A Simple Modal Encoding of Propositional Finite Many-Valued Logics

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Abstract

We present a method for testing the validity for any finite many-valued logic by using simple transformations into the validity problem for von Wright's logic of elsewhere. The method provides a new original viewpoint on finite many-valuedness. Indeed, we present a uniform modal encoding of any finite many-valued logic that views truth-values as nominals. Improvements of the transformations are discussed and the translation technique is extended to any finite annotated logic. Using similar ideas, we conclude the paper by defining transformations from the validity problem for any finite many-valued logic into TAUT (the validity problem for the classical propositional calculus). As already known, this sharply illustrates that reasoning within a finite many-valued logic can be naturally and easily encoded into a two-valued logic. All the many-one reductions in the paper are tight since they require only time in $\mathcal{O}(n.log\ n)$ and space in $\mathcal{O}(log\ n)$.

Key-words: finite many-valued logic, modal logic of elsewhere, many-one reduction

1 Introduction

Background. Many-valued logics devised as formal systems have been extensively studied since [Łuk20, Pos21]. Several applications exist in various areas such as in Mathematical Logic (see e.g. [Łuk63]), in Quantum Physics (see e.g. [Rei44]), in Theoretical Computer Science (see e.g. [Ras77, Fit91]) and in Artificial Intelligence (see e.g. [Gin88]). The proof theory of finite (propositional and first-order) many-valued logics is quite well understood (see e.g. [Tak68, Mor76, Sur77, Car87, Wój88, Zac93, Häh93, Zab93, BF95, Sal96]) and various publicly available running systems exist (see e.g. [BHOS96, fMVL96]).

Our contribution. In this work, we concentrate on the relationships between finite propositional many-valued logics and modal logic, proposing an original approach about the essence of finite many-valuedness. The need for such a comparison is mainly motivated by Urquhart's statement in the survey paper [Urq86, Section 3.1]:

> *"..., the semantical methods involving relational model structures, 'possible worlds' and the like, which [...] have proved so fruitful in areas like modal logic, seem to have no clear connection with traditional many-valued logic."*

Our goal here is to clarify such a connection. A relational semantics for finite many-valued logics can be found in [KMO98] (see also [MO93]). In the present paper, we show that there exists a Sahlqvist modal logic [Sah75], say $\mathtt{L}_{\neq}$, such that any finite many-valued logic has a *natural embedding* into $\mathtt{L}_{\neq}$. Extensions are also given for finite annotated logics. The result is of interest especially because it can be then argued that any finite many-valued logic (and any finite annotated logic) can be viewed as a modal system. Furthermore, we also define transformations ("many-one reductions", see e.g. [Pap94]) from the validity problem for any finite many-valued logic into $\mathtt{TAUT}$ (the validity problem for the propositional calculus PC) using similar ideas. This clearly illustrates that reasoning within a finite many-valued logic can be easily and naturally encoded into a two-valued logic. Although this fact has been known at least since [RT52] (see also [Zin63]), we also provide evidence here that such an encoding can be effectively done into a modal logic that is enough expressive to allow nominals (atomic propositions that hold true in a unique world of the Kripke-style models) and the truth-values are viewed as nominals.

When $\mathtt{L}$ is a co-**NP**-hard language, for any finite many-valued logic $\mathcal{L}$, there exists a polynomial-time transformation f from the set of $\mathcal{L}$-valid formulae into $\mathtt{L}$ such that for any $\mathcal{L}$-formula ϕ, ϕ is $\mathcal{L}$-valid iff $f(\phi) \in \mathtt{L}$. This is just by definition of co-**NP**-hardness with respect to polynomial-time many-one reductions (see e.g. [Pap94]) since $\mathcal{L}$-validity is in co-**NP**. However,

1. there is no guarantee that the transformation f is natural (it might be necessary to apply first principles about Turing machines and therefore it is difficult to implement);
2. f is in the complexity class $\mathbf{P}$, that is there is a polynomial $p(n)$ such that computing $f(\phi)$ requires time in $\mathcal{O}(p(|\phi|))$ where $|\phi|$ is the length of ϕ for some reasonably -unspecified- succinct encoding. However, we ignore the value of $p(n)$ and whether the main exponent of $p(n)$ increases with the number of truth values and/or with the number of connectives;
3. a uniform definition of the transformations for all the finite many-valued logics is not immediate.

By constrast, herein, the transformations into $\mathtt{L}_{\neq}$ are natural, they require only time in $\mathcal{O}(n.log\ n)$ and $\mathtt{L}_{\neq}$ is precisely co-$\mathbf{NP}$-complete. So even with respect to the worst-case computational complexity, the transformations can be hardly optimized. An interesting potential application of our results is that mechanization of the finite many-valued logics can be done with a *unique theorem prover* dedicated to the logic $\mathtt{L}_{\neq}$ (or dedicated to PC) and it is guaranteed that the transformations do not introduce any significant complexity blow-up. Experimental analyses are needed, even though the transformations given in the paper are theoretically appealing.

The logic of elsewhere The point here is that the logic $\mathtt{L}_{\neq}$ is not some *ad hoc* modal logic made up for the purpose of the paper but it is the well-known von Wright's logic of elsewhere [Wri79] defined as the smallest normal modal logic (extension of the minimal normal modal logic K -see e.g. [Che80]) containing all instances of the axiom schemes: $\phi \Rightarrow \Box\Diamond\phi$ and $\Diamond\Diamond\phi \Rightarrow (\phi \vee \Diamond\phi)$. Hence, $\mathtt{L}_{\neq}$ is an extension of the modal logic K by addition of Sahlqvist axiom schemes. In order to make this paper self-contained, let us recall some known facts about $\mathtt{L}_{\neq}$. In [Seg81], it is shown that $\mathtt{L}_{\neq}$ is characterized by the class $\mathcal{C}_{\neq}$ of modal frames $\langle W, R\rangle$ such that the binary relation R is the complement of the diagonal relation of W (this was conjectured in [Wri79]). This means that for any modal formula ϕ, $\phi \in \mathtt{L}_{\neq}$ iff ϕ is valid in all the frames belonging to $\mathcal{C}_{\neq}$. In the present paper, we assume that the modal formulae are built from a countably infinite set $\{\mathtt{q}_0, \ldots, \mathtt{q}_n, \ldots\}$ of atomic propositions[1] and closed under $\Box$, $\neg$, $\Rightarrow$, $\wedge$ (the

[1]The metavariables for atomic propositions are $\mathtt{p}, \mathtt{q}, \ldots$. When $\mathtt{q}$ is subscripted by some natural number, we mean the exact member in the enumeration of the atomic propositions.

