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Relative Nondeterministic Information Logic is EXPTIME-complete

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Abstract. We define a relative version of the logic NIL introduced by Orłowska, Pawlak and Vakarelov and we show that satisfiability is not only decidable but also EXPTIME-complete. Such a logic combines two ingredients that are seldom present simultaneously in information logics: frame conditions involving more than one information relation and relative frames. The EXPTIME upper bound is obtained by designing a well-suited decision procedure based on the nonemptiness problem of Büchi automata on infinite trees. The paper provides evidence that Büchi automata on infinite trees are crucial language acceptors even for relative information logics with multiple types of relations.

Keywords: information logic, relative frame, computational complexity, Büchi tree automaton

1. Introduction

Logics for information systems. A formal model of information systems was proposed in a series of papers by Wiktor Marek and Zdzisław Pawlak ([16] and [17] published later as [18]) and next in the papers [25] and [26] by Zdzisław Pawlak. In these papers a logic is considered obtained from the classical

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propositional calculus by assuming that atomic formulas have the form of what is called descriptors which in the later papers have been identified with attribute-value pairs. An extension of this formalism to first-order logic is presented in [19]. A modal approach to reasoning in information systems resulted in various modal systems which are now called information logics. The first logics of that family are defined in [20] published later as [21], and in [22] and [24]. We refer the reader to [23, 7] for a comprehensive survey of information logics and to [32, 1, 12, 33, 30, 2] for some more recent examples of these logics. Due to their modal character information logics provide a formal specification language for expressing properties of relations among the objects from information systems.

In this paper we employ automata-theoretic decision procedures to prove complexity results for the very expressive information logic RNIL that is introduced in the paper. An *information system* S can be viewed as a structure $S = \langle OB, AT \rangle$ such that OB is a non-empty set of *objects*, AT is a non-empty set of *attributes*, and every attribute $a \in AT$ is a mapping $a : OB \rightarrow \mathcal{P}(VAL_a) \setminus \{\emptyset\}$, where VAL_a is a non-empty set of *values* and $\mathcal{P}(\cdot)$ denotes the powerset operator. For every object x and for every attribute a, $a(x)$ can be read as the set of possible values of the attribute a for the object x. In that setting, various derived relations between objects can be defined. We recall below some standard relations (see e.g. [23]). For all $x_1, x_2 \in OB$, for every $A \subseteq AT$,

(I) $\langle x_1, x_2 \rangle \in ind_A$ iff for every $a \in A$, $a(x_1) = a(x_2)$ (indiscernibility),

(II) $\langle x_1, x_2 \rangle \in fin_A$ iff for every $a \in A$, $a(x_1) \subseteq a(x_2)$ (forward inclusion),

(III) $\langle x_1, x_2 \rangle \in bin_A$ iff for every $a \in A$, $a(x_2) \subseteq a(x_1)$ (backward inclusion),

(IV) $\langle x_1, x_2 \rangle \in sim_A$ iff for every $a \in A$, $a(x_1) \cap a(x_2) \neq \emptyset$ (similarity).

Given an information system $S = \langle OB, AT \rangle$, we can define a structure $\langle OB, (\mathcal{R}_A)_{A \subseteq AT} \rangle$, where $(\mathcal{R}_A)_{A \subseteq AT}$ is a family of relations derived from S. In a more abstract setting, an *information frame* is a pair $\langle W, (\mathcal{R}_P)_{P \subseteq PAR} \rangle$ such that W and PAR are non-empty sets and $(\mathcal{R}_P)_{P \subseteq PAR}$ is a family of binary relations indexed by subsets of PAR. More generally, an information frame can be defined as a set equipped with families of (non necessarily binary) relations. An information logic is usually defined as a multi-modal logic characterised by a class of information frames. Since the relations derived from information systems are constrained between each other, the information frames generally satisfy additional conditions. For example, for every $\mathcal{R} \in \{ind, fin, bin, sim\}$, we have

$$\mathcal{R}_{P \cup Q} = \mathcal{R}_P \cap \mathcal{R}_Q \text{ for all } P, Q \subseteq PAR, \text{ and} \tag{1}$$

$$\mathcal{R}_{\emptyset} = W \times W. \tag{2}$$

Moreover, every relation $\mathcal{R}_P$ satisfies certain local conditions: for instance, the forward inclusion relations are reflexive and transitive, and the similarity relations are reflexive and symmetric.

The logic NIL. The information logic NIL (introduced in [24]) is one of the information logics proposed for reasoning with incomplete information. In 1987 Vakarelov [31] provides the first result about first-order characterization of structures derived from information systems. This has been done in the case of the semantical structures of NIL. The NIL semantical structures contain forward and backward inclusions and the similarity relation. More precisely, the NIL frames are all the structures

$$\langle OB, fin(AT), bin(AT), sim(AT) \rangle$$

derived from some information system $\langle OB, AT \rangle$. In [31], it has been shown that in order to appropriately characterize the relationships between these relations an additional condition (not present in [24]) between forward inclusion and similarity needs to be taken into account (the forthcoming condition (N4)). In [4], NIL satisfiability is shown PSPACE-complete by designing a Ladner-like algorithm [14] and it can be viewed as a well-identified fragment of PDL [7, Section 11.4].

Our contribution. In this paper, we introduce and study the relative version of NIL, called RNIL, and we show that the satisfiability problem for the logic RNIL is not only decidable but also EXPTIME-complete. RNIL extends NIL by allowing families of information relations instead of a single relation per family. This substantial extension leads to technical difficulties. However, as a gain, the logical formalism can express more complex properties about information systems. The EXPTIME lower bound is a consequence of more general results, since RNIL contains a universal modal connective with a family of modal connectives of logic B (see e.g., [29, 3, 11]). The EXPTIME upper bound is established by an exponential reduction into the nonemptiness problem for Büchi automata on infinite trees that is known to be in PTIME (see e.g., [34, 9]). This technique is nowadays standard for logics of programs, and it has been applied once before to information logics in [8]. Hence, this paper follows the automata-based approach introduced in [8] for information logics and applies it to an information logic with information frames equipped with more than one family of relations. Moreover, we believe that the proof techniques presented in the paper can be reused for other information logics with different families of relative frames. That is why, we plan to explain the technical developments in full details to facilitate further extensions. The complexity jump from PSPACE-completeness of NIL to EXPTIME-completeness of its relative extension RNIL is the best we can hope for, since RNIL contains universal modality with B modal connectives.

Plan of the paper. The paper is structured as follows. Sect. 2 introduces the logic RNIL and normal forms for RNIL formulae. In Sects. 3 and 4 we provide a notion of Hintikka trees for RNIL-models preparing the automata construction. The satisfiability problem for the logic RNIL is reduced to the nonemptiness problem for Büchi automata on infinite trees in Sect. 5. Finally, we give some concluding remarks in Sect. 6.

2. Relative Nondeterministic Information Logic RNIL

2.1. Syntax and Semantics

In this section, we introduce syntax and semantics of the logic RNIL. The set of primitive symbols of the language for RNIL is composed of

- a countably infinite set $\mathrm{VARPROP} = \{\mathrm{p}_1, \mathrm{p}_2, \ldots\}$ of *propositional variables*,
- a set P of *parameter expressions*, which is the smallest set containing a countably infinite set $\mathrm{PVAR} = \{\mathrm{C}_1, \mathrm{C}_2, \ldots\}$ of *parameter variables* and is closed under the set-theoretical operators $\cap, \cup, -$.

