Approximating Traces*

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Abstract

In order to give a semantics to concurrent processes, one needs a powerful model which enjoys many important mathematical properties. We generalize Mazurkiewicz (infinite) traces by adding an information concerning the possible continuations of a process. This allows to define an approximation order and a composition. We obtain a prime algebraic coherently complete domain where the compacts are exactly the finite approximations of actual processes. The composition is shown to be monotone and $\sqcup$-continuous. We define a suitable metric which induces the Lawson topology and yields a complete and compact metric space. The finite approximations of processes form a dense subset and the composition is uniformly continuous.

Résumé

Un modèle doit avoir de nombreuses propriétés mathématiques importantes afin d'être utilisable en sémantique des systèmes concurrents. Dans ce papier, on généralise les traces (infinies) de Mazurkiewicz en ajoutant une information sur les extensions possibles d'un processus. Ceci permet de définir un ordre d'approximation et une composition ayant de bonnes interactions. On obtient un domaine cohéremment complet et premier algébrique dans lequel les éléments compacts sont exactement les approximations finies des processus réels. La composition est monotone et continue par rapport à l'ordre d'approximation. On définit une distance ultramétrique naturelle qui engendre la topologie de Lawson. On obtient donc un espace métrique complet et compact dans lequel l'ensemble des approximations finies des processus est dense. De plus la composition des processus est en fait uniformément continue.

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1 Introduction

The aim of this paper is to provide a powerful semantic domain for concurrent systems. As argued in the discussion below, such a semantic domain must enjoy many algebraic, topological and order theoretic properties.

We want to be able to describe different stages of a behaviour, from its initialization to its termination. This will be abstracted through an approximation order with the following semantics : x approximates y if y describes a more recent state of the process than x does. This might be very useful, especially when dealing with recursion. In this case, each unfolding of the recursion corresponds to a new stage of the process. We thus obtain an increasing sequence of approximations of the actual behaviour. It is then natural to require that this increasing sequence of approximations converges to the semantics of the terminated behaviour. In the partial order setting, this limit will be a least upper bound and naturally, we want a complete partial order (cpo) to ensure the existence of these least upper bounds. On the other hand, we don't want our cpo's too big : every element has to be the limit of finite approximations. This is indeed an algebraic property and explains why we are looking for a domain (an algebraic cpo) where the compact elements are the finite approximations.

Now, we want to be able to compute the semantics of a system *modularly* : the semantics of a composed process $P \odot Q$ should be the composition of the semantics of P and Q. More precisely, considering that the semantics of a process is the limit of its finite approximations, this means that the following diagram should commute.

$$\begin{array}{ccc}
(P, Q) & \xrightarrow{\odot} & P \odot Q \\
| & & | \\
\textit{approximation} & & \textit{approximation} \\
\downarrow & & \downarrow \\
(\mathrm{approx}(P), \mathrm{approx}(Q)) & \xrightarrow{\odot} & \mathrm{approx}(P \odot Q) = \mathrm{approx}(P) \odot \mathrm{approx}(Q) \\
| & & | \\
\textit{limit} & & \textit{limit} \\
\downarrow & & \downarrow \\
(\sqcup \mathrm{approx}(P), \sqcup \mathrm{approx}(Q)) & \xrightarrow{\odot} & \sqcup \mathrm{approx}(P \odot Q) = \sqcup \mathrm{approx}(P) \odot \sqcup \mathrm{approx}(Q)
\end{array}$$

The first commutation requires in particular that the composition is monotone with respect to the approximation, but it is actually stronger. The second commutation says that composition should be continuous with respect to the approximation.

Finally, all these properties may be rephrased in the framework of topological spaces. The limit of a sequence of approximations is indeed the topological limit. We then require the metric space to be complete and the set of finite approximations to be dense. Finally, the composition has to be continuous in the topological sense.

In the sequential case, one might be tempted to use the concatenation for the composition of processes and the prefix relation for the approximation relation. But it is well known that the concatenation of strings is not monotone with respect to the prefix ordering. The classical solution to this problem is to consider strings together with a non termination symbol $\perp$ (or equivalently a termination symbol $\top$ or $\surd$) in order to get all the above properties [BHR84]. The domain is thus $\Sigma^*\perp \cup \Sigma^\infty$ and $u\perp$ means that u has already been observed and that the process is not yet terminated. The concatenation is defined almost as usual but the non-termination symbol $\perp$ is right absorbent: the composition of u and v is uv if $u \in \Sigma^*$ and is u otherwise. The approximation relation is defined by $u \sqsubseteq v$ if either $u = v$ or $u = v'\perp$ and $v = v'v''$. For instance,

$$1\perp \sqsubseteq a\perp \sqsubseteq ab\perp \sqsubseteq aba\perp \sqsubseteq aba.$$

It is easy to verify that the composition operation is monotone with respect to the approximation relation.

The situation is much more interesting when concurrency comes into play since some components may be terminated while others may still go on. As an easy example, consider a system composed of k fully independent processes. A terminated behaviour is a tuple $p = (u_1, \ldots, u_k)$ where $u_i \in \Sigma^\infty$ for all $1 \leq i \leq k$. In an approximation p' of p, some components may be terminated, say $i+1, \ldots, k$, while others may still go on. Thus, $p' = (u'_1\perp, \ldots, u'_i\perp, u_{i+1}, \ldots, u_k)$ where u'_j is a prefix of u_j for all $1 \leq j \leq i$. Now, even if the first process has not yet reached its global termination, another process $q = (v_1, \ldots, v_k)$ may be (partially) started according to the following "sequential" composition :

$$p' \odot q = (u'_1\perp, \ldots, u'_i\perp, u_{i+1}v_{i+1}, \ldots, u_kv_k)$$

Our model will be a generalization of this situation. Note that, the approximation of a composed process will effectively use some informations concerning the second process. This is not the case for sequential processes: when the first process is not terminated, the approximation of a composed process makes no use of the second process.

Since we want to address concurrent systems, we need a truly concurrent model, as opposed to interleaving models. We have chosen to extend Mazurkiewicz traces [Maz77] which form one of the most studied truly concurrent model, see [DR95] for the state of the art or overviews [Maz87, AR88, Per89, Die90]. Note that several other generalizations of Mazurkiewicz traces have already been considered but none of them have all desired properties. First, infinite real traces have been introduced [Gas90, Kwi91, GR93, GP95]. Real traces can be seen as the ideal completion of finite traces with the prefix ordering. But the composition of real traces is only partially defined and is not monotone with respect to the prefix order. To obtain a fully defined concatenation, complex traces were introduced as the topological completion of finite traces for a suitable metric [Die93]. The composition of complex traces is thus fully defined and uniformly continuous. But once again it is not monotone with respect to the prefix order and the poset properties of complex traces are not fully satisfactory [GP92].

The main problem with all these approaches is that they use the prefix ordering as the approximation relation and that they do not differentiate between terminated (finite) processes and finite approximations. Our model takes the opposite view by extending the idea of non termination exposed above to traces and by considering an approximation relation which is orthogonal to the prefix ordering.

An α-trace describes an approximation of a process by a pair (t, A) where t is a Mazurkiewicz (possibly infinite) trace describing what has already been observed and A is the set of actions which may still occur. If A is the empty set this means global termination. One should note the difference with failure (or ready) semantics in process algebra like CSP or CCS [Hoa85, Mil80], where the behavior in the next step only is specified. Here the finite alphabetic information A is given for the whole future. It should be noted that defining composition laws using dependencies from the alphabetic information about the future has been done in process algebra too. Let us refer to [RW94, Weh94].

We define an approximation ordering mainly by letting the "real" part t of the α-trace grow with actions specified in A and by possibly reducing uncertainty about the future by dropping letters from A. We then define a composition which allows to start the second process even if the first one has not yet reached its global termination.

We obtain a prime algebraic coherently complete domain with the finite α-traces as compact elements. The composition is monotone and $\sqcup$-continuous. We then define a metric and show that it induces the Lawson topology [Law88] on α-traces. Hence, we also have a complete and compact metric space with the set of finite α-traces being dense. Moreover, the composition is uniformly continuous. We also show how the main topological results can be obtained using projective limits.

In the fully concurrent (commutative) case, we do obtain a direct product of strings with non termination. But in the sequential case, α-traces are a bit more concrete than strings with non termination. An α-trace does not merely express non termination, it tells which actions may still occur. In order to have a more abstract model, we introduce δ-traces which differ from α-traces only by their information about the future : a δ-trace specifies in fact which actions may be started without waiting for the global termination of the current process. Unfortunately, δ-traces are a bit too abstract. Some elementary informations are missing resulting in a domain which fails to be prime algebraic and distributive in general.

A preliminary version of this work appeared in [DG95].

2 Preliminaries

In this section, we first recall some definitions concerning poset theory. Then, we introduce real traces and recall the results on real traces which will be used throughout. The reader is referred to [GR93] which contains most results on real traces or to [GP95] for a more complete survey on infinite traces.

2.1 Poset theory

Let $(Z, \leq)$ be a poset, i.e., a partially ordered set. A subset $Y \subseteq Z$ is *coherent* if for all $x, y \in Y$, there exists some $z \in Z$ such that $x \leq z$ and $y \leq z$. A subset $Y \subseteq Z$ is *directed* if it is not empty and for all $x, y \in Y$, there exists some $z \in Y$ such that $x \leq z$ and $y \leq z$. A subset $Y \subseteq Z$ is *bounded above* if there exists $z \in Z$ such that $x \leq z$ for all $x \in Y$. The poset $(Z, \leq)$ is a *coherently complete partial order* (a *ccpo* for short) if every coherent subset of Z has a least upper bound in Z. Similarly, the poset $(Z, \leq)$ is a *directed complete partial order* (a *dcpo* for short) if every directed subset of Z has a least upper bound in Z. Finally, the poset $(Z, \leq)$ is a *bounded complete partial order* (a *bcpo* for short) if every bounded subset of Z has a least upper bound in Z. Note that a ccpo is indeed both a dcpo and a bcpo. Whenever they exist, the least upper bound and the greatest lower bound of a subset $Y \subseteq Z$ will be denoted by $\sqcup Y$ and $\sqcap Y$ respectively.

Let $(Z, \leq)$ be a ccpo. An element $x \in Z$ is *prime* if for all *coherent* subsets $Y \subseteq Z$, $x \leq \bigsqcup Y$ implies $x \leq y$ for some $y \in Y$. For all $z \in Z$, we denote by $P(z) = \{x \in Z \mid x \leq z \text{ and } x \text{ is prime}\}$ the set of prime elements of Z less than or equal to z. A ccpo $(Z, \leq)$ is *prime algebraic* if $z = \bigsqcup P(z)$ for all $z \in Z$.

Similarly, let $(Z, \leq)$ be a dcpo. An element $x \in Z$ is *compact* if for all *directed* subsets $Y \subseteq Z$, the inequality $x \leq \bigsqcup Y$ implies $x \leq y$ for some $y \in Y$. For $z \in Z$, we denote by $K(z) = \{x \in Z \mid x \leq z \text{ and } x \text{ is compact}\}$ the set of compact elements less than or equal to z. A dcpo $(Z, \leq)$ is *algebraic* if $K(z)$ is directed and $z = \bigsqcup K(z)$ for all $z \in Z$. Note that, if any coherent pair of elements of Z admits a least upper bound (in particular if $(Z, \leq)$ is a ccpo or a bcpo) then $K(z)$ is directed for all $z \in Z$. Finally, an algebraic dcpo is called a *domain*.

For a poset $(Z, \leq)$ and $z \in Z$ we use the following standard notations:

$$\downarrow z = \{y \in Z \mid y \leq z\} \quad \text{and} \quad \uparrow z = \{y \in Z \mid y \geq z\}.$$

Let $(X, \leq)$ and $(Y, \leq)$ be two posets. A mapping $f : X \longrightarrow Y$ is monotone if for all $x \in X$ and $y \in Y$, $x \leq y$ implies $f(x) \leq f(y)$. Let $(X, \leq)$ and $(Y, \leq)$ be two dcpos. A monotone mapping $f : X \longrightarrow Y$ is continuous if $f(\bigsqcup Z) = \bigsqcup f(Z)$ for all directed sets $Z \subseteq X$.

2.2 Real traces

We recall in this subsection basic definitions and results of trace theory which will be used throughout. A detailed presentation of traces and dependence graphs appears in [GP95].