standard abbreviations include $\vee, \Leftrightarrow, \Diamond$). "$\Box\phi$" can be read as "everywhere else ϕ holds". It is known that $\mathcal{C}_{\neq}$ is not modally definable since it is not closed under *disjoint union* [GT75]. $\mathcal{C}_{\neq}$-satisfiability (or equivalently $\mathtt{L}_{\neq}$-consistency) is known to be **NP**-complete [SEB90] and a formula ϕ is $\mathcal{C}_{\neq}$-satisfiable iff it has a $\mathcal{C}_{\neq}$-model of cardinality at most $2.mw(\phi)+1$ [Dem96] where $mw(\phi)$ is the number of occurrences of the modal operator $\Box$ in ϕ. Hence, $\mathtt{L}_{\neq}$ is co-**NP**-complete. $\mathtt{L}_{\neq}$ has been recently equipped with analytic calculi since prefixed tableaux-style proof systems for $\mathtt{L}_{\neq}$ have been defined in [Dem96, BD97] as well as decision procedures building models with the best known model's size [Dem96, BD97]. All those previous facts are evidence that $\mathtt{L}_{\neq}$ is a quite well-understood modal logic.

Related work. The idea of interpreting finite many-valued logics into modal logics is not a new one and even the three-valued logic Ł of Łukasiewicz had a modal flavour from the beginning [Łuk20] (see also [Tur63] [Urq86, Section 1.2]). In [Seg67], a (poly)modal logic is defined based on a three-valued logic (see also in [Tho78, Mor89, Fit92] the definition of many-valued modal logics). It is argued in [Seg67] that a generalization to any finite many-valued logic is possible. Hence, a family of (poly)modal logics can be defined on the basis of a family of finite many-valued logics. However, the main concern in [Seg67] is not to embed many-valued logics into modal logics and *a fortiori* not into a *unique* modal logic. In [Duf79] (see also [Woo74]), the three-valued logic Ł of Łukasiewicz is translated into the standard modal logic S5 (eight modal interpretations are explored). The second part of [Duf79] contains 3042 mappings from Słupecki's functionnally complete version of Ł into a modified version of S5. Every model of the modified S5 has at least two worlds and the language is augmented with a unique nominal. This is the first work we are aware of that relates a many-valued logic with a nominal modal logic. As known, $\mathtt{L}_{\neq}$ is also a nominal modal logic in disguise and this fact shall be quite useful in what follows. In [Zab93, Chapter 9], Łukasiewicz's logics $Ł_n$ are also translated into modal logics with a minimal models semantics (see e.g. [Che80]). Another way to relate modal logics with finite many-valued logics consists in translating modal logics into finite many-valued logics. For instance, in [CZ90], using similar ideas from [Kri63, Section 5.2], the modal logic S5 is translated into finite many-valued logics. Each S5-formula of size n is translated into a finite many-valued logic having 2^{n+1} truth values, which does not necessarily guarantee the best computational behaviour (see [Häh93, Section 7.2.4] for improvements of the method).

2 Propositional finite many-valued logics

In this section, we just recall a few basic notions about finite many-valued logics. A *signature* Σ is a (non-empty) finite set $\{c_1, \ldots, c_K\}$ of propositional connectives augmented with an arity function $ar : \{c_1, \ldots, c_K\} \to \omega$. Given a countably infinite set $\mathtt{For}_0 = \{\mathtt{p}_0, \ldots, \mathtt{p}_n, \ldots\}$ of atomic propositions, the Σ-formulae $\phi \in \mathtt{For}$ are inductively defined as follows:

$$\phi ::= \mathtt{p}_i \ \mid \ c_1(\phi_1, \ldots, \phi_{ar(c_1)}) \ \mid \ \ldots \ \mid \ c_K(\phi_1, \ldots, \phi_{ar(c_K)})$$

where $\mathtt{p}_i$ is an atomic proposition. A Σ-*logic* $\mathcal{L}$ is a structure

$$\langle \Sigma, \{1, \ldots, N\}, N', f_{c_1}, \ldots, f_{c_K} \rangle$$

such that

1. $N \geq 2$;
2. $1 < N' \leq N$ ($\{1, \ldots, N'-1\}$ is the non-empty set of *designated values*);
3. for $i \in \{1, \ldots, K\}$ such that $ar(c_i) \neq 0$, f_{c_i} is a map $f_i : \{1, \ldots, N\}^{ar(c_i)} \to \{1, \ldots, N\}$;
4. for $i \in \{1, \ldots, K\}$ such that $ar(c_i) = 0$, f_{c_i} is a truth-value, that is a member of $\{1, \ldots, N\}$.

An $\mathcal{L}$-*interpretation* $\mathcal{I}$ is a map $\mathcal{I} : \mathtt{For}_0 \to \{1, \ldots, N\}$ and the set of $\mathcal{L}$-interpretations is denoted $Int_{\mathcal{L}}$. The satisfiability relation $\models$ is a ternary relation subset of $Int_{\mathcal{L}} \times \{1, \ldots, N\} \times \mathtt{For}$. $(\mathcal{I}, v, \phi) \in \models$ ($(\mathcal{I}, v, \phi) \in \models$ shall be written $\mathcal{I}, v \models \phi$)) is understood as "the formula ϕ takes the truth-value v under the interpretation $\mathcal{I}$". The relation $\models$ is inductively defined as follows:

- $\mathcal{I}, v \models \mathtt{p}_i \stackrel{\text{def}}{\Leftrightarrow} \mathcal{I}(\mathtt{p}_i) = v$;
- for $i \in \{1, \ldots, K\}$ such that $ar(c_i) \neq 0$, $\mathcal{I}, v \models c_i(\phi_1, \ldots, \phi_{ar(c_i)}) \stackrel{\text{def}}{\Leftrightarrow}$ there exist $v_1, \ldots, v_{ar(c_i)}$ such that for $1 \leq j \leq ar(c_i)$, $\mathcal{I}, v_j \models \phi_j$ and $f_{c_i}(v_1, \ldots, v_{ar(c_i)}) = v$;
- for $i \in \{1, \ldots, K\}$ such that $ar(c_i) = 0$, $\mathcal{I}, v \models c_i \stackrel{\text{def}}{\Leftrightarrow} f_{c_i} = v$.

