Weighted strategy logic with boolean goals over one-counter games

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Abstract

Strategy Logic is a powerful specification language for expressing non-zero-sum properties of multi-player games. SL conveniently extends the logic ATL with explicit quantification and assignment of strategies. In this paper, we consider games over one-counter automata, and a quantitative extension 1cSL of SL with assertions over the value of the counter. We prove two results: we first show that, if decidable, model checking the so-called *Boolean-goal* fragment of 1cSL has non-elementary complexity; we actually prove the result for the Boolean-goal fragment of SL over finite-state games, which was an open question in [MMPV14]. As a first step towards proving decidability, we then show that the Boolean-goal fragment of 1cSL over one-counter games enjoys a nice periodicity property.

Keywords and phrases Temporal logics, multi-player games, strategy logic, quantitative games

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1 Introduction

Model checking. Model checking [CGP00] has been developed for almost 40 years as a formal method for verifying correctness of computerized systems: this technique first consists in representing the system under study as a mathematical model (a finite-state transition system (a.k.a. Kripke structure), in the most basic setting), expressing the correctness property in some logical formalism (usually, using various *temporal logics* such as LTL [Pnu77] or CTL [CE82, QS82]), and running an algorithm that exhaustively explores the set of behaviours of the model for proving or disproving the property.

Over the years, model checking has been extended in various directions, in order to take into account richer models and more precise properties. Several families of quantitative models (e.g. weighted Kripke structures [CC95], counter automata [EN94], timed automata [AD90]) and temporal logics [Koy90, EMSS92, AH94, BCHK14, BGM14, among others] have been defined and studied. Those formalisms conveniently extend the qualitative setting; they provide powerful ways of representing quantities, while in several cases keeping reasonably efficient model-checking algorithms.

Multi-agent systems (a.k.a. graph games [Tho02, AG11]) form another direction where model checking has been extended for reasoning about the interactions between components of a computerized system. Temporal logics have been extended accordingly [AHK02, CHP07, MMV10, DLM10], in order to express the existence of *winning strategies* in multi-player games. Among the most popular approaches, the logic ATL [AHK02] has limited expressive power but enjoys reasonably efficient model-checking algorithms, while the more expressive Strategy Logic (SL) [CHP07, MMV10] extends LTL with explicit manipulation of strategies, and can express very rich non-zero-sum properties of games, including equilibria; however, model checking SL is non-elementary. Several fragments of SL have recently been introduced in order to mitigate the complexity of the model-checking problem while retaining the interesting aspects of SL [MMS14, ČLM15].

Quantitative games, combining both extensions, have also been widely considered. This includes games on weighted graphs [EM79, CdAHS03, LMO06, BFL+08], games on counter systems or VASS [Ser06, BJK10], or timed games [AMPS98, dAFH+03]. A large part of these works have focused on "simple" objectives, such as mean-payoff objectives [EM79], energy constraints [CdAHS03, BFL+08], or combinations thereof [CDHR10, FJLS11].

Our contribution. In this paper, we consider a quantitative extension of SL over quantitative games. While such extensions have already been proven decidable for ATL [LMO06, Ves15], we focus here on a quantitative extension of the richer logic SL, more specifically, its so-called *Boolean-goal* fragment SL[BG] [MMPV14]. SL with Boolean goals restricts SL by preventing arbitrary nesting of strategy quantifiers within temporal modalities. This and several other fragments of SL have been introduced in [MMPV14] with the aim of getting more efficient model-checking algorithms. However, while several fragments have been shown to have *2*-EXPTIME model-checking algorithms, the exact complexity of SL[BG] remained open.

We prove that model checking (the flat fragment of) SL[BG] is Tower-complete, thus negatively answering the open question whether SL[BG] would enjoy more efficient model-checking algorithm than SL. This hardness result obviously extends to the quantitative version 1cSL[BG] of SL[BG] over one-counter games. On the way to proving decidability of the model-checking problem for this logic, we then show that 1cSL[BG] over one-counter games enjoys a nice periodicity property: for any given formula, there is a threshold above which truth value of the formula is periodic (w.r.t. the value of the counter).

Related works. Several works have focused on one-counter models: two-player games with parity objectives have been proven PSPACE-complete [Ser06]; this was recently extended to a quantitative extension of ATL [Ves15], which is thus closely related to our paper. Model checking LTL and CTL over one-counter automata is also PSPACE-complete [GL10, GHOW10]. Quantitative extensions of those logics have been studied in [DG09, BCHK14, BGM14]. In many cases, they lead to undecidability of the model-checking problem. Games on VASS have also been considered, but reachability is only decidable in restricted cases [BJK10, Rei13].

Games over integer-weighted graphs have a different flavour, as the behaviours do not depend on the value of the accumulated weight. Those games have been extensively considered with various quantitative objectives (e.g. mean-payoff [EM79, ZP96], energy [CdAHS03, BFL+08], and combinations thereof [CDHR10, CRR12]), and with objectives expressed in temporal logics [LMO06, BBFR13].

2 Definitions

▶ **Definition 1.** Let AP be a set of atomic propositions, and Agt be a set of agents. A *1-counter concurrent game structure* (1cCGS for short) is a tuple $\mathcal{G} = \langle \mathsf{Loc}, \mathsf{label}, \mathsf{Act}, \mathsf{Tab}_{\{0,1\}}, \mathsf{Wgt}_{\{0,1\}} \rangle$ where

- Loc is a finite set of locations;
- $\mathsf{label}\colon \mathsf{Loc} \to 2^{\mathsf{AP}}$ labels locations with atomic propositions;
- Act is a finite set of actions;
- $\mathsf{Tab}_0\colon \mathsf{Loc} \times \mathsf{Act}^{\mathsf{Agt}} \to \mathsf{Loc}$ and $\mathsf{Tab}_1\colon \mathsf{Loc} \times \mathsf{Act}^{\mathsf{Agt}} \to \mathsf{Loc}$ are two transition tables;
- $\mathsf{Wgt}_0\colon \mathsf{Loc} \times \mathsf{Act}^{\mathsf{Agt}} \to \{0, 1\}$ and $\mathsf{Wgt}_1\colon \mathsf{Loc} \times \mathsf{Act}^{\mathsf{Agt}} \to \{-1, 0, 1\}$ assign a weight to each transition of the transition tables.

A *finite path* in a 1cCGS $\mathcal{G}$ is a finite non-empty sequence of configurations $\rho = \gamma_0\gamma_1\gamma_2 \ldots \gamma_k$, where for all $0 \le i \le k$, the configuration γ_i is a pair (ℓ_i, c_i) with $\ell_i \in \mathsf{Loc}$ and $c_i \in \mathbb{N}$. For such a path, we denote by $\mathsf{last}(\rho)$ its last element γ_k, and we let $|\rho| = k$. An *infinite path* is an infinite sequence of configurations with the same property. We denote by Path (resp. $\mathsf{InfPath}$) the set of finite (resp. infinite) paths. The length of an infinite path is $+\infty$. For $0 \le i < |\rho|$, $\rho(i)$ represents the $i+1$-th element γ_i of ρ. For a path ρ and $0 \le i < |\rho|$, we denote by $\rho_{\le i}$ the prefix of ρ until position i, i.e. the finite path $\rho(0)\rho(1)\ldots\rho(i)$.

A *strategy* for some agent $a \in \mathsf{Agt}$ is a function $\sigma_a\colon \mathsf{Path} \to \mathsf{Act}$. We write Strat for the set of strategies. Given a finite path (or *history*) in the game, a strategy σ_a returns the action that agent a will play next. A strategy σ_A for a coalition of agents $A \subseteq \mathsf{Agt}$ is a function assigning a strategy $\sigma_A(a)$ to each agent $a \in A$. Given a strategy σ_A for coalition A, we say that a path ρ respects σ_A after a finite prefix π if, writing $\rho(i) = (\ell_i, c_i)$ for all $0 \le i \le |\rho|$, the following two conditions hold:

- for all $0 \le i < |\pi|$, we have $\rho(i) = \pi(i)$
- for all $|\pi| \le i < |\rho| - 1$, we have that $\ell_{i+1} = \mathsf{Tab}_s(\ell_i, \mathbf{m})$ and $c_{i+1} = c_i + \mathsf{Wgt}_s(\rho_{\le i}, \mathbf{m})$, where $s = 0$ if $c_i = 0$ and $s = 1$ otherwise, and $\mathbf{m}$ is an action vector satisfying $\mathbf{m}(a) = \sigma_A(a)(\rho_{\le i})$ for all $a \in A$.

Notice that the value of the counter always remains nonnegative as Wgt_0 only returns nonnegative values. Given a finite path π, we denote by $\mathsf{Out}(\pi, \sigma_A)$ the set of paths that respect the strategy σ_A after prefix π. Notice that if σ_A assigns a strategy to all the agents, then $\mathsf{Out}(\pi, \sigma_A)$ contains a single path, which we write $\mathsf{out}(\pi, \sigma_A)$.

▶ Remark. Several semantics have been given to quantitative games, see [Rei13]. The semantics chosen here, with zero tests (using $\mathsf{Tab}_0, \mathsf{Tab}_1$), allows to easily express the three semantics studied in [Rei13]. Hence our algorithms apply in all these settings. It is worth noticing that the hardness proof holds for the non-quantitative setting, hence also for all three semantics mentioned above.

We now define our weighted extension of Strategy Logic [CHP07, MMV10]:

▶ **Definition 2.** Let AP be a set of atomic propositions, Agt be a set of agents, and Var be a finite set of strategy variables. Formulas in $\mathsf{1cSL}$ are built from the following grammar:

$$\mathsf{1cSL} \ni \phi ::= p \mid \mathsf{cnt} \in S \mid \neg\,\phi \mid \phi \vee \phi \mid \mathbf{X}\,\phi \mid \phi\,\mathbf{U}\,\phi \mid \exists x.\ \phi \mid \mathsf{bind}(a \mapsto x).\ \phi$$

where p ranges over AP, S is a subset of $\mathbb{N}$ that can be described as $S^1_{\mathsf{fin}} \cup \big(S^2_{\mathsf{fin}} + k\cdot\mathbb{N}\big)$, where S^i_{fin} are finite subsets of $\mathbb{N}$ and $k \in \mathbb{N}$ is a period[1], x ranges over Var, and a ranges over Agt. The logic SL is the fragment of $\mathsf{1cSL}$ where no counter constraint $\mathsf{cnt} \in S$ or $\mathsf{cnt} \in S_{[k]}$ is used. The logic $\mathsf{1cLTL}$ is the fragment of $\mathsf{1cSL}$ where no strategy quantifiers $\exists x.\ \phi$ and no strategy bindings $\mathsf{bind}(a \mapsto x).\ \phi$ are used. Finally, LTL is the intersection of SL and $\mathsf{1cLTL}$.

The set of *free agents and variables* of a formula ϕ of $\mathsf{1cSL}$, which we write $\mathsf{free}(\phi)$, contains the agents and variables that have to be associated with a strategy before ϕ can be evaluated. It is defined inductively as follows:

$$\begin{array}{rll} \mathsf{free}(p) = \varnothing & \text{for all } p \in \mathsf{AP} & \mathsf{free}(\mathbf{X}\,\phi) = \mathsf{Agt} \cup \mathsf{free}(\phi) \\ \mathsf{free}(\mathsf{cnt} \in S) = \varnothing & \text{for all } n \in \mathbb{N} & \mathsf{free}(\phi\,\mathbf{U}\,\psi) = \mathsf{Agt} \cup \mathsf{free}(\phi) \cup \mathsf{free}(\psi) \\ \mathsf{free}(\neg\,\phi) = \mathsf{free}(\phi) & & \mathsf{free}(\phi \vee \psi) = \mathsf{free}(\phi) \cup \mathsf{free}(\psi) \end{array}$$

$$\mathsf{free}(\exists x.\ \phi) = \mathsf{free}(\phi) \setminus \{x\} \qquad \mathsf{free}(\mathsf{bind}(a \mapsto x).\ \phi) = \begin{cases} \mathsf{free}(\phi) & \text{if } a \notin \mathsf{free}(\phi) \\ (\mathsf{free}(\phi) \cup \{x\}) \setminus \{a\} & \text{otherwise} \end{cases}$$

A formula ϕ is *closed* if $\mathsf{free}(\phi) = \emptyset$.