A dependence alphabet is a pair (Σ, D) where Σ is a finite set of actions, called the alphabet, and $D \subseteq \Sigma \times \Sigma$ is a reflexive and symmetric relation over Σ, called the dependence relation. Its complement $I = \Sigma \times \Sigma \setminus D$ is the independence relation. We will represent a dependence alphabet by an undirected graph over Σ where self loops are omitted. For instance, the dependence alphabet (Σ, D) defined by $\Sigma = \{a, b, c\}$ and $D = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ will be represented by the following graph:

$$(\Sigma, D) = a \longrightarrow b \qquad c.$$

For $A \subseteq \Sigma$, we denote by $I(A) = \{b \in \Sigma \mid (a, b) \in I \text{ for all } a \in A\}$ the set of letters independent from all letters of A and we denote by $D(A) = \{b \in \Sigma \mid (a, b) \in D \text{ for some } a \in A\}$ the set of letters dependent from some letter of A.

A *real trace* over the dependence alphabet (Σ, D) is (an isomorphism class of) a labelled directed acyclic graph $[V, E, \lambda]$, where the set V of vertices is at most countable, $E \subseteq V \times V$ is the set of edges, $\lambda : V \to \Sigma$ is the labelling function and it holds

- edges are between dependent vertices: for all $p, q \in V$,

$$(\lambda(p), \lambda(q)) \in D \iff p = q \text{ or } (p,q) \in E \text{ or } (q,p) \in E$$

- vertices have finite pasts: for all $p \in V$, $\downarrow p = \{q \in V \mid (q,p) \in E^*\}$ is finite (where E^* denotes the reflexive and transitive closure of the edge relation E).

The set of real traces is denoted by $\mathbb{R}(\Sigma, D)$, or simply by $\mathbb{R}$. A finite trace is a real trace with a finite set of vertices. The set of finite traces is denoted by $\mathbb{M}(\Sigma, D)$, or simply by $\mathbb{M}$.

Let $t = [V, E, \lambda]$ be a real trace. The *alphabet* of t is the set of letters which occur in t: $\mathrm{alph}(t) = \lambda(V)$. The *alphabet at infinity* of t is the set of letters which occur infinitely often in t: $\mathrm{alphinf}(t) = \{a \in \Sigma \mid \lambda^{-1}(a) \text{ is infinite}\}$. The *length* of t is defined by $|V|$ and denoted by $|t|$ and the number $|\lambda^{-1}(a)|$ of occurrences of a in t is denoted by $|t|_a$.

the concatenation of two real traces is their disjoint union together with new edges from the first trace to the second one. Formally, let $r = [V_1, E_1, \lambda_1]$ and $s = [V_2, E_2, \lambda_2]$ be two real traces (we may assume that $V_1 \cap V_2 = \emptyset$). The concatenation of r and s is defined only if $\mathrm{alphinf}(r) \times \mathrm{alph}(s) \subseteq I$. Then we set $r \cdot s = [V, E, \lambda]$ where

$$\begin{aligned}
V &= V_1 \mathbin{\dot\cup} V_2 \\
E &= E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} \{(p_1, p_2) \in V_1 \times V_2 \mid (\lambda_1(p_1), \lambda_2(p_2)) \in D\} \\
\lambda &= \lambda_1 \mathbin{\dot\cup} \lambda_2
\end{aligned}$$

The concatenation of real traces is associative and the empty trace $1 = [\emptyset, \emptyset, \emptyset]$ is the unity. The concatenation of two finite traces is always defined and is a finite trace. Hence, $(\mathbb{M}, \cdot)$ is a monoid. The canonical morphism $\varphi : \Sigma^* \longrightarrow \mathbb{M}$ defined by $\varphi(a) = [\{x\}, \emptyset, x \mapsto a]$ is surjective. In fact, φ defines an isomorphism between the monoid of finite traces and the quotient of the free monoid Σ^* by the congruence generated by $\{(ab, ba) \mid (a,b) \in I\}$.

To underline the fact that the concatenation on real traces is only partially defined, we say that $(\mathbb{R}, \cdot)$ is a partial monoid. This concatenation extends easily to an infinite product. Let $(r_i)_{i<\omega}$ be a sequence of real traces such that $\mathrm{alphinf}(r_i) \times \mathrm{alph}(r_j) \subseteq I$ for all $i < j$, then $\prod_{i<\omega} r_i \in \mathbb{R}$ is defined as the disjoint union of the graphs (r_i) with new arcs from vertices of r_i to vertices of r_j whenever $i < j$ and the vertices have dependent labels. The canonical morphism $\varphi : \Sigma^* \longrightarrow \mathbb{M}$ extends to a mapping $\varphi : \Sigma^\infty \longrightarrow \mathbb{R}$ which turns out to be surjective.

A trace $r \in \mathbb{R}$ is a prefix of a trace $t \in \mathbb{R}$, denoted by $r \leq t$, if $t = r \cdot s$ for some $s \in \mathbb{R}$. The partial monoid $(\mathbb{R}, \cdot)$ is left cancellative: let $r, s, t \in \mathbb{R}$ be such that $r \cdot s$ and $r \cdot t$ are well-defined, if $r \cdot s = r \cdot t$ then $s = t$. Hence, if $r \leq t$, there exists a unique trace $s \in \mathbb{R}$ such that $t = r \cdot s$. This unique trace will be denoted by $r^{-1}t$. Note that $(\mathbb{R}, \cdot)$ is right cancellative by finite traces only.

The prefix relation is a partial order on real traces. The poset $(\mathbb{R}, \leq)$ is prime algebraic and coherently complete, in particular, it is a bounded complete domain. A real trace is compact if and only if it is a finite trace. A real trace r is prime if and only if it is finite and has exactly one maximal vertex.

The following result is a generalization of Levi's lemma to real traces. Let x, y, z, t be four real traces. Then (all products are assumed to be well-defined), $xy = zt$ if and only if there exist four traces $r, u, v, s \in \mathbb{R}$ such that $x = ru$, $y = vs$, $z = rv$, $t = us$ and $\text{alph}(u) \times \text{alph}(v) \subseteq I$.

Let $s = [V, E, \lambda] \in \mathbb{R}$ be a real trace and let $A \subseteq \Sigma$ be a subalphabet. We denote by $\mu_A(s)$ the maximal prefix of s being independent of A, i.e., the restriction of the graph s to the set $\{p \in V \mid \lambda(\downarrow p) \times A \subseteq I\}$. We denote the alphabet of the corresponding suffix by $\sigma_A(s) = \text{alph}(\mu_A(s)^{-1}s)$. Let $s, t \in \mathbb{R}$ be real traces and let $A, B \subseteq \Sigma$ be subalphabet, then,

$$\mu_A(st) = \mu_A(s)\mu_{A\cup\sigma_A(s)}(t) \tag{1}$$

$$\sigma_A(st) = \sigma_A(s) \cup \sigma_{A\cup\sigma_A(s)}(t) \tag{2}$$

$$\mu_{A\cup B}(s) = \mu_A(\mu_B(s)) \tag{3}$$

$$\sigma_{A\cup B}(s) = \sigma_A(\mu_B(s)) \cup \sigma_B(s) \tag{4}$$

3 Alpha-Traces

In this section, we introduce α-traces which form our new semantic model and generalize Mazurkiewicz's traces [Maz77] and their infinite extensions [GP95]. As explained in the introduction, our main motivation is to deal both with the approximation of processes and with the composition of processes in a satisfactory way. To this purpose, it is necessary to abstract differently in our model approximations of processes and terminated processes. By approximation of a process, we mean something which gives only partial informations about the actual process. Hence, it is natural to require that the approximation relation is a partial order. On the other hand, the composition of processes will be an associative operation. Therefore, our model will be both a poset (partially ordered set) for the approximation relation and a monoid for the composition of processes. Moreover, we will require good interconnections between these two structures. For instance, the composition of processes will be monotone with respect to the approximation order. In other words, if x and y are approximations of processes p and q, then the composition of x and y will be an approximation of the composition of the processes p and q. Finally, we introduce and ω-product on α-traces and establish some of its properties.

Definition 3.1 *Let (Σ, D) be a dependence alphabet. An α-trace is an ordered pair (r, A) where $r \in \mathbb{R}(\Sigma, D)$ is a real trace and $A \subseteq \Sigma$ is such that* $\text{alphinf}(r) \subseteq A$. *The set of α-traces is denoted by $\mathbb{F}(\Sigma, D)$ or simply by $\mathbb{F}$.*

Intuitively, the first component of an α-trace represents what has already occurred whereas the second component contains the actions which still may occur in the future. For instance, let $(\Sigma, D) = a — b — c$. The α-trace $x = (acbaacbc, \{a\})$ tells us that the process has already executed $acbaacbc$ but it may be not terminated, the only action that the process may still perform is a. Note that this gives us more informations than the non termination symbol $\perp$. Here, we know that the process is not terminated but, in addition, we know that all occurrences of b and c have been performed while some a's may still occur.

Let $x = (r, A)$ be an α-trace. The first component r of x is called its *real part* and is denoted by $\mathrm{Re}(x)$. The second component A of x is called its *imaginary part* and is denoted by $\mathrm{Im}(x)$. The terminology used is explained by the previous remarks: we know for sure that the real part has been performed while the actions contained in the imaginary part may only possibly be executed in the future. This imaginary part is not visible but it has influence on the composition of α-traces. The length of the α-trace x is the length of its real part and is denoted by $|x|$. Finally, the alphabet of the α-trace x is the set $\mathrm{alph}(x) = \mathrm{alph}(r) \cup A$.

Definition 3.2 *Let $x = (r, A)$ and $y = (s, B)$ be α-traces. We say that x is an approximation of y, denoted by $x \sqsubseteq y$, if $r \leq s$ and $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$.*

Intuitively, x is an approximation of y if y contains more information about the actual process than x. In our setting, this is described by three different properties: first, the behaviour actually observed in x is a prefix of the behaviour actually observed in y, i.e., the real part r of x is a prefix of the real part s of y; second, all actions which occur in $r^{-1}s$ have to be non terminated in x, i.e., $\mathrm{alph}(r^{-1}s)$ is contained in the imaginary part of x; third, all actions which are non terminated in y are also non terminated in x, i.e., the imaginary part of y is contained in the imaginary part of x. these three conditions yield precisely Definition 3.2.

For instance, let $(\Sigma, D) = a$ —— b —— c. Then,

$$(1, \Sigma) \sqsubseteq (acba, \Sigma) \sqsubseteq (acbaac, \{b, c\}) \sqsubseteq (acbaaccb, \{c\}) \sqsubseteq (acbaaccb, \emptyset)$$

and

$$(1, \Sigma) \sqsubseteq (acb, \Sigma) \sqsubseteq ((acb)^2, \Sigma) \sqsubseteq \cdots \sqsubseteq ((acb)^\omega, \Sigma)$$

The bottom element with respect to $\sqsubseteq$ is the α-trace $(1, \Sigma)$. It signifies that nothing has happened and everything is possible.

Proposition 3.3 $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ *is a poset (partially ordered set) with* $(1, \Sigma)$ *as least element.*

Proof: The approximation relation is reflexive and antisymmetric. Now assume that $(r, A) \sqsubseteq (s, B) \sqsubseteq (t, C)$. From, $r \leq s \leq t$ we deduce $r \leq t$ and $r^{-1}t = (r^{-1}s)(s^{-1}t)$. therefore,

$$\mathrm{alph}(r^{-1}t) = \mathrm{alph}(r^{-1}s) \cup \mathrm{alph}(s^{-1}t) \subseteq A \cup B \subseteq A.$$

Finally, $C \subseteq B \subseteq A$ and we obtain $(r, A) \sqsubseteq (s, B)$. thus, the approximation relation is transitive and is then a partial order relation on the set $\mathbb{F}$ of α-traces. $(1, \Sigma)$ is clearly the least element of the poset $(\mathbb{F}, \sqsubseteq)$. □

Note that a real trace $r \in \mathbb{R}$ may be identified with the α-trace $(r, \mathrm{alphinf}(r))$. More precisely, there is an embedding $\chi : \mathbb{R} \hookrightarrow \mathbb{F}$ from the set of real traces into the set of α-traces defined by $\chi(r) = (r, \text{alphinf}(r))$. Therefore, we will identify a real trace $r \in \mathbb{R}$ with the corresponding α-trace $\chi(r)$ and we will write $\mathbb{M} \subseteq \mathbb{R} \subseteq \mathbb{F}$.