We take here some freedom with the usual presentation of the semantics in order to facilitate the presentation of the forthcoming transformations. Indeed, there is a *unique* truth-value v such that $\mathcal{I}, v \models \phi$ given an $\mathcal{L}$-interpretation $\mathcal{I}$ and a Σ-formula ϕ. A Σ-formula ϕ is $\mathcal{L}$-*valid* $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-interpretation $\mathcal{I}$, there is a designated truth-value $v \in \{1, \ldots, N' - 1\}$ such that $\mathcal{I}, v \models \phi$. A Σ-formula ϕ is $\mathcal{L}$-*satisfiable* $\stackrel{\text{def}}{\Leftrightarrow}$ there exist an $\mathcal{L}$-interpretation $\mathcal{I}$ and $v \in \{1, \ldots, N' - 1\}$ such that $\mathcal{I}, v \models \phi$. It is easy to see that $\mathcal{L}$-satisfiability is in the complexity class **NP** ($\mathcal{L}$-satisfiability can be solved by a non-deterministic Turing machine in polynomial-time) whereas $\mathcal{L}$-validity is in co-**NP** (the set of non $\mathcal{L}$-valid formulae is in **NP**).

Example 2.1. Let $\mathcal{L} = \langle \Sigma, \{1, \ldots, 3\}, 2, \{f_{\mathtt{T}}, f_{\mathtt{N}}, f_{\supset}\} \rangle$ be Słupecki's functionnally complete version of Łukasiewicz three-valued logic Ł.

- $\Sigma = \{\mathtt{T}, \mathtt{N}, \supset\}$, $ar(\mathtt{T}) = 0$, $ar(\mathtt{N}) = 1$, $ar(\supset) = 2$;
- 1, 2 and 3 respectively corresponds to the more usual truth-values 1, 1/2 and 0; the unique designated truth-value is 1;
- $f_{\mathtt{T}} = 2$ and $f_{\mathtt{N}}(v) = 4 - v$ for $v \in \{1, 2, 3\}$;
- $f_{\supset}(v, v') = 1$ if $v' \leq v$; $f_{\supset}(v, v') = 2$ if $\langle v, v' \rangle = \langle 1, 2 \rangle$ or $\langle v, v' \rangle = \langle 2, 3 \rangle$; $f_{\supset}(v, v') = 3$ if $\langle v, v' \rangle = \langle 1, 3 \rangle$.

3 Embeddings

In this section, we assume that $\mathcal{L}$ is a finite many-valued logic and we use the notations from Section 2.

3.1 Finite many-valued logics into $\mathtt{L}_{\neq}$

The main idea of the embedding from any finite many-valued logic $\mathcal{L}$ into $\mathtt{L}_{\neq}$ is quite simple. Roughly speaking, any $\mathcal{L}$-interpretation can be seen as a partition of cardinality N of the set of Σ-formulae. Indeed, an $\mathcal{L}$-interpretation

$$\mathcal{I} : \mathtt{For}_0 \rightarrow \{1, \ldots, N\}$$

can be viewed as a $\mathcal{C}_{\neq}$-model

$$\mathcal{M}_{\mathcal{I}} = \langle \{1, \ldots, N\}, \{1, \ldots, N\}^2 \setminus \{\langle v, v \rangle : v \in \{1, \ldots, N\}\}, m \rangle$$

such that for any atomic proposition $\mathtt{p}$,

$$\mathcal{I}(\mathtt{p}) = v \text{ iff } v \in m(\mathtt{p}')$$

where $\mathtt{p}'$ is an atomic proposition associated to $\mathtt{p}$. In the paper, if $\mathtt{p} = \mathtt{p}_i$, then $\mathtt{p}' \stackrel{\text{def}}{=} \mathtt{q}_{2.i}$. Since finite cardinality can be defined in $\mathtt{L}_{\neq}$ [Koy92, Chapter 4], we can enforce that $\mathcal{M}_{\mathcal{I}}$ is of cardinality N. Indeed, there exists a modal formula ϕ_n ($n \geq 1$) such that for any frame $\langle W, R\rangle \in \mathcal{C}_{\neq}$, ϕ_n is valid in $\langle W, R\rangle$ iff $card(W) = n$. Moreover, to each world, say v, of $\mathcal{M}_{\mathcal{I}}$ we can attach a *name*, that is an atomic proposition, say $\mathtt{q}_{2.v+1}$, that holds true only in the world v. In this way, each world of $\mathcal{M}_{\mathcal{I}}$ can be viewed as a truth-value and its name allows to know which one. Observe that $\mathtt{p}'$ associated to $\mathtt{p}$ (see above) also behaves as a name. Until now, we have only treated atomic propositions from $\mathcal{L}$. To each subformula ψ of ϕ (the formula to be translated), we associate an atomic proposition, say $\mathtt{q}_{2.[\psi]+1}$, in $\mathtt{L}_{\neq}$ where $[\psi]$ is a natural number associated to ψ and $\mathtt{q}_{2.[\psi]+1}$ in $\mathtt{L}_{\neq}$ behaves as ψ in $\mathcal{L}$. We have also to take care that the assignment of the atomic propositions does not interfer with the names or between each other. A similar *renaming* technique is also used in [Häh94] to define normal forms for many-valued logics. Our purpose here is different although the renaming technique is a standard tool also used for classical, modal and intuitionistic logics, to quote a few logics.

Let us give now the formal definitions. Let ϕ be a Σ-formula from the Σ-logic $\mathcal{L}$. Let $\phi_1, \ldots, \phi_m$ be an arbitrary enumeration (without repetition) of the subformulae of ϕ such that the n first formulae are all the atomic propositions occurring in ϕ. For $i \in \{1, \ldots, m\}$, we write $[\phi_i]$ to denote i. Moreover, for $i \in \{n+1, \ldots, m\}$, we write $c(i)$ to denote the outermost connective of ϕ_i. We shall build a formula $g(\phi)$ upon $\{\mathtt{q}_2, \ldots, \mathtt{q}_{2.N}\} \cup \{\mathtt{q}_3, \ldots, \mathtt{q}_{2.m+1}\}$ such that

$$\phi \text{ is } \mathcal{L}\text{-valid iff } g(\phi) \in \mathtt{L}_{\neq}.$$

The atomic propositions from $\{\mathtt{q}_2, \ldots, \mathtt{q}_{2.N}\}$ shall behave as nominals that encodes truth-values whereas the atomic propositions from $\{\mathtt{q}_3, \ldots, \mathtt{q}_{2.m+1}\}$ shall respectively behave as the subformulae $\phi_1, \ldots, \phi_m$.