The set $\mathrm{FOR}(\mathrm{RNIL})$ of RNIL-formulae is defined by the grammar below:

$$\mathrm{p} \mid \neg\phi \mid \phi \wedge \phi \mid [\mathrm{sim}(\mathrm{A})]\phi \mid [\mathrm{fin}(\mathrm{A})]\phi \mid [\mathrm{bin}(\mathrm{A})]\phi.$$

where $\mathrm{p} \in \mathrm{VARPROP}$ and $\mathrm{A} \in \mathrm{P}$. Throughout the paper, we freely use standard abbreviations. For all r in $\{\mathrm{sim}, \mathrm{bin}, \mathrm{fin}\}$ and $\mathrm{A} \in \mathrm{P}$, an $[\mathrm{r}(\mathrm{A})]$-*formula* is a formula of the form $[\mathrm{r}(\mathrm{A})]\phi$. The following is an example of a (valid) RNIL-formula:

$$[\mathrm{sim}(\mathrm{C}_2 \cap -\mathrm{C}_2)]\mathrm{p} \Rightarrow [\mathrm{fin}(\mathrm{C}_1)]\mathrm{p}.$$

Moreover, for every syntactic object O, we write $|\mathrm{O}|$ to denote its *length* (or *size*), that is the number of symbol occurrences in O viewed as a string. As usual, $\mathrm{sub}(\phi)$ denotes the set of *subformulae* of the formula ϕ (including ϕ itself). For every $\mathrm{X} \in \{\mathrm{PVAR}, \mathrm{P}\}$, we write $\mathrm{X}(\phi)$ to denote the elements of X occurring in the formula ϕ. Obviously, $\mathrm{card}(\mathrm{X}(\phi)) < |\phi|$. We write $\mathrm{md}(\phi)$ to denote the *modal degree* of the formula ϕ, that is the maximum nesting of modal operators in ϕ.

Definition 2.1. Let PAR be a non-empty set. A P-*interpretation* m is a map $m : \mathrm{P} \rightarrow \mathcal{P}(PAR)$ such that, for all $\mathrm{A}_1, \mathrm{A}_2 \in \mathrm{P}$,

- $m(\mathrm{A}_1 \cap \mathrm{A}_2) = m(\mathrm{A}_1) \cap m(\mathrm{A}_2)$,
- $m(\mathrm{A}_1 \cup \mathrm{A}_2) = m(\mathrm{A}_1) \cup m(\mathrm{A}_2)$,
- $m(-\mathrm{A}_1) = PAR \setminus m(\mathrm{A}_1)$.

PAR is referred to as a set of parameters, it is a counterpart to the set of attributes in information systems. Given parameter expressions A and B, we write $\mathrm{A} \equiv \mathrm{B}$ [resp. $\mathrm{A} \sqsubseteq \mathrm{B}$] iff for every P-interpretation m, we have $m(\mathrm{A}) = m(\mathrm{B})$ [resp. $m(\mathrm{A}) \subseteq m(\mathrm{B})$].

Definition 2.2. An *RNIL-model* $\mathcal{M}$ is a structure

$$\mathcal{M} = \langle W, (\mathcal{R}^{\mathrm{fin}}_P)_{P \subseteq PAR}, (\mathcal{R}^{\mathrm{bin}}_P)_{P \subseteq PAR}, (\mathcal{R}^{\mathrm{sim}}_P)_{P \subseteq PAR}, m\rangle,$$

where W and PAR are non-empty sets (set of objects and set of parameters, respectively) and the three families of binary relations on W satisfy the conditions below:

(uni) $\mathcal{R}^{\mathrm{r}}_{\emptyset} = W \times W$ for $\mathrm{r} \in \{\mathrm{fin}, \mathrm{bin}, \mathrm{sim}\}$,

(inter) $\mathcal{R}^{\mathrm{r}}_{P \cup Q} = \mathcal{R}^{\mathrm{r}}_P \cap \mathcal{R}^{\mathrm{r}}_Q$ for all $P, Q \subseteq PAR$ and $\mathrm{r} \in \{\mathrm{fin}, \mathrm{bin}, \mathrm{sim}\}$,

(NIL) for every $P \subseteq PAR$,

- **(N1)** $\mathcal{R}^{\mathrm{fin}}_P = (\mathcal{R}^{\mathrm{bin}}_P)^{-1}$, that is $\mathcal{R}^{\mathrm{fin}}_P$ is the converse of $\mathcal{R}^{\mathrm{bin}}_P$,
- **(N2)** $\mathcal{R}^{\mathrm{fin}}_P$ is reflexive and transitive,
- **(N3)** $\mathcal{R}^{\mathrm{sim}}_P$ is reflexive and symmetric,
- **(N4)** if $\langle x, y\rangle \in \mathcal{R}^{\mathrm{sim}}_P$ and $\langle y, z\rangle \in \mathcal{R}^{\mathrm{fin}}_P$, then $\langle x, z\rangle \in \mathcal{R}^{\mathrm{sim}}_P$.

Moreover, m is a mapping $m : \mathrm{VARPROP} \cup \mathrm{P} \rightarrow \mathcal{P}(W) \cup \mathcal{P}(PAR)$ such that $m(\mathrm{p}) \subseteq W$ for every $\mathrm{p} \in \mathrm{VARPROP}$, and the restriction of m to P is a P-interpretation.

A specific feature of RNIL-models is that the relations are constrained in two ways. For instance, the binary relation $\mathcal{R}_Q^{\rm fin}$ is related to $\mathcal{R}_Q^{\rm bin}$ and $\mathcal{R}_Q^{\rm sim}$, but $\mathcal{R}_Q^{\rm fin}$ is also constrained with the relations in $(\mathcal{R}_P^{\rm fin})_{P \subseteq PAR}$ via the condition **(inter)**.

Consequently, two levels of interpretation are used to define the relations in the RNIL-models. On the one hand, the parameter expressions are interpreted within the Boolean algebra

$$\mathcal{B} = \langle \mathcal{P}(PAR), \cup, \cap, -, PAR, \emptyset \rangle,$$

where the non-empty set PAR and the empty set are the unit element and zero element, respectively. On the other hand, the conditions on $(\mathcal{R}_P)_{P \subseteq PAR}$ induce a semi-lattice structure of $\mathcal{L} = \langle \{\mathcal{R}_P : P \in \mathcal{B}\}, \cap \rangle$ with the zero element $W \times W$. All these facts are well-known, see e.g. [7, 5].

Similarly, $(\mathcal{R}_P^{\rm fin})_{P \subseteq PAR}$ is an abstraction of the family $(fin_A)_{A \subseteq AT}$ derived from information systems. The adequacy between the class of RNIL-models and information systems follows from the developments of [31].

Let $\mathcal{M} = \langle W, (\mathcal{R}_P^{\rm fin})_{P \subseteq PAR}, (\mathcal{R}_P^{\rm bin})_{P \subseteq PAR}, (\mathcal{R}_P^{\rm sim})_{P \subseteq PAR}, m \rangle$ be a model. As usual, we say that a formula ϕ is *satisfied by* $w \in W$ *in* $\mathcal{M}$ (written $\mathcal{M}, w \models \phi$) if the following conditions are satisfied.

$\mathcal{M}, w \models {\rm p}$	iff	$w \in m({\rm p})$ for ${\rm p} \in {\rm VARPROP}$,
$\mathcal{M}, w \models \neg\phi$	iff	not $\mathcal{M}, w \models \phi$,
$\mathcal{M}, w \models \phi \wedge \psi$	iff	$\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$,
$\mathcal{M}, w \models [{\rm r}({\rm A})]\phi$	iff	for every $w' \in W$, if $\langle w, w' \rangle \in \mathcal{R}^{\rm r}_{m({\rm A})}$, then $\mathcal{M}, w' \models \phi$,
		where $r \in \{{\rm fin}, {\rm bin}, {\rm sim}\}$.

A formula ϕ is *true* in a RNIL-model $\mathcal{M}$ (written $\mathcal{M} \models \phi$) iff for every $w \in W$, $\mathcal{M}, w \models \phi$. A formula ϕ is said to be *RNIL-valid* iff ϕ is true in every RNIL-model. A formula ϕ is said to be *RNIL-satisfiable* iff $\neg\phi$ is not RNIL-valid. By way of example, for all ϕ and ${\rm A}$, the formulae below are valid:

- $[{\rm bin}({\rm A})][{\rm sim}({\rm A})][{\rm fin}({\rm A})]\phi \Leftrightarrow [{\rm sim}({\rm A})]\phi$,
- $[{\rm fin}({\rm A})][{\rm fin}({\rm A})]\phi \Leftrightarrow [{\rm fin}({\rm A})]\phi$,
- $\phi \Rightarrow [{\rm bin}({\rm A})]\neg[{\rm fin}({\rm A})]\neg\phi$.