An α-trace is maximal for the approximation relation if and only if it is a real trace. Indeed, let $x = (r, \text{alphinf}(r))$ be a real trace and assume that x is an approximation of the α-trace $y = (s, B)$. Since r is a prefix of s, it holds $\text{alph}(r^{-1}s) \times \text{alphinf}(r) \subseteq I$. From $\text{alph}(r^{-1}s) \subseteq \text{Im}(x) = \text{alphinf}(r)$, we deduce that $\text{alph}(r^{-1}s) = \emptyset$, whence $r = s$. Finally, the third condition of the approximation relation implies $\text{alphinf}(r) = \text{alphinf}(s) \subseteq B \subseteq \text{alphinf}(r)$. Therefore, $x = y$ which proves that all real traces are maximal for the approximation order. Conversely, assume that the α-trace $x = (r, A)$ is maximal. Clearly, x is an approximation of the real trace $y = (r, \text{alphinf}(r))$. Hence, $x = y$ is a real trace.

We turn now to the definition of the concatenation (composition) of α-traces. This definition uses the maximal prefix $\mu_A(s)$ of a real trace s independent from a subalphabet A and the alphabet of the corresponding suffix $\sigma_A(s)$ which were introduced in Section 2.

Definition 3.4 *Let $x = (r, A)$ and $y = (s, B)$ be two α-traces. The concatenation of x and y is*

$$x \cdot y = (r\mu_A(s), A \cup B \cup \sigma_A(s))$$

Let us explain the semantics of this concatenation. Assume that $x = (r, A)$ and $y = (s, B)$ are approximations of actual processes p and q. The composition of x and y should give us all possible informations concerning the composition of p and q. We know that r is a prefix of p and s is a prefix of q. Since the actions in A may still occur in p, only actions of q independent from A may be started. Hence, we know for sure that $r\mu_A(s)$ is a prefix of the composition of p and q. Finally, actions which may still be performed in the composition of p and q after $r\mu_A(s)$ are actions which may still be performed in p after r, i.e., A; and actions which may still be performed in q after $\mu_A(s)$, i.e., $B \cup \sigma_A(s)$.

For instance, let $(\Sigma, D) = a — b — c$. Then,

$$\begin{aligned}
(acbac, \{c\}) \cdot (aaccb, \emptyset) &= (acbaaac, \{b, c\}) \\
(acbac, \Sigma) \cdot (aaccb, \emptyset) &= (acbac, \Sigma) \\
(acbac, \emptyset) \cdot (aaccb, \{a\}) &= (acbaaacccb, \{a\}) \\
(acbac, \{a\}) \cdot (aaccb, \{c\}) &= (acbaccc, \Sigma)
\end{aligned}$$

Note that the bottom element $(1, \Sigma)$ for the approximation relation is right-absorbent with respect to the concatenation and we have $(r, A) \cdot (1, \Sigma) = (r, \Sigma)$ for all $(r, A) \in \mathbb{F}$.

Proposition 3.5 $(\mathbb{F}(\Sigma, D), \cdot)$ *is a monoid with neutral element* $(1, \emptyset)$. *Moreover, the embedding* $\chi : \mathbb{R} \hookrightarrow \mathbb{F}$ *from the set of real traces into the set of α-traces is a partial morphism.*

Proof: Clearly, $(1, \emptyset)$ is a neutral for the concatenation. Now, let $x = (r, A)$, $y = (s, B)$ and $z = (t, C)$ be three α-traces. Using Equations 1–4 of Section 2, we deduce

$$\begin{aligned}
\mathrm{Re}(x(yz)) &= r\mu_A(s\mu_B(t)) \\
&= r\mu_A(s)\mu_{A\cup\sigma_A(s)}(\mu_B(t)) \\
&= r\mu_A(s)\mu_{A\cup B\cup\sigma_A(s)}(t) \\
&= \mathrm{Re}((xy)z)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Im}(x(yz)) &= A \cup (B \cup C \cup \sigma_B(t)) \cup \sigma_A(s\mu_B(t)) \\
&= A \cup B \cup C \cup \sigma_B(t) \cup \sigma_A(s) \cup \sigma_{A\cup\sigma_A(s)}(\mu_B(t)) \\
&= (A \cup B \cup \sigma_A(s)) \cup C \cup \sigma_{A\cup B\cup\sigma_A(s)}(t) \\
&= \mathrm{Im}((xy)z)
\end{aligned}$$

Therefore, the concatenation is associative and $(\mathbb{F}, \cdot)$ is a monoid.

Finally, let $r, s \in \mathbb{R}$ be two real traces such that $\mathrm{alphinf}(r) \times \mathrm{alph}(s) \subseteq I$. It holds, $\mu_{\mathrm{alphinf}(r)}(s) = s$. Hence, $\chi(r)\chi(s) = (rs, \mathrm{alphinf}(r) \cup \mathrm{alphinf}(s)) = \chi(rs)$. $\square$

Remark 3.6 Let $(\Sigma, D) = a$ —— b —— c. A first try to define a concatenation of real traces might ask for the following laws: $a^\omega b^\omega = a^\omega$, $b^\omega c^\omega = b^\omega$, and $a^\omega c^\omega = (ac)^\omega$. However, then we have:

$$(a^\omega b^\omega)c^\omega = a^\omega c^\omega = (ac)^\omega \neq a^\omega = a^\omega b^\omega = a^\omega(b^\omega c^\omega) \ .$$

Since for obvious reasons (a and c are independent) we want to keep the law that $a^\omega c^\omega \neq a^\omega$, we have to change the law for $a^\omega b^\omega$ (and, of course, for $b^\omega c^\omega$) in order to have an associative operation. This is what happens in our calculus: The interpretation as an α-trace yields $x^\omega = (x^\omega, \{x\})$ for $x \in \Sigma$. Hence $a^\omega b^\omega = (a^\omega, \{a\})(b^\omega, \{b\}) = (a^\omega, \{a, b\}) \neq a^\omega$. Here are some more lines of the associative calculus of α-traces:

$$\begin{aligned}
(a^\omega, \{a\})(b^\omega, \{b\})(c^\omega, \{c\}) &= (a^\omega, \Sigma) \\
(a^\omega, \{a\})(c^\omega, \{c\})(b^\omega, \{b\}) &= ((ac)^\omega, \Sigma) \\
(a^\omega, \{a\})((ac)^\omega, \{a, c\}) &= ((ac)^\omega, \{a, c\})
\end{aligned}$$

The monoid of α-traces is not cancellative. For all real traces $r \in \mathbb{R}$, the α-trace (r, Σ) is a left- zero. It is not right cancellative either since, for instance, $a \cdot a^\omega = a^2 \cdot a^\omega$. Nevertheless, the following proposition holds.

Proposition 3.7 *The monoid* $(\mathbb{F}(\Sigma, D), \cdot)$ *is left-cancellative by finite real traces:* $\forall f \in \mathbb{M}(\Sigma, D), \forall x, y \in \mathbb{F}(\Sigma, D)$,

$$f \cdot x = f \cdot y \Longrightarrow x = y$$

Hence, if a finite trace $f \in \mathbb{M}$ is a prefix of an α-trace $x \in \mathbb{F}$, the corresponding suffix is well defined and will be denoted by $f^{-1}x$.

Proof: Let $f = (f, \emptyset) \in \mathbb{M}$ be a finite trace and let $x = (r, A)$ and $y = (s, B)$ be two α-traces. Since $\mathbb{R}$ is left-cancellative by finite traces, we deduce from $(f \cdot r, A) = f \cdot x = f \cdot y = (f \cdot s, B)$ that $r = s$ and $A = B$, i.e., $x = y$. $\square$

Besides the approximation order, the prefix relation, which may be defined in any monoid, is also of interest. It will allow us to define an infinite product on α-traces.

Definition 3.8 *Let $x, y \in \mathbb{F}(\Sigma, D)$ be α-traces. x is a prefix of y, denoted by $x \leq y$, if $xz = y$ for some α-trace $z \in \mathbb{F}(\Sigma, D)$.*

Remark 3.9 Let $x = (r, A)$ and $y = (s, B)$ be two α-traces. Then,

$$x \leq y \Longleftrightarrow r \leq s, A \subseteq B \text{ and } \operatorname{alph}(r^{-1}s) \subseteq I(A)$$

As in any monoid, the prefix relation is reflexive and transitive. From Remark 3.9, we deduce that the prefix relation on α-traces is also antisymmetric, whence is a partial order relation. The set of α-traces provided with the prefix relation is thus a poset with $1 = (1, \emptyset)$ as least element. Note that the prefix ordering $\leq$ and the approximation ordering $\sqsubseteq$ are disjoint in the following sense: If $(r, A) \leq (s, B)$ and $(r, A) \sqsubseteq (s, B)$ then we have $(r, A) = (s, B)$.

In order to define the infinite product on α-traces we first need to prove that every $\leq$-increasing sequence of α-traces has a $\leq$-least upper bound. This is a corollary of the following stronger result.

Proposition 3.10 *The poset $(\mathbb{F}(\Sigma, D), \leq)$ with the prefix ordering is a a coherently complete partial order. Moreover, if $X \subseteq \mathbb{F}(\Sigma, D)$ is an $\leq$-coherent set then we have $\bigsqcup_{\leq} X = (t, C)$ where $t = \bigsqcup \operatorname{Re}(X)$ and $C = \operatorname{alphinf}(t) \cup (\bigcup_{x \in X} \operatorname{Im}(x))$.*

Proof: Let $X \subseteq \mathbb{F}$ be a coherent set. For all $x, y \in \mathbb{F}$, there exists $z \in \mathbb{F}$ such that $x \leq z$ and $y \leq z$, whence $\operatorname{Re}(x) \leq \operatorname{Re}(z)$ and $\operatorname{Re}(y) \leq \operatorname{Re}(z)$. Therefore, the set $\operatorname{Re}(X) \subseteq \mathbb{R}$ is coherent and $t = \bigsqcup \operatorname{Re}(X)$ exists. Let $C = \operatorname{alphinf}(t) \cup (\bigcup_{x \in X} \operatorname{Im}(x))$ and let $z = (t, C) \in \mathbb{F}$. We claim that $z = \bigsqcup_{\leq} X$.

Let $x = (r, A) \in X$. By definition, $r \leq t$ and $A \subseteq C$. In order to obtain $x \leq z$, it remains to show that $\operatorname{alph}(r^{-1}t) \subseteq I(A)$ (Remark 3.9). Let $a \in \operatorname{alph}(r^{-1}t)$. There exists $y = (s, B) \in X$ such that $|r|_a < |s|_a \leq |t|_a$. Since X is coherent, there exists $z' = (t', C') \in \mathbb{F}$ such that $x \leq z'$ and $y \leq z'$. Hence, $|s|_a \leq |t'|_a$ and $a \in \operatorname{alph}(r^{-1}t') \subseteq I(A)$. Therefore, z is an upper bound of X.

Let $y = (s, B) \in \mathbb{F}$ and assume that y is an upper bound for X. For all $x \in X$, $\operatorname{Re}(x) \leq \operatorname{Re}(y) = s$ and $\operatorname{Im}(x) \subseteq \operatorname{Im}(y) = B$. Hence, we obtain $t = \bigsqcup \operatorname{Re}(X) \leq s$ and $C = \operatorname{alphinf}(t) \cup (\bigcup_{x \in X} \operatorname{Im}(x)) \subseteq \operatorname{alphinf}(s) \cup B = B$. Since $t \leq s$, it holds $\operatorname{alph}(t^{-1}s) \subseteq I(\operatorname{alphinf}(t))$. On the other hand, for all $x \in X$, $\operatorname{Re}(x) \leq t \leq s$, whence $\operatorname{alph}(t^{-1}s) \subseteq \operatorname{alph}(\operatorname{Re}(x)^{-1}s) \subseteq I(\operatorname{Im}(x))$. Hence, $\operatorname{alph}(t^{-1}s) \subseteq \bigcap_{x \in X} I(\operatorname{Im}(x)) = I(\bigcup_{x \in X} \operatorname{Im}(x))$. Finally, $\operatorname{alph}(t^{-1}s) \subseteq I(\operatorname{alphinf}(t)) \cap I(\bigcup_{x \in X} \operatorname{Im}(x)) = I(C)$. Therefore, $z \leq y$ which concludes the proof. $\square$

Thanks to the previous proposition, we can define the ω-product on α-traces.