We use the following abbreviations:

$$\langle U\rangle\psi \stackrel{\text{def}}{=} \Diamond\psi \vee \psi \quad [U]\psi \stackrel{\text{def}}{=} \Box\psi \wedge \psi \quad [!]\psi \stackrel{\text{def}}{=} \langle U\rangle(\psi \wedge \Box\neg\psi)$$

"$[U]\psi$" can be read as "everywhere ψ holds" whereas "$[!]\psi$" can be read as "there is exactly one world where ψ holds". $[U]$ is the standard universal modal operator (see e.g. [GP92]). Let $\phi_1^{\neq}$ be the formula stating that the

$\mathcal{C}_{\neq}$-model has exactly N worlds and each world is labelled by a different atomic proposition $\mathsf{q}_{2.k}$ for $k \in \{1, \dots, N\}$:

$$\phi_1^{\neq} \stackrel{\text{def}}{=} (\bigwedge_{k=1}^{N} [!]\mathsf{q}_{2.k}) \wedge ([U] \bigvee_{k=1}^{N} \mathsf{q}_{2.k}) \wedge ([U] \bigwedge_{1 \leq k < k' \leq N} \neg(\mathsf{q}_{2.k} \wedge \mathsf{q}_{2.k'}))$$

The size of $\phi_1^{\neq}$ is constant in $|\phi|$ since it does not depend on ϕ.

Lemma 3.1. Let $\mathcal{M} = \langle W, R, m \rangle$ be a Kripke-style model based on a frame in $\mathcal{C}_{\neq}$, that is $R = \{\langle w, w' \rangle \in W^2 : w \neq w'\}$. Then, there is $w \in W$ such that[2] $\mathcal{M}, w \models \phi_1^{\neq}$ iff the conditions below holds:

1. $card(W) = N$;
2. for $w \in W$ there is a unique $v \in \{1, \dots, N\}$ such that $\mathcal{M}, w \models \mathsf{q}_{2.v}$.

Let $\phi_2^{\neq}$ be the formula stating that each atomic proposition in ϕ is assigned a unique truth value in the interpretation:

$$\phi_2^{\neq} \stackrel{\text{def}}{=} \bigwedge_{k=1}^{n} [!]\mathsf{q}_{2.k+1}$$

The size of $\phi_2^{\neq}$ is in $\mathcal{O}(|\phi|.log\ |\phi|)$. As usual in complexity theory, the extra $log\ |\phi|$ factor in the size of ϕ is due to the fact that we need an index of size $\mathcal{O}(log\ |\phi|)$ for these different atomic propositions. The natural numbers are indeed represented in binary writing.

Lemma 3.2. Let $\mathcal{M} = \langle W, R, m \rangle$ be a Kripke-style model based on a frame in $\mathcal{C}_{\neq}$. Then, there is $w \in W$ such that $\mathcal{M}, w \models \phi_2^{\neq}$ iff for $k \in \{1, \dots, n\}$ there is a unique $w \in W$ such that $\mathcal{M}, w \models \mathsf{q}_{2.k+1}$.

Let $\phi_i = c(i)(\varphi_1, \dots, \varphi_{ar(c(i))})$ be a non atomic subformula of ϕ such that $ar(c(i)) \neq 0$. For any $v \in \{1, \dots, N\}$, we write $f_{c(i)}^{-1}(v)$ to denote the set

$$f_{c(i)}^{-1}(v) \stackrel{\text{def}}{=} \{\langle v_1, \dots, v_{ar(c(i))} \rangle \in \{1, \dots, N\}^{ar(c(i))} : f_{c(i)}(v_1, \dots, v_{ar(c(i))}) = v\}$$

[2]We hasten to point out that the symbol $\models$ is being overloaded here. We'll continue this overloading throughout the paper, counting on context to make clear which satisfiability relation we have in mind.

Let $\phi_3^{i,v}$ be the formula stating that $\mathtt{q}_{2.i+1}$ behaves in $\mathtt{L}_{\neq}$ as ϕ_i in $\mathcal{L}$ with respect to the truth value v:

$$\phi_3^{i,v} \stackrel{\text{def}}{=} [U](\mathtt{q}_{2.v} \Rightarrow \mathtt{q}_{2.i+1}) \Leftrightarrow \bigvee_{\langle v_1,\ldots,v_{ar(c(i))}\rangle \in f_{c(i)}^{-1}(v)} \bigwedge_{j=1}^{ar(c(i))} [U](\mathtt{q}_{2.v_j} \Rightarrow \mathtt{q}_{2.[\varphi_j]+1})$$

When $f_{c(i)}^{-1}(v) = \emptyset$, the generalized disjunction should be read as the logical constant $\perp$ (*falsum*). The size of $\phi_3^{i,v}$ is also in $\mathcal{O}(log\ |\phi|)$. Remember that N, K, $M = max\{ar(i) : 1 \leq i \leq K\}$ are constants of the logic $\mathcal{L}$. Let $\phi_i = c(i)$ be such that $ar(c(i)) = 0$. The formula ϕ_3^i is defined as follows:

$$\phi_3^i \stackrel{\text{def}}{=} [\mathtt{U}](\mathtt{q}_{2.i+1} \Leftrightarrow \mathtt{q}_{2.f_{c(i)}})$$

Let $g(\phi)$ be the formula:

$$g(\phi) \stackrel{\text{def}}{=} (\phi_1^{\neq} \wedge \phi_2^{\neq} \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))\neq 0} \bigwedge_{v\in X_{\mathcal{L}}^{c(i)}} \phi_3^{i,v}) \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_3^i)) \Rightarrow [U](\mathtt{q}_{2.[\phi]+1} \Rightarrow \bigvee_{v=1}^{N'-1} \mathtt{q}_{2.v})$$

where

$$X_{\mathcal{L}}^{c(i)} \stackrel{\text{def}}{=} \{v \in \{1,\ldots,N\} : f_{c(i)}^{-1}(v) \neq \emptyset\}$$

So $|g(\phi)|$ is in $\mathcal{O}(|\phi|.log\ |\phi|)$ and computing $g(\phi)$ requires time $\mathcal{O}(|\phi|.log\ |\phi|)$ and space[3] $\mathcal{O}(log\ |\phi|)$. Furthermore, the modal depth of $g(\phi)$ is only two, that is, two is the maximum number of nested modal operator $\Box$ in $g(\phi)$.

Lemma 3.3. Let $\mathcal{M} = \langle W, R, m\rangle$ be a Kripke-style model based on a frame in $\mathcal{C}_{\neq}$ such that $\mathcal{M}, w \models \phi_1^{\neq} \wedge \phi_2^{\neq} \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))\neq 0} \bigwedge_{v\in X_{\mathcal{L}}^{c(i)}} \phi_3^{i,v}) \wedge \bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_3^i$. Then, for $k \in \{1,\ldots,m\}$ there is a unique $w \in W$ such that $\mathcal{M}, w \models \mathtt{q}_{2.k+1}$.

