{f_p^{n_p} \cdots f_1^{n_1}(U) \mid n_1, \ldots, n_p \in \mathbb{N}\}$ is undecidable.*

Proof: To show this last result, we re-use the strong form of Hilbert's tenth problem: theorem 3.10 of [8], cited here as Theorem 4.2.

We associate with the polynomial $\mathcal{P}$ (defined in Theorem 4.2), affine functions C_x, C_{y_i}, $f_{m_i,j}$, $A^+_{m_i}$, $A^-_{m_i}$, $D \in \mathcal{A}_{N+1}(\mathbb{N})$ as in section 4.4. Let a be the constant term of $\mathcal{P}$. Then:

given n_0, $\mathcal{P}(n_0, y_1, \ldots, y_m) = 0$ has a solution in nonnegative integers $y_1, \ldots, y_m$ iff

$$V \in \left\{ D^{n_D} \cdots C_{y_1}^{n_1} \cdots C_{y_m}^{n_m} U \,\middle|\, n_D, \ldots, n_1, \ldots, n_m \in \mathbb{N} \right\},$$

where $U = C_x^{n_0} O_{N+1}$ and $V = |a|\, E_{N+1}^{N+1}$. This is not decidable, so we complete the proof. $\square$

5. Conclusion

We have proved that the original reachability problem is undecidable for some fixed number p of functions and that it is decidable for $p = 1$ and $p = 2$. There are some restrictions on the f_i which restore the decidability: for example, if $\forall i\, \forall X \quad |f_i(X)| \geq X$ (where $|V|$ is the sum of the absolute values of its coordinates) or if each B_i is nonnegative. These minor results are not shown here (see [4]).

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