We can now define the semantics of $\mathsf{1cSL}$. Let $\mathcal{G}$ be a $\mathsf{1cCGS}$, π be a path, i be a position along π, and $\chi\colon \mathsf{Var} \cup \mathsf{Agt} \dashrightarrow \mathsf{Strat}$ be a partial valuation (or context) with domain $\mathsf{dom}(\chi)$. Let $\phi \in \mathsf{SL}$ such that $\mathsf{free}(\phi) \subseteq \mathsf{dom}(\chi)$. Whether ϕ holds true at position i along π within context χ is defined inductively as follows:

$$\begin{array}{lcll} \mathcal{G},\pi,i \models_\chi p & \text{iff} & p \in \mathsf{label}(\ell_i) & (\text{writing } \pi(i) = (\ell_i, c_i)) \\ \mathcal{G},\pi,i \models_\chi \mathsf{cnt} \in S & \text{iff} & c_i \in S & (\text{writing } \pi(i) = (\ell_i, c_i)) \\ \mathcal{G},\pi,i \models_\chi \neg\,\phi_1 & \text{iff} & \mathcal{G},\pi,i \not\models_\chi \phi_1 & \\ \mathcal{G},\pi,i \models_\chi \phi_1 \vee \phi_2 & \text{iff} & \mathcal{G},\pi,i \models_\chi \phi_1 \text{ or } \mathcal{G},\pi,i \models_\chi \phi_2 & \\ \mathcal{G},\pi,i \models_\chi \mathbf{X}\,\phi_1 & \text{iff} & \mathcal{G},\rho,i+1 \models_\chi \phi_1 & (\text{writing } \rho = \mathsf{out}(\pi_{\le i}, \chi_{|\mathsf{Agt}})) \\ \mathcal{G},\pi,i \models_\chi \phi_1\,\mathbf{U}\,\phi_2 & \text{iff} & \exists k \ge i.\ \mathcal{G},\rho,k \models_\chi \phi_2 \text{ and} & \\ & & \forall i \le j < k.\ \mathcal{G},\rho,j \models_\chi \phi_1 & (\text{writing } \rho = \mathsf{out}(\pi_{\le i}, \chi_{|\mathsf{Agt}})) \\ \mathcal{G},\pi,i \models_\chi \exists x.\ \phi_1 & \text{iff} & \exists \sigma \in \mathsf{Strat}.\ \mathcal{G},\pi,i \models_{\chi[x\mapsto\sigma]} \phi_1 & \\ \mathcal{G},\pi,i \models_\chi \mathsf{bind}(a \mapsto x).\ \phi_1 & \text{iff} & \mathcal{G},\pi,i \models_{\chi[a\mapsto\chi(x)]} \phi_1 & \end{array}$$

Notice that the constraint that $\mathsf{free}(\phi) \subseteq \mathsf{dom}(\chi)$ is preserved at each step.

[1] This allows to express standard counter constraints like $\mathsf{cnt} \ge 5$ (using negation) or periodic constraint like $\mathsf{cnt} = 4 \bmod 7$. Notice that our periodicity result is not a consequence of the periodicity of the quantitative assertions, and would also hold with assertions of the form $\mathsf{cnt} \sim n$.

▶ Remark. One may notice that the relation $\mathcal{G}, \pi, i \models_\chi \phi$ does not depend on the suffix of π after position i. Moreover, writing $\sigma_{\overrightarrow{\pi_{\leq i}}}$ for the strategy σ' such that $\sigma'(\rho) = \sigma(\pi_{\leq i} \cdot \rho)$, it is easily proved that $\mathcal{G}, \pi, i \models_\chi \phi$ if, and only if, $\mathcal{G}, \pi', 0 \models_{\chi'} \phi$, where $\chi'(x) = \chi(x)_{\overrightarrow{\pi_{\leq i}}}$ for all $x \in \mathsf{Var} \cup \mathsf{Agt}$ (we will later write $\chi_{\overrightarrow{\pi_{\leq i}}}$ for χ'). As the satisfaction relation does not depend on the suffix of π after position i, we may also write $\mathcal{G}, \gamma \models_{\chi'} \phi$, where $\gamma = \pi(i)$. In the sequel, we may even omit to mention $\mathcal{G}$ when it is clear from the context, and simply write $\gamma \models_\chi \phi$.

▶ Remark. We write $\langle\!\cdot a \cdot\!\rangle\, \phi$ as a shorthand for $\exists \sigma_a.\ \mathsf{bind}(a \mapsto \sigma_a).\ \phi$, when we do not need to have hands on σ_a in the rest of the formula. Similarly, $[\cdot a \cdot]\, \phi$ stands for $\neg \langle\!\cdot a \cdot\!\rangle \neg \phi$. This construct $\langle\!\cdot a \cdot\!\rangle\, \phi$ precisely corresponds to the strategy quantification used in the logic ATL_{sc} [LM15], but it should be noticed that it does *not* correspond to the strategy quantifier of ATL [AHK02].

In the sequel, we also use other classical shorthands such as $\top$, defined as $p \vee \neg p$ for some p (hence it is always true); $\mathbf{F}\, \phi$ as a shorthand for $\top\, \mathbf{U}\, \phi$, meaning that ϕ holds at a later position; and $\mathbf{G}\, \phi$, defined as $\neg\, \mathbf{F} \neg \phi$, meaning that ϕ holds true at every future position.

Several fragments of SL have recently been defined and studied [MMPV14]. Those fragments restrict the use of strategy bindings and quantifications. In the present paper, we are mainly interested in the quantitative extension of the fragment SL[BG]. Before defining 1cSL[BG], we first introduce its *flat* fragment $\mathsf{1cSL}^0$[BG]:

$$\begin{aligned} \mathsf{1cSL}^0[\mathsf{BG}] \ni \phi &::= \neg \phi \mid \phi \vee \phi \mid \exists x.\ \phi \mid \mathsf{bind}(a \mapsto x).\ \phi \mid \psi \\ \psi &::= p \mid \mathsf{cnt} \in S \mid \neg \psi \mid \psi \vee \psi \mid \mathbf{X}\, \psi \mid \psi\, \mathbf{U}\, \psi \end{aligned}$$

▶ Remark. Any closed formula φ in $\mathsf{1cSL}^0$[BG] can be written in *prenex form* as

$$\wp(\mathsf{Var}).\ f\Big((\beta_i(\mathsf{Agt}, \mathsf{Var}).\ \psi_i)_{1 \leq i \leq n}\Big)$$

where $\wp(\mathsf{Var})$ is a series of strategy quantifiers involving all variables in Var, f is a Boolean combination over n atoms, and for every $1 \leq i \leq n$, β_i assigns a strategy from Var to each agent of Agt, and ψ_i is a 1cLTL formula.

1cSL[BG] then extends $\mathsf{1cSL}^0$[BG] by allowing nesting *closed* formulas at the level of atomic propositions. Formally, we defined the depth-i fragment as

$$\begin{aligned} \mathsf{1cSL}^i[\mathsf{BG}] \ni \phi &::= \neg \phi \mid \phi \vee \phi \mid \exists x.\ \phi \mid \mathsf{bind}(a \mapsto x).\ \phi \mid \psi \\ \psi &::= p \mid \phi_{i-1} \mid \mathsf{cnt} \in S \mid \neg \psi \mid \psi \vee \psi \mid \mathbf{X}\, \psi \mid \psi\, \mathbf{U}\, \psi \end{aligned}$$

where ϕ_{i-1} ranges over closed formulas of $\mathsf{1cSL}^{i-1}$[BG]. We let 1cSL[BG] be the union of the fragments $\mathsf{1cSL}^i$[BG] for all $i \in \mathbb{N}$. It can be checked that if we drop the quantitative constraints from 1cSL[BG], we precisely get the logic SL[BG] of [MMPV14].

3 Hardness of SL[BG] model checking

In this section, we prove that the model-checking problem for SL[BG] is Tower-hard (the complexity class Tower is the union of all classes k-EXPTIME when k ranges over $\mathbb{N}$ [Sch13]). We actually prove the result for (the flat fragment of) SL[BG], closing a question left open in [MMPV14].

▶ **Theorem 3.** *Model checking SL[BG], and hence 1cSL[BG], is* Tower*-hard.*

We give a sketch of the proof here, and develop the full proof in Appendix A.

Sketch of proof. We prove this result by encoding the satisfiability problem for QLTL into the model-checking problem for SL[BG]. QLTL is the extension of LTL with quantification over atomic propositions [Sis83]: formulas in QLTL are of the form $\Phi = \forall p_1 \exists p_2 \ldots \forall p_{n-1} \exists p_n.\ \varphi$ where ϕ is in LTL. Notice that we only consider strictly alternating formulas for the sake of readability. The general case can be handled similarly. Formula $\exists p.\ \varphi$ holds true over a word $w \colon \mathbb{N} \to 2^{\mathsf{AP}}$ if there exists a word $w' \colon \mathbb{N} \to 2^{\mathsf{AP}}$ with $w'(i) \cap (\mathsf{AP} \setminus \{p\}) = w(i) \cap (\mathsf{AP} \setminus \{p\})$ and $w' \models \varphi$ for all i. Universal quantification is defined similarly. It is well-known that model checking (and satisfiability) of QLTL is Tower-complete [SVW85]. We reduce the satisfiability of QLTL into a model-checking problem for a SL[BG] formula involving $n+4$ players (where n is the number of quantifiers in the QLTL formula), and three additional quantifier alternations.

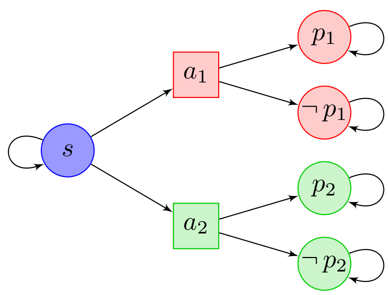


Figure 1 The 3-player turn-based game for the reduction to SL model checking

Before developing this technical encoding, we first present an example of a reduction to plain SL, which already contains most of the intuitions of our reduction to SL[BG]. Consider the QLTL formula

$$\Phi = \forall p_1.\ \exists p_2.\ \mathbf{G}\,(p_2 \Leftrightarrow \mathbf{X}\, p_1).$$

To solve the satisfiability problem of this formula via SL, we use the three-player turn-based game depicted on Fig. 1. In that game, Player Blue controls the blue state s, while Players Red and Green control the square states a_1 and a_2, respectively. Fix a strategy of Player Red: this strategy will be evaluated only in red state a_1, hence after histories of the form $s^n \cdot a_1$. Hence a strategy of Player Red can be seen as associating with each integer n a value for p_1. In other words, a strategy for Player Red defines a labeling of the time line with atomic proposition p_1. Similarly for Player Green and proposition p_2.

It remains to use this correspondence for encoding our QLTL formula. We have to express that for any strategy σ_{Red} of Player Red, there is a strategy σ_{Green} of Player Green under which, at each step along the path that stays in s forever, Player Blue can enforce $\mathbf{X}\,\mathbf{X}\, p_2$ if, and only if, he can enforce $\mathbf{X}\,\mathbf{X}\, p_1$ one step later. In the end, the formula reads as follows:

$$[\cdot\mathsf{Red}\cdot]\ \langle\cdot\mathsf{Green}\cdot\rangle\ \langle\cdot\mathsf{Blue}\cdot\rangle\ \mathbf{G}\left(s \wedge (\langle\cdot\mathsf{Blue}\cdot\rangle\, \mathbf{X}\,\mathbf{X}\, p_2) \Leftrightarrow (\mathbf{X}\ \langle\cdot\mathsf{Blue}\cdot\rangle\, \mathbf{X}\,\mathbf{X}\, p_1)\right) \tag{1}$$

One may notice that the above property is not in SL[BG]: for instance, the subformula $\langle\cdot\mathsf{Blue}\cdot\rangle\, \mathbf{X}\,\mathbf{X}\, p_2$ is not closed. We provide a different construction, refining the ideas above, in order to reduce QLTL satisfiability to SL[BG] model checking.

In order to do so, we take another approach for encoding the LTL formula, since our technique of encoding p_i with $\langle\cdot\mathsf{Blue}\cdot\rangle\, \mathbf{X}\,\mathbf{X}\, p_i$ is not compatible with getting a formula in SL[BG]. Instead, we will use a Büchi automaton encoding the formula; another player, say Player Black, will be in charge of selecting states of the Büchi automaton at each step. Using the same trick as above in the game structure on the left of Fig. 3, a strategy for Player Black can be seen as a mapping from $\mathbb{N}$ to states of the Büchi automaton. Our formula will ensure that this sequence of states is in accordance with the atomic propositions selected by the square players in states a_i, and that it forms an accepting run of the Büchi automaton.

For our example, an eight-state Büchi automaton associated with the (LTL part of the) QLTL formula is depicted on Fig. 2. Notice that smaller automata exist for this property (for

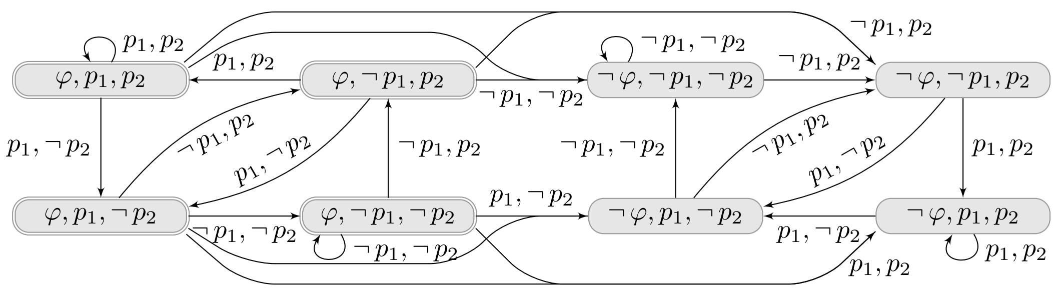


Figure 2 Büchi automaton for $\mathbf{G}\,(p_2 \Leftrightarrow \mathbf{X}\,p_1)$

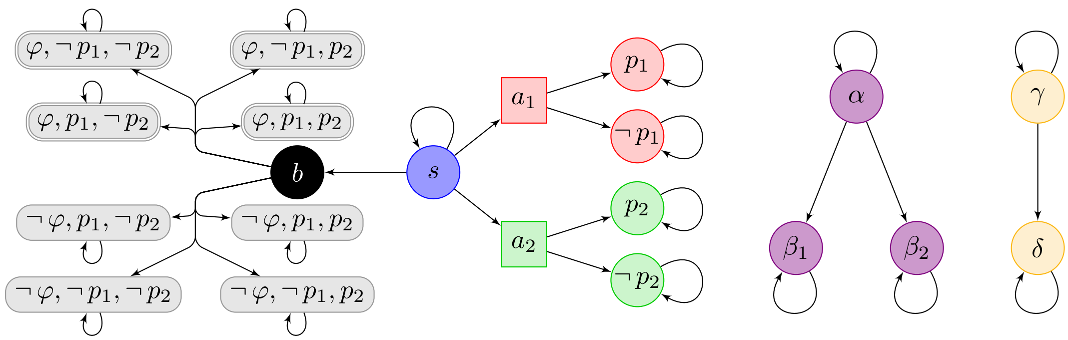


Figure 3 The concurrent game for the reduction to SL[BG] model checking.

instance, the four states on the right could be merged into a single one), but for technical reasons in our construction, we require that each state of the Büchi automaton corresponds to a single valuation of the atomic propositions, hence the number of states must be a multiple of $2^{|\mathsf{AP}|}$. Accordingly, we augment our game structure of Fig. 1 with eight extra states, as depicted on the left of Fig. 3. Again, a strategy of Player Black (controlling state b) defines a sequence of states of the Büchi automaton.