Definition 3.11 *Let $(x_i)_{i<\omega}$ be an ω-sequence of α-traces. The ω-product of $(x_i)_{i<\omega}$ is the $\leq$-least upper bound of the $\leq$-increasing sequence $(\prod_{j<i} x_j)_{i<\omega}$:*

$$\prod_{i<\omega} x_i = \bigsqcup_{i<\omega}^{\leq} \left(\prod_{j<i} x_j \right)$$

For instance, let $(\Sigma, D) = a \text{ --- } b \text{ --- } c \text{ --- } d$ and let $x_i = (a^i bd, \{d\})$ for all $i < \omega$. Then, $\prod_{i<\omega} x_i = (dbaba^2ba^3b\cdots, \{a, b, d\})$.

The monoid $\mathbb{M}(\Sigma, D)$ of finite traces is strictly embedded in the monoid of *finite* α-traces, denoted by $\mathbb{F}_f(\Sigma, D)$ or simply by $\mathbb{F}_f$. The alphabet Σ is a finite set of generators of $\mathbb{M}(\Sigma, D)$, hence, Σ cannot also generate the monoid $\mathbb{F}_f(\Sigma, D)$. In order to get a finite set of generators of $\mathbb{F}_f(\Sigma, D)$, we will use another copy of Σ.

Let $\xi : \mathcal{P}(\Sigma) \longrightarrow \mathbb{F}_f(\Sigma, D)$ defined by $\xi(A) = (1, A)$ for all $A \subseteq \Sigma$. Clearly, ξ is an injective morphism, hence defines an embedding of the power set monoid $(\mathcal{P}(\Sigma), \cup)$ into the monoid $\mathbb{F}_f(\Sigma, D)$ of finite α-traces. For $A \subseteq \Sigma$, we will also simply denote the α-trace $\xi(A)$ by A.

Clearly, a finite α-trace $x = (r, A) \in \mathbb{F}_f(\Sigma, D)$ is the concatenation of the finite trace r by the α-trace A:

$$\mathbb{F}_f(\Sigma, D) = \mathbb{M}(\Sigma, D) \cdot \mathcal{P}(\Sigma)$$

Let $\mathcal{S}(\Sigma) = \{\{a\} \mid a \in \Sigma\}$ be the set of singletons from Σ. Since $\mathbb{M}(\Sigma, D)$ is generated by Σ and $(\mathcal{P}(\Sigma), \cup)$ is generated by $\mathcal{S}(\Sigma)$ we deduce that $\Sigma \cup \mathcal{S}(\Sigma)$ is a finite set of generators of $(\mathbb{F}_f(\Sigma, D), \cdot)$.

In order to generate infinite α-traces, we will use the ω-product defined above. Let $x = (r, A) \in \mathbb{F}(\Sigma, D)$ be an arbitrary α-trace. As above, we have $x = r \cdot A$ therefore,

$$\mathbb{F}(\Sigma, D) = \mathbb{R}(\Sigma, D) \cdot \mathcal{P}(\Sigma)$$

Since any real trace is a finite product or an ω-product of letters from Σ, we deduce that $\Sigma \cup \mathcal{S}(\Sigma)$ is a finite set of generators of the set $\mathbb{F}(\Sigma, D)$ provided both with the concatenation and with the ω-product.

Note that, following Definition 3.11, we may also define γ-products for arbitrary ordinals γ. We restrict ourselves to ω-products for two reasons. First, ω- products are sufficient to generate α-traces from $\Sigma \cup \mathcal{S}(\Sigma)$. Second, in an arbitrary γ-product, at most $\omega \cdot |\Sigma|$ factors are significant. Hence, any γ-product may be obtained with ω-products and finitely many concatenations.

The concatenation on finite traces is not monotone with respect to the prefix order. For instance, if $(a, b) \in D$ then $a \leq ab$ while $a^2 \not\leq (ab)^2$. Since the monoid of finite traces is embedded in the monoid of α-traces, the concatenation on α-traces is not monotone either with respect to the prefix order. As another example, let $(\Sigma, D) = a \text{ --- } b \text{ --- } c$. Then, $(a, \{c\}) \leq (a, \Sigma)$ while $(a, \{c\})^2 = (a^2, \{c\}) \not\leq (a, \Sigma)^2 = (a, \Sigma)$. This is in part to have a concatenation which is monotone with respect to the partial order that we have introduced α-traces and the approximation order.

Theorem 3.12 *The concatenation on α-traces is monotone for the approximation order: Let $x, x', y, y' \in \mathbb{F}(\Sigma, D)$ be α-traces. If $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x \cdot x' \sqsubseteq y \cdot y'$.*

Proof: Let $x = (r, A)$, $x' = (r', A')$, $y = (s, B)$ and $y' = (s', B')$ be α-traces. Assume that $x \sqsubseteq x'$ and $y \sqsubseteq y'$, i.e., $r \leq r'$, $\mathrm{alph}(r^{-1}r') \cup A' \subseteq A$ and $s \leq s'$, $\mathrm{alph}(s^{-1}s') \cup B' \subseteq B$. We have

$$\begin{aligned} x \cdot y &= (r\mu_A(s), A \cup B \cup \sigma_A(s)) \\ x' \cdot y' &= (r'\mu_{A'}(s'), A' \cup B' \cup \sigma_{A'}(s')) \end{aligned}$$

Since $s \leq s'$ and $A' \subseteq A$, it holds $\mu_A(s) \leq \mu_A(s') \leq \mu_{A'}(s')$. Hence,

$$r'\mu_{A'}(s') = r(r^{-1}r')\mu_A(s)(\mu_A(s)^{-1}\mu_{A'}(s'))$$

From $\mathrm{alph}(r^{-1}r') \subseteq A$, we deduce

$$r'\mu_{A'}(s') = r\mu_A(s)(r^{-1}r')(\mu_A(s)^{-1}\mu_{A'}(s'))$$

Therefore, $\mathrm{Re}(xy) = r\mu_A(s) \leq r'\mu_{A'}(s') = \mathrm{Re}(x'y')$ and

$$\begin{aligned} &\mathrm{Im}(x'y') \cup \mathrm{alph}(\mathrm{Re}(xy)^{-1}\,\mathrm{Re}(x'y')) \\ &\quad= A' \cup B' \cup \sigma_{A'}(s') \cup \mathrm{alph}((r^{-1}r')(\mu_A(s)^{-1}\mu_{A'}(s'))) \\ &\quad= A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}((\mu_A(s)^{-1}\mu_{A'}(s'))(\mu_{A'}(s')^{-1}s')) \\ &\quad= A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}(\mu_A(s)^{-1}s') \\ &\quad= A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}((\mu_A(s)^{-1}s)(s^{-1}s')) \\ &\quad= A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}(s^{-1}s') \cup \sigma_A(s) \\ &\quad\subseteq A \cup B \cup \sigma_A(s) = \mathrm{Im}(xy) \end{aligned}$$

□

In order to prove that the ω-product is also monotone with respect to the approximation order , we first establish that the $\leq$-least upper bound is $\sqsubseteq$-monotone.

Let $X, Y \subseteq \mathbb{F}$ be two sets of α-traces. We say that X approximates Y, denoted by $X \sqsubseteq Y$, if for all $x \in X$ there exists $y \in Y$ such that $x \sqsubseteq y$ and for all $y \in Y$ there exists $x \in X$ such that $x \sqsubseteq y$.

Lemma 3.13 *Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be two $\leq$-coherent sets of α-traces such that X approximates Y. Then, $\bigsqcup_{\leq} X \sqsubseteq \bigsqcup_{\leq} Y$.*

Proof: Let $x = \bigsqcup_{\leq} X = (r, A)$ and let $y = \bigsqcup_{\leq} Y = (s, B)$. For all $x' \in X$ there exists $y' \in Y$ such that $x' \sqsubseteq y'$, whence $\mathrm{Re}(x') \leq \mathrm{Re}(y') \leq s$. Therefore, $r = \bigsqcup \mathrm{Re}(X) \leq s = \bigsqcup \mathrm{Re}(Y)$.

We will now prove that $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$. Recall that by Proposition 3.10, $A = \mathrm{alphinf}(r) \cup (\bigcup_{x' \in X} \mathrm{Im}(x'))$ and $B = \mathrm{alphinf}(s) \cup (\bigcup_{y' \in Y} \mathrm{Im}(y'))$. First, let $y' \in Y$. There exists $x' \in X$ such that $x' \sqsubseteq y'$. Hence, $\mathrm{Im}(y') \subseteq \mathrm{Im}(x') \subseteq A$. Second, let $a \in \mathrm{alph}(r^{-1}s)$. There exists $y' = (s', B') \in Y$ such that $|r|_a < |s'|_a \leq |s|_a$. There exists $x' = (r', A') \in X$ such that $x' \sqsubseteq y'$. Since $x' \leq x$, we have $r' \leq r$ and $|r'|_a \leq |r|_a < |s'|_a$. Therefore, $a \in \mathrm{alph}(r'^{-1}s') \subseteq A' = \mathrm{Im}(x') \subseteq A$. Third, $\mathrm{alphinf}(s) \setminus \mathrm{alphinf}(r) \subseteq \mathrm{alph}(r^{-1}s)$ and we obtain $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$. □

Proposition 3.14 *The ω-product on α-traces is $\sqsubseteq$-monotone: Let $(x_i)_{i<\omega}$ and $(y_i)_{i<\omega}$ be two sequences of α-traces such that $x_i \sqsubseteq y_i$ for all $i < \omega$. Then,*

$$\prod_{i<\omega} x_i \sqsubseteq \prod_{i<\omega} y_i$$

Proof: From Theorem 3.12, we deduce that $\prod_{j<i} x_j \sqsubseteq \prod_{j<i} y_j$ for all $i < \omega$. Therefore, $X = \{\prod_{j<i} x_j \mid i < \omega\}$ approximates $Y = \{\prod_{j<i} y_j \mid i < \omega\}$. It follows from Lemma 3.13 that

$$\prod_{i<\omega} x_i = \bigsqcup_{\leq} X \sqsubseteq \bigsqcup_{\leq} Y = \prod_{i<\omega} y_i$$

□

4 Poset Properties

In this section, we will show that α-traces form a good semantic domain. We first investigate completeness properties of the Poset $(\mathbb{F}, \sqsubseteq)$. We show that $(\mathbb{F}, \sqsubseteq)$ is a coherently complete partial order (ccpo), whence a directed complete partial order (dcpo) and a bounded complete partial order (bcpo). Then, we show that the compact elements of $(\mathbb{F}, \sqsubseteq)$ are the finite α-traces and that the poset is algebraic, whence a domain. Next, we address the question of prime algebraicity and the characterization of prime α-traces. Finally, we show that the concatenation of α-traces is $\sqsubseteq$-continuous. The definitions concerning poset theory which are used in this section are recalled is Section 2.1.

Theorem 4.1 *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is a coherently complete partial order. Moreover, if $X \subseteq \mathbb{F}(\Sigma, D)$ is a coherent set of α-traces, then*

$$\bigsqcup X = (\bigsqcup \mathrm{Re}(X), \bigcap_{x \in X} \mathrm{Im}(x))$$

Proof: Let $X \subseteq \mathbb{F}$ be $\sqsubseteq$-coherent. For all $x, y \in X$, there exists $z \in \mathbb{F}$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, whence $\mathrm{Re}(x) \leq \mathrm{Re}(z)$ and $\mathrm{Re}(y) \leq \mathrm{Re}(z)$. Therefore, the set $\mathrm{Re}(X) \subseteq \mathbb{R}$ is $\leq$-coherent and admits a $\leq$-least upper bound (see Section 2.2). Let $t = \bigsqcup \mathrm{Re}(X)$ and let $C = \bigcap_{x \in X} \mathrm{Im}(x)$.

Claim: For all $x = (r, A) \in X$, it holds $\mathrm{alph}(r^{-1}t) \subseteq A$.

Indeed, let $a \in \mathrm{alph}(r^{-1}t)$. There exists $y = (s, B) \in X$ such that $|r|_a < |s|_a \leq |t|_a$. Since X is coherent, there exists $z' = (t', C') \in \mathbb{F}$ such that $x \sqsubseteq z'$ and $y \sqsubseteq z'$. Hence, $|r|_a < |s|_a \leq |t'|_a$ and $a \in \mathrm{alph}(r^{-1}t') \subseteq A$.

The first consequence of this claim is that $z = (t, C)$ is an α-trace. Indeed, for all $x = (r, A) \in X$, we have $\mathrm{alphinf}(t) \subseteq \mathrm{alphinf}(r) \cup \mathrm{alph}(r^{-1}t) \subseteq A$. Therefore, $\mathrm{alphinf}(t) \subseteq C$.