Observe that due to condition **(inter)**, RNIL captures intersection of relations. Indeed, let us write $R_{\rm A}$ for some $\mathcal{R}^{\rm r}_{m({\rm A})}$. Then, for all parameter expressions ${\rm A}, {\rm B}$, we have $R_{{\rm A} \cup {\rm B}} = R_{\rm A} \cap R_{\rm B}$. By contrast, complement and union cannot be expressed in a similar fashion. Additionally, RNIL contains the universal modality since $R_{{\rm A} \cap -{\rm A}}$ is precisely the product $W \times W$.

The logic NIL [24, 31] can be seen as a fragment of RNIL restricted to formulae with the unique parameter expression ${\rm C}_1 \cup -{\rm C}_1$ (interpreted as the full set of parameters).

2.2. Normal Forms for Parameter Expressions

In this section, we recall the notion of a normal form for parameter expressions inspired by the canonical disjunctive normal form for propositional logic. Such normal forms play a special role for the relative information logics and they are introduced in [13]. This technique has been also useful in showing decidability of SIM [6] and, EXPTIME-completeness of SIM [8] and of some fragments of Boolean modal logic BML [15, Sect. 5].

Let $X = \{\mathrm{C}_1, \ldots, \mathrm{C}_n\}$ be a set of distinct parameter variables for some $n \geq 1$. For every integer $k \in \{0, \ldots, 2^n - 1\}$, we denote by B_k the parameter expression $\mathrm{B}_k \stackrel{\text{def}}{=} \mathrm{A}_1 \cap \cdots \cap \mathrm{A}_n$ where, for every $s \in \{1, \ldots, n\}$, $\mathrm{A}_s = \mathrm{C}_s$ if $bit_s(k) = 0$ and $\mathrm{A}_s = -\mathrm{C}_s$ otherwise, where $bit_s(k)$ denotes the sth bit in the binary representation of k with n bits. So $\mathrm{B}_3 = \mathrm{C}_1 \cap -\mathrm{C}_2 \cap -\mathrm{C}_3$ with $n = 3$. Although not essential, the use of binary representation will facilitate the presentation of technical developments. The set $\mathrm{Comp}(X)$ of X-*components*, is defined as follows:

$$\mathrm{Comp}(X) \stackrel{\text{def}}{=} \{\mathrm{B}_k \mid k \in \{0, \ldots, 2^n - 1\}\}.$$

The set $\mathrm{Comp}(X)$ of X-components enables us to partition every set of parameters. Indeed, for every P-interpretation $m : \mathrm{P} \to \mathcal{P}(PAR)$, the family $\{m(\mathrm{A}) \mid \mathrm{A} \in \mathrm{Comp}(X)\}$ is a partition of PAR [13]. As a consequence, we obtain the following property.

Lemma 2.1. Let A be a parameter expression built over the parameter variables of X. Then either $\mathrm{A} \equiv -\mathrm{A} \cap \mathrm{A}$ or there is a unique non-empty subset $\{\mathrm{A}'_1, \ldots, \mathrm{A}'_u\}$ of $\mathrm{Comp}(X)$ such that $\mathrm{A} \equiv \mathrm{A}'_1 \cup \cdots \cup \mathrm{A}'_u$.

Lemma 2.1 enables us to define normal forms of parameter expressions. Let A be a parameter expression built from the resources in X. The normal form of A, $\mathrm{N}_X(\mathrm{A})$, is defined as follows:

$$\mathrm{N}_X(\mathrm{A}) \stackrel{\text{def}}{=} \begin{cases} \emptyset \text{ if } \mathrm{A} \equiv (\mathrm{A} \cap -\mathrm{A}); \\ \{\mathrm{B}_{k_1}, \ldots, \mathrm{B}_{k_u}\} \text{ if } \mathrm{A} \equiv \mathrm{B}_{k_1} \cup \ldots \cup \mathrm{B}_{k_u}. \end{cases}$$

Observe that there exists an effective procedure that computes $\mathrm{N}_X(\mathrm{A})$ in exponential-time in $|\mathrm{A}| + n$. Moreover, it is known that, for all parameter expressions A, B built from the resources in X, we have $\mathrm{A} \equiv \mathrm{B}$ iff $\mathrm{N}_X(\mathrm{A}) = \mathrm{N}_X(\mathrm{B})$. This normal form is not thought to be applied to all parameter expressions in a RNIL-formula to be tested for satisfiability (since this would obviously yield an exponential blow-up), but it is used in the following section to decide the implication relation between parameter expressions, as also done in [8]. Obviously, for all $\mathrm{r} \in \{\mathrm{fin}, \mathrm{bin}, \mathrm{sim}\}$, ϕ and $\mathrm{A}, \mathrm{B} \in \mathrm{P}$, $\mathrm{A} \equiv \mathrm{B}$ implies that $[\mathrm{r}(\mathrm{A})]\phi \Leftrightarrow [\mathrm{r}(\mathrm{B})]\phi$ is valid.

3. Symbolic Representation of States

In this section, we define the notion of symbolic states which represent objects in RNIL-models. In Definition 3.1 below, we introduce a closure operator for sets of RNIL-formulae as it is done for Propositional Dynamic Logic PDL by Fischer and Ladner in [10].

Definition 3.1. Given a set X of RNIL-formulae, let $\mathrm{cl}(X)$ be the smallest set of formulae such that:

(CL1) $X \subseteq \mathrm{cl}(X)$,

(CL2) if $\neg\phi \in \mathrm{cl}(X)$, then $\phi \in \mathrm{cl}(X)$,

(CL3) if $\phi_1 \wedge \phi_2 \in \mathrm{cl}(X)$, then $\phi_1, \phi_2 \in \mathrm{cl}(X)$,

(CL4) if $[\mathrm{r}(\mathrm{A})]\phi \in \mathrm{cl}(X)$, then $\phi \in \mathrm{cl}(X)$,

(CL5) if $[\mathrm{sim}(\mathrm{A})]\phi \in \mathrm{cl}(X)$ and ϕ is not a $[\mathrm{fin}(\mathrm{A})]$-formula, then $[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\phi \in \mathrm{cl}(X)$,

(CL6) if $[\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\phi \in \mathrm{cl}(X)$, then $[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\phi \in \mathrm{cl}(X)$.

Consequently, if $[\mathrm{sim}(\mathrm{A})]\phi \in \mathrm{cl}(X)$ and ϕ is not a $[\mathrm{fin}(\mathrm{A})]$-formula, then $[\mathrm{fin}(\mathrm{A})]\phi \in \mathrm{cl}(X)$. A set X of formulae is said to be *closed* iff $\mathrm{cl}(X) = X$. For any finite set X of formulae, we have $\mathrm{md}(\mathrm{cl}(X)) \leq \mathrm{md}(X) + 2$.

Lemma 3.1. Let ϕ be a RNIL-formula. Then, $\mathrm{card}(\mathrm{cl}(\{\phi\})) < 4 \times |\phi|$.

Proof:
Let $\mathrm{sub}(\phi)$ be the set of subformulae of the formula ϕ. Obviously, $\mathrm{sub}(\phi) \subseteq \mathrm{cl}(\{\phi\})$. Moreover, $\mathrm{cl}(\{\phi\})$ is the union of the following sets:

1. $\mathrm{sub}(\phi)$,
2. $\{[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi : [\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi \in \mathrm{sub}(\phi)\}$,
3. $\{[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi : [\mathrm{sim}(\mathrm{A})]\psi \in \mathrm{sub}(\phi), \psi \neq [\mathrm{fin}(\mathrm{A})]\psi'\}$,
4. $\{[\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi : [\mathrm{sim}(\mathrm{A})]\psi \in \mathrm{sub}(\phi), \psi \neq [\mathrm{fin}(\mathrm{A})]\psi'\}$,
5. $\{[\mathrm{fin}(\mathrm{A})]\psi : [\mathrm{sim}(\mathrm{A})]\psi \in \mathrm{sub}(\phi), \psi \neq [\mathrm{fin}(\mathrm{A})]\psi'\}$.