It then remains to "synchronize" the run of the Büchi automaton with the valuations of the atomic propositions, selected by the players controlling the square states. This is achieved by taking the product of the game we just built with two extra one-player structures, as depicted on the right of Fig. 3. The product gives rise to a concurrent game, where one transition is taken simultaneously in the main structure and in the Purple and Orange structures. In this product, as long as Player Blue remains in s and Player Purple remains in α, a strategy of Player Orange (controlling state γ) either remains in γ forever, or it can be characterized by a value $n \in \mathbb{N}$. Similarly, as long as Player Blue remains in s and Player Orange remains in γ, a strategy of Player Purple (controlling state α) either loops forever in α, or can be uniquely characterized by a pair (k, p_l), where k is the number of times the loop over α is taken before entering state β_l corresponding to $p_l \in \mathsf{AP}$.

Our construction can then be divided in two steps:

- first, with any strategy of Player Purple (characterized by (k, p_l) for the interesting cases), we associate auxiliary strategies of Players Blue, Purple and Orange satisfying certain properties, that can be enforced by an SL[BG] formula Ψ_{aux}; Fig. 4 should help visualizing the associated strategies; in particular, strategies $\sigma_{+}^{\mathsf{orange}}$, $\sigma_{+}^{\mathsf{blue}}$ and $\sigma_{+}^{\mathsf{purple}}$ characterize position $k+1$ (which will be useful for checking transitions of the Büchi automaton), while $\sigma_{\mathsf{Büchi}}^{\mathsf{blue}}$ and $\sigma_{\mathsf{AP}}^{\mathsf{blue}}$ are Player-Blue strategies that either go to the Büchi part or to the proposition part of the main part of the game.

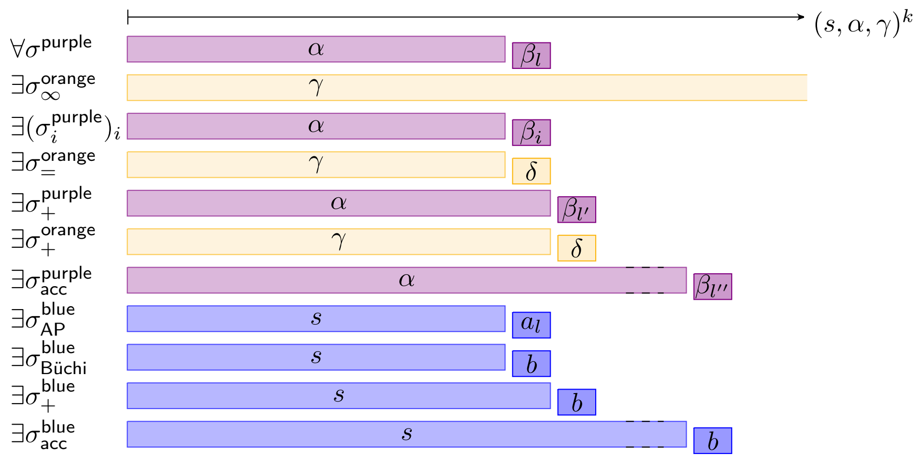


Figure 4 Visualization of the strategies selected by Ψ_{aux} on history $(s,\alpha,\gamma)^k$.

- then, using those strategies, we write another SL[BG] formula to enforce that the transitions of the Büchi automaton are correctly applied, following the valuations of the atomic propositions selected in the square states, and that an accepting state is visited infinitely many times.

The construction of the game structure $\mathcal{G}_\Phi$ depicted on Fig. 3 is readily extended to any number of atomic propositions, and to any Büchi automaton. We now explain how we build our SL[BG] formula replacing Formula (1), and ensuring correctness of our reduction.

We do not detail the first step mentioned above and assume that a formula Ψ_{aux} has been written, which properly generates auxiliary strategies, as depicted on Fig. 4 (see Appendix A, on page 15). Instead we focus on the Büchi automaton simulation. We look for a strategy of Player Black that will mimic the run of the Büchi automaton, following the valuation of the atomic propositions selected by the square players A_1 to A_n. We also require that the run of the Büchi automaton be accepting.

The formula Ψ enforcing these constraints is as follows[2]:

$$\forall\sigma^{A_1}.\ \exists\sigma^{A_2}.\ \ldots\forall\sigma^{A_{n-1}}.\ \exists\sigma^{A_n}.\ \exists\sigma^{\mathsf{black}}.\ \mathsf{bind}(\sigma^{A_1},\sigma^{A_2},\ldots,\sigma^{A_{n-1}},\sigma^{A_n},\sigma^{\mathsf{black}},\sigma_\infty^{\mathsf{orange}}).\ \Psi_{\mathsf{aux}}$$

$$\wedge \bigwedge_{p_i,p_j\in\mathsf{AP}}\ \bigwedge_{q\in Q} (\mathsf{bind}(\sigma_{\mathsf{Büchi}}^{\mathsf{blue}},\sigma_i^{\mathsf{purple}})\mathbf{F}\,q) \Leftrightarrow (\mathsf{bind}(\sigma_{\mathsf{Büchi}}^{\mathsf{blue}},\sigma_j^{\mathsf{purple}})\mathbf{F}\,q) \qquad (\varphi_1)$$

$$\wedge \bigwedge_{p_i\in\mathsf{AP}} \Big(\big(\mathsf{bind}(\sigma_{\mathsf{AP}}^{\mathsf{blue}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\,p_i\big) \Rightarrow \big(\mathsf{bind}(\sigma_{\mathsf{Büchi}}^{\mathsf{blue}},\sigma^{\mathsf{purple}}).\ \bigvee_{q\in Q\mid p_i\in\mathsf{label}(q)} \mathbf{F}\,q\big)\Big) \qquad (\varphi_2)$$

$$\wedge \bigwedge_{p_i\in\mathsf{AP}} \Big(\big(\mathsf{bind}(\sigma_{\mathsf{AP}}^{\mathsf{blue}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\,\neg p_i\big) \Rightarrow \big(\mathsf{bind}(\sigma_{\mathsf{Büchi}}^{\mathsf{blue}},\sigma^{\mathsf{purple}}).\ \bigvee_{q\in Q\mid p_i\notin\mathsf{label}(q)} \mathbf{F}\,q\big)\Big) \qquad (\varphi_3)$$

$$\wedge \bigwedge_{q\in Q} \mathsf{bind}(\sigma_{\mathsf{Büchi}}^{\mathsf{blue}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\,q \Rightarrow \bigvee_{q'\in\mathsf{succ}(q)} \mathsf{bind}(\sigma_+^{\mathsf{blue}},\sigma_+^{\mathsf{purple}}).\ \mathbf{F}\,q' \qquad (\varphi_4)$$

$$\wedge\ \mathsf{bind}(\sigma_{\mathsf{acc}}^{\mathsf{blue}},\sigma_{\mathsf{acc}}^{\mathsf{purple}}).\ \bigvee_{q\in\mathsf{accept}(Q)} \mathbf{F}\,q \qquad (\varphi_5)$$

[2] We notice that Ψ is not syntactically in SL[BG], as some bindings appear before quantifications in Ψ_{aux}. However, quantifiers in Ψ_{aux} could be moved before the bindings of Ψ.

We now analyze formula Ψ:

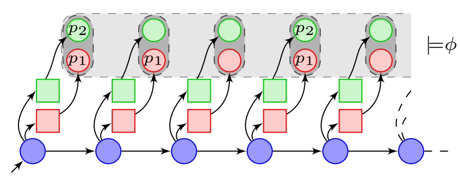
- Formula (φ_1) requires that strategy σ^{black} returns the same move after any history of the form $(s, \alpha, \gamma)^k(b, \beta_i, \gamma)$, whichever β_i has been selected by σ^{purple};
- Formulas (φ_2) and (φ_3) constrain the state of the Büchi automaton to correspond to the valuation of the atomic propositions selected. Because of the universal quantification over σ^{purple}, this property will be enforced at all positions and for all atomic propositions;
- Formula (φ_4) additionally requires that two consecutive states of the run of the Büchi automaton indeed correspond to a transition;
- finally, Formula (φ_5) states that for any position (selected by σ^{purple}), there exists a later position (given by $\sigma^{\mathsf{purple}}_{\mathsf{acc}}$) at which the run of the Büchi automaton visits an accepting state.

The correctness of the construction is then stated in the next lemma, whose proof can be found in Appendix A, page 17.

▶ **Lemma 4.** *Formula Φ in* QLTL *is satisfiable if, and only if, Formula Ψ in SL[BG] holds true in state (s, α, γ) of the game $\mathcal{G}_\Phi$.* ◀

▶ Remark. SL[BG] and several other fragments were defined in [MMPV14, MMS14] with the aim of getting more tractable fragments of SL. In particular, the authors advocate for the restriction to *behavioural strategies*: this forbids strategies that prescribe actions depending of what other strategies would prescribe later on, or after different histories. Non-behavioural strategies are thus claimed to have limited interest in practice; moreover, they are suspected of being responsible for the non-elementary complexity of SL model-checking. Our hardness result strengthens the latter claim, as SL[BG] is known for not having behavioral strategies.

▶ Remark. We had to rely on a Büchi automaton instead of directly using the original LTL formula directly in the SL[BG] formula. This is because we need to evaluate the formula not on a real path of our game structure, but on a sequence of "unions" of states. The figure on the right represents this situation for the game structure of Fig. 1: the path on which the LTL formula is given by the red and green circle states, which define the valuations for p_1 and p_2.



4 Periodicity of 1cSL[BG] model checking

In this section we prove our periodicity property for 1cSL[BG]. We inductively define the function $\mathsf{tower}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as $\mathsf{tower}\,(a, 0) = a$ and $\mathsf{tower}\,(a, b+1) = 2^{\mathsf{tower}(a,b)}$. This encodes *towers of exponentials* of the form $2^{2^{\cdot^{\cdot^{\cdot^{a}}}}}$.

▶ **Theorem 5.** *Let $\mathcal{G}$ be a 1cCGS, and φ be a 1cSL[BG] formula. Then there exist a threshold $h \geq 0$ and a period $\Lambda \geq 0$ for the truth value of φ over $\mathcal{G}$. That is, for every configuration (q, c) of $\mathcal{G}$ with $c \geq h$, for every $k \in \mathbb{N}$, $\mathcal{G}, (q, c) \models \varphi$ if, and only if, $\mathcal{G}, (q, c + k \cdot \Lambda) \models \varphi$.*

Furthermore the order of magnitude for $h + \Lambda$ is bounded by

$$\mathsf{tower}\left(\max_{\theta \in \mathsf{Subf}(\varphi)} n_\theta, \max_{\theta \in \mathsf{Subf}(\varphi)} k_\theta + 1\right)^{|Q| \cdot 2^{2^{|\varphi|}}}$$

where $\mathsf{Subf}(\varphi)$ is the set of 1cSL[BG] formulas of φ, k_θ is the number of quantifier alternations in θ, and n_θ is the number of different bindings used in θ.

The rest of this section is devoted to developing the proof of this result, though not with full details. Detailed proofs of intermediate results are given in Appendix B. We first prove this property for the flat fragment $\mathsf{1cSL}^0[\mathsf{BG}]$, and then extend it to the full $\mathsf{1cSL}[\mathsf{BG}]$.

4.1 The flat fragment $\mathsf{1cSL}^0[\mathsf{BG}]$

We fix a $\mathsf{1cCGS}$ $\mathcal{G}$ and a formula $\varphi = Q_1 x_1 \ldots Q_k x_k.\ f((\beta_i \phi_i)_{1\leq i\leq n})$ in $\mathsf{1cSL}^0[\mathsf{BG}]$, where for every $1 \leq j \leq k$, we have $Q_j \in \{\exists, \forall\}$ (assuming quantifiers strictly alternate), f is a Boolean formula over n atoms, and for every $1 \leq i \leq n$, β_i is a complete binding for the players' strategies, and ϕ_i is a $\mathsf{1cLTL}$ formula. We write M for the maximal constant appearing in one of the finite sets describing a counter constraint S appearing in φ.