The second consequence is that z is an upper bound of X.

Now let $y = (s, B) \in \mathbb{F}$ be an upper bound of X. Clearly, $t = \bigsqcup \mathrm{Re}(X) \leq s$ and $B \subseteq \bigcap_{x \in X} \mathrm{Im}(x) = C$. For all $x = (r, A) \in X$, we have $r \leq t \leq s$. Hence, $\mathrm{alph}(t^{-1}s) \subseteq \mathrm{alph}(r^{-1}s) \subseteq A$. Therefore, $\mathrm{alph}(t^{-1}s) \subseteq \bigcap_{x \in X} \mathrm{Im}(x) = C$ and $z \sqsubseteq y$. □

From Theorem 4.1, we deduce that the poset $(\mathbb{F}, \sqsubseteq)$ is a dcpo and a bcpo and that every non empty set of α-traces admits a greatest lower bound. The following corollary gives, in addition, the expression of the greatest lower bound of a non empty set of α-traces.

Corollary 4.2 *Let $Y \subseteq \mathbb{F}(\Sigma, D)$ be a non empty set of α-traces. Then, $\sqcap Y = (r, A)$ where*

$$r = \sqcap \mathrm{Re}(Y) \qquad \textit{and} \qquad A = \bigcup_{y \in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r^{-1}\,\mathrm{Re}(y)))$$

Proof: Let $y \in Y$ (Y is non empty). We have $\mathrm{alphinf}(r) \subseteq \mathrm{alphinf}(\mathrm{Re}(y)) \subseteq \mathrm{Im}(y) \subseteq A$. Hence, $x = (r, A)$ is an α-trace. Moreover, by definition of r and A, it follows directly that x is a lower bound of Y.

Now, let $x' = (r', A')$ be a lower bound of Y. For all $y \in Y$, we have $x' \sqsubseteq y$, i.e., $r' \leq \mathrm{Re}(y)$ and $\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y)) \subseteq A'$. Hence, $r' \leq \sqcap \mathrm{Re}(Y) = r$ and $\bigcup_{y \in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y))) \subseteq A'$. Since $r'^{-1}\,\mathrm{Re}(y) = (r'^{-1}r)(r^{-1}\,\mathrm{Re}(y))$, it follows that

$$\begin{aligned} A \cup \mathrm{alph}(r'^{-1}r) &= \bigcup_{y \in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r^{-1}\,\mathrm{Re}(y)) \cup \mathrm{alph}(r'^{-1}r)) \\ &= \bigcup_{y \in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y))) \\ &\subseteq A'. \end{aligned}$$

Therefore, $x' \sqsubseteq x$ which concludes the proof. □

We turn now to the study of the algebraicity of the poset $(\mathbb{F}, \sqsubseteq)$. To this purpose, we first characterize compact α-traces.

Theorem 4.3

1. *An α-trace is compact if and only if it is finite.*
2. *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is algebraic.*

Proof: 1. Let $y = (s, B) \in \mathbb{F}_f$ be a finite α-trace and let $X \subseteq \mathbb{F}$ be a directed set such that $y \sqsubseteq \bigsqcup X$. The set $\mathrm{Re}(X)$ is $\leq$-directed and $s \leq \bigsqcup \mathrm{Re}(X) = \mathrm{Re}(\bigsqcup X)$. Since $s \in \mathbb{M}$ is a finite trace, s is a compact element of the dcpo $(\mathbb{R}, \leq)$ (see Section 2.2). Hence, there exists $x_0 \in X$ such that $s \leq \mathrm{Re}(x_0)$. Now, for all $a \notin \mathrm{Im}(\bigsqcup X) = \bigcap_{x \in X} \mathrm{Im}(x)$, there exists $x_a \in X$ such that $a \notin \mathrm{Im}(x_a)$. Since X is directed, there exists $x \in X$ which is an upper bound of x_0 and $(x_a)_{a \notin \mathrm{Im}(\bigsqcup X)}$. We will show that $y \sqsubseteq x$. First, $\mathrm{Re}(y) = s \leq \mathrm{Re}(x_0) \leq \mathrm{Re}(x)$. Second,

$$\mathrm{Im}(x) \subseteq \bigcap_{a \notin \mathrm{Im}(\bigsqcup X)} \mathrm{Im}(x_a) \subseteq \mathrm{Im}(\bigsqcup X) \subseteq \mathrm{Im}(y).$$

Third, $\mathrm{alph}(\mathrm{Re}(y)^{-1}\,\mathrm{Re}(x)) \subseteq \mathrm{alph}(\mathrm{Re}(y)^{-1}\,\mathrm{Re}(\bigsqcup X)) \subseteq \mathrm{Im}(y)$. Therefore, $y \sqsubseteq x$ which proves that y is compact.

In order to prove the converse and the second part of the theorem, we will use the following claim.

Claim: Let $y = (s, B) \in \mathbb{F}$ be an arbitrary α-trace, then $X = \downarrow y \cap \mathbb{F}_f$ is directed and $\bigsqcup X = y$.

Let (r', A') and (r'', A'') be two finite α-traces bounded above by y. Since r' and r'' are bounded above by s, the least upper bound $r = r' \vee r''$ exists, is finite and bounded above by s. From $r' \le r \le s$, we deduce that $r'^{-1}s = (r'^{-1}r)(r^{-1}s)$. Since $(r', A') \sqsubseteq y$, it follows that $(r', A') \sqsubseteq (r', B \cup \mathrm{alph}(r'^{-1}s)) \sqsubseteq (r, B \cup \mathrm{alph}(r^{-1}s))$. Similarly, $(r'', A'') \sqsubseteq (r, B \cup \mathrm{alph}(r^{-1}s))$ which proves that X is directed.

Note that, $\mathrm{Re}(X)$ is the set of compact real traces lower than s. Since $\mathbb{R}$ is algebraic, it follows $s = \bigsqcup \mathrm{Re}(X) = \mathrm{Re}(\bigsqcup X)$. Let $r \in \mathbb{M}$ be such that $r \le s$ and $\mathrm{alph}(r^{-1}s) = \mathrm{alphinf}(s)$. Since $\mathrm{alphinf}(s) \subseteq B$, we deduce that $(r, B \cup \mathrm{alph}(r^{-1}s)) = (r, B) \in X$. Therefore, $\mathrm{Im}(\bigsqcup X) = B$ which concludes the proof of the claim.

The first consequence of this claim is that a compact α-trace must be finite. Indeed, if $y = (s, B) \in \mathbb{F} \setminus \mathbb{F}_f$ is an infinite α-trace, then $y = \bigsqcup X$ but, being infinite, y is not lower than any member of $X \subseteq \mathbb{F}_f$.

2. From 1., we deduce that for all α-traces $y \in \mathbb{F}$, the set $K(y)$ of compact α-traces lower than y is precisely the set $\downarrow y \cap \mathbb{F}_f$. Therefore, the claim states in fact that the poset $(\mathbb{F}, \sqsubseteq)$ is algebraic. $\square$

From Theorems 4.1 and 4.3, we deduce

Corollary 4.4 *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is a domain, i.e., an algebraic dcpo.*

We address now the more complex problem of prime algebraicity of the poset $(\mathbb{F}, \sqsubseteq)$. As above, we will first characterize the prime α-traces. We introduce a notation. Let $r \in \mathbb{R}$ be a real trace. We denote by $\max(r)$ the set of letters which occur at maximal vertices of the graph r. For instance, let $(\Sigma, D) = a$ —— b —— c. Then, $\max(acb) = \{b\}$, $\max(acbac) = \{a, c\}$ and $\max((acb)^\omega) = \emptyset$. Recall that a non empty real trace r is prime if and only if it is finite and has exactly one maximal vertex ($|\max(r)| = 1$).

Proposition 4.5 *An α-trace $x = (r, A)$ is prime if and only if $r = 1$ and $|\Sigma \setminus A| = 1$ or r is a prime real trace and $A \cup \max(r) = \Sigma$.*

Proof: The only if part is a rephrasing of the following necessary conditions :

1. If $(r, A) \in \mathbb{F}$ is prime, then r is a prime real trace or $r = 1$.
2. If $(r, A) \in \mathbb{F}$ is prime, then $|\Sigma \setminus A| \le 1$.
3. If $(r, A) \in \mathbb{F}$ is prime and r is a prime real trace, then $A \cup \max(r) = \Sigma$.

Indeed, assume that $r = sa = tb$ for some $s, t \in \mathbb{M}$, $a, b \in \Sigma$, $(a, b) \in I$. Then the α-trace (r, A) is not prime since $(r, A) = (s, A \cup \{a\}) \vee (t, A \cup \{b\})$ but $(r, A) \not\sqsubseteq (s, A \cup \{a\})$ and $(r, A) \not\sqsubseteq (t, A \cup \{b\})$. Since a prime α-trace is also compact, its real part must be finite and the first point follows.

Now, assume that $a, b \in \Sigma \setminus A$ with $a \neq b$. It holds $(r, A) = (r, A \cup \{a\}) \vee (r, A \cup \{b\})$. Hence, (r, A) is not prime which proves 2.

Finally, assume that there exists a letter $b \in \Sigma \setminus (A \cup \{a\})$ where $a = \max(r)$. We may write $r = sa$ and we see that (r, A) cannot be prime since we have $(r, A) = (s, A \cup \{a\}) \vee (r, A \cup \{b\})$. Hence the last condition.

Conversely, assume that $x = (r, A) \in \mathbb{F}$ satisfies the conditions of the proposition. Let $Y \subseteq \mathbb{F}$ be a coherent set such that $x \sqsubseteq \bigsqcup Y = z = (t, C)$. Note that since $(1, \Sigma)$ does not satisfies the conditions of the proposition, the set Y must be nonempty.

First, assume that $A = \Sigma$. We have $r \leq t = \bigsqcup \mathrm{Re}(Y)$. Since $r = 1$ or r is a prime real trace, there exists $y \in Y$ such that $r \leq \mathrm{Re}(y)$. It follows that $x = (r, \Sigma) \sqsubseteq y$ which proves that x is prime.

Second, assume that $\Sigma \setminus A = \{a\}$. Since $C = \bigcap_{y \in Y} \mathrm{Im}(y) \subseteq A$, there exists $y = (s, B) \in Y$ such that $a \notin B$, i.e., $B \subseteq A$. We claim that $x \sqsubseteq y$ which concludes the proof of the proposition. Indeed, assume that $r \not\leq s$. Then, r is prime and $\max(r) = \{a\}$. Since $x \sqsubseteq z$ and $y \sqsubseteq z$, it follows $a \in \mathrm{alph}(s^{-1}t) \subseteq B$ which forms a contradiction. Therefore, $r \leq s$ and from $x \sqsubseteq z$ we deduce that $\mathrm{alph}(r^{-1}s) \subseteq \mathrm{alph}(r^{-1}t) \subseteq A$. Thus, the claim $x \sqsubseteq y$. $\square$

Thanks to this characterization of prime α-traces, we will now prove:

Theorem 4.6 *The coherently complete domain $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is prime algebraic.*

Proof: Let $x = (r, A) \in \mathbb{F}$ and let $P(x) = \{y \in \mathbb{F} \mid y \sqsubseteq x \text{ and } y \text{ is prime}\}$. Let $Y_1 = \{(s, \Sigma) \mid s \leq r \text{ and } s \text{ is prime}\}$. By Proposition 4.5, we have $Y_1 \subseteq P(x)$. Since $\mathbb{R}$ is prime algebraic, we deduce that $r = \bigsqcup \mathrm{Re}(Y_1) \leq \bigsqcup \mathrm{Re}(P(x)) \leq r$. From Theorem 4.1, we obtain $\mathrm{Re}(\bigsqcup P(x)) = \bigsqcup \mathrm{Re}(P(x)) = r$.