Each set above is of the cardinality at most $\mathrm{card}(\mathrm{sub}(\phi))$ and a formula in $\mathrm{sub}(\phi)$ can generate at most three other formulae in $\mathrm{cl}(\{\phi\})$. So $\mathrm{card}(\mathrm{cl}(\{\phi\})) < 4 \times |\phi|$, since $\mathrm{card}(\mathrm{sub}(\phi)) < |\phi|$. □

Only consistent subsets of $\mathrm{cl}(\phi)$ are useful for checking satisfiability. Definition 3.2 introduces local consistency whose modal part is based on the valid formulae below ($\mathrm{r} \in \{\mathrm{sim}, \mathrm{fin}, \mathrm{bin}\}$, ψ and $\mathrm{A} \in \mathrm{P}$):

- $[\mathrm{r}(\mathrm{A})]\psi \Rightarrow \psi$,
- $[\mathrm{sim}(\mathrm{A})]\psi \Rightarrow [\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi$.

Definition 3.2. Let X be a subset of $\mathrm{cl}(\{\phi\})$ for some formula ϕ. The set X is said to be *locally RNIL-consistent* iff each $\psi \in \mathrm{sub}(\phi)$ satisfies the following conditions:

(LOC1) if $\psi = \neg\varphi$, then $\varphi \in X$ iff $\psi \notin X$,

(LOC2) if $\psi = \varphi_1 \wedge \varphi_2$, then $\{\varphi_1, \varphi_2\} \subseteq X$ iff $\psi \in X$,

(LOC3) if $\psi = [\mathrm{r}(\mathrm{A})]\varphi$ and $\psi \in X$, then $\varphi \in X$,

(LOC4) if $\psi = [\mathrm{sim}(\mathrm{A})]\varphi$, $\varphi \neq [\mathrm{fin}(\mathrm{A})]\varphi'$ and $\psi \in X$, then $[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\varphi \in X$,

(LOC5) if $\psi = [\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\varphi$ and $\psi \in X$, then $[\mathrm{bin}(\mathrm{A})][\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\varphi \in X$.

Definition 3.1 has been designed so that local consistency does not require formulae outside $\mathrm{cl}(\{\phi\})$. The binary relation $\sim_{\mathrm{A}}^{\mathrm{sim}}$ on subsets of $\mathrm{cl}(\{\phi\})$ is defined as follows: $X \sim_{\mathrm{A}}^{\mathrm{sim}} Y$ iff

1. for every $[\mathrm{sim}(\mathrm{B})]\psi \in X$ with $\mathrm{B} \sqsubseteq \mathrm{A}$, $\psi \in Y$,
2. for every $[\mathrm{sim}(\mathrm{B})]\psi \in Y$ with $\mathrm{B} \sqsubseteq \mathrm{A}$, $\psi \in X$,
3. for every $[\mathrm{r}(\mathrm{B})]\psi \in \mathrm{cl}(\{\phi\})$ with $\mathrm{N}_X(\mathrm{B}) = \emptyset$, $[\mathrm{r}(\mathrm{B})]\psi \in X$ iff $[\mathrm{r}(\mathrm{B})]\psi \in Y$.

Clause 3. encodes that $[\mathrm{r}(\mathrm{B})]$ is a universal modality whereas clauses 1. and 2. are standard in order to deal with modal operators based on symmetrical relations. The binary relation $\sim_{\mathrm{A}}^{\mathrm{fin}}$ is defined as follows: $X \sim_{\mathrm{A}}^{\mathrm{fin}} Y$ iff

1. for all $[\mathrm{fin}(\mathrm{B})]\psi \in X$ with $\mathrm{B} \sqsubseteq \mathrm{A}$, $[\mathrm{fin}(\mathrm{B})]\psi, \psi \in Y$,
2. for all $[\mathrm{bin}(\mathrm{B})]\psi \in Y$ with $\mathrm{B} \sqsubseteq \mathrm{A}$, $[\mathrm{bin}(\mathrm{B})]\psi, \psi \in X$,
3. for every $[\mathrm{r}(\mathrm{B})]\psi \in \mathrm{cl}(\{\phi\})$ with $\mathrm{N}_X(\mathrm{B}) = \emptyset$, $[\mathrm{r}(\mathrm{B})]\psi \in X$ iff $[\mathrm{r}(\mathrm{B})]\psi \in Y$.

As expected, the third binary relation $\sim_{\mathrm{A}}^{\mathrm{bin}}$ is defined as the converse of $\sim_{\mathrm{A}}^{\mathrm{fin}}$. The relation $\sim_{\mathrm{A}}^{\mathrm{sim}}$ [resp. $\sim_{\mathrm{A}}^{\mathrm{fin}}$] is the abstract counterpart of the relation $\mathcal{R}_{m(\mathrm{A})}^{\mathrm{sim}}$ [resp. $\mathcal{R}_{m(\mathrm{A})}^{\mathrm{fin}}$] in RNIL-models.

We are now ready to define symbolic states. Each such state contains information on the relation between the associated node and its (unique) predecessor, A *symbolic state for* ϕ is either $\perp$ or a pair $q = \langle \mathrm{r}(\mathrm{A}), X \rangle$ such that $\mathrm{r}(\mathrm{A})$ occurs in ϕ and $\mathrm{A} \in \mathrm{P}(\phi)$, X is a locally RNIL-consistent set (subset of $\mathrm{cl}(\phi)$). In $q = \langle \mathrm{r}(\mathrm{A}), X \rangle$, $\mathrm{r}(\mathrm{A})$ refers to the relation $\mathcal{R}_{m(\mathrm{A})}^{\mathrm{r}}$ which relates q's (unique) predecessor to q, X is the set of formulae satisfied by q. We write $\psi \in q = \langle \mathrm{r}(\mathrm{A}), X \rangle$ whenever $\psi \in X$. The "dummy" value $\perp$ is used for those nodes in a tree not representing objects, and we call a symbolic state q *dummy* if $q = \perp$. We use $\mathrm{SYMB}(\phi)$ to denote the set of symbolic states of ϕ.

Since each condition **(LOCi)** in Definition 3.2 is quite easy to check, we can establish the following result.

Lemma 3.2. Deciding whether a subset of $\mathrm{cl}(\phi)$ is locally RNIL-consistent can be done in polynomial time.

4. Tree Model Property

We are now ready to introduce Hintikka trees for RNIL. Such trees are abstractions of RNIL-models that allow a further treatment with Büchi automata on infinite trees (see Section 5). We will show that each RNIL-model can be unravelled into a Hintikka tree, and thus prove a tree model property for RNIL. This is the key property to use then automata accepting trees.

Given an RNIL-formula ϕ, a Hintikka-tree for ϕ is labelled with symbolic states. We recall that, given $K \geq 1$ and a finite alphabet Σ, an infinite Σ, K-tree $\mathcal{T}$ is a mapping $\mathcal{T} : \{1, \ldots, K\}^* \to \Sigma$ where, as usual, for a set X, X^* denotes the family of all the finite sequences with the elements taken in X. Let ϕ be a RNIL-formula with $K = |\mathrm{cl}(\{\phi\})|$, and $\mathrm{PVAR}(\phi)$ the set of parameter variables occurring in ϕ with $n = \mathrm{card}(\mathrm{PVAR}(\phi)) \geq 1$.