For every $1 \leq i \leq n$, we let $\mathcal{D}_i$ be a deterministic (counter) parity automaton that recognizes formula ϕ_i (this is the standard LTL-to-(deterministic parity) automata construction in which quantitative constraints are seen as atoms). A run of $\mathcal{G}$ is read in a standard way, with the additional condition that quantitative constraints labelling a state should be satisfied by the counter value when the state is traversed (a state can be labelled by a constraint $\mathsf{cnt} \in S$, with S arbitrarily complex—it does not impact the description of the automaton).

The proof proceeds by showing that, above some threshold, the truth value of φ is periodic w.r.t. counter values. To prove this, we define an equivalence relation over counter values that generates identical strategic possibilities (in a sense that will be made clear later on).

4.1.1 Definition of an equivalence relation

Fix a configuration $\gamma = (\ell, c)$ in $\mathcal{G}$, pick for every $1 \leq i \leq n$ a state d_i in the automaton $\mathcal{D}_i$, and define the tuple $D = (d_1, \ldots, d_n)$. For every context χ_k for variables $\{x_1, \ldots, x_k\}$, we define the level-0 identifier $\mathsf{Id}_{\chi_k}(\gamma, D)$ as:

$$\mathsf{Id}_{\chi_k}(\gamma, D) = \big\{ i \;\big|\; 1 \leq i \leq n \text{ and } \mathsf{out}(\gamma, \beta_i[\chi_k]) \text{ is accepted by } \mathcal{D}_i \text{ from } d_i \big\}$$

where $\beta_i[\chi_k]$ assigns a strategy from χ_k to each player in Agt following β_i.

Assuming we have defined level-$(k-j+1)$ identifiers $\mathsf{Id}_{\chi_{j+1}}(\gamma, D)$ for every partial context χ_{j+1} for variables $\{x_1, \ldots, x_{j+1}\}$, we define the level-$(k-j)$ identifier $\mathsf{Id}_{\chi_j}(\gamma, D)$ for every partial context χ_j for variables $\{x_1, \ldots, x_j\}$ as follows:

$$\mathsf{Id}_{\chi_j}(\gamma, D) = \big\{ \mathsf{Id}_{\chi_{j+1}}(\gamma, D) \;\big|\; \chi_{j+1} \text{ is a context for } \{x_1, \ldots, x_{j+1}\} \text{ that extends } \chi_j \big\}.$$

There is a unique level-k identifier for every configuration $\gamma = (\ell, c)$ and every D, which corresponds to the empty context. It somehow contains full information about what kinds of strategies can be used in the game (this is a hierarchical information set, which contains all level-j identifiers for $j < k$).

Let P be the least common multiple of all the periods appearing in periodic quantitative assertions used in formula φ. We define the following equivalence on counter values:

$$c \sim c' \quad \text{if, and only if,} \quad c = c' \bmod P \text{ and } \forall D.\ \forall \ell.\ \mathsf{Id}_\emptyset((\ell, c), D) = \mathsf{Id}_\emptyset((\ell, c'), D).$$

Combinatorics. Given a configuration (ℓ, c) and a tuple D, the number of possible values for the level-0 identifier is $\mathsf{tower}\,(n, 1)$, and for the level-j identifier it is $\mathsf{tower}\,(n, j+1)$. Hence, the number $\mathsf{ind}_\sim$ of equivalence classes of the relation $\sim$ satisfies

$$\mathsf{ind}_\sim \leq P \cdot (\mathsf{tower}\,(n, k+1))^{\left(|Q| \cdot \prod_{1\leq i\leq n} 2^{2^{|\phi_i|}}\right)} \leq P \cdot (\mathsf{tower}\,(n, k+1))^{\left(|Q| \cdot 2^{2^{|\varphi|}}\right)}$$

with $|Q|$ the number of states in $\mathcal{G}$. We let $\overline{M} = M + \mathsf{ind}_\sim + 1$. By the pigeon-hole principle, there must exist $M < h < h' \leq \overline{M}$ such that $h \sim h'$.

4.1.2 Periodicity property

We define $\Lambda = h' - h$, and now prove that it is a period for φ for counter values larger than or equal to h. Assume that $\gamma = (\ell, c)$ is a configuration such that $c \geq h$, and define $\gamma' = (\ell, c + \Lambda)$ (note that $c + \Lambda \geq h'$). We show that $\mathcal{G}, \gamma \models \varphi$ if, and only if, $\mathcal{G}, \gamma' \models \varphi$.

▶ Notations. For the rest of this proof, we fix the following notations:

1. if ρ is a run starting with counter value $a > c$, then either the counter always remains above c along ρ (in which case we say that ρ is fully above c), or it eventually hits value c, and we define $\rho_{\searrow c}$ for the smallest prefix of ρ such that $\mathsf{last}(\rho_{\searrow c})$ has counter value c;
2. let ρ be a run that is fully above M, and let c be the least counter value appearing in ρ. For every $\nu \geq M - c$, we write $\mathsf{Shift}_\nu(\rho)$ for the run ρ' obtained from ρ by shifting the counter value by ν. It is a real run since the counter values along ρ' are also all above M.
3. if D is a tuple of states of the deterministic automata $\mathcal{D}_i$, and if ρ is a finite run of $\mathcal{G}$ that is fully above M, then we write $D_{+\rho}$ for the image of D after reading ρ.

Let $0 \leq j \leq k$. We assume that χ_j and χ'_j are two contexts for $\{x_1, \ldots, x_j\}$, and D is a tuple of states of the $\mathcal{D}_i$'s. We write $\mathbb{R}^{D,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ if the following property holds for any run ρ from γ:

(i) if ρ is fully above h (or equivalently, if $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$, which starts from γ', is fully above h'), then for every $1 \leq g \leq j$, $\chi_j(x_g)(\rho) = \chi'_j(x_g)(\rho')$;

(ii) if ρ is not fully above h (equivalently, if $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$ is not fully above h'), then we decompose ρ (resp. ρ') w.r.t. h (resp. h') and write $\rho = \rho_{\searrow h} \cdot \overline{\rho}$ and $\rho' = \rho'_{\searrow h'} \cdot \overline{\rho}'$. Then:

$$\mathsf{Id}_{\chi_{j\overrightarrow{\rho_{\searrow h}}}}(\mathsf{last}(\rho_{\searrow h}), \widetilde{D}) = \mathsf{Id}_{\chi'_{j\overrightarrow{\rho'_{\searrow h'}}}}(\mathsf{last}(\rho'_{\searrow h'}), \widetilde{D})$$

with $\widetilde{D} = D_{+\rho_{\searrow h}} = D_{+\rho'_{\searrow h'}}$. Recall that $\chi_{j\overrightarrow{\rho_{\searrow h}}}$ shifts all strategies in context χ_j after the prefix $\rho_{\searrow h}$ (that is, χ_j is the strategy such that $\chi_{j\overrightarrow{\rho_{\searrow h}}}(\pi) = \chi_j(\rho_{\searrow h} \cdot \pi)$ for every π).

We then have:

▶ **Lemma 6.** *Fix $0 \leq j < k$, and assume that $\mathbb{R}^{D,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ holds true. Then:*

1. *for every strategy v for x_{j+1} from γ, one can build a strategy $\mathcal{T}(v)$ for x_{j+1} from γ' such that $\mathbb{R}^{D,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{v\}, \chi'_j \cup \{\mathcal{T}(v)\})$ holds true;*
2. *for every strategy v' for x_{j+1} from γ', one can build a strategy $\mathcal{T}^{-1}(v')$ for x_{j+1} from γ such that $\mathbb{R}^{D,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{\mathcal{T}^{-1}(v')\}, \chi'_j \cup \{v'\})$ holds true.*

Sketch of proof. The idea is the following: either we are in case (i), in which case identical (but shifted) strategies can be applied; or we are in case (ii), in which case identical (but shifted) strategies can be applied until counter value h (resp. h') is hit, in which case equality of identifiers allows to apply equivalent strategies. The construction is illustrated in Fig. 5. ◀

We use this lemma to transfer a proof that $\gamma \models_\emptyset \varphi$ to a proof that $\gamma' \models_\emptyset \varphi$. We decompose the proof of this equivalence into two lemmas:

▶ **Lemma 7.** *Fix D^0 for the tuple of initial states of the $\mathcal{D}_i$'s. Assume that $\mathbb{R}^{D^0,k}_{(\gamma,\gamma')}(\chi, \chi')$ holds (for full contexts χ and χ'). Let $1 \leq i \leq n$, and write $\rho = \mathsf{Out}(\gamma, \beta_i[\chi])$ and $\rho' = \mathsf{Out}(\gamma', \beta_i[\chi'])$. Then $\rho \models \phi_i$ if and only if $\rho' \models \phi_i$. In particular, $\gamma \models_\chi f((\beta_i\phi_i)_{1\leq i\leq n})$ if and only if $\gamma' \models_{\chi'} f((\beta_i\phi_i)_{1\leq i\leq n})$.*

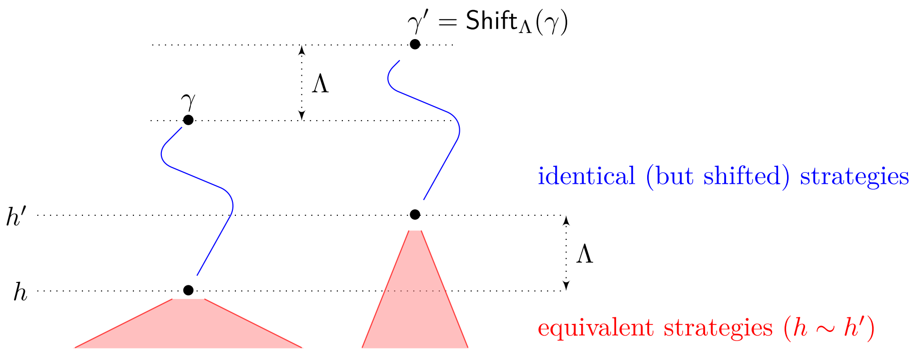


Figure 5 Construction in Lemma 6 (case (ii))

Sketch of proof. As long as runs are above h (resp. h') they visit states that satisfy exactly the same atomic properties (atomic propositions and counter constraints), hence they progress in each $\mathcal{D}_i$ along the same run. When value h (resp. h') is hit, they are generated by strategies that have the same level-0 id, which precisely means they are equivalently accepted by each $\mathcal{D}_i$. Hence both outcomes satisfy the same formulas ϕ_i under binding $\beta_i[\chi]$ (resp. $\beta_i[\chi']$). ◀

We finally show the following lemma, by induction on the context, and by noticing that $h \sim h'$ precisely implies the induction property at level 0.

▶ **Lemma 8.** *$\gamma \models_\emptyset \varphi$ if and only if $\gamma' \models_\emptyset \varphi$.*

This allows to conclude with the following corollary:

▶ **Corollary 9.** *Λ is a period for the satisfiability of φ for configurations with counter values larger than or equal to h.*

Furthermore, $h + \Lambda$ is bounded by $M + P \cdot (\mathsf{tower}\,(n, k+1))^{|Q| \cdot \prod_{1 \leq i \leq} 2^{2^{|\varphi|}}} + 1$.

▶ Remark. Note that the above proof of existence of a period, though effective (a period can be computed by computing the truth of identifier predicates), does not allow for an algorithm to decide the model-checking problem. One possible idea to lift that periodicity result to an effective algorithm would be to bound the counter values; however things are not so easy: in Fig. 5, equivalent strategies from h and h' might generate runs with (later on) counter values larger than h or h'. The decidability status of $\mathsf{1cSL}^1[\mathsf{BG}]$ (and of $\mathsf{1cSL}[\mathsf{BG}]$) model checking remains open.

4.2 Extension to 1cSL[BG]

We explain how we can extend the previous periodicity analysis to the full logic $\mathsf{1cSL}[\mathsf{BG}]$. We fix a formula of $\mathsf{1cSL}^{k+1}[\mathsf{BG}]$

$$\varphi = Q_1 x_1 \ldots Q_k x_k \cdot f((\beta_i \phi_i)_{1 \leq i \leq n})$$

with the same notations than the ones at the beginning of the previous subsection, but ϕ_i can use closed formulas of $\mathsf{1cSL}^k[\mathsf{BG}]$ as subformulas.

Let Ψ_φ be the set of closed subformulas of $\mathsf{1cSL}^k[\mathsf{BG}]$ that appear directly under the scope of some ϕ_i. We will replace subformulas of Ψ_φ by other formulas involving only (new) atomic propositions and counter constraints. Pick $\psi \in \Psi_\varphi$. Let h_ψ and Λ_ψ be the threshold and the period mentioned in Corollary 9. For every location ℓ of the game, the set of counter values c such that $(\ell, c) \models \psi$ can be written as S^{ψ}_{ℓ} (we use a non-periodic set for the values smaller than h_ψ and a periodic set of period Λ_ψ for the values above h_ψ)—note that we know such a set exists, even though there is (for now) no effective procedure to express it. The size of formula S^{ψ}_{ℓ} is 1 (we do not take into account the complexity of writing the precise sets used in the constraint). Expand the set of atomic propositions AP with an extra atomic proposition for each location, say p_ℓ for location ℓ, which holds only at location ℓ. For every $\psi \in \Psi_\varphi$, replace that occurrence of ψ in φ by formula $\bigwedge_{\ell\in L} p_\ell \to (\mathsf{cnt} \in S^{\psi}_{\ell})$. This defines formula φ', which is now a $\mathsf{1cSL}^0[\mathsf{BG}]$ formula, and holds equivalently (w.r.t. φ) from every configuration of $\mathcal{G}$. The size of φ' is that of φ. We apply the result of the previous subsection and get a proof of periodicity of the satisfaction relation for φ', hence for φ.