The proof of Lemma 3.3 is based on the fact that for any $i \in \{1,\ldots,K\}$ such that $ar(c_i) \neq 0$,

$$\{f_{c_i}^{-1}(v) : v \in \{1,\ldots,N\}, f_{c_i}^{-1}(v) \neq \emptyset\}$$

is a partition of $\{1,\ldots,N\}^{ar(c_i)}$.

[3]As usual, the space (number of cells) occupied by the input and output tapes of the Turing machine is not counted for the space complexity.

Proposition 3.4. ϕ is $\mathcal{L}$-valid iff $g(\phi) \in \mathtt{L}_{\neq}$.

Proof: Let ϕ be a Σ-formula. For any $\mathcal{L}$-interpretation $\mathcal{I}$, we write $\mathcal{M}_{\mathcal{I}} \stackrel{\text{def}}{=} \langle W, R, m\rangle$ (depending also on ϕ) to denote the $\mathcal{C}_{\neq}$-model such that

1. $W \stackrel{\text{def}}{=} \{1, \ldots, N\}$; $R \stackrel{\text{def}}{=} W^2 \setminus \{\langle v, v\rangle : v \in W\}$;
2. for $v \in \{1, \ldots, N\}$, $m(\mathtt{q}_{2.v}) \stackrel{\text{def}}{=} \{v\}$;
3. for $i \in \{1, \ldots, m\}$, $m(\mathtt{q}_{2.i+1}) \stackrel{\text{def}}{=} \{v \in W : \mathcal{I}, v \models \psi_i\}$;
4. the interpretation of the other atomic propositions is not constrained.

One can check that for any $v \in W$, $\mathcal{M}_{\mathcal{I}}, v \models \phi_1^{\neq} \wedge \phi_2^{\neq} \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))\neq 0} \bigwedge_{v \in X_{\mathcal{L}}^{c(i)}} \phi_3^{i,v}) \wedge \bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_3^{i}$. For $i \in \{1, \ldots, m\}$, $m(\mathtt{q}_{2.i+1})$ is also a singleton by Lemma 3.3. Then, one can prove that for any $i \in \{1, \ldots, m\}$ and for any $v \in \{1, \ldots, N\}$, $\mathcal{I}, v \models \phi_i$ iff $\mathcal{M}_{\mathcal{I}}, v \models \mathtt{q}_{2.i+1}$ (by an induction on the structure of the formulae). Consequently, if $g(\phi) \in \mathtt{L}_{\neq}$, then ϕ is $\mathcal{L}$-valid.

Now, let $\mathcal{M} \stackrel{\text{def}}{=} \langle W, R, m\rangle$ be a $\mathcal{C}_{\neq}$-model such that for some $w \in W$, $\mathcal{M}, w \models \phi_1^{\neq} \wedge \phi_2^{\neq} \wedge \bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))\neq 0} \bigwedge_{v \in X_{\mathcal{L}}^{c(i)}} \phi_3^{i,v} \wedge \bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_3^{i}$. Let $\mathcal{I}_{\mathcal{M}}$ be the $\mathcal{L}$-interpretation such that for any $i \in \{1, \ldots, n\}$, $\mathcal{I}_{\mathcal{M}}(\phi_i) \stackrel{\text{def}}{=} v$ where v is the *unique index* in $\{1, \ldots, N\}$ such that there is world $w \in W$ verifying $\mathcal{M}, w \models \mathtt{q}_{2.v} \wedge \mathtt{q}_{2.i+1}$. One can show that for any $i \in \{1, \ldots, m\}$ and for any $v \in \{1, \ldots, N\}$, $\mathcal{I}_{\mathcal{M}}, v \models \phi_i$ iff $\mathcal{M}, v \models \mathtt{q}_{2.i+1}$ (by an induction on the structure of the formulae). Consequently, if ϕ is $\mathcal{L}$-valid, then $g(\phi) \in \mathtt{L}_{\neq}$. **Q.E.D.**

Let $g'(\phi)$ be the formula:

$$g'(\phi) \stackrel{\text{def}}{=} (\phi_1^{\neq} \wedge \phi_2^{\neq} \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))\neq 0} \bigwedge_{v \in X_{\mathcal{L}}^{c(i)}} \phi_3^{i,v}) \wedge (\bigwedge_{i\in\{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_3^{i})) \wedge \langle U\rangle (\mathtt{q}_{2.[\phi]+1} \wedge \bigvee_{v=1}^{N'-1} \mathtt{q}_{2.v})$$

It is now easy to show that ϕ is $\mathcal{L}$-satisfiable iff $g'(\phi)$ is $\mathcal{C}_{\neq}$-satisfiable (or equivalently, iff $g'(\phi)$ is $\mathtt{L}_{\neq}$-consistent). $g(\phi)$ and $g'(\phi)$ share the same complexity measures described previously.

The definition of $\phi_3^{i,v}$ can be improved in order to significantly reduce the size of $g(\phi)$. Let $i \in \{1, \ldots, K\}$ be such that $ar(c_i) \neq 0$ and $v \in \{1, \ldots, N\}$. The set $X_v^{c_i}$ is defined as a set of strings over $\{1, \ldots, ar(c_i)\} \times \{1, \ldots, N\}$ of length at most $ar(c_i)$. $\langle l_1, v_1\rangle \ldots \langle l_s, v_s\rangle \in X_v^{c_i}$ is understood as a compact representation for "$f_i(w_1, \ldots, w_{ar(c_i)}) = v$ when $w_{l_1} = v_1, \ldots, w_{l_s} = v_s$".