Definition 4.1. A $\mathrm{SYMB}(\phi), K$-tree $\mathcal{T}$ is a *Hintikka tree for* ϕ iff

(H1) $\phi \in \mathcal{T}(\varepsilon)$ where ε is the empty sequence,

(H2) if $\mathcal{T}(s)$ is dummy, then $\mathcal{T}(s \cdot 1), \ldots, \mathcal{T}(s \cdot K)$ are also dummy,

(H3) if $\mathcal{T}(s) = \langle \mathrm{r}'(\mathrm{A}), X \rangle$ is not dummy and $[\mathrm{r}(\mathrm{B})]\psi \in \mathrm{sub}(\phi) \setminus X$, then there is $i \in \{1, \ldots, K\}$ with $\mathcal{T}(s \cdot i) = \langle \mathrm{r}(\mathrm{B}), X' \rangle$, $\mathcal{T}(s \cdot i)$ is not dummy, and $\psi \notin X'$,

(H4) for every $i \in \{1, \ldots, K\}$, if both $\mathcal{T}(s) = \langle \mathrm{r}(\mathrm{A}), X \rangle$ and $\mathcal{T}(s \cdot i) = \langle \mathrm{r}'(\mathrm{B}), X' \rangle$ are not dummy, then $X \sim_{\mathrm{B}}^{\mathrm{r}'} X'$.

The preliminary developments presented so far yield Lemma 4.1 below.

Lemma 4.1. For every RNIL-formula ϕ, (I) ϕ is RNIL-satisfiable iff (II) ϕ has a Hintikka tree.

Lemma 4.1 is the main technical lemma of the paper and its proof takes advantage of all preliminary definitions and properties. The main difficulty is in the construction of an RNIL-model from an Hintikka tree. Indeed, one needs to build from a tree structure three families of relations with interacting constraints.

Proof:
(II) $\rightarrow$ (I). Let $\mathcal{T}$ be a Hintikka tree for ϕ.

The construction of $\mathcal{M}$.

We construct an RNIL-model $\mathcal{M} = \langle W, (\mathcal{R}_P^{\mathrm{fin}})_{P \subseteq PAR}, (\mathcal{R}_P^{\mathrm{bin}})_{P \subseteq PAR}, (\mathcal{R}_P^{\mathrm{sim}})_{P \subseteq PAR}, m\rangle$ of ϕ as follows:

- $W \stackrel{\text{def}}{=} \{s \in \{1, \ldots, K\}^* : \mathcal{T}(s) \text{ is not dummy}\}$.
- $PAR \stackrel{\text{def}}{=} \{0, \ldots, 2^n - 1\}$. Observe that PAR is finite.
- For every $i \in \{1, \ldots, n\}$, $m(\mathrm{C}_i) \stackrel{\text{def}}{=} \{k \in \{0, \ldots, 2^n - 1\} \mid bit_i(k) = 0\}$. This guarantees that $m(\mathrm{B}_k) = \{k\}$ where $\mathrm{B}_k \in \mathrm{Comp}(\mathrm{P}(\phi))$.
- For every $s \in W$, for every $\mathrm{p} \in \mathrm{VARPROP}$, $s \in m(\mathrm{p})$ iff $\mathrm{p} \in \mathcal{T}(s)$.
- For every $\mathrm{A} \in \mathrm{P}(\phi)$, and r in {fin, bin, sim}, let $S_{\mathrm{A}}^{\mathrm{r}}$ be the binary relation $\{\langle s, s \cdot i\rangle \in W^2 \mid s \in \{1, \ldots, K\}^*,\ i \in \{1, \ldots, K\}, \mathcal{T}(s \cdot i) = \langle \mathrm{r}(\mathrm{A}), X\rangle\}$. We write $S_{\mathrm{A}}^{\mathrm{r}^{-1}}$ to denote the converse of $S_{\mathrm{A}}^{\mathrm{r}}$. These relations are the building blocks to define the families in $\mathcal{M}$.
- For every $i \in \{0, \ldots, 2^n - 1\}$,
  - $\mathcal{R}_{\{i\}}^{\mathrm{fin}}$ is the reflexive and transitive closure of $\bigcup\{S_{\mathrm{A}}^{\mathrm{fin}}, S_{\mathrm{A}}^{\mathrm{bin}^{-1}} \mid \mathrm{A} \in \mathrm{P}(\phi),\ i \in m(\mathrm{A})\}$,
  - $\mathcal{R}_{\{i\}}^{\mathrm{bin}}$ is the converse of $\mathcal{R}_{\{i\}}^{\mathrm{fin}}$,
  - $\mathcal{R}_{\{i\}}^{\mathrm{sim}}$ is $\mathcal{R}_{\{i\}}^{\mathrm{bin}} \circ \mathcal{S}_{\{i\}}^{\mathrm{sim}} \circ \mathcal{R}_{\{i\}}^{\mathrm{fin}}$ where $\mathcal{S}_{\{i\}}^{\mathrm{sim}}$ is the reflexive closure of $\bigcup\{S_{\mathrm{A}}^{\mathrm{sim}}, S_{\mathrm{A}}^{\mathrm{sim}^{-1}} \mid \mathrm{A} \in \mathrm{P}(\phi),\ i \in m(\mathrm{A})\}$.

  Reflexivity and transitivity are obtained by closure. Symmetry is guaranteed thanks to converse relations.
- For all $P \subseteq PAR$ such that $\mathrm{card}(P) \geq 2$ and r in {fin, bin, sim}, $\mathcal{R}_P^{\mathrm{r}} \stackrel{\text{def}}{=} \bigcap_{i \in P} \mathcal{R}_{\{i\}}^{\mathrm{r}}$.
- For every r in {fin, bin, sim}, $\mathcal{R}_{\emptyset}^{\mathrm{r}} \stackrel{\text{def}}{=} W \times W$.

Basic properties of $\mathcal{M}$. In order to establish that $\mathcal{M}$ is a RNIL-model and to complete the proof by induction, we can show the properties below.

1. For every $k \in \{0, \ldots, 2^n - 1\}$, $m(\mathrm{B}_k) = \{k\}$.

2. For every path $s_0 S^{\mathrm{r}_1}_{\mathrm{A}_1} s_1 S^{\mathrm{r}_2}_{\mathrm{A}_2} \ldots S^{\mathrm{r}_\mathrm{N}}_{\mathrm{A}_N} s_N$ such that $\{r_1, \ldots, r_N\} \subseteq \{\mathrm{fin}, \mathrm{bin}^{-1}\}$ and for $i > 0$, $r_i = \mathrm{bin}^{-1}$ implies $r_{i-1} \neq \mathrm{fin}$, there is no other path between s_0 and s_N satisfying these properties.
3. For every path $s_0 S^{\mathrm{r}_1}_{\mathrm{A}_1} s_1 S^{\mathrm{r}_2}_{\mathrm{A}_2} \ldots S^{\mathrm{r}_\mathrm{N}}_{\mathrm{A}_N} s_N$ such that $\{r_1, \ldots, r_N\} \subseteq \{\mathrm{bin}, \mathrm{fin}^{-1}\}$ and for $i > 0$, $r_i = \mathrm{fin}^{-1}$ implies $r_{i-1} \neq \mathrm{bin}$, there is no other path between s_0 and s_N satisfying these properties.
4. For all s, s', if $\langle s, s' \rangle \in S^{\mathrm{r}}_{\mathrm{A}}$ for some $r \in \{\mathrm{sim}, \mathrm{sim}^{-1}\}$ and $\mathrm{A} \in \mathrm{P}$, then r and A are unique.
5. For every $P \subseteq PAR$ such that $\mathrm{card}(P) \geq 2$, $\mathcal{R}^{\mathrm{sim}}_P = \mathcal{R}^{\mathrm{bin}}_P \circ \mathcal{S}^{\mathrm{sim}}_P \circ \mathcal{R}^{\mathrm{fin}}_P$.
6. If $\langle s, s' \rangle \in \mathcal{R}^{\mathrm{r}}_{m(\mathrm{A})}$ and $[\mathrm{r}(\mathrm{A})]\psi \in \mathcal{T}(s)$, then $\psi \in \mathcal{T}(s')$.