It remains to compute bounds on the overall period Λ_φ and threshold h_φ. The modulo constraints in φ' involve periods Λ_ψ ($\psi \in \Psi_\varphi$), and the constants used are bounded by h_ψ. So the bound $M_{\varphi'}$ is bounded by $\max(\max_{\psi\in\Psi}(h_\psi), M_\varphi)$ where M_φ is the maximal constant used in φ, and the value $P_{\varphi'}$ is the l.c.m. of the periods used in φ (call it P_φ) and of the Λ_ψ's (for $\psi \in \Psi_\varphi$): hence $P_{\varphi'} \leq P_\varphi \cdot \max_{\psi\in\Psi_\varphi}(\Lambda_\psi)^{|\varphi|}$ Hence for formula φ', we get

$$h_{\varphi'} + \Lambda_{\varphi'} \quad\leq\quad M_{\varphi'} + P_{\varphi'} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi'|}}} + 1$$

We infer the following order of magnitude for $h_\varphi + \Lambda_\varphi$, where $\omega_{\Psi_\varphi} = \max_{\psi\in\Psi_\varphi} \omega_\psi$:

$$\begin{aligned}\omega_\varphi &\approx \omega_{\Psi_\varphi} + M_\varphi^{|\varphi|} \cdot (\max \Lambda_\psi)^{|\varphi|} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi|}}}\\ &\approx M_\varphi^{|\varphi|} \cdot \omega_{\Psi_\varphi}^{|\varphi|} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi|}}}\end{aligned}$$

Using notations of Theorem 5, the order of magnitude can therefore be bounded by

$$\mathsf{tower}\left(\max_{\theta\in\mathsf{Subf}(\varphi)} n_\theta, \max_{\theta\in\mathsf{Subf}(\varphi)} k_\theta + 1\right)^{|Q|\cdot 2^{2^{|\varphi|}}}.$$

▶ Remark. Note that this proof is non-constructive, even for the period and the threshold, since it relies on the model-checking of subformulas, which we don't know how to do. We can nevertheless effectively compute a threshold and a period by taking the l.c.m. of all the integers up to the bound over the period and threshold given in this proof.

5 Conclusion

In this paper, we investigated a quantitative extension of Strategy Logic (and more precisely, of its *Boolean-Goal* fragment) over games played on one-counter games. We proved that the corresponding model-checking problem enjoys a nice periodicity property, which we see as a first step towards proving decidability of the problem. We proved however that, if decidable, the problem is hard; this is proved by showing that model checking the fragment SL[BG] over finite-state games is Tower-hard, hence answering an open question from [MMPV14].

We are now trying to see how our periodicity property can be used to prove decidability of the model-checking problem. While such a periodicity property helps getting effective algorithms for model checking CTL over one-counter machines [GL10], the game setting used here makes things much harder. Other further works also include the more general logic 1cSL, whose decidability status (and complexity) is also open. Finally, we did not manage to extend our hardness proof to turn-based games. It would be nice to understand whether the restriction to turn-based games would make 1cSL[BG] (and SL[BG]) model checking easier.

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A Appendix for Section 3

We give here details that are missing in the proof of Theorem 3. To ease the reading, we copy here the figure describing the game (Fig. 6)).

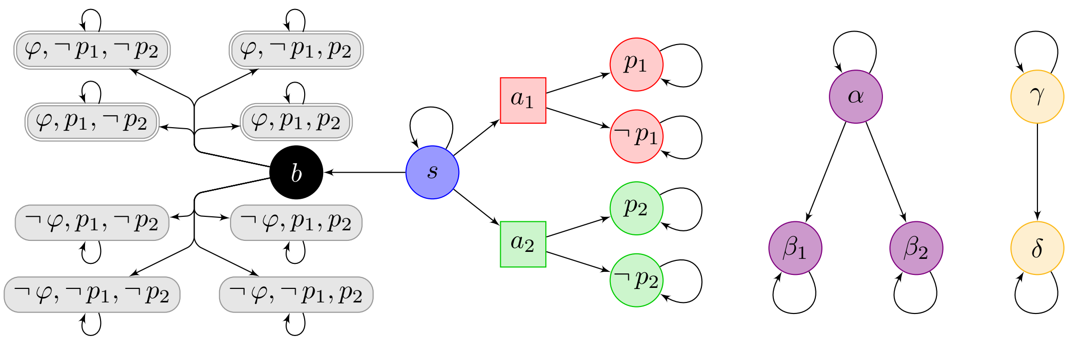

Figure 6 The concurrent game for the reduction to SL[BG] model checking.

Characterization of auxiliary strategies (definition of formula Ψ_{aux}).

We begin with simple constructions to select specific strategies, independently of the choice of the valuations for the atomic propositions. In the rest of this proof, we name strategy variables after the name of the player who will be assigned that strategy; for instance, $\sigma^{\mathsf{orange}}_{\infty}$ (that we use below) is the strategy of Player Orange that always plays to γ. Our assignment operator will thus take only a strategy as argument, as the associated player will be clear from the name of the strategy. Also, players with no assigned strategies may have any strategy.

We consider the following formula, which we denote Ψ_{aux} in the sequel, in which we write β as a shorthand for $\bigvee_{p_i\in\mathsf{AP}} \beta_i$:

$$\forall\sigma^{\mathsf{purple}}.\ \exists\sigma^{\mathsf{orange}}_{\infty}.\ \exists(\sigma^{\mathsf{purple}}_{i})_{p_i\in\mathsf{AP}}.\ \exists\sigma^{\mathsf{orange}}_{=}.\ \exists\sigma^{\mathsf{purple}}_{+}.\ \exists\sigma^{\mathsf{orange}}_{+}.\ \exists\sigma^{\mathsf{purple}}_{\mathsf{acc}}.$$
$$\exists\sigma^{\mathsf{blue}}_{\mathsf{AP}}.\ \exists\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}.\ \exists\sigma^{\mathsf{blue}}_{+}.\ \exists\sigma^{\mathsf{blue}}_{\mathsf{acc}}.\ \forall\sigma^{\mathsf{blue}}_{\forall}.\ \forall\sigma^{\mathsf{purple}}_{\forall}.$$

$$\mathsf{bind}(\sigma^{\mathsf{blue}}_{\forall}, \sigma^{\mathsf{purple}}_{\forall}, \sigma^{\mathsf{orange}}_{\infty}).\ \mathbf{G}\,(\neg\,\delta) \qquad (\varphi'_1)$$
$$\wedge$$
$$\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}, \sigma^{\mathsf{orange}}_{\infty}).\ \big[\mathbf{G}\,(\alpha\wedge s)\ \vee (\alpha\wedge s)\,\mathbf{U}\,(\beta\wedge b)\big] \qquad (\varphi'_2)$$
$$\wedge$$
$$\mathsf{bind}(\sigma^{blue}_{+}, \sigma^{\mathsf{purple}}_{+}, \sigma^{\mathsf{orange}}_{\infty}).\ \big[\mathbf{G}\,(\alpha\wedge s)\ \vee (\alpha\wedge s)\,\mathbf{U}\,(\beta\wedge b)\big] \qquad (\varphi'_3)$$
$$\wedge$$
$$\mathsf{bind}(\sigma^{blue}_{acc}, \sigma^{\mathsf{purple}}_{\mathsf{acc}}, \sigma^{\mathsf{orange}}_{\infty}).\ \big[\mathbf{G}\,(\alpha\wedge s)\ \vee (\alpha\wedge s)\,\mathbf{U}\,(\beta\wedge b)\big] \qquad (\varphi'_4)$$
$$\wedge$$
$$\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{AP}}, \sigma^{\mathsf{purple}}, \sigma^{\mathsf{orange}}_{\infty}).\ \big[\mathbf{G}\,(\alpha\wedge s)\ \vee (\alpha\wedge s)\,\mathbf{U}\,(\textstyle\bigvee_{p_i\in\mathsf{AP}} \beta_i\wedge a_i)\big] \qquad (\varphi'_5)$$
$$\wedge$$
$$\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}, \sigma^{\mathsf{orange}}_{=}).\ \mathbf{G}\,(\beta\Leftrightarrow\delta) \qquad (\varphi'_6)$$
$$\wedge$$
$$\bigwedge_{p_i\in\mathsf{AP}}\Big[\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}_{i}, \sigma^{\mathsf{orange}}_{=}).\ \mathbf{G}\,(\beta_i\Leftrightarrow\delta)\wedge\bigwedge_{p_j\neq p_i}\mathbf{G}\,\neg\,\beta_j\Big] \qquad (\varphi'_7)$$
$$\wedge$$
$$\Big(\big[\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}, \sigma^{\mathsf{orange}}_{\infty}).\ \mathbf{F}\,\beta\big]\Rightarrow$$
$$\big[(\mathsf{bind}(\sigma^{\mathsf{blue}}_{+}, \sigma^{\mathsf{purple}}_{+}, \sigma^{\mathsf{orange}}_{\infty}).\ \mathbf{F}\,\beta\wedge\mathsf{bind}(\sigma^{\mathsf{purple}}_{+}, \sigma^{\mathsf{orange}}_{=}).\ \mathbf{F}\,(\delta\wedge\neg\,\beta))\big]\Big) \qquad (\varphi'_8)$$

$$\begin{aligned}
\wedge\quad & \Big(\big[\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}, \sigma^{\mathsf{orange}}_{\infty}).\ \mathbf{F}\,\beta\big] \Rightarrow \\
& \qquad \big[(\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{acc}}, \sigma^{\mathsf{purple}}_{\mathsf{acc}}, \sigma^{\mathsf{orange}}_{\infty}).\ \mathbf{F}\,\beta \wedge \mathsf{bind}(\sigma^{\mathsf{purple}}_{\mathsf{acc}}, \sigma^{\mathsf{orange}}_{=}).\ \mathbf{F}\,(\delta \wedge \neg\,\beta))\big]\Big) && (\varphi'_9)\\
\wedge\quad & \mathsf{bind}(\sigma^{\mathsf{blue}}_{+}, \sigma^{\mathsf{purple}}_{+}, \sigma^{\mathsf{orange}}_{+}).\ \mathbf{G}\,(\beta \Leftrightarrow \delta) && (\varphi'_{10})\\
\wedge\quad & \mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}_{\forall}, \sigma^{\mathsf{orange}}_{=}).\ \mathbf{F}\,(\delta \wedge \alpha) \Rightarrow \mathsf{bind}(\sigma^{\mathsf{blue}}_{+}, \sigma^{\mathsf{purple}}_{\forall}, \sigma^{\mathsf{orange}}_{+}).\ \alpha\,\mathbf{U}\,\delta && (\varphi'_{11})
\end{aligned}$$

We now explain how this formula holds true in the product game of Fig. 6, whichever strategies are assigned to the square players and to Player Black. Actually, it only associates, with each strategy σ^{purple} (quantified universally at the beginning of the formula) a set of strategies that "synchronize" with σ^{purple}:

- strategy $\sigma^{\mathsf{orange}}_{\infty}$ always plays to γ, whatever the history of the game (Formula (φ'_1));
- strategy $\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}$ must be such that, for all ρ of the form $(s,\alpha,\gamma)^j$, it holds $\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}(\rho) = s$ as long as $\sigma^{\mathsf{purple}}(\rho) = \alpha$, and $\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}(\rho) = b$ when (if ever) $\sigma^{\mathsf{purple}}(\rho) \neq \alpha$ (Formula (φ'_2)). Likewise, for any ρ of the form $(s,\alpha,\gamma)^j$, strategy $\sigma^{\mathsf{blue}}_{+}$ (resp. $\sigma^{\mathsf{blue}}_{\mathsf{acc}}$) must be such that $\sigma^{\mathsf{blue}}_{+}(\rho) = s$ (resp. $\sigma^{\mathsf{blue}}_{\mathsf{acc}}(\rho) = s$) as long as $\sigma^{\mathsf{purple}}_{+}(\rho) = \alpha$ (resp. $\sigma^{\mathsf{purple}}_{\mathsf{acc}}(\rho) = \alpha$), and such that $\sigma^{\mathsf{blue}}_{+}(\rho) = b$ (resp. $\sigma^{\mathsf{blue}}_{\mathsf{acc}}(\rho) = b$) when (if ever) $\sigma^{\mathsf{purple}}_{+}(\rho) \neq \alpha$ (resp. $\sigma^{\mathsf{purple}}_{\mathsf{acc}}(\rho) \neq \alpha$) (Formulas (φ'_3) and (φ'_4));
- similarly, strategy $\sigma^{\mathsf{blue}}_{\mathsf{AP}}(\rho) = s$ when $\sigma^{\mathsf{purple}}(\rho) = \alpha$, and $\sigma^{\mathsf{blue}}_{\mathsf{AP}}(\rho) = a_i$ when (if ever) $\sigma^{\mathsf{purple}}(\rho) = \beta_i$ (Formula (φ'_5));
- similarly, strategy $\sigma^{\mathsf{orange}}_{=}(\rho) = \gamma$ as long as $\sigma^{\mathsf{purple}}(\rho) = \alpha$, and $\sigma^{\mathsf{orange}}_{=}(\rho) = \delta$ when (if ever) $\sigma^{\mathsf{purple}}(\rho) = \beta_i$ for some i (Formula (φ'_6));
- using similar ideas, it must be the case that $\sigma^{\mathsf{purple}}_{i}(\rho) = \alpha$ as long as $\sigma^{\mathsf{orange}}_{=}(\rho) = \gamma$, and $\sigma^{\mathsf{purple}}_{i}(\rho) = \beta_i$ when (if ever) $\sigma^{\mathsf{orange}}_{=}(\rho) = \delta$ (Formula (φ'_7));
- if $\sigma^{\mathsf{purple}}(\rho) = \beta_i$ for some $\rho = (s,\alpha,\gamma)^j$ and for some i, then $\sigma^{\mathsf{purple}}_{+}(\rho') = \beta_l$ for some $\rho' = (s,\alpha,\gamma)^k$ and some l. Moreover, the last part of Formula (φ'_8) imposes that $k > j$;
- strategy $\sigma^{\mathsf{purple}}_{\mathsf{acc}}$ satisfies the same condition as above (possibly for a different value of k) (Formula (φ'_9));
- strategy $\sigma^{\mathsf{orange}}_{+}(\rho) = \gamma$ as long as $\sigma^{\mathsf{purple}}_{+}(\rho) = \alpha$, and $\sigma^{\mathsf{orange}}_{+}(\rho) = \delta$ when (if ever) $\sigma^{\mathsf{purple}}_{+}(\rho) = \beta_i$ for some i (Formula (φ'_{10}));
- finally, Formula (φ'_{11}) imposes that $\sigma^{\mathsf{orange}}_{+}$ plays δ (for the first time) exactly one step after σ^{purple} has played β (for the first time). This also imposes the same property for the first time at which $\sigma^{\mathsf{purple}}_{+}$ plays β_l.

Figure 7 summarizes the constraints imposed by the formulas above on the selected strategies.

Acceptance by the Büchi automaton

We also copy here the formulas defined for the Büchi automaton (though it already appears in the core of the paper), this will help reading the correctness lemma.

Using the auxiliary strategies selected above, we now look for a strategy of Player Black that will mimic the run of the Büchi automaton, following the valuation of the atomic propositions selected by the square players A_1 to A_n. We also require that the run of the Büchi automaton be accepting, which will conclude the reduction.

The formula Ψ enforcing these constraints is as follows:

$$\begin{aligned}
& \forall\sigma^{A_1}.\ \exists\sigma^{A_2}.\ \ldots\forall\sigma^{A_{n-1}}.\ \exists\sigma^{A_n}.\ \exists\sigma^{\mathsf{black}}.\ \mathsf{bind}(\sigma^{A_1}, \sigma^{A_2}, \ldots, \sigma^{A_{n-1}}, \sigma^{A_n}, \sigma^{\mathsf{black}}, \sigma^{\mathsf{orange}}_{\infty}).\ \Psi_{\mathsf{aux}}\\
\wedge\quad & \bigwedge_{p_i,p_j \in \mathsf{AP}}\ \bigwedge_{q \in Q}\ (\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}_{i})\mathbf{F}\,q) \Leftrightarrow (\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}, \sigma^{\mathsf{purple}}_{j})\mathbf{F}\,q) && (\varphi_1)
\end{aligned}$$

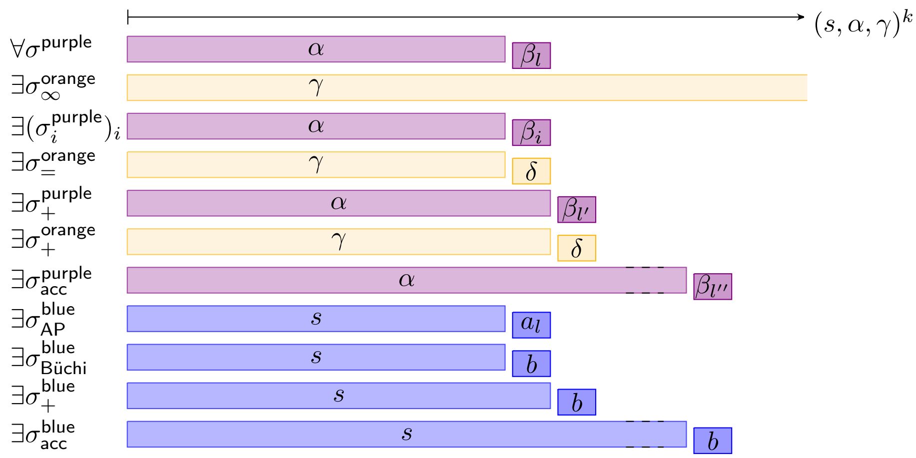

Figure 7 Visualization of the strategies selected by Ψ_{aux} on history $(s,\alpha,\gamma)^k$.

$$\wedge \bigwedge_{p_i\in\mathsf{AP}} \Big(\big(\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{AP}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\, p_i\big) \Rightarrow \big(\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}},\sigma^{\mathsf{purple}}).\ \bigvee_{q\in Q\mid p_i\in\mathsf{label}(q)} \mathbf{F}\, q\big)\Big) \qquad (\varphi_2)$$

$$\wedge \bigwedge_{p_i\in\mathsf{AP}} \Big(\big(\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{AP}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\, \neg p_i\big) \Rightarrow \big(\mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}},\sigma^{\mathsf{purple}}).\ \bigvee_{q\in Q\mid p_i\notin\mathsf{label}(q)} \mathbf{F}\, q\big)\Big) \qquad (\varphi_3)$$

$$\wedge \bigwedge_{q\in Q} \mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}},\sigma^{\mathsf{purple}}).\ \mathbf{F}\, q \Rightarrow \bigvee_{q'\in\mathsf{succ}(q)} \mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}},\sigma^{\mathsf{purple}}_{+}).\ \mathbf{F}\, q' \qquad (\varphi_4)$$

$$\wedge\ \mathsf{bind}(\sigma^{\mathsf{blue}}_{\mathsf{Büchi}},\sigma^{\mathsf{purple}}_{\mathsf{acc}}).\ \bigvee_{q\in\mathsf{accept}(Q)} \mathbf{F}\, q \qquad (\varphi_5)$$

Notice that because of Ψ_{aux} after the assignments on the first line, this formula is syntactically not in SL[BG]. However, the quantifiers in Ψ_{aux} can obviously be switched with the assignments.

Now, analyzing formula Ψ:

- Formula (φ_1) requires that strategy σ^{black} returns the same move after any history of the form $(s,\alpha,\gamma)^k(b,\beta_i,\gamma)$, whichever β_i has been selected by σ^{purple};
- Formulas (φ_2) and (φ_3) constrain the state of the Büchi automaton to correspond to the valuation of the atomic propositions selected. Because of the universal quantification over σ^{purple}, this property will be enforced at all positions and for all atomic propositions;
- Formula (φ_4) additionally requires that two consecutive states of the run of the Büchi automaton indeed correspond to a transition;
- finally, Formula (φ_5) states that for any position (selected by σ^{purple}), there exists a later position (given by $\sigma^{\mathsf{purple}}_{\mathsf{acc}}$) at which the run of the Büchi automaton visits an accepting state.

It now simply remains to state the correctness of our construction:

▶ **Lemma 4.** *Formula Φ in* QLTL *is satisfiable if, and only if, Formula Ψ in SL[BG] holds true in state (s,α,γ) of the game $\mathcal{G}_\Phi$.*

Proof. Write $\Phi = \forall p_1.\ \exists p_2 \ldots \forall p_{n-1}.\ \exists p_n.\ \varphi$. Assume this formula is satisfied, and pick a valuation $w\colon \mathbb{N}\to 2^{\{p_1,\ldots,p_n\}}$ such that $w\models\varphi$. With this valuation, we associate the following strategies for players A_i and for Player Black:

- let $0 \leq i \leq n$. We define σ^{A_i} over finite prefixes ending in states containing a_i. Let ρ_k be any path of the form $(s,\star,\star)^k \cdot (a_i,\star,\star)$, where we use $\star$ to represent any symbol in $\{\alpha,\beta_i,\gamma,\delta \mid p_i \in \mathsf{AP}\}$. Then we define $\sigma^{A_i}(\rho_k) = p_i$ if $p_i \in w(k)$, and $\sigma^{A_i}(\rho_k) = \neg p_i$ otherwise. Similarly, let π be an accepting run over w of the Büchi automaton $\mathcal{B}_\varphi$ for φ used in the construction of $\mathcal{G}_\Phi$. Again, this run can be associated with a strategy σ^{black} for Player Black, which maps any run of the form $(s,\star,\star)^k \cdot (b,\star,\star)$ to π_k.
  We now prove that, under suitable strategies defined by Ψ_{aux}, all formulas (φ_1) to (φ_5) hold. This is easily seen for (φ_1), as σ^{black} only depends on the number of visits to locations of the form $(s,\star,\star)$. As already explained, any strategy σ^{purple} can be associated with a pair (n,p_i), for $n \in \mathbb{N}$ and $p_i \in \mathsf{AP}$. Since $\sigma^{\mathsf{blue}}_{\mathsf{AP}}$ and $\sigma^{\mathsf{blue}}_{\mathsf{Büchi}}$ synchronize with σ^{purple}, formulas (φ_2) and (φ_3) simply encode the fact that the n-th state of π in the Büchi automaton corresponds to the valuation $w(n) \subseteq \mathsf{AP}$. Since $\sigma^{\mathsf{purple}}_{+}$ corresponds to $(n+1,p_j)$ for some $p_j \in \mathsf{AP}$, Formula (φ_4) simply enforces that there is indeed a transition in the Büchi automaton from the n-th state of π to the $n+1$-th state. Finally, for each σ^{purple}, we can build a strategy $\sigma^{\mathsf{purple}}_{\mathsf{acc}}$ corresponding to a subsequent visit to an accepting location, so that Formula (φ_5) is also fulfilled.
  This correspondence holds true for any word $w \colon \mathbb{N} \to 2^{\mathsf{AP}}$. Hence if Φ is true, so does Ψ in $\mathcal{G}_\Phi$.
- The converse direction follows the same lines: if Ψ holds true in $\mathcal{G}_\Phi$, then with any family of strategies for Players $(A_i)_{1\leq i\leq n}$, we can associate a word $w \colon \mathbb{N} \to 2^{\mathsf{AP}}$ such that $p_k \in w(n)$ if, and only if, $\sigma^{A_k}(\rho_{n,k}) = p_k$, where $\rho_{n,k} = (s,\alpha,\gamma)^n \cdot (a_k,\beta_k,\delta)$. Similarly, strategy σ^{black} defines a sequence of states of the Büchi automaton $\mathcal{B}_\varphi$ in the same way. Formula (φ_1) is used to prove that this sequence does not depend on which location β_k has been visited. Formulas (φ_2) to (φ_5) then enforce that the sequence of states is an accepting run over w. ◀

B Appendix for Section 4

We give full proof of the periodicity theorem (stated page 8).

▶ **Theorem 5.** *Let $\mathcal{G}$ be a 1cCGS, and φ be a 1cSL[BG] formula. Then there exist a threshold $h \geq 0$ and a period $\Lambda \geq 0$ for the truth value of φ over $\mathcal{G}$. That is, for every configuration (q,c) of $\mathcal{G}$ with $c \geq h$, for every $k \in \mathbb{N}$, $\mathcal{G},(q,c) \models \varphi$ if, and only if, $\mathcal{G},(q,c+k\cdot\Lambda) \models \varphi$.*

Furthermore the order of magnitude for $h+\Lambda$ is bounded by

$$\mathsf{tower}\left(\max_{\theta\in\mathsf{Subf}(\varphi)} n_\theta, \max_{\theta\in\mathsf{Subf}(\varphi)} k_\theta + 1\right)^{|Q|\cdot 2^{2^{|\varphi|}}}$$

where $\mathsf{Subf}(\varphi)$ is the set of 1cSL[BG] formulas of φ, k_θ is the number of quantifier alternations in θ, and n_θ is the number of different bindings used in θ.

The rest of this section is devoted to the proof of this result. We first prove this property for the flat fragment $\mathsf{1cSL}^0[\mathsf{BG}]$, and then extend it to the full $\mathsf{1cSL}[\mathsf{BG}]$.

B.1 The flat fragment 1cSL^0[BG]

We first focus on the flat fragment $\mathsf{1cSL}^0[\mathsf{BG}]$. We fix a 1cCGS $\mathcal{G}$ and a formula φ in $\mathsf{1cSL}^0[\mathsf{BG}]$ written as:

$$\varphi = Q_1 x_1 \ldots Q_k x_k.\ f((\beta_i \phi_i)_{1\leq i\leq n})$$

where for every $1 \le j \le k$, we have $Q_j \in \{\exists, \forall\}$ (assuming quantifiers strictly alternate), f is a Boolean formula over n atoms, and for every $1 \le i \le n$, β_i is a complete binding for the players' strategies, and ϕ_i is a 1cLTL formula. We write M for the maximal constant appearing in one of the finite sets describing a counter constraint S appearing in φ.