For all $a \in \Sigma \setminus A$, let $s_a \in \mathbb{R}$ be the least prefix of r such that $|s_a|_a = |r|_a$. Since $\mathrm{alphinf}(r) \subseteq A$, it holds $|r|_a < \infty$ and s_a is a finite real trace. Either $s_a = 1$, or s_a is a prime real trace with $\max(s_a) = \{a\}$. Let $y_a = (s_a, \Sigma \setminus \{a\}) \in \mathbb{F}$. By Proposition 4.5, the α-trace y_a is prime. Moreover, $y_a \sqsubseteq x$ and we obtain $Y_2 = \{y_a \mid a \in \Sigma \setminus A\} \subseteq P(x)$. Using Theorem 4.1, we deduce $A \subseteq \mathrm{Im}(\bigsqcup P(x)) \subseteq \bigcap_{a \notin A} \mathrm{Im}(y_a) = A$. Therefore, as desired $x = \bigsqcup P(x)$. $\square$

We conclude this section by showing that the concatenation on α-traces is continuous with respect to the approximation order. Note that, if $X, Y \subseteq \mathbb{F}$ are two coherent sets then it follows from Theorem 3.12 that $X \cdot Y$ is coherent and that $\bigsqcup(X \cdot Y) \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$. This inequality is strict in general as shown by the following example. Let $(\Sigma, D) = a$ — b — c, let $X = \{(1, \{a\}), (1, \{c\})\}$ and let $Y = \{(b, \emptyset)\}$. Then, $X \cdot Y = \{(1, \{a, b\}), (1, \{b, c\})\}$ and $\bigsqcup(X \cdot Y) = (1, \{b\})$ while $(\bigsqcup X) \cdot (\bigsqcup Y) = (1, \emptyset)(b, \emptyset) = (b, \emptyset) \neq \bigsqcup(X \cdot Y)$.

Proposition 4.7 *The concatenation on $\mathbb{F}(\Sigma, D)$ is continuous with respect to the approximation order: Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be two directed sets, then $X \cdot Y$ is directed and $\bigsqcup(X \cdot Y) = (\bigsqcup X) \cdot (\bigsqcup Y)$.*

Proof: Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be directed. As above, it follows directly from Theorem 3.12 that $X \cdot Y$ is directed and that $\bigsqcup(X \cdot Y) \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$. Since $\mathbb{F}$ is algebraic (Theorem 4.3), the converse inequality follows directly from the claim: $z \sqsubseteq \bigsqcup(X \cdot Y)$ for all compact z such that $z \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$.

Let $x = \bigsqcup X = (r, A)$, $y = \bigsqcup Y = (s, B)$ and $z = (t, C) \sqsubseteq x \cdot y$ be compact. We have $t \leq r\mu_A(s)$ and $A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1} r\mu_A(s)) \subseteq C$. Using the generalization of Levi's lemma to real traces (see Section 2.2), we deduce that $t = r's'$ where $r' \leq r$, $s' \leq \mu_A(s)$ and $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r) \subseteq I$. Note that this implies $t^{-1} r\mu_A(s) = (r'^{-1}r)(s'^{-1}\mu_A(s))$ and therefore $\sigma_A(s) \cup \mathrm{alph}(t^{-1} r\mu_A(s)) = \mathrm{alph}(r'^{-1}r) \cup \mathrm{alph}(s'^{-1}s)$.

Let $x' = (r', A \cup \mathrm{alph}(r'^{-1}r))$ and $y' = (s', C)$. We have $x' \sqsubseteq x$ and $y' \sqsubseteq y$. Since x' and y' are compact, it follows that $x' \sqsubseteq x''$ and $y' \sqsubseteq y''$ for some $x'' \in X$ and $y'' \in Y$. Now, $x' \cdot y' = (r's', A \cup \mathrm{alph}(r'^{-1}r) \cup C) = (t, C) = z$. Using Theorem 3.12, we obtain $z = x' \cdot y' \sqsubseteq x'' \cdot y'' \in X \cdot Y$ which concludes the proof of the claim. $\square$

Let us add a final remark. Recall that the partial monoid $(\mathbb{R}, \cdot)$ is embedded in the monoid $(\mathbb{F}, \cdot)$ by the partial morphism $\chi : r \mapsto (r, \mathrm{alphinf}(r))$. In fact, the poset $(\mathbb{R}, \leq)$ may also be embedded in the poset $(\mathbb{F}, \sqsubseteq)$ but the embedding is quite different. This time, we associate with a real trace $r \in \mathbb{R}$ the α-trace $\Xi(r) = (r, \Sigma)$. Clearly, for all real traces $r, s \in \mathbb{R}$, we have $r \leq s$ if and only if $\Xi(r) \sqsubseteq \Xi(s)$. In other words, Ξ is a poset morphism which embeds the poset $(\mathbb{R}, \leq)$ into the poset $(\mathbb{F}, \sqsubseteq)$. Note that the poset morphism Ξ is continuous. In fact, if $R \subseteq \mathbb{R}$ is $\leq$-coherent then $\Xi(R)$ is $\sqsubseteq$-coherent and $\Xi(\bigsqcup R) = \bigsqcup \Xi(R)$. Moreover, a real trace $r \in \mathbb{R}$ is $\leq$-compact ($\leq$-prime resp.) if and only if the α-trace $\Xi(r)$ is $\sqsubseteq$-compact ($\sqsubseteq$-prime resp.).

5 Topology

In this section, we introduce a very natural ultra-metric d on α-traces. We show that the topology induced by this metric is precisely the Lawson topology [Law88] associated with the approximation order. Using the results of Section 4, we deduce that the metric space $(\mathbb{F}, d)$ is compact and complete. Moreover, the set $\mathbb{F}_f$ of finite α-traces is discrete and dense in $(\mathbb{F}, d)$. Then, we prove that the concatenation is uniformly continuous. Finally, we show that the main topological properties of α-traces (compactness, concatenation uniformly continuous) can be obtained using projective limits.

In order to define the ultra metric, we first introduce the n^{th} approximation of a real trace and that of an α-trace.

Definition 5.1 *Let $n \in \mathbb{N}$.*

1. *Let $r \in \mathbb{R}(\Sigma, D)$, the n^{th} approximation of r is the finite trace*

$$r[n] = \bigsqcup \{p \in \mathbb{M}(\Sigma, D) \mid p \leq r \text{ and } |p| \leq n\}$$

Moreover, we denote by $r[-n] = r[n]^{-1}r$ the corresponding suffix.

2. *Let $x \in \mathbb{F}(\Sigma, D)$, the n^{th} approximation of x is the finite α-trace*

$$x[n] = \bigsqcup\{y \in \mathbb{F}_f(\Sigma, D) \mid y \sqsubseteq x \text{ and } |y| \leq n\}$$

Clearly, if $r \in \mathbb{R}$ is a real trace, then $(r[n])_{n\geq 0}$ is an $\leq$-increasing sequence bounded above by r. Hence, $\bigsqcup_{n\geq 0} r[n] \sqsubseteq r$. Conversely, let $K(r) \subseteq \mathbb{M}$ be the set of compact real traces lower than r. For all $k \in K(r)$, we have $k \leq r[|k|]$. Hence, $\bigsqcup K(r) \sqsubseteq \bigsqcup_{n\geq 0} r[n]$. Since $(\mathbb{R}, \leq)$ is algebraic we deduce that $r = \bigsqcup_{n\geq 0} r[n]$.

Similarly, for all α-traces $x \in \mathbb{F}$, the sequence $(x[n])_{n\geq 0}$ is $\sqsubseteq$-increasing and we have $x = \bigsqcup_{n\geq 0} x[n]$.

Proposition 5.2 *Let $x = (r, A) \in \mathbb{F}(\Sigma, D)$ be an α-trace and let $n \in \mathbb{N}$. Then,*

$$x[n] = (r[n], A \cup \mathrm{alph}(r[-n]))$$

Proof: Let $Y = \{y \in \mathbb{F} \mid y \sqsubseteq x \text{ and } |y| \leq n\}$. Let $R = \{p \in \mathbb{M} \mid p \leq r \text{ and } |p| \leq n\}$. Clearly, $\mathrm{Re}(Y) \subseteq R$. Conversely, let $p \in R$, then $y = (p, A \cup \mathrm{alph}(p^{-1}r)) \in Y$. Hence, $R = \mathrm{Re}(Y)$ and $\mathrm{Re}(x[n]) = \mathrm{Re}(\bigsqcup Y) = \bigsqcup \mathrm{Re}(Y) = \bigsqcup R = r[n]$.

Since $x[n] \sqsubseteq x$, we have $A \cup \mathrm{alph}(r[-n]) \subseteq \mathrm{Im}(x[n])$. Conversely, we have seen above that $\{(p, A \cup \mathrm{alph}(p^{-1}r)) \mid p \in R\} \subseteq Y$. Therefore,

$$\begin{aligned}
\mathrm{Im}(x[n]) &= \mathrm{Im}(\bigsqcup Y) = \bigcap_{y \in Y} \mathrm{Im}(y) \\
&\subseteq \bigcap_{p \in R} (A \cup \mathrm{alph}(p^{-1}r)) = A \cup \left(\bigcap_{p \in R} \mathrm{alph}(p^{-1}r) \right).
\end{aligned}$$

Hence, the result follows from the claim:

Claim:

$$\mathrm{alph}(r[-n]) = \bigcap_{p \in R} \mathrm{alph}(p^{-1}r)$$

For all $p \in R$, we have $p \leq r[n] \leq r$. Hence, $\mathrm{alph}(r[-n]) \subseteq \bigcap_{p\in R} \mathrm{alph}(p^{-1}r)$. Conversely, let $a \notin \mathrm{alph}(r[-n])$ and let q be the least prefix or r such that $|q|_a = |r|_a$. Since $\mathrm{alphinf}(r) \subseteq \mathrm{alph}(r[-n])$, we have $|q|_a = |r|_a < \infty$ and the trace q is either empty or prime. Moreover, $q \leq r[n] = \bigsqcup R$. Since $R = \downarrow R$ is downward closed, we deduce that $q \in R$. Therefore, $\bigcap_{p\in R} \mathrm{alph}(p^{-1}r) \subseteq \mathrm{alph}(q^{-1}r)$ and the claim follows from $a \notin \mathrm{alph}(q^{-1}r)$. □

Using these n^{th} approximations, we give now the definition of the metric on α-traces.

Definition 5.3 *Let $x, y \in \mathbb{F}(\Sigma, D)$ be two α-traces. The distance of x and y is $d(x, y) = 2^{-\ell(x,y)}$ where*

$$\ell(x, y) = \inf\{n \in \mathbb{N} \mid x[n] \neq y[n]\} \in \mathbb{N} \cup \{\infty\}$$

Let $x \in \mathbb{F}(\Sigma, D)$ and let $n \in \mathbb{N}$. The n^{th} open ball centered on x is

$$B(x, n) = \{y \in \mathbb{F}(\Sigma, D) \mid \ell(x, y) > n\}.$$

By convention, $\inf(\emptyset) = \infty$ and $2^{-\infty} = 0$. Hence, for all $x \in \mathbb{F}$, $d(x, x) = 0$. Conversely, let $x, y \in \mathbb{F}$ and assume that $d(x, y) = 0$. Then for all $n \in \mathbb{N}$, $x[n] = y[n]$ and we deduce that $x = \bigsqcup_{n\geq 0} x[n] = \bigsqcup_{n\geq 0} y[n] = y$. Clearly, the distance d is symmetric and for all $x, y, z \in \mathbb{F}$, it holds $d(x, z) \leq \max(d(x, y), d(y, z))$. Hence d is an ultra-metric distance on $\mathbb{F}$. Moreover, the family of open balls

$$\mathcal{B} = \{B(x, n) \mid x \in \mathbb{F} \text{ and } n \in \mathbb{N}\}$$

is an open basis for $(\mathbb{F}, d)$.

We will now show that the d-topology on $\mathbb{F}$ coincides with the $\sqsubseteq$-Lawson topology on $\mathbb{F}$. We first recall some definitions. Let $(T, \mathcal{O})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{O}$ is a basis of the topology if any open set can be written as an arbitrary union of opens from the basis $\mathcal{B}$. A family $\mathcal{S} \subseteq \mathcal{O}$ is a subbasis of open sets if the family of finite intersections of opens from $\mathcal{S}$ is a basis of the topological space $(T, \mathcal{O})$.

Let $(Z, \leq)$ be a domain, i.e., an algebraic dcpo. The family

$$\mathcal{S} = \{\uparrow k \mid k \text{ is compact}\} \cup \{Z \setminus \uparrow k \mid k \text{ is compact}\}$$

is a subbasis of the Lawson topology on $(Z, \leq)$. We have given the definition of the Lawson topology for domains only. For a more general definition, the reader is referred to [GHK+80].