A nice consequence of the point (1) is that reasoning about the normal form of A can be reduced to reasoning on the elements in $m(\mathrm{A})$. Point (5) combined with the fact that reflexivity, symmetry and transitivity are closed under intersection, guarantees that $\mathcal{M}$ is an RNIL-model. The points (2)–(4) are direct consequences of the fact that $\mathcal{T}$ has a tree structure. Let us start by showing Property (6).
Case 1: r = fin.
Suppose $\langle s, s' \rangle \in \mathcal{R}^{\mathrm{fin}}_{m(\mathrm{A})}$ and $[\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s)$. If $m(\mathrm{A}) = \emptyset$, then for all s', s'' that are not dummy in $\mathcal{T}$, $[\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s')$ iff $[\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s'')$. Hence, $[\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s')$ and since $\mathcal{T}(s')$ is RNIL-consistent, by **(LOC3)**, we have $\psi \in \mathcal{T}(s')$. Now suppose that $m(\mathrm{A})$ is non-empty and equal to $\{i_1, \ldots, i_k\}$. By Property (2) and since $\langle s, s' \rangle \in \mathcal{R}^{\mathrm{bin}}_{m(\mathrm{A})}$, there exists $s_0 S^{\mathrm{r}_1}_{\mathrm{A}_1} s_1 S^{\mathrm{r}_2}_{\mathrm{A}_2} \ldots S^{\mathrm{r}_\mathrm{N}}_{\mathrm{A}_N} s_N$ with $s_0 = s$, $s_N = s'$, and for $1 \leq l \leq N$, $r_l \in \{\mathrm{fin}, \mathrm{bin}^{-1}\}$ and $\{i_1, \ldots, i_k\} \subseteq m(\mathrm{A}_l)$. Hence for every l, $\mathrm{A} \sqsubseteq \mathrm{A}_l$. By definition of $\sim^{\mathrm{bin}}_{\mathrm{A}_l}$, $\sim^{\mathrm{fin}}_{\mathrm{A}_l}$ and by RNIL-consistency of non dummy nodes of $\mathcal{T}$, we get $\psi \in \mathcal{T}(s')$.

Case 2: r = bin.
Similar to the above case.

Case 3: r = sim.
Suppose $\langle s, s' \rangle \in \mathcal{R}^{\mathrm{sim}}_{m(\mathrm{A})}$ and $[\mathrm{sim}(\mathrm{A})]\psi \in \mathcal{T}(s)$. We treat below the case when ψ is not a $[\mathrm{fin}(\mathrm{A})]$-formula, otherwise the arguments are analogous. We know how to deal with $m(\mathrm{A}) = \emptyset$ from the previous cases. Suppose that $m(\mathrm{A})$ is non-empty and is equal to $\{i_1, \ldots, i_k\}$. By slightly extending Properties (2)–(4) and since $\langle s, s' \rangle \in \mathcal{R}^{\mathrm{sim}}_{m(\mathrm{A})}$, there exist two paths

- $s_0 S^{\mathrm{r}_1}_{\mathrm{A}_1} s_1 S^{\mathrm{r}_2}_{\mathrm{A}_2} \ldots S^{\mathrm{r}_\mathrm{N}}_{\mathrm{A}_N} s_N$ with $\{r_1, \ldots, r_N\} \subseteq \{\mathrm{bin}, \mathrm{fin}^{-1}\}$,
- $s'_0 S^{\mathrm{s}_1}_{\mathrm{A}'_1} s'_1 S^{\mathrm{s}_2}_{\mathrm{A}'_2} \ldots S^{\mathrm{s}_\mathrm{M}}_{\mathrm{A}'_M} s'_M$ with $\{s_1, \ldots, s_M\} \subseteq \{\mathrm{fin}, \mathrm{bin}^{-1}\}$,

such that

1. $s_0 = s$, $s'_M = s'$,
2. either $s_N = s'_0$ or $s_N S^{\mathrm{sim}}_{\mathrm{A}'} s'_0$ or $s'_0 S^{\mathrm{sim}}_{\mathrm{A}'} s_N$,
3. for all l, l', we have $\{i_1, \ldots, i_k\} \subseteq m(\mathrm{A}_l) \cap m(\mathrm{A}'_{l'}) \cap m(\mathrm{A}')$.

By definition of $\sim^{\mathrm{bin}}_{\mathrm{A}_l}$, $\sim^{\mathrm{fin}}_{\mathrm{A}_l}$ and since $\mathcal{T}(s)$ is locally RNIL-consistent (see **(LOC4)**) $[\mathrm{sim}(\mathrm{A})]\psi \in \mathcal{T}(s)$ implies $[\mathrm{sim}(\mathrm{A})][\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s_N)$. By definition of $\sim^{\mathrm{sim}}_{\mathrm{A}_l}$ and by **(LOC3)** (if $s_N = s'_0$), $[\mathrm{fin}(\mathrm{A})]\psi \in \mathcal{T}(s'_0)$. By definition of $\sim^{\mathrm{fin}}_{\mathrm{A}_l}$, we obtain $\psi \in \mathcal{T}(s')$.

It remains to prove Property (5). Let $P \subseteq PAR$ such that $\mathrm{card}(P) \geq 2$. By definition, $\mathcal{R}_P^{\mathrm{sim}} = \bigcap_{i \in P} \mathcal{R}_{\{i\}}^{\mathrm{sim}}$. First suppose $\langle s, s' \rangle \in \mathcal{R}_P^{\mathrm{sim}}$ with $P = \{i_1, \ldots, i_k\}$. By definition of $\mathcal{R}_{\{i\}}^{\mathrm{sim}}$, for every $j \in \{1, \ldots, k\}$, there exist two paths

- $s_0^j S_{\mathrm{A}_1^j}^{\mathrm{r}_1^{\mathrm{j}}} s_1^j S_{\mathrm{A}_2^j}^{\mathrm{r}_2^{\mathrm{j}}} \ldots S_{\mathrm{A}_{N^j}}^{\mathrm{r}_{\mathrm{N}}^{\mathrm{j}}} s_{N^j}$ with $\{r_1^j, \ldots, r_{N^j}^j\} \subseteq \{\mathrm{bin}, \mathrm{fin}^{-1}\}$,
- $s_0^{\prime j} S_{\mathrm{A}_1^{\prime j}}^{\mathrm{s}_1^{\mathrm{j}}} s_1^{\prime j} S_{\mathrm{A}_2^{\prime j}}^{\mathrm{s}_2^{\mathrm{j}}} \ldots S_{\mathrm{A}_{Mj}^{\prime j}}^{\mathrm{s}_{\mathrm{M_j}}^{\mathrm{j}}} s_{M^j}^{\prime j}$ with $\{s_1^j, \ldots, s_{M^j}^j\} \subseteq \{\mathrm{fin}, \mathrm{bin}^{-1}\}$,

such that

1. $s_0^j = s$, $s_{M_j}^{\prime j} = s'$,
2. either $s_N^j = s_0^{\prime j}$ or $s_N^j S_{\mathrm{A}^{\prime j}}^{\mathrm{sim}} s_0^{\prime j}$ or $s_0^{\prime j} S_{\mathrm{A}^{\prime j}}^{\mathrm{sim}} s_N^j$,
3. for all l, l', we have $i_j \in m(\mathrm{A}_l^j) \cap m(\mathrm{A}_{l'}^{\prime j}) \cap m(\mathrm{A}^{\prime j})$.