For every $1 \le i \le n$, we let $\mathcal{D}_i$ be a deterministic (counter) parity automaton that recognizes formula ϕ_i (this is the standard LTL-to-(deterministic parity) automata construction in which quantitative constraints are seen as atoms). A run of $\mathcal{G}$ is read in a standard way, with the additional condition that quantitative constraints labelling a state should be satisfied by the counter value when the state is traversed (a state can be labelled by a constraint $\mathsf{cnt} \in S$, with S arbitrarily complex—it does not impact the description of the automaton).

The proof proceeds by showing that, above some threshold, the truth value of φ is periodic w.r.t. counter values. To prove this, we define an equivalence relation over counter values that generates identical strategic possibilities (in a sense that will be made clear later on).

B.1.1 Definition of an equivalence relation

Fix a configuration $\gamma = (\ell, c)$ in $\mathcal{G}$, pick for every $1 \le i \le n$ a state d_i in the automaton $\mathcal{D}_i$, and define the tuple $D = (d_1, \ldots, d_n)$. For every context χ_k for variables $\{x_1, \ldots, x_k\}$, we define the level-0 identifier $\mathsf{Id}_{\chi_k}(\gamma, D)$ as:

$$\mathsf{Id}_{\chi_k}(\gamma, D) = \left\{ i \;\middle|\; 1 \le i \le n \text{ and } \mathsf{out}(\gamma, \beta_i[\chi_k]) \text{ is accepted by } \mathcal{D}_i \text{ from } d_i \right\}$$

where $\beta_i[\chi_k]$ assigns a strategy from χ_k to each player in Agt following β_i.

Assuming we have defined level-$(k-j+1)$ identifiers $\mathsf{Id}_{\chi_{j+1}}(\gamma, D)$ for every partial context χ_{j+1} for variables $\{x_1, \ldots, x_{j+1}\}$, we define the level-$(k-j)$ identifier $\mathsf{Id}_{\chi_j}(\gamma, D)$ for every partial context χ_j for variables $\{x_1, \ldots, x_j\}$ as follows:

$$\mathsf{Id}_{\chi_j}(\gamma, D) = \left\{ \mathsf{Id}_{\chi_{j+1}}(\gamma, D) \;\middle|\; \chi_{j+1} \text{ is a context for } \{x_1, \ldots, x_{j+1}\} \text{ that extends } \chi_j \right\}.$$

There is a unique level-k identifier for every configuration $\gamma = (\ell, c)$ and every D, which corresponds to the empty context. It somehow contains full information about what kinds of strategies can be used in the game (this is a hierarchical information set, which contains all level-j identifiers for $j < k$).

We will first give a characterization of the construction of identifiers, which will help understand how it can be used; we then count how many values the level-k identifier can take, from which our period will be derived.

Characterization

We inductively define the following Boolean property:

$$\mathbb{P}_0^{\ell,D}(\chi_k, \chi'_k)(c, c')\colon \quad \text{(truth value of) } \mathsf{Id}_{\chi_k}((\ell, c), D) = \mathsf{Id}_{\chi'_k}((\ell, c'), D)$$

and for every $0 \le j < k$,

$$\mathbb{P}_{k-j}^{\ell,D}(\chi_j, \chi'_j)(c, c')\colon \quad \begin{cases} \forall v_{j+1}.\ \exists v'_{j+1}.\ \mathbb{P}_{k-j-1}^{\ell,D}(\chi_j \cup \{v_{j+1}\}, \chi'_j \cup \{v'_{j+1}\})(c, c') & \text{and} \\ \forall v'_{j+1}.\ \exists v_{j+1}.\ \mathbb{P}_{k-j-1}^{\ell,D}(\chi_j \cup \{v_{j+1}\}, \chi'_j \cup \{v'_{j+1}\})(c, c') \end{cases}$$

This property allows to characterize equivalent configurations w.r.t. the identifier predicate.

▶ **Lemma 10.** *Fix some $0 \le j \le k$ and some partial contexts χ_j and χ'_j for variables $\{x_1, \ldots, x_j\}$ from (ℓ, c) and (ℓ, c'), respectively. The following two properties are equivalent:*

- $\mathit{Id}_{\chi_j}((\ell,c),D) = \mathit{Id}_{\chi'_j}((\ell,c'),D)$
- $\mathbb{P}^{\ell,D}_{k-j}(\chi_j,\chi'_j)(c,c')$

Proof. We show the equivalence by induction on $0 \leq j \leq k$, starting from $j = k$, where the equivalence precisely corresponds to the definition.

Assume $\mathsf{Id}_{\chi_j}((\ell,c),D) = \mathsf{Id}_{\chi'_j}((\ell,c'),D)$ for some $0 \leq j < k$. By definition, it means that for every extended context χ_{j+1} of χ_j, there exists an extended context χ'_{j+1} of χ'_j (and conversely) such that $\mathsf{Id}_{\chi_{j+1}}((\ell,c),D) = \mathsf{Id}_{\chi'_{j+1}}((\ell,c'),D)$. By induction hypothesis, it holds $\mathbb{P}^{\ell,D}_{k-j-1}(\chi_{j+1},\chi'_{j+1})(c,c')$. Since this holds for all appropriate quantifications, we get that $\mathbb{P}^{\ell,D}_{k-j}(\chi_j,\chi'_j)(c,c')$ holds as well. The converse is similar. ◀

We then define the following equivalence on counter values:

$$c \sim c' \quad \text{if, and only if,} \quad c \equiv c' \bmod P \text{ and } \forall D.\ \forall \ell.\ \mathsf{Id}_{\emptyset}((\ell,c),D) = \mathsf{Id}_{\emptyset}((\ell,c'),D)$$

where P is the lcm of all the periods appearing in modulo atoms used in formula φ.

Combinatorics

We inductively define the function $\mathsf{tower}\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ as $\mathsf{tower}\,(a,0) = a$ and $\mathsf{tower}\,(a,b+1) = 2^{\mathsf{tower}(a,b)}$. This encodes *towers of exponentials* of the form $2^{2^{\cdot^{\cdot^{\cdot^{a}}}}}$. Given a configuration (ℓ,c) and a tuple D, the number of possible values for the level-0 identifier is $\mathsf{tower}\,(n,1)$, and for the level-j identifier it is $\mathsf{tower}\,(n,j+1)$.

Hence, the number $\mathsf{ind}_\sim$ of equivalence classes of the relation $\sim$ satisfies

$$\mathsf{ind}_\sim \leq P \cdot (\mathsf{tower}\,(n,k+1))^{\left(|Q|\cdot\prod_{1\leq i\leq n} 2^{2^{|\phi_i|}}\right)} \leq P \cdot (\mathsf{tower}\,(n,k+1))^{\left(|Q|\cdot 2^{2^{|\varphi|}}\right)}$$

We let $\overline{M} = M + \mathsf{ind}_\sim + 1$. By the pigeon-hole principle, there must exist $M < h < h' \leq \overline{M}$ such that $h \sim h'$.

B.1.2 Periodicity property

We define $\Lambda = h' - h$, and now prove that it is a period for φ for counter values larger than or equal to h. Assume that $\gamma = (\ell,c)$ is a configuration such that $c \geq h$, and define $\gamma' = (\ell, c+\Lambda)$ (note that $c + \Lambda \geq h'$). We show that $\mathcal{G},\gamma \models \varphi$ if, and only if, $\mathcal{G},\gamma' \models \varphi$.

▶ Notations. For the rest of this proof, we fix the following notations:

1. if ρ is a (finite or infinite) run of $\mathcal{G}$ starting with counter value $a > c$, then either the counter always remains above c along ρ (in which case we say that ρ is fully above c), or it eventually hits value c, and we define $\rho_{\searrow c}$ for the smallest prefix of ρ such that $\mathsf{last}(\rho_{\searrow c})$ has counter value c;
2. let ρ be a run that is fully above M, and let c be the least counter value appearing in ρ. For every $\nu \geq M - c$ (where $M - c$ can be negative), we write $\mathsf{Shift}_\nu(\rho)$ for the run ρ' obtained from ρ by shifting the counter value by ν. It is a real run since the counter values along ρ' are also all above M.
3. if D is a tuple of states of the deterministic automata $\mathcal{D}_i$, and if ρ is a finite run of $\mathcal{G}$ that is fully above M, then we write $D_{+\rho}$ for the image of D after reading ρ.

We first show an easy result:

▶ **Lemma 11.** *Let ρ be a finite run, and $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$. Let D be a tuple of states of the automata $(\mathcal{D}_i)_{1\le i\le n}$. Then, ρ is fully above h iff ρ' is fully above h'. In case both are wrong, it holds that $\rho'_{\searrow h'} = \mathsf{Shift}_{+\Lambda}(\rho_{\searrow h})$. Furthermore, in the first case, $D_{+\rho} = D_{+\rho'}$ whereas in the second case, $D_{+\rho_{\searrow h}} = D_{+\rho'_{\searrow h'}}$.*

Proof. The two first properties are obvious by definition of $\mathsf{Shift}_{+\Lambda}$ (since $h' = h + \Lambda$).

Since $h > M$, all counter values along both paths are larger than M, and hence, two corresponding configurations along ρ and ρ' satisfy the same non-modulo counter constraints. The period Λ is a multiple of P, the lcm of all the periods, hence two corresponding configurations along ρ and ρ' also satisfy the same modulo constraints. Finally, all atomic propositions are equivalently satisfied at two corresponding positions along ρ and ρ'. Fix $1 \le i \le n$. Since $\mathcal{D}_i$ is deterministic, using the above arguments, we get the last results. ◀

Let $0 \le j \le k$. We assume that χ_j and χ'_j are two contexts for $\{x_1, \ldots, x_j\}$, and D is a tuple of states of the $\mathcal{D}_i$'s. We write $\mathbb{R}^{D,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ if the following property holds for any run ρ from γ:

(i) if ρ is fully above h (or equivalently, if $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$, which starts from γ', is fully above h'), then for every $1 \le g \le j$, $\chi_j(x_g)(\rho) = \chi'_j(x_g)(\rho')$;

(ii) if ρ is not fully above h (equivalently, if $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$ is not fully above h'), then we decompose ρ (resp. ρ') w.r.t. h (resp. h') and write $\rho = \rho_{\searrow h} \cdot \overline{\rho}$ and $\rho' = \rho'_{\searrow h'} \cdot \overline{\rho}'$. Then:

$$\mathsf{Id}_{\chi_{j\overrightarrow{\rho_{\searrow h}}}}(\mathsf{last}(\rho_{\searrow h}), \widetilde{D}) = \mathsf{Id}_{\chi'_{j\overrightarrow{\rho'_{\searrow h'}}}}(\mathsf{last}(\rho'_{\searrow h'}), \widetilde{D})$$

with $\widetilde{D} = D_{+\rho_{\searrow h}} = D_{+\rho'_{\searrow h'}}$. Recall that $\chi_{j\overrightarrow{\rho_{\searrow h}}}$ shifts all strategies in context χ_j after the prefix $\rho_{\searrow h}$ (that is, χ_j is the strategy such that $\chi_{j\overrightarrow{\rho_{\searrow h}}}(\pi) = \chi_j(\rho_{\searrow h} \cdot \pi)$ for every π).

▶ **Lemma 6.** *Fix $0 \le j < k$, and assume that $\mathbb{R}^{D,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ holds true. Then:*

1. *for every strategy v for x_{j+1} from γ, one can build a strategy $\mathcal{T}(v)$ for x_{j+1} from γ' such that $\mathbb{R}^{D,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{v\}, \chi'_j \cup \{\mathcal{T}(v)\})$ holds true;*
2. *for every strategy v' for x_{j+1} from γ', one can build a strategy $\mathcal{T}^{-1}(v')$ for x_{j+1} from γ such that $\mathbb{R}^{D,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{\mathcal{T}^{-1}(v')\}, \chi'_j \cup \{v'\})$ holds true.*

Proof. We prove the first property. The second property is proven similarly by replacing $\mathsf{Shift}_{-\Lambda}$ with $\mathsf{Shift}_{+\Lambda}$.

We fix a new strategy v for variable x_{j+1} from γ. We define the lifted strategy $\mathcal{T}(v)$ for variable x_{j+1} (which we will add to context χ'_j) from γ' as follows:

- for every (finite) ρ' from γ' that is a prefix along which the counter is always larger than h', we define $\mathcal{T}(v)(\rho') = v(\rho)$, where $\rho = \mathsf{Shift}_{-\Lambda}(\rho')$ (note that in that case, the counter is always larger than h along ρ, and ρ starts at configuration γ), so this is well-defined;
- if ρ' hits value h', then decompose ρ' w.r.t. h' as $\rho'_{\searrow h'} \cdot \overline{\rho}'$. Similarly, decompose $\rho = \mathsf{Shift}_{-\Lambda}(\rho')$ w.r.t. h, yielding $\rho = \rho_{\searrow h} \cdot \overline{\rho}$. It is not difficult to see that $\rho_{\searrow h} = \mathsf{Shift}_{-\Lambda}(\rho'_{\searrow h'})$. By hypothesis, it holds

$$\mathsf{Id}_{\chi_{j\overrightarrow{\rho_{\searrow h}}}}(\mathsf{last}(\rho_{\searrow h}), \widetilde{D}) = \mathsf{Id}_{\chi'_{j\overrightarrow{\rho'_{\searrow h'}}}}(\mathsf{last}(\rho'_{\searrow h'}), \widetilde{D}))$$

(with $\widetilde{D} = D_{+\rho_{\searrow h}} = D_{+\rho'_{\searrow h'}}$).