The improvement in size is due to the fact that the length s can be less than (or equal to) $ar(c_i)$. Let σ, σ' be two strings over $\{1, \ldots, ar(c_i)\} \times \{1, \ldots, N\}$. We write $\sigma \preceq \sigma' \stackrel{\text{def}}{\Leftrightarrow}$ σ can be obtained from σ' be deleting some pairs in the sequence σ'. For instance,

$$\langle 1, 3\rangle.\langle 2, 3\rangle.\langle 4, 3\rangle \preceq \langle 1, 3\rangle.\langle 2, 3\rangle.\langle 3, 3\rangle.\langle 4, 3\rangle$$

The empty string is denoted Λ. $X_v^{c_i}$ should satisfy the properties:

- (arity) any string $\langle l_1, v_1\rangle.\ldots.\langle l_s, v_s\rangle \in X_v^{c_i}$ of length $s \geq 1$ satisfies $1 \leq l_1 < \ldots < l_s \leq ar(c_i)$;
- (maximality) any $\sigma \in X_v^{c_i}$ is maximal in $X_v^{c_i}$ with respect to $\preceq$;
- (minimality) if $\sigma.\sigma'$ is a string over $\{1, \ldots, ar(c_i)\} \times \{1, \ldots, N\}$ of length at most $ar(c_i)$, and for some $\langle j, v'\rangle \in \{1, \ldots, ar(c_i)\} \times \{1, \ldots, N\}$, $\sigma.\langle j, v'\rangle.\sigma' \in X_v^{c_i}$, then there is $v'' \in \{1, \ldots, N\}$ such that $\sigma.\langle j, v''\rangle.\sigma' \notin X_v^{c_i}$;
- (correctness)

  $$f_{c_i}^{-1}(v) = \{\langle v_1, \ldots, v_{ar(c_i)}\rangle : \sigma' \preceq \langle 1, v_1\rangle.\ldots.\langle ar(c_i), v_{ar(c_i)}\rangle, \sigma' \in X_v^{c_i}\}$$

Lemma 3.5. Let $v \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, K\}$ such that $ar(c_i) \neq 0$.

1. $card(X_v^{c_i}) \leq card(f_i^{-1}(v))$;
2. $X_v^{c_i}$ is uniquely determined.

Lemma 3.5 ensures that the definition of $\phi_3^{i,v}$ below is more compact that the first one. The definition of $\phi_3^{i,v}$ is modified as follows.

$$\phi_3^{i,v} \stackrel{\text{def}}{=} [U](\mathsf{q}_{2.v} \Rightarrow \mathsf{q}_{2.i+1}) \Leftrightarrow \bigvee_{\langle l_1, v_1\rangle.\ldots.\langle l_s, v_s\rangle \in X_v^{c(i)}} \bigwedge_{j=1}^{s} [U](\mathsf{q}_{2.v_j} \Rightarrow \mathsf{q}_{2.[\varphi_{l_j}]+1})$$

In the case when $X_v^{c(i)} = \{\Lambda\}$ (that is, f_{c_i} is the constant map returning the unique truth-value v), $\phi_3^{i,v}$ is replaced by $[U](\mathsf{q}_{2.i+1} \Leftrightarrow \mathsf{q}_{2.v})$ which corresponds exactly to ϕ_3^j when $ar(c(j)) = 0$. The worst-case complexity of the new map g remains unchanged since the computation of the $X_v^{c_i}$'s for $v \in \{1, \ldots, N\}$ and for $i \in \{1, \ldots, K\}$ (such that $ar(c_i) \neq 0$) can be done *once and for all*, and therefore its cost is constant in the computation of g. It is also a routine task to check that Proposition 3.4 still holds true with the new definition. Although the above improvement is unessential for the soundness of the embeddings, it can considerably reduce the size of the translated formula and therefore it is well-suited for mechanization. By the way, this optimization is comparable, for instance to refinements in the calculi from [Car87, Zab93, Häh93, Zac93]. This provides another reading for some of those calculi: these are calculi for fragments of von Wright's logic of elsewhere obtained by optimally translating finite many-valued logics. Indeed, every subformula of the form $[U](\mathtt{q}_{2.v} \Rightarrow \mathtt{q}_{2.i+1})$ occurring in $g(\phi)$ corresponds to the *signed* formula $v : \phi_i$. More importantly, techniques for analysis of a finite many-valued logic at the metalevel can be embedded into the object level (in $\mathtt{L}_{\neq}$ for instance). Finally, the proof of Proposition 3.4 shows that there exists a natural correspondence between $\mathcal{C}_{\neq}$-models and $\mathcal{L}$-interpretations. This allows the construction of $\mathcal{L}$-interpretations by using proof systems for $\mathtt{L}_{\neq}$ that admit model-building mechanisms, as done in [BD97] for instance.

Example 3.1. (Example 2.1 continued) Let ϕ be the formula $\mathtt{Np}_{200} \supset \mathtt{T}$. The sets $X_j^{\mathtt{N}}$ and $X_j^{\supset}$ for $j \in \{1,2,3\}$ takes the following values:

- $X_1^{\mathtt{N}} = \{\langle 1,3\rangle\}$ (the unary connective $\mathtt{N}$ returns the truth-value 1 with input the truth-value 3); $X_2^{\mathtt{N}} = \{\langle 1,2\rangle\}$; $X_3^{\mathtt{N}} = \{\langle 1,1\rangle\}$;
- $X_1^{\supset} = \{\langle 2,1\rangle, \langle 1,2\rangle.\langle 2,2\rangle, \langle 1,3\rangle.\langle 2,2\rangle, \langle 1,3\rangle.\langle 2,3\rangle\}$ (for instance the "first" element in $X_1^{\supset}$ means that if the second argument of $v \supset v'$ is 1 then $v \supset v'$ is evaluated to 1) ;
- $X_2^{\supset} = \{\langle 1,1\rangle.\langle 2,2\rangle, \langle 1,2\rangle.\langle 2,3\rangle\}$; $X_3^{\supset} = \{\langle 1,1\rangle.\langle 2,3\rangle\}$.

The sequence $\phi_1, \ldots, \phi_4$ of subformulae of ϕ can be for instance $\mathtt{p}_{200}, \mathtt{T}, \mathtt{Np}_{200}, \phi$. Let us present the different formulae that constitute the formula $g(\phi)$.

$$\phi_1^{\neq} \stackrel{\text{def}}{=} (\bigwedge_{k=1}^{3}[!]\mathtt{q}_{2.k}) \wedge ([U] \bigvee_{k=1}^{3} \mathtt{q}_{2.k}) \wedge ([U] \bigwedge_{1 \leq k < k' \leq 3} \neg(\mathtt{q}_{2.k} \wedge \mathtt{q}_{2.k'}))$$

$$\phi_2^{\neq} \stackrel{\text{def}}{=} [!]\mathtt{q}_3 \qquad \phi_3^{2} \stackrel{\text{def}}{=} [U](\mathtt{q}_4 \Leftrightarrow \mathtt{q}_5)$$

$$\phi_3^{3,1} \stackrel{\text{def}}{=} [U](\mathtt{q}_2 \Rightarrow \mathtt{q}_7) \Leftrightarrow [U](\mathtt{q}_6 \Rightarrow \mathtt{q}_3)$$

$$\phi_3^{3,2} \stackrel{\text{def}}{=} [U](\mathtt{q}_4 \Rightarrow \mathtt{q}_7) \Leftrightarrow [U](\mathtt{q}_4 \Rightarrow \mathtt{q}_3) \quad \phi_3^{3,3} \stackrel{\text{def}}{=} [U](\mathtt{q}_6 \Rightarrow \mathtt{q}_7) \Leftrightarrow [U](\mathtt{q}_2 \Rightarrow \mathtt{q}_3)$$

$$\phi_3^{4,2} \stackrel{\text{def}}{=} [U](\mathtt{q}_4 \Rightarrow \mathtt{q}_9) \Leftrightarrow ([U](\mathtt{q}_2 \Rightarrow \mathtt{q}_7) \wedge [U](\mathtt{q}_4 \Rightarrow \mathtt{q}_5)) \vee ([U](\mathtt{q}_4 \Rightarrow \mathtt{q}_7) \wedge [U](\mathtt{q}_6 \Rightarrow \mathtt{q}_5))$$