By the unicity property of a slight extension of Properties (2)–(4), for all $j \neq j'$ and l, l', $N_j = N_{j'}$, $M_j = M_{j'}$, $s_l^j = s_l^{j'}$, $s_{l'}^{\prime j} = s_{l'}^{\prime j'}$. Hence, by definition of $\mathcal{R}_{\{i\}}^{\mathrm{sim}}$, for every $j \in \{1, \ldots, k\}$, there exist two paths

(I) $s_0 S_{\mathrm{A}_1}^{\mathrm{r}_1} s_1 S_{\mathrm{A}_2}^{\mathrm{r}_2} \ldots S_{\mathrm{A}_N}^{\mathrm{r}_{\mathrm{N}}} s_N$ with $\{r_1, \ldots, r_N\} \subseteq \{\mathrm{bin}, \mathrm{fin}^{-1}\}$,

(II) $s_0' S_{\mathrm{A}_1'}^{\mathrm{s}_1} s_1' S_{\mathrm{A}_2'}^{\mathrm{s}_2} \ldots S_{\mathrm{A}_M'}^{\mathrm{s}_{\mathrm{M}}} s_M'$ with $\{s_1, \ldots, s_M\} \subseteq \{\mathrm{fin}, \mathrm{bin}^{-1}\}$,

such that

1. $s_0 = s$, $s_M' = s'$,
2. either $s_N = s_0'$ or $s_N S_{\mathrm{A}'}^{\mathrm{sim}} s_0'$ or $s_0' S_{\mathrm{A}'}^{\mathrm{sim}} s_N$,
3. for all l, l', we have $P \subseteq m(\mathrm{A}_l) \cap m(\mathrm{A}_{l'}') \cap m(\mathrm{A}')$.

By (I), $\langle s_0, s_N \rangle \in \mathcal{R}_P^{\mathrm{bin}}$, by 2. $\langle s_N, s_0' \rangle \in \mathcal{S}_P^{\mathrm{sim}}$ and by (II), $\langle s_0', s_M' \rangle \in \mathcal{R}_P^{\mathrm{fin}}$. So $\langle s, s' \rangle \in \mathcal{R}_P^{\mathrm{bin}} \circ \mathcal{S}_P^{\mathrm{sim}} \circ \mathcal{R}_P^{\mathrm{fin}}$.

For the converse, suppose that $\langle s, s' \rangle \in \mathcal{R}_P^{\mathrm{bin}} \circ \mathcal{S}_P^{\mathrm{sim}} \circ \mathcal{R}_P^{\mathrm{fin}}$. There are s_0, s_1, s_2, s_3 with $s_0 = s$, $s_3 = s'$, $\langle s_0, s_1 \rangle \in \mathcal{R}_P^{\mathrm{bin}}$, $\langle s_1, s_2 \rangle \in \mathcal{R}_P^{\mathrm{sim}}$ and $\langle s_2, s_3 \rangle \in \mathcal{R}_P^{\mathrm{fin}}$. So by definition, for every $j \in P$, $\langle s_0, s_1 \rangle \in \mathcal{R}_{\{j\}}^{\mathrm{bin}}$, $\langle s_1, s_2 \rangle \in \mathcal{R}_{\{j\}}^{\mathrm{sim}}$ and $\langle s_2, s_3 \rangle \in \mathcal{R}_{\{j\}}^{\mathrm{fin}}$. This means that

$$\langle s, s' \rangle \in \bigcap_{j \in P} \mathcal{R}_{\{j\}}^{\mathrm{bin}} \circ \mathcal{S}_{\{j\}}^{\mathrm{sim}} \circ \mathcal{R}_{\{j\}}^{\mathrm{fin}}.$$

Hence, $\langle s, s' \rangle \in \bigcap_{j \in P} \mathcal{R}_{\{j\}}^{\mathrm{sim}} = \mathcal{R}_P^{\mathrm{sim}}$.

The induction. Since $\mathcal{T}$ is a Hintikka tree, $\phi \in \mathcal{T}(\varepsilon)$. In order to show that $\mathcal{M}, i \models \phi$ (and therefore $\mathcal{M}$ is a model for ϕ), we prove by structural induction that, for every $\psi \in \mathrm{sub}(\phi)$, for every $s \in W$, we have $\psi \in \mathcal{T}(s)$ iff $\mathcal{M}, s \models \psi$. The base case with propositional variables and the induction steps for conjunction and negation are by an easy verification (see conditions **(LOC1)** and **(LOC2)**). Let us treat in detail the remaining case. Let $[\mathrm{r}(\mathrm{A})]\psi$ be a subformula of ϕ and assume that $[\mathrm{r}(\mathrm{A})]\psi \in \mathcal{T}(s)$. As we have seen above, by Property (6) this implies that, for every $\langle s, s' \rangle \in \mathcal{R}^{\rm r}_{m({\rm A})}$, we have $\psi \in \mathcal{T}(s')$. By the induction hypothesis, we have $\mathcal{M}, s' \models \psi$. So, $\mathcal{M}, s \models [{\rm r}({\rm A})]\psi$.

Now let $[{\rm r}({\rm A})]\psi$ be a subformula of ϕ and assume that $\mathcal{M}, s \models [{\rm r}({\rm A})]\psi$ and that $[{\rm r}({\rm A})]\psi \not\in \mathcal{T}(s)$. Due to **(H3)**, $\psi \not\in \mathcal{T}(s \cdot i)$ for some $i \in \{1, \ldots, K\}$ and $\mathcal{T}(s \cdot i)$ is not dummy. By the induction hypothesis, $\mathcal{M}, s \cdot i \not\models \psi$. However, one can show that $\langle s, s' \rangle \in \mathcal{R}^{\rm r}_{m({\rm A})}$. Consequently, $\mathcal{M}, s \not\models [{\rm r}({\rm A})]\psi$ which leads to a contradiction.

(I) $\rightarrow$ (II) Let $\mathcal{M} = \langle W, (\mathcal{R}^{\rm fin}_P)_{P \subseteq PAR}, (\mathcal{R}^{\rm bin}_P)_{P \subseteq PAR}, (\mathcal{R}^{\rm sim}_P)_{P \subseteq PAR}, m \rangle$ be a RNIL-model and $w_0 \in W$ such that $\mathcal{M}, w_0 \models \phi$. We define a Hintikka tree $\mathcal{T}$ for ϕ. In the construction of $\mathcal{T}$, we use an auxiliary mapping $\tau : \{1, \ldots, K\}^* \rightarrow W \cup \{\perp\}$ which is defined inductively together with $\mathcal{T}$ as follows (this is quite standard). Indeed, τ can be viewed as a tree skeleton of $\mathcal{M}$.

Let $[{\rm r}_1({\rm A}_1)]\psi_1, \ldots, [{\rm r}_\beta({\rm A}_\beta)]\psi_\beta$ be all box formulae in ${\rm cl}(\phi)$. For every $w \in W$, we write X_w to denote the set $\{\psi \in {\rm cl}(\phi) \mid \mathcal{M}, w \models \psi\}$. As usual, $\mathcal{T}$ is obtained by unravelling $\mathcal{M}$. We define τ and $\mathcal{T}$ as follows.

- $\tau(\varepsilon) \stackrel{\rm def}{=} w_0$ and $\mathcal{T}(\varepsilon) \stackrel{\rm def}{=} \langle {\rm r}({\rm A}), X_{w_0} \rangle$ for some arbitrary ${\rm r}({\rm A})$ occurring in ϕ.
- For every $s \in \{1, \ldots, K\}^+$,
  - for every $i \in \{\beta + 1, \ldots, K\}$, $\tau(s \cdot i) \stackrel{\rm def}{=} \perp$ and $\mathcal{T}(s \cdot i) \stackrel{\rm def}{=} \perp$,
  - if $\tau(s) = \perp$, then for every $i \in \{1, \ldots, \beta\}$, $\tau(s \cdot i) \stackrel{\rm def}{=} \perp$ and $\mathcal{T}(s \cdot i) \stackrel{\rm def}{=} \perp$,
  - otherwise, if $[{\rm r}_i({\rm A}_i)]\psi_i \not\in \mathcal{T}(s)$ for some $i \in \{1, \ldots, \beta\}$, then there is $w' \in W \setminus \{w_1, \ldots, w_\alpha\}$ such that $\langle \tau(s), w' \rangle \in \mathcal{R}^{{\rm r}_i}_{m({\rm A}_i)}$ and $\mathcal{M}, w' \not\models \psi_i$. In that case $\tau(s \cdot i) \stackrel{\rm def}{=} w'$ and $\mathcal{T}(s \cdot i) \stackrel{\rm def}{=} \langle {\rm r}_i({\rm A}_i), X_{w'} \rangle$.