By Lemma 10 we find a strategy v' from $\mathsf{last}(\rho'_{\searrow h'})$ for variable x_{j+1} such that

$$\mathsf{Id}_{(\chi_j \cup \{v\})_{\overrightarrow{\rho_{\searrow h}}}}(\mathsf{last}(\rho_{\searrow h}), \widetilde{D}) = \mathsf{Id}_{\chi'_{j\,\overrightarrow{\rho_{\searrow h}}} \cup \{v'\}}(\mathsf{last}(\rho'_{\searrow h'}), \widetilde{D}).$$

We then define $\mathcal{T}(v)(\rho') = v'(\overline{\rho}')$. The property $\mathbb{R}^{D,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{v\}, \chi'_j \cup \{\mathcal{T}(v)\})$ holds. ◀

The construction made in the above lemma is illustrated in Fig. 8.

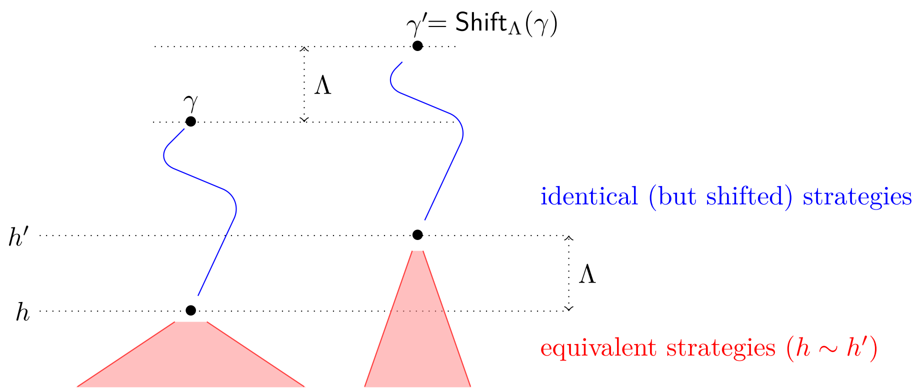

Figure 8 Construction in Lemma 6 (case (ii))

We use this lemma to transfer a proof that $\gamma \models_\emptyset \varphi$ to a proof that $\gamma' \models_\emptyset \varphi$. We decompose the proof of this equivalence into two lemmas:

▶ **Lemma 7.** *Fix D^0 for the tuple of initial states of the $\mathcal{D}_i$'s. Assume that $\mathbb{R}^{D^0,k}_{(\gamma,\gamma')}(\chi,\chi')$ holds (for full contexts χ and χ'). Let $1 \le i \le n$, and write $\rho = \mathit{Out}(\gamma, \beta_i[\chi])$ and $\rho' = \mathit{Out}(\gamma', \beta_i[\chi'])$. Then $\rho \models \phi_i$ if and only if $\rho' \models \phi_i$. In particular, $\gamma \models_\chi f((\beta_i\phi_i)_{1\le i\le n})$ if and only if $\gamma' \models_{\chi'} f((\beta_i\phi_i)_{1\le i\le n})$.*

Proof. We distinguish between two cases:

- Assume that ρ is fully above h. By definition of property $\mathbb{R}^{D^0,k}_{(\gamma,\gamma')}(\chi,\chi')$, it holds that $\rho' = \mathsf{Shift}_{+\Lambda}(\rho)$. Applying Lemma 11 to all prefixes of ρ and ρ' (which are all above h, resp. h'), we get that they follow the same paths in all automata $\mathcal{D}_i$'s, hence for each $1 \le i \le n$, $\rho \models \phi_i$ iff $\rho' \models \phi_i$.
- Assume that ρ is not fully above h. Then, by definition of property $\mathbb{R}^{D^0,k}_{(\gamma,\gamma')}(\chi,\chi')$, ρ' is not fully above either, and $\rho'_{\searrow h'} = \mathsf{Shift}_{+\Lambda}(\rho_{\searrow h})$. Also,

  $$\mathsf{Id}_{\chi_{\overrightarrow{\rho_{\searrow h}}}}(\mathsf{last}(\rho_{\searrow h}), D) = \mathsf{Id}_{\chi'_{\overrightarrow{\rho'_{\searrow h'}}}}(\mathsf{last}(\rho'_{\searrow h'}), D)$$

  with $D = D^0_{+\rho_{\searrow h}} = D^0_{+\rho'_{\searrow h'}}$ (by Lemma 11).

  There exists some location $\hat{\ell}$ such that $\mathsf{last}(\rho_{\searrow h}) = (\hat{\ell}, h)$ and $\mathsf{last}(\rho'_{\searrow h'}) = (\hat{\ell}, h')$, and by definition of the id, this means that for every $1 \le i \le n$, the two following properties are equivalent:
  - $\overline{\rho} = \mathsf{out}((\hat{\ell}, h), \beta_i[\chi_{\overrightarrow{\rho_{\searrow h}}}])$ is accepted by $\mathcal{D}_i$ from d_i
  - $\overline{\rho}' = \mathsf{out}((\hat{\ell}, h'), \beta'_i[\chi'_{\overrightarrow{\rho'_{\searrow h'}}}])$ is accepted by $\mathcal{D}_i$ from d_i

  We conclude by noticing that $\rho = \rho_{\searrow h} \cdot \overline{\rho}$ and $\rho' = \rho'_{\searrow h'} \cdot \overline{\rho}'$, which are then equivalently accepted or rejected by each of the $\mathcal{D}_i$'s.

◀

▶ **Lemma 8.** *$\gamma \models_\emptyset \varphi$ if and only if $\gamma' \models_\emptyset \varphi$.*

Proof. For every $0 \leq j \leq k$, we write $\varphi_j = Q_{j+1}x_{j+1} \dots Q_l x_l \; f((\beta_i \phi_i)_{1 \leq i \leq n})$.

We show by induction that if χ_j and χ'_j are contexts such that $\mathbb{R}^{D^0,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ holds, then $\gamma \models_{\chi_j} \varphi_j$ if and only if $\gamma' \models_{\chi'_j} \varphi_j$.

This holds for full contexts χ_k and χ'_k by applying Lemma 7. Assume it holds at rank $j+1$ with $1 \leq j < k$; we show it for j. Assume χ_j and χ'_j are contexts such that $\mathbb{R}^{D^0,j}_{(\gamma,\gamma')}(\chi_j, \chi'_j)$ holds and $\gamma \models_{\chi_j} \varphi_j$. We distinsguish two cases:

- **Case $Q_{j+1} = \exists$.** Pick a strategy v_{j+1} for variable x_{j+1} such that $\gamma \models_{\chi_j \cup \{v_{j+1}\}} \varphi_{j+1}$. Applying Lemma 6, choose strategy v'_{j+1} such that $\mathbb{R}^{D^0,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{v_{j+1}\}, \chi'_j \cup \{v'_{j+1}\})$ holds. Applying the induction hypothesis, we deduce that $\gamma' \models_{\chi'_j \cup \{v'_{j+1}\}} \varphi_{j+1}$.
- **Case $Q_{j+1} = \forall$.** Pick a strategy v'_{j+1} for variable x_{j+1} from γ'. Applying Lemma 6, choose strategy v_{j+1} such that $\mathbb{R}^{D^0,j+1}_{(\gamma,\gamma')}(\chi_j \cup \{v_{j+1}\}, \chi'_j \cup \{v'_{j+1}\})$ holds. Applying the induction hypothesis, we deduce that $\gamma' \models_{\chi'_j \cup \{v'_{j+1}\}} \varphi_{j+1}$ iff $\gamma \models_{\chi_j \cup \{v_{j+1}\}} \varphi_{j+1}$. This is actually the case. Hence $\gamma' \models_{\chi'_j \cup \{v'_{j+1}\}} \varphi_{j+1}$.

We conclude the proof by noticing that $\mathbb{R}^{D^0,0}_{(\gamma,\gamma')}(\emptyset, \emptyset)$ holds since $h \sim h'$. ◀

▶ **Corollary 9.** *Λ is a period for the satisfiability of φ for configurations with counter values larger than or equal to h.*

Furthermore, $h + \Lambda$ is bounded by $M + P \cdot (\mathsf{tower}\,(n, k+1))^{|Q| \cdot \prod_{1 \leq i \leq} 2^{2^{|\varphi|}}} + 1$.

▶ Remark. Note that the above proof of existence of a period, though effective (a period can be computed by computing the truth of identifier predicates), does not allow for an algorithm to decide the model-checking problem. One possible idea to lift that periodicity result to an effective algorithm would be to bound the counter values; however things are not so easy: in Fig. 5, equivalent strategies from h and h' might generate runs with (later on) counter values larger than h or h'. The decidability status of $\mathsf{1cSL}^1[\mathsf{BG}]$ (and of $\mathsf{1cSL}[\mathsf{BG}]$) model checking remains open.

B.2 Extension to 1cSL[BG]

We explain how we can extend the previous periodicity analysis to the full logic $\mathsf{1cSL}[\mathsf{BG}]$. We fix a formula of $\mathsf{1cSL}^{k+1}[\mathsf{BG}]$

$$\varphi = Q_1 x_1 \dots Q_k x_k \cdot f((\beta_i \phi_i)_{1 \leq i \leq n})$$

with the same notations than the ones at the beginning of the previous subsection, but ϕ_i can use closed formulas of $\mathsf{1cSL}^k[\mathsf{BG}]$ as subformulas.

Let Ψ_φ be the set of closed subformulas of $\mathsf{1cSL}^k[\mathsf{BG}]$ that appear directly under the scope of some ϕ_i. We will replace subformulas of Ψ_φ by other formulas involving only (new) atomic propositions and counter constraints. Pick $\psi \in \Psi_\varphi$. Let h_ψ and Λ_ψ be the threshold and the period mentioned in Corollary 9. For every location ℓ of the game, the set of counter values c such that $(\ell, c) \models \psi$ can be written as S^ψ_ℓ (we use a non-periodic set for the values smaller than h_ψ and a periodic set of period Λ_ψ for the values above h_ψ) – note that we know such a set exists, even though there is (for now) no effective procedure to express it. The size of formula S^ψ_ℓ is 1 (we do not take into account the complexity of writing the precise sets used in the constraint). Expand the set of atomic propositions AP with an extra atomic proposition for each location, say p_ℓ for location ℓ, which holds only at location ℓ. For every $\psi \in \Psi_\varphi$, replace that occurrence of ψ in φ by formula $\bigwedge_{\ell \in L} p_\ell \to (\mathsf{cnt} \in S_\ell^\psi)$. This defines formula φ', which is now a $\mathsf{1cSL}^0[\mathsf{BG}]$ formula, and holds equivalently (w.r.t. φ) from every configuration of $\mathcal{G}$. The size of φ' is that of φ. We apply the result of the previous subsection and get a proof of periodicity of the satisfaction relation for φ', hence for φ.

It remains to compute bounds on the overall period Λ_φ and threshold h_φ. The modulo constraints in φ' involve periods Λ_ψ ($\psi \in \Psi_\varphi$), and the constants used are bounded by h_ψ. So the bound $M_{\varphi'}$ is bounded by $\max(\max_{\psi\in\Psi}(h_\psi), M_\varphi)$ where M_φ is the maximal constant used in φ, and the value $P_{\varphi'}$ is the lcm of the periods used in φ (call it P_φ) and of the Λ_ψ's (for $\psi \in \Psi_\varphi$): hence $P_{\varphi'} \le P_\varphi \cdot \max_{\psi\in\Psi_\varphi}(\Lambda_\psi)^{|\varphi|}$

Hence for formula φ', we get

$$h_{\varphi'} + \Lambda_{\varphi'} \quad \le \quad M_{\varphi'} + P_{\varphi'} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi'|}}} + 1$$

We infer the following order of magnitude for $h_\varphi + \Lambda_\varphi$:

$$\begin{aligned}\omega_\varphi &\approx \omega_{\Psi_\varphi} + M_\varphi^{|\varphi|} \cdot (\max \Lambda_\psi)^{|\varphi|} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi|}}} \\ &\approx M_\varphi^{|\varphi|} \cdot \omega_{\Psi_\varphi}^{|\varphi|} \cdot \mathsf{tower}\,(n_\varphi, k_\varphi + 1)^{|Q|\cdot 2^{2^{|\varphi|}}}\end{aligned}$$

where $\omega_{\Psi_\varphi} = \max_{\psi\in\Psi_\varphi} \omega_\psi$ Using notations of Theorem 5, the order of magnitude can therefore be bounded by

$$\mathsf{tower}\left(\max_{\theta\in\mathsf{Subf}(\varphi)} n_\theta, \max_{\theta\in\mathsf{Subf}(\varphi)} k_\theta + 1\right)^{|Q|\cdot 2^{2^{|\varphi|}}}.$$

► Remark. Note that this is a non-constructive proof, even for the periods and the thresholds, since it relies on the model-checking of subformulas, which we don't know how to do. We can effectively compute a threshold and a period by taking the lcm of all the integers up to the bound over the (non-constructive) period and threshold.