The values for $\phi_3^{4,1}$ and $\phi_3^{4,3}$ can be also easily computed.

3.2 Finite annotated logics into $\mathtt{L}_{\neq}$

The treatment of "truth values as nominals" can be easily carried out to particular paraconsistent logics, namely to the finite annotated logics (see e.g. [Sub87, dCHLS90, LSC91, KL92]) as shown below. First, let us briefly recall some basic definitions. Let $\langle\{1,\ldots,N\},\cap,\cup\rangle$ be a finite lattice and $\sim$ be an 1-1 mapping $\sim\colon \{1,\ldots,N\} \to \{1,\ldots,N\}$. Here, we do not need to require any particular property for $\sim$.

Example 3.2. The standard complete lattice FOUR (see e.g. [dCHLS90]) is composed of the truth values $\{\bot,\top,\mathtt{t},\mathtt{f}\}$ with $\top$ the greatest element, $\bot$ the lowest element and $\mathtt{t}$ and $\mathtt{f}$ are not comparable.

Given a countably infinite set $\mathtt{For}_0 = \{\mathtt{p}_0,\ldots,\mathtt{p}_n,\ldots\}$ of atomic propositions, the formulae ϕ are inductively defined as follows:

$$\phi ::= \langle \mathtt{p}_i, v\rangle \ \mid\ \neg'\langle \mathtt{p}_i, v\rangle \ \mid\ \neg\phi \ \mid\ \phi_1 \wedge \phi_2$$

for $\mathtt{p}_i$ an atomic proposition and $v \in \{1,\ldots,N\}$. An interpretation $\mathcal{I}$ is a map $\mathcal{I} : \mathtt{For}_0 \to \{1,\ldots,N\}$. The satisfiability relation $\models$ is inductively defined as follows:

- $\mathcal{I} \models \langle \mathtt{p}_i, v\rangle \overset{\text{def}}{\Leftrightarrow} \mathcal{I}(\mathtt{p}_i) \geq' v$ ($\geq'$ is defined from the lattice structure $\langle\{1,\ldots,N\},\cap,\cup\rangle$);
- $\mathcal{I} \models \neg'\langle \mathtt{p}_i, v\rangle \overset{\text{def}}{\Leftrightarrow} \mathcal{I} \models \langle \mathtt{p}_i, \sim v\rangle$;
- $\mathcal{I} \models \neg\phi \overset{\text{def}}{\Leftrightarrow}$ not $\mathcal{I} \models \phi$;
- $\mathcal{I} \models \phi_1 \wedge \phi_2 \overset{\text{def}}{\Leftrightarrow} \mathcal{I} \models \phi_1$ and $\mathcal{I} \models \phi_2$.

An annotated logic $\mathcal{L}$ can be defined as the structure $\langle\langle\{1,\ldots,N\},\cap,\cup\rangle, \sim, \mathtt{For}_0\rangle$. A formula ϕ is $\mathcal{L}$-valid $\overset{\text{def}}{\Leftrightarrow}$ for all interpretations $\mathcal{I}$ (based on $\langle\{1,\ldots,N\},\cap,\cup\rangle$ and $\sim$), $\mathcal{I} \models \phi$. Without any loss of generality[4], let ϕ be a $\neg'$-free annotated formula. Let $\phi_1,\ldots,\phi_m$ be an arbitrary enumeration (without repetition) of the subformulae of ϕ such that the n first formulae are all the annotated atomic propositions occurring in ϕ. For $i \in \{1,\ldots,m\}$, we write $[\phi_i]$ to denote i. This time, we build a formula $g(\phi)$ upon $\{\mathtt{q}_0\} \cup \{\mathtt{q}_2,\ldots,\mathtt{q}_{2.N},\mathtt{q}_3,\ldots,\mathtt{q}_{2.m+1}\}$. The atomic proposition $\mathtt{q}_0$

[4] For any annotated formula ϕ', there is a logically equivalent $\neg'$-free annotated formula ϕ computable in time $\mathcal{O}(|\phi'|)$.

shall also behave as a name and it refers to the world where the (binary) satisfiability relation is encoded.

Let $\phi_1^{\neq}$ be the formula stating that the model has exactly $N+1$ worlds and names are attached to the worlds:

$$\phi_1^{\neq} \stackrel{\text{def}}{=} (\bigwedge_{k=0}^{N} [!]\mathtt{q}_{2.k} \wedge [U] \bigvee_{k=0}^{N} \mathtt{q}_{2.k} \wedge [U] \bigwedge_{0 \leq k < k' \leq N} \neg(\mathtt{q}_{2.k} \wedge \mathtt{q}_{2.k'}))$$

Let $\phi_2^{\neq}$ be the formula stating that the atomic propositions are interpreted by a value in $\{0, \ldots, N\} \setminus \{0\}$:

$$\phi_2^{\neq} \stackrel{\text{def}}{=} \bigwedge_{k=1}^{n} [!]\mathtt{q}_{2.k+1} \wedge [U](\bigwedge_{k=1}^{n} (\mathtt{q}_{2.k+1} \Rightarrow \neg \mathtt{q}_0))$$

For each subformula ϕ_i of ϕ we associate a formula ϕ_3^i as shown in Figure 1 and let

$$g(\phi) \stackrel{\text{def}}{=} (\phi_1^{\neq} \wedge \phi_2^{\neq} \wedge \bigwedge_{i=1}^{m} \phi_3^i) \Rightarrow [U](\mathtt{q}_0 \Rightarrow \mathtt{q}_{2.[\phi]+1})$$

Form of ϕ_i	ϕ_3^i
$\langle \mathrm{p}, v \rangle$	$[U](\mathtt{q}_0 \Rightarrow (\mathtt{q}_{2.i+1} \Leftrightarrow \langle U \rangle(\mathtt{q}_{2.i+1} \wedge \bigvee_{v' \geq' v} \mathtt{q}_{2.v'}))$
$\psi' \wedge \psi''$	$[U](\mathtt{q}_0 \Rightarrow (\mathtt{q}_{2.i+1} \Leftrightarrow (\mathtt{q}_{2.[\psi']+1} \wedge \mathtt{q}_{2.[\psi'']+1}))$
$\neg \psi'$	$[U](\mathtt{q}_0 \Rightarrow (\mathtt{q}_{2.i+1} \Leftrightarrow \neg \mathtt{q}_{2.[\psi']+1}))$

Figure 1: Definition of ϕ_3^i

Proposition 3.6. ϕ is $\mathcal{L}$-valid iff $g(\phi) \in \mathtt{L}_{\neq}$.