Proposition 5.4 *The topology on $\mathbb{F}(\Sigma, D)$ induced by d is the Lawson topology on $\mathbb{F}(\Sigma, D)$ induced by the approximation order.*

Proof: Let $x = (r, A) \in \mathbb{F}_f$ be a compact α-trace and let $n = |x| = |r| < \infty$. Let $y, z \in \mathbb{F}$ be such that $\ell(y, z) > n$. Then,

$$y \in \uparrow x \iff x \sqsubseteq y[n] \iff x \sqsubseteq z[n] \iff z \in \uparrow x$$

Hence, for all $y \in \mathbb{F}$, either $B(y, n) \subseteq \uparrow x$ or $B(y, n) \subseteq \mathbb{F} \setminus \uparrow x$. Therefore, $\uparrow x$ and $\mathbb{F} \setminus \uparrow x$ are both d-open and d-closed. The d-topology on $\mathbb{F}$ is thus finer than the $\sqsubseteq$-Lawson topology on $\mathbb{F}$.

Claim: Let $x \in \mathbb{F}$ and let $n \in \mathbb{N}$. Then,

$$B(x, n) = \uparrow x[n] \setminus \uparrow \{z[n] \mid z[n] \not\sqsubseteq x[n]\}.$$

Let $y \in B(x, n)$. We have $x[n] = y[n] \sqsubseteq y$. Hence $B(x, n) \subseteq \uparrow x[n]$. Now assume that $y \in \uparrow z[n]$ for some $z \in \mathbb{F}$. From $z[n] \sqsubseteq y$, we deduce easily that $z[n] \sqsubseteq y[n] = x[n]$. Therefore, $B(x, n) \cap \uparrow \{z[n] \mid z[n] \not\sqsubseteq x[n]\} = \emptyset$.

Conversely, let $y \in \uparrow x[n] \setminus \uparrow \{z[n] \mid z[n] \not\sqsubseteq x[n]\}$. From $x[n] \sqsubseteq y$, we deduce that $x[n] \sqsubseteq y[n]$. Now, since $y \in \uparrow y[n]$ and $y \notin \uparrow \{z[n] \mid z[n] \not\sqsubseteq x[n]\}$ we deduce that $y[n] \sqsubseteq x[n]$. Therefore, $x[n] = y[n]$, i.e., $y \in B(x, n)$ and the claim is proved.

From the claim and the definition of the Lawson topology, we deduce that the d-open balls are also $\sqsubseteq$-Lawson open. $\square$

From Proposition 5.4, we will infer several properties which are general facts for the Lawson topology. First, let $(Z, \leq)$ be a dcpo and let $(x_n)_{n\geq 0} \subseteq Z$ be an increasing sequence. Then, the sequence $(x_n)_{n\geq 0}$ converges to $\bigsqcup_{n\geq 0} x_n$ for the Lawson topology (see [GHK+80]). Hence,

Corollary 5.5 *Let $(x_n)_{n\geq 0} \subseteq \mathbb{F}(\Sigma, D)$ be an $\sqsubseteq$-increasing sequence of α-traces. Then, the sequence $(x_n)_{n\geq 0}$ converges in $(\mathbb{F}(\Sigma, D), d)$ and*

$$\lim_{n\to\infty} x_n = \bigsqcup_{n\geq 0} x_n$$

We give now the most important corollary of Proposition 5.4. Let $(Z, \leq)$ be a poset which is both bounded complete and directed complete. Then, the Lawson topology is quasi-compact and T_1. If in addition, $(Z, \leq)$ is algebraic then the Lawson topology is also T_2 (Hausdorff), whence compact, and the set $K(Z)$ of compact elements of Z is dense in Z. For all these properties, the reader is referred to [GHK+80] where they are stated and proved for slightly more general posets, namely continuous complete semi-lattices.

Theorem 5.6 *The metric space $(\mathbb{F}(\Sigma, D), d)$ is compact and complete. The set $\mathbb{F}_f(\Sigma, D)$ of finite α-traces is dense, discrete and open in $(\mathbb{F}(\Sigma, D), d)$.*

Proof: By Theorems 4.1 and 4.3, the poset $(\mathbb{F}, \sqsubseteq)$ is a bounded complete domain. Since the d-topology is the $\sqsubseteq$-Lawson topology, it follows that $(\mathbb{F}, d)$ is a compact (whence complete) metric space and that the set $\mathbb{F}_f$ of compact α-traces (Theorem 4.3) is dense in $(\mathbb{F}, d)$.

Hence, it remains to prove that $\mathbb{F}_f$ is discrete (this is not a general fact for the Lawson topology). Let $x = (r, A) \in \mathbb{F}_f$ be a finite α-trace and let $n = |x| = |r|$. We claim that $B(x, n+1) = \{x\}$. Indeed, let $y = (s, B) \in B(x, n+1)$. We deduce from Proposition 5.2 that $r[n] = s[n]$ and $r[n+1] = s[n+1]$. Since $|r| = n$, we have $r = r[n] = r[n+1]$. Hence, $s[n] = s[n+1]$ and $|y| = |s| \leq n$. Therefore, $y = y[n] = x[n] = x$ which proves the claim.

We deduce directly from this claim that $\mathbb{F}_f$ is discrete, whence open. $\square$

We have seen in the previous section (Proposition 4.7) that the concatenation of α-traces is continuous with respect to the approximation order. We prove here a stronger result: the concatenation is uniformly continuous for the Lawson topology induced by the approximation order. This will be an immediate consequence of the following lemma.

Lemma 5.7 *For all α-traces $x, y \in \mathbb{F}$ and for all $n \geq 0$, it holds*

$$(x \cdot y)[n] = (x[n] \cdot y[n])[n]$$

Proof: Let $x, y \in \mathbb{F}$ be α-traces. Since $x[n] \sqsubseteq x$ and $y[n] \sqsubseteq y$, we have by Theorem 3.12 $x[n] \cdot y[n] \sqsubseteq x \cdot y$ and it follows $(x[n] \cdot y[n])[n] \sqsubseteq (x \cdot y)[n]$. Conversely, we claim that $z \sqsubseteq x[n] \cdot y[n]$ for all $z \sqsubseteq x \cdot y$ such that $|z| \leq n$. Note that this claim allows to conclude the proof of the lemma since clearly it implies $(x \cdot y)[n] \sqsubseteq (x[n] \cdot y[n])[n]$.

In order to prove this claim, assume that $x = (r, A)$, $y = (s, B)$ and let $z = (t, C)$ be a finite α-trace of length at most n such that $z \sqsubseteq x \cdot y$. By definition of the approximation order and of the concatenation on α-traces, we have $t \leq r\mu_A(s)$ and

$A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1}r\mu_A(s)) \subseteq C$. Using the generalization of Levi's lemma to real traces (Section 2.2) we obtain $t = r's'$ for some $r', s' \in \mathbb{M}$ such that $r' \leq r$, $s' \leq \mu_A(s)$ and $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r) \subseteq I$.

To simplify the notations, we let $A' = A \cup \mathrm{alph}(r[-n])$ and $B' = B \cup \mathrm{alph}(s[-n])$ so that $x[n] = (r[n], A')$ and $y[n] = (s[n], B')$. Note that, since $|t| = |r's'| \leq n$, we have $r' \leq r[n] \leq r$ and $A' \subseteq A \cup \mathrm{alph}(r'^{-1}r)$. Similarly, $s' \leq s[n] \leq s$ and $B' \subseteq B \cup \mathrm{alph}(s'^{-1}s)$.

We have to prove that $z \sqsubseteq x[n] \cdot y[n]$, i.e.,

$$t \leq r[n]\mu_{A'}(s[n]) \tag{5}$$

and

$$A' \cup B' \cup \sigma_{A'}(s[n]) \cup \mathrm{alph}(t^{-1}r[n]\mu_{A'}(s[n])) \subseteq C. \tag{6}$$

To prove (5) we use again the generalization of Levi's lemma to real traces (Section 2.2). We thus have to prove that

$$r' \leq r[n], s' \leq \mu_{A'}(s[n]) \text{ and } \mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r[n]) \subseteq I.$$

We have already seen that $r' \leq r[n] \leq r$ and $s' \leq s[n] \leq s$. From $A' \subseteq A \cup \mathrm{alph}(r'^{-1}r)$ and $\mathrm{alph}(s') \times (A \cup \mathrm{alph}(r'^{-1}r)) \subseteq I$, we deduce that $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r[n]) \subseteq I$ and $s' \leq \mu_{A'}(s[n])$, which completes the proof of (5).

Now, we have $r' \leq r[n] \leq r$ and $s' \leq \mu_{A'}(s[n]) \leq s[n] \leq s$. Using in addition $t^{-1}r\mu_A(s) = (r'^{-1}r)(s'^{-1}\mu_A(s))$ and $t^{-1}r[n]\mu_{A'}(s[n]) = (r'^{-1}r[n])(s'^{-1}\mu_{A'}(s[n]))$ we deduce that

$$\begin{aligned} A' \cup B' \cup \sigma_{A'}(s[n]) \cup \mathrm{alph}(t^{-1}r[n]\mu_{A'}(s[n])) & \\ \subseteq \;& A \cup \mathrm{alph}(r'^{-1}r) \cup B \cup \mathrm{alph}(s'^{-1}s) \\ = \;& A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1}r\mu_A(s)) \\ \subseteq \;& C \end{aligned}$$

which proves (6). $\square$

Proposition 5.8 *The concatenation on α-traces is uniformly continuous. More precisely, let $x, x', y, y' \in \mathbb{F}(\Sigma, D)$ be α-traces. Then*

$$d(xy, x'y') \leq \max(d(x, x'), d(y, y')).$$

Proof: From Lemma 5.7, we deduce that the set of approximations of length at most n of xy depends only on the n^{th} approximations of x and y. Therefore, for all $n \leq 0$, $x[n] = x'[n]$ and $y[n] = y'[n]$ imply $(xy)[n] = (x'y')[n]$ which proves the proposition. $\square$

Note that Proposition 4.7 is in fact a corollary of Proposition 5.8. Indeed, a Lawson continuous monotone mapping between two domains is necessarily $\bigsqcup$-continuous with respect to the partial orders of the posets. However, the converse is false in general. Hence, we cannot infer the uniform continuity of the concatenation from its $\bigsqcup$-continuity and the compactness of $(\mathbb{F}, d)$.

Most of the results of this section can also be obtained by means of projective limits. Let us consider the sets $\mathbb{F}[n] = \{x[n] \mid x \in \mathbb{F}\}$. Thanks to Lemma 5.7, we see that the operation $\cdot_n$ defined on $\mathbb{F}[n]$ by $x[n] \cdot_n y[n] = (x \cdot y)[n]$ is well defined. It is clearly associative with $(1, \emptyset)$ as neutral, whence $(\mathbb{F}[n], \cdot_n)$ is a monoid. Now, for $n \geq m$, let $\varphi_{n,m} : \mathbb{F}[n] \longrightarrow \mathbb{F}[m]$ be defined by $\varphi_{n,m}(x[n]) = x[m]$. This mapping is well defined since $x[m] = x[n][m]$ for $n \geq m$. Moreover, for $n \geq m \geq p$ we clearly have $\varphi_{n,p} = \varphi_{m,p} \circ \varphi_{n,m}$ hence these mappings define a projective system. Finally, it is easy to see that the mappings $\varphi_{n,m}$ for $n \geq m$ are morphisms. Therefore, the projective limit

$$\varprojlim \mathbb{F}[n] = \{(x_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{F}[n] \mid \varphi_{n,m}(x_n) = x_m \text{ for all } n \geq m\}$$

is a monoid with the operation $(x_n)_{n\geq 0} \cdot (y_n)_{n\geq 0} = (x_n \cdot_n y_n)_{n\geq 0}$. We claim that this projective monoid is actually isomorphic to the monoid of α-traces. Indeed, we have clearly a morphism $\varphi : (\mathbb{F}, \cdot) \longrightarrow (\varprojlim \mathbb{F}[n], \cdot)$ defined by $\varphi(x) = (x[n])_{n\geq 0}$ for $x \in \mathbb{F}$. Since $x = \bigsqcup_{n\geq 0} x[n]$, the morphism φ is injective. To show that it is also onto, let $(x_n)_{n\geq 0} \in \varprojlim \mathbb{F}[n]$. Since this is a projective sequence, we have $x_m = \varphi_{n,m}(x_n) = x_n[m] \sqsubseteq x_n$ for all $n \geq m$. Hence, the sequence $(x_n)_{n\geq 0}$ is increasing and we can consider $x = \bigsqcup_{n\geq 0} x_n \in \mathbb{F}$. We claim that $\varphi(x) = (x_n)_{n\geq 0}$ which shows that φ is surjective. Indeed, let $m \geq 0$, the α-trace $x[m]$ is compact and $x[m] \sqsubseteq x = \bigsqcup_{n\geq 0} x_n$. Hence, $x[m] \sqsubseteq x_n$ for some $n \geq m$ and we obtain $x[m] \sqsubseteq x_n[m] = x_m$. Conversely, $x_m \sqsubseteq x$ and we get $x_m = x_m[m] \sqsubseteq x[m]$. Therefore, $x[m] = x_m$ which proves the claim.