    If $[{\rm r}_i({\rm A}_i)]\psi_i \in \mathcal{T}(s)$ for some $i \in \{1, \ldots, \beta\}$, then $\tau(s \cdot i) \stackrel{\rm def}{=} \perp$ and $\mathcal{T}(s \cdot i) \stackrel{\rm def}{=} \perp$.

We can easily check that $\mathcal{T}$ is a Hintikka tree for ϕ. $\Box$

5. Tree Automata for RNIL Formulae

In this section, we will exploit the tree model property for RNIL and describe a decision procedure based on automata on infinite trees, so-called *Büchi tree automata*. For a given RNIL-formula ϕ, we construct a Büchi tree automaton $\mathcal{A}_\phi$ that accepts exactly all Hintikka trees for ϕ.

A *Büchi tree automaton* $\mathcal{A} = \langle \Sigma, Q, \delta, I, F \rangle$ for Σ, K-trees is an operational model where Q is a non-empty, finite set of states, Σ is a finite alphabet, $\delta \subseteq Q \times \Sigma \times Q^K$ is a transition relation, I and F are non-empty subsets of Q, the set of initial states and the set of terminal states, respectively. A *run* r on a Σ, K-tree $\mathcal{T}$ is a Q, K-tree such that, for every $s \in \{1, \ldots, K\}^*$, $\langle r(s), \mathcal{T}(s), r(s \cdot 1), \ldots, r(s \cdot K) \rangle \in \delta$. A run r on $\mathcal{T}$ is *accepting* iff for every path in $\mathcal{T}$ there is a state in F that occurs infinitely often. Deciding whether a Büchi tree automaton for Σ, K-trees has an accepting run can be done in polynomial-time [34] (see also [28, 9]).

5.1. The Construction

Before giving the formal definition of $\mathcal{A}_\phi$, we give an intuitive description of it and the conditions it imposes on the trees it accepts.

- Each state consists of a symbolic state.
- If a node is labelled with $\perp$, then so are all its descendants.
- Successors of a node s satisfy conditions imposed by the box formulae in s's label.
- Diamond formulae in a node s's label (i.e., box formulae not in s's label) are witnessed by one of s's successors.

The above conditions are all local and can thus be "encoded" in the transition function of a tree automaton. Let us now give the formal definition for $\mathcal{A}_\phi$ when ϕ is a RNIL-formula satisfying the hypotheses at the beginning of Sect. 4. $\mathcal{A}_\phi$ is the Büchi tree automaton $\langle \Sigma, Q, \delta, I, Q \rangle$ defined as follows.

1. $\Sigma \stackrel{\text{def}}{=} \mathrm{SYMB}(\phi)$.
2. $Q \stackrel{\text{def}}{=} \Sigma$.
3. $I \stackrel{\text{def}}{=} \{\langle \mathrm{r}(\mathrm{A}), X \rangle : \phi \in X\}$,
4. $\langle q', a, q'_1, \ldots, q'_K \rangle \in \delta$ iff either

   (H$2'$) $q' = a = q'_1 = \ldots = q'_K = \perp$, or

   (witnesses) $q' = a$, q' is of the form $\langle \mathrm{r}(\mathrm{A}), X \rangle$ and, for every $i \in \{1, \ldots, K\}$, $q'_i = \langle \mathrm{r}_i(\mathrm{A}_i), X_i \rangle$ and the following conditions are satisfied:

   (H$3'$) if $[\mathrm{r}(\mathrm{B})]\psi \in \mathrm{cl}(\phi) \setminus q$, then there is $i \in \{1, \ldots, K\}$ such that $q_i = \langle \mathrm{r}(\mathrm{B}), X' \rangle$ is not dummy and $\psi \notin q_i$;

   (H$4'$) for every $i \in \{1, \ldots, K\}$, if q_i is not dummy, then $X \sim^{\mathrm{r}_i}_{\mathrm{A}_i} X_i$.

The conditions **(Hi')** are the obvious counterparts of the conditions **(Hi)**.

Lemma 5.1. A $\mathrm{SYMB}(\phi), K$-tree $\mathcal{T}$ is a Hintikka tree for ϕ iff $\mathcal{A}_\phi$ has an accepting run on $\mathcal{T}$.

Proof:
Let $\mathcal{T}$ be a Hintikka tree for ϕ and $r : \{1, \ldots, K\}^* \rightarrow Q$ be the Q, K-tree such that for every $s \in \{1, \ldots, K\}^*$, $r(s) = \mathcal{T}(s)$. One can check easily that r is an accepting run for $\mathcal{T}$.

For the converse, let $\mathcal{T}$ be an infinite tree accepted by $\mathcal{A}_\phi$. By construction, $\mathcal{T}$ is a Hintikka tree for ϕ. □

We are now in the position to establish the main result of the paper which extends EXPTIME-completeness of the logic FORIN in [8] and PSPACE-completeness of NIL in [4].

Theorem 5.1. The satisfiability problem for the logic RNIL is EXPTIME-complete.

Proof:
The arguments below are standard (see e.g., [8]) but we recall them for the sake of completeness. The lower bound is by an easy verification from the results in [3] and [11, Theorem 5.1]. The EXPTIME-complete bimodal logic with B modality $\Box$ and universal modality $[U]$ can be translated into RNIL by replacing $\Box$ by $[\mathrm{sim}(\mathrm{C}_1 \cup -\mathrm{C}_1)]$ and $[U]$ by $[\mathrm{sim}(\mathrm{C}_1 \cap -\mathrm{C}_1)]$. Let us establish the EXPTIME upper bound. Lemma 4.1 and Lemma 5.1 imply that every RNIL-formula ϕ is RNIL-satisfiable iff $\mathcal{A}_\phi$ accepts

at least one tree. Since $\mathrm{card}(\mathrm{SYMB}(\phi)) \leq |\phi| \times 2^{4|\phi|}$, $\mathcal{A}_\phi$ has $2^{\mathcal{O}(|\phi|)}$ states. Moreover, $\mathrm{card}(\delta)$ is in $2^{\mathcal{O}(|\phi|^2)}$ and checking whether $\langle q, a, q_1, \ldots, q_K \rangle \in \delta$ can be done in time $2^{\mathcal{O}(|\phi|)}$ (using Lemma 3.2). Consequently, computing $\mathcal{A}_\phi$ requires time in $2^{\mathcal{O}(|\phi|^4)}$. Since the nonemptiness problem for Büchi tree automata of the form $\mathcal{A}_\phi$ can be checked in time $\mathcal{O}(|\delta|^2)$, RNIL-satisfiability can be checked in time $2^{\mathcal{O}(|\phi|^4)}$. □

EXPTIME-hardness of RNIL holds even for its restriction with a unique parameter variable and modalities $[\mathrm{sim}(\mathrm{C}_1 \cup -\mathrm{C}_1)]$ and $[\mathrm{sim}(\mathrm{C}_1 \cap -\mathrm{C}_1)]$.

6. Concluding remarks

On the basis of existing automata-theoretic techniques for logical problems, we have shown that the newly introduced logic RNIL, the relative version of the logic NIL introduced by Orłowska, Pawlak and Vakarelov has an EXPTIME-complete satisfiability problem. This new logic combines two ingredients that are rarely present in information logics: its semantic structures include several families of relations parameterized by the subsets of attributes and, moreover, both local constraints (i.e., the constraints on individual relations) and global constraints (the constraints on the subfamilies of a family of relations) are assumed for their relations. The proof is by a reduction to the nonemptiness problem for Büchi automata on infinite trees combining advantageously the distinct proof techniques developed in [4, 8]. By contrast, the decidability status of RNIL augmented with object nominals is a challenging open question.

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