Proof: Similar to the proof of Proposition 3.4. **Q.E.D.**

3.3 Finite many-valued logics into PC

In Section 3.1 and in Section 3.2, we have seen that the concept of "truth-values as nominals" is essential to define natural translations into $\mathtt{L}_{\neq}$. However, by close examination of the proof in Section 3.1, it is possible to observe that the names are very useful but they are absolutely not necessary. In what follows, we show how to define natural transformations (in time $\mathcal{O}(n.log\ n)$) from the validity problem for any finite many-valued logic into TAUT. We assume that the formula of PC are built over the set $\{\mathtt{q}_{i,j} : i, j \in \omega\}$ of atomic propositions. We could also just consider the set $\{\mathtt{q}_i : i \in \omega\}$ of atomic propositions and use a 1-1 mapping from $\omega^2 \rightarrow \omega$ but for the sake of simplicity, the present option is the most convenient.

Let ϕ be a Σ-formula from the Σ-logic $\mathcal{L}$ (we use again the notations of Section 3.1). We shall build a formula $g''(\phi)$ over $\{\mathtt{q}_{i,v} :\ 1 \leq i \leq m,\ 1 \leq v \leq N\}$ such that ϕ is $\mathcal{L}$-valid iff $g''(\phi)$ is a tautology. The simple idea of the translation is the following. Since the $\mathcal{C}_{\neq}$-models involved in the proof of Proposition 3.4 are finite and only a finite amount of atomic propositions are considered for a given formula ϕ, we introduce atomic propositions such that $\mathcal{M}_{\mathcal{I}}, v \models \mathtt{q}_{2.i+1}$ iff $\mathtt{q}_{i,v}$ holds true in some interpretation for PC.

Let ϕ_1^{PC} be the formula stating that each atomic proposition in ϕ is assigned a unique truth value in the interpretation:

$$\phi_1^{PC} \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n}((\bigvee_{v=1}^{N} \mathtt{p}_{i,v}) \wedge \bigwedge_{1 \leq v < v' \leq N} \neg(\mathtt{p}_{i,v} \wedge \mathtt{p}_{i,v'}))$$

The size of ϕ_1^{PC} is in $\mathcal{O}(|\phi|.log\ |\phi|)$. Let $\phi_i = c(i)(\varphi_1, \ldots, \varphi_{ar(c(i))})$ be a non atomic subformula of ϕ such that $ar(c(i)) \neq 0$. Let $\phi_{2,PC}^{i,v}$ be the formula stating that $\mathtt{q}_{i,v}$ behaves in PC as ϕ_i in $\mathcal{L}$ with respect to the truth value v:

$$\phi_{2,PC}^{i,v} \stackrel{\text{def}}{=} \mathtt{q}_{i,v} \Leftrightarrow \bigvee_{\langle l_1, v_1 \rangle \ldots . \langle l_s, v_s \rangle \in X_v^{c(i)}} (\bigwedge_{j=1}^{s} \mathtt{q}_{[\varphi_{l_j}], v_j})$$

In the case when $X_v^{c(i)} = \{\Lambda\}$, $\phi_{2,PC}^{i,v} \stackrel{\text{def}}{=} \mathtt{q}_{i,v}$. Now let $\phi_i = c(i)$ be a subformula of ϕ such that $ar(c(i)) = 0$. $\phi_{2,PC}^{i}$ is defined as the atomic formula $\mathtt{q}_{i,f_{c(i)}}$. Let $g''(\phi)$ be the formula:

$$g''(\phi) \stackrel{\text{def}}{=} (\phi_1^{PC} \wedge (\bigwedge_{i \in \{n+1,\ldots,m\}}^{ar(c(i)) \neq 0} \bigwedge_{v \in X_{\mathcal{L}}^{c(i)}} \phi_{2,PC}^{i,v}) \wedge (\bigwedge_{i \in \{n+1,\ldots,m\}}^{ar(c(i))=0} \phi_{2,PC}^{i})) \Rightarrow \bigvee_{v=1}^{N'-1} \mathtt{q}_{[\phi],v})$$

So $|g''(\phi)|$ is also in $\mathcal{O}(|\phi|.log\ |\phi|)$ and computing $g''(\phi)$ requires time $\mathcal{O}(|\phi|.log\ |\phi|)$ and space $\mathcal{O}(log\ |\phi|)$.

Proposition 3.7. ϕ is $\mathcal{L}$-valid iff $g''(\phi) \in$ TAUT.

Proof: Similar to the proof of Proposition 3.4. **Q.E.D.**

g'' is a very natural map. It requires only time in $\mathcal{O}(n.log\ n)$ and it is not difficult to see that similar many-one reductions from any finite annotated logic in PC can be defined.

4 Concluding remarks

In this paper, the modal encoding of propositional finite many-valued logics are based on the notion of transformations ("many-one reductions") used in complexity theory. All the transformations in this paper are tight since they require only time in $\mathcal{O}(n.log\ n)$ and space in $\mathcal{O}(log\ n)$. Futhermore, the target problems (either $\mathtt{L}_{\neq}$ or TAUT) are in co-**NP** which is an evidence of the efficiency of the transformations since validity for the propositional finite many-valued logics and validity for the propositional annotated logics are also in co-**NP**. Apart from the modal viewpoint on finite many-valued logics we provided here, it is an open question whether any efficient theorem prover dedicated to either $\mathtt{L}_{\neq}$ or PC could be used to mechanize the propositional finite many-valued logics and the propositional annotated logics. Experimental analyses are needed and this is part of future work.

Acknowlegments: The author thanks Ricardo Caferra and Ewa Orłowska for their comments on an earlier version of this work.

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