Now, the sets $\mathbb{F}[n]$ for $n \geq 0$ are finite, whence compact for the discrete topology and the morphisms $\varphi_{n,m}$ for $n \geq m$ are continuous. Therefore, the projective limit $\mathbb{F} = \varprojlim \mathbb{F}[n]$ is compact for the product topology. Moreover, the concatenations $\cdot_n$ for $n \geq 0$ are also continuous and it follows that the concatenation $\cdot$ on $\mathbb{F} = \varprojlim \mathbb{F}[n]$ is continuous, whence uniformly continuous since $\mathbb{F} = \varprojlim \mathbb{F}[n]$ is compact. Finally, we have seen that $(\mathbb{F}, \cdot) = (\varprojlim \mathbb{F}[n], \cdot)$ is a compact monoid for the product topology, with a uniformly continuous concatenation.

Now, the discrete topology on $\mathbb{F}[n]$ is induced by the ultra-metric d_n defined by

$$d_n(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Hence, the product topology on $\mathbb{F} = \varprojlim \mathbb{F}[n]$ is induced by the ultra-metric distance δ defined for $x, y \in \mathbb{F}$ by

$$\delta(x, y) = \frac{1}{2} \sum_{n\geq 0} 2^{-n} d_n(x[n], y[n]) = \frac{1}{2} \sum_{n\geq \ell(x,y)} 2^{-n} = 2^{-\ell(x,y)}.$$

Therefore, the product topology on $\mathbb{F} = \varprojlim \mathbb{F}[n]$ is precisely the topology induced by the metric d of Definition 5.3 which was studied in this section.

6 Delta-Traces

One might object that the domain of α-traces is not abstract enough. If we consider the full dependence relation $D = \Sigma \times \Sigma$ then we are in the sequential case and the real part of an α-trace is simply a word : $\mathbb{F}(\Sigma, \Sigma \times \Sigma) \subseteq \Sigma^\infty \times \mathcal{P}(\Sigma)$. Indeed, it has all the good properties seen in the previous sections but this is also the case of the more abstract sequential model $\Sigma^*\bot \cup \Sigma^\infty$ discussed at the beginning of Section 3 : $(\Sigma^*\bot \cup \Sigma^\infty, \sqsubseteq)$ is a prime algebraic coherently complete poset; the compacts, which coincide with the primes, are the finite behaviours; the concatenation is monotone and continuous with respect to the approximation ordering.

Intuitively, in the sequential case, when a behaviour is not terminated it is not necessary to know which actions may still occur because, whatever these possible actions are, we cannot start the execution of another process. In other words, in $\mathbb{F}(\Sigma, \Sigma \times \Sigma)$ the relation $(r, A) \equiv (s, B)$ if $r = s$ and either $A = B = \emptyset$ or $A \neq \emptyset \neq B$ is a congruence with respect to the concatenation.

This remark can be extended to the non sequential case. If a behaviour x is not terminated, one can immediately start all actions independent from the actions which may still occur in x. Therefore, if a process p has already performed the real behaviour r and A contains all actions which may still occur in p, then it is sufficient to know $D(A) = \{b \in \Sigma \mid (b, a) \in D \text{ for some } a \in A\}$ to decide which actions can be started concurrently to the end of the process p. In other words, the relation on $\mathbb{F}(\Sigma, D)$ defined by $(r, A) \equiv (s, B)$ if $r = s$ and $D(A) = D(B)$ is a congruence for the concatenation. This yields to the following definition.

Definition 6.1 *Let (Σ, D) be a dependence alphabet. A δ-trace over (Σ, D) is an ordered pair $(r, D(A))$ where $r \in \mathbb{R}(\Sigma, D)$ is a real trace and $A \subseteq \Sigma$ is such that $D(\mathrm{alphinf}(r)) \subseteq D(A)$. The set of δ-traces is denoted by $\mathbb{F}_\delta(\Sigma, D)$ or simply by $\mathbb{F}_\delta$. The concatenation on $\mathbb{F}_\delta(\Sigma, D)$ is defined by*

$$(r, D(A)) \cdot (s, D(B)) = (r\mu_A(s), D(A \cup B \cup \sigma_A(s)))$$

and the approximation relation is defined by

$$(r, D(A)) \sqsubseteq (s, D(B)) \text{ if } r \leq s \text{ and } D(B \cup \mathrm{alph}(r^{-1}s)) \subseteq D(A)$$

Note that the concatenation is well defined since for all real traces s, both $\mu_A(s)$ and $\sigma_A(s)$ depend on $D(A)$ only. $(\mathbb{F}_\delta(\Sigma, D), \cdot)$ is a monoid with $(1, \emptyset)$ as neutral and $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is a poset with $(1, \Sigma)$ as bottom element. Moreover, the concatenation of δ-traces is monotone for the approximation order.

If $D = \Sigma \times \Sigma$ is the full dependence relation then the consideration of $\mathbb{F}_\delta(\Sigma, \Sigma \times \Sigma)$ is equivalent to the formalism of sequential words together with the non-termination symbol $\bot$. In fact, for $D(A)$ we only have two possibilities: either $D(A) = \Sigma$, which means non-termination, or $D(A) = \emptyset$, which means explicit termination. A δ-trace in this special case either is a finite non-terminated word $v\bot$, with $v \in \Sigma^*$, or it is a finite terminated word $v \in \Sigma^*$, or it is an infinite word $v \in \Sigma^\omega$ (where the non-termination symbol is redundant and may be omitted). Therefore $\mathbb{F}_\delta(\Sigma, \Sigma \times \Sigma) = \Sigma^*\bot \cup \Sigma^\infty$.

Moreover, the calculus of δ-traces is as usual such that $v\bot$ or infinite words are right-absorbent. Furthermore, for all $u, v \in \Sigma^\infty$, we have $u\bot \sqsubseteq v\bot \sqsubseteq v$ if and only if $u \le v$.

Even if $\mathbb{F}_\delta(\Sigma, D)$ is more abstract than $\mathbb{F}(\Sigma, D)$, we have concentrated our attention on the latter because we believe that it is more intuitive and easier to deal with. Moreover, most properties (but not all) can be transfered from $\mathbb{F}(\Sigma, D)$ to $\mathbb{F}_\delta(\Sigma, D)$ using the canonical surjective and monotone morphism $\Phi : \mathbb{F}(\Sigma, D) \longrightarrow \mathbb{F}_\delta(\Sigma, D)$ defined by $\Phi(r, A) = (r, D(A))$. Finally, as we will see below, the domain $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is less satisfactory than $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ since it is neither distributive nor prime algebraic in general.

Let us sketch the main steps which allow to transfer results from α-traces to δ-traces. We mainly use general facts from poset theory which are easy to prove. If needed, the reader should be able to reconstruct the detailed proofs from the skeleton below.

Note that, we can also embed $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ into $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ with the monotone mapping Ψ defined by $\Psi(r, D(A)) = (r, A_{\max})$ where $A_{\max} = \{a \in \Sigma \mid D(a) \subseteq D(A)\}$. We have $\Phi \circ \Psi = \mathrm{Id}$ but Ψ is not a morphism. For instance, if $(\Sigma, D) = a — b — c$ then $\Psi(1, D(a)) \cdot \Psi(1, D(c)) = (1, \{a, c\}) \ne (1, \Sigma) = \Psi((1, D(a)) \cdot (1, D(c)))$.

Let $Y \subseteq \mathbb{F}_\delta(\Sigma, D)$ be a coherent set of δ-traces. Then, $X = \Psi(Y) \subseteq \mathbb{F}(\Sigma, D)$ is coherent and we can verify that $\sqcup Y$ exists and

$$\sqcup Y = \Phi(\sqcup X) = (\sqcup \mathrm{Re}(Y), D(\{a \in \Sigma \mid D(a) \subseteq \mathrm{Im}(y) \text{ for all } y \in Y\})).$$

Therefore, $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is a coherently complete poset.

The mappings Φ and Ψ are both $\sqcup$-continuous. This allows to deduce that a δ-trace y is compact if and only if $\Psi(y)$ is a compact α-trace, i.e., if and only if y is finite. Then, we easily obtain that $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is algebraic, whence a coherently complete domain.

The situation is not as good for the prime algebraicity. First, the mapping Ψ is $\sqcup$-continuous even for coherent sets : We have $\Psi(\sqcup X) = \sqcup \Psi(X)$ for all coherent sets $X \subseteq \mathbb{F}_\delta(\Sigma, D)$. This is not the case for the mapping Φ (consider $\Sigma = \{a, b, c\}$, $D = \Sigma \times \Sigma$ and $X = \{(1, a), (1, b)\}$). Therefore if $y \in \mathbb{F}_\delta(\Sigma, D)$ is such that $\Psi(y)$ is a prime α-trace, then we can deduce that y is a prime δ-trace. But we cannot infer the converse. In fact, the converse is false in general : let $(\Sigma, D) = a — b — c$, then $y = (1, D(a)) \in \mathbb{F}_\delta(\Sigma, D)$ is a prime δ-trace while $\Psi(y) = (1, a)$ is not a prime α-trace. Moreover, the domain of δ-trace is not prime algebraic in general. We have the following counterexample: Let $(\Sigma, D) = a — b — c — d$ be the line of four letters. Then the finite δ-trace $(1, D(a))$ is not the least upper bound of its prime approximations. The approximations of $(1, D(a))$ are in fact $(1, \Sigma) \sqsubset (1, D(a, b)) \sqsubset (1, D(a))$, and its only prime approximation is $(1, D(a, b))$. Note that $(1, D(a))$ is not prime since $(1, D(a)) \sqsubseteq (1, D(b)) \vee (1, D(c)) = (1, \emptyset)$ but $(1, D(a)) \not\sqsubseteq (1, D(b))$ and $(1, D(a)) \not\sqsubseteq (1, D(c))$.

The characterization of the dependence alphabets for which $\mathbb{F}_\delta(\Sigma, D)$ is prime algebraic is still an open question. Moreover, it would be interesting to give also a characterization of prime δ-traces.

Finally, since they do not use the fact that $\mathbb{F}(\Sigma, D)$ is prime algebraic, all results of Section 5 extend easily to δ-traces. The n^{th} approximation of a δ-trace $x = (r, D(A))$ is

$$\begin{aligned} x[n] &= \sqcup\{y \in \mathbb{F}_\delta(\Sigma, D) \mid y \sqsubseteq x \text{ and } |y| \leq n\} \\ &= (r[n], A \cup D(\{a \in \Sigma \mid D(a) \subseteq D(p^{-1}r) \text{ for all } p \leq r, |p| \leq n\})) \end{aligned}$$

The ultra-metric distance $d_\delta(x, y) = 2^{-\ell_\delta(x,y)}$ where

$$\ell_\delta(x, y) = \inf\{n \in \mathbb{N} \mid x[n] \neq y[n]\} \in \mathbb{N} \cup \{\infty\}$$

induces the Lawson topology on $\mathbb{F}_\delta(\Sigma, D)$. Hence, $(\mathbb{F}_\delta(\Sigma, D), d_\delta)$ is a compact and complete metric space. The set of finite δ-traces is dense, discrete and open in $\mathbb{F}_\delta(\Sigma, D)$. Finally, the concatenation of δ-traces is uniformly continuous for d_δ, whence it is also $\sqcup$-continuous. The proofs of all these results are quite similar to the proofs of the corresponding results in Section 5.

7 Conclusion

We have seen that α-traces enjoy all desired mathematical properties needed for modular semantics of concurrent systems with recursive constructs. We believe that this model will be very convenient and powerful and we plan to experience it by giving semantics of classical concurrent languages like CSP or CCS [Hoa85, Mil80].

On the other hand, we have seen that α-traces are a bit too concrete while δ-traces are a bit too abstract. The lack of abstractness of α-traces is not a big problem since it concerns only the imaginary part which is finite and in fact bounded by the size of the alphabet. Still, it would be interesting to find the right intermediary domain between α-traces and δ-traces.

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