

# From local to global temporal logics over Mazurkiewicz traces

Volker Diekert<sup>a</sup>, Paul Gastin<sup>b,\*</sup>

<sup>a</sup>*Institut für Formale Methoden der Informatik (FMI), Universität Stuttgart, Universitätsstr. 38, D-70569 Stuttgart, Germany*

<sup>b</sup>*Laboratoire Spécification et Vérification (LSV), ENS de Cachan & CNRS 61, Av. du Président Wilson, F-94235 Cachan Cedex, France*

The paper is dedicated to the 60th birthday of Christian Choffrut

## Abstract

We review some results on global and local temporal logic on Mazurkiewicz traces. Our main contribution is to show how to derive the expressive completeness of global temporal logic with respect to first-order logic [V. Diekert, P. Gastin, LTL is expressively complete for Mazurkiewicz traces, *J. Comput. System Sci.* 64 (2002) 396–418] from the similar result on local temporal logic [V. Diekert, P. Gastin, Pure future local temporal logics are expressively complete for Mazurkiewicz traces, in: M. Farach-Colton (Ed.), *Proc. LATIN'04, Lecture Notes in Computer Science*, Vol. 2976, Springer, Berlin, 2004, pp. 232–241, Full version available as Research Report LSV-05-22, Laboratoire Spécification et Vérification, ENS Cachan, France].

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## 1. Introduction

Trace theory has a long history in computer science. It started with the classical works of Keller [16] and Mazurkiewicz [17,18]. Contributions to trace theory include combinatorial properties, formal languages, automata, and logic. Christian Choffrut participated in the development of trace theory in all these areas with papers [2–4,6–8] and his survey in [5]. In the present paper, we focus on linear temporal logics which have received quite an attention, see [1,19–27]. In [9], we have shown that a pure future linear temporal logic is powerful enough to express all first-order properties, if a global semantics is used, where the formulae are interpreted at configurations. This means traces are viewed as labelled partial orders and the interpretation is done along cuts as e.g., in [25]. The drawback of this approach is however that the satisfiability problem and the model checking problem become non-elementary, [26]. On the other hand, a local interpretation of linear temporal logics at vertices leads to polynomially space bounded satisfiability and model checking algorithms. In fact, all local temporal logics over traces where the modalities are definable in monadic second-order logic are decidable in PSPACE [14]. Therefore, local logics are much more of interest. In [11], we were able to prove that a pure future local linear temporal logic is also powerful enough to express all first-order properties.

The contribution here is a direct translation of local temporal logic formulae into equivalent global temporal logic formulae. From this translation, using the expressive completeness of local temporal logics with respect to first-order logic [11] we get as a corollary the expressive completeness of global temporal logics, which is the main result of [9].

\* Corresponding author.

*E-mail addresses:* [Volker.Diekert@fmi.uni-stuttgart.de](mailto:Volker.Diekert@fmi.uni-stuttgart.de) (V. Diekert), [Paul.Gastin@lsv.ens-cachan.fr](mailto:Paul.Gastin@lsv.ens-cachan.fr) (P. Gastin).

Actually, we strengthen the result of [9] in two respects. First, we show that we can use the basic *next* modality ( $\text{EX } \varphi$ ) instead of the more precise  $\langle a \rangle \varphi$  modality to get the expressive completeness of global temporal logics. Second, we show that this expressive completeness can be obtained using *robust* global formulae whose evaluation does not depend on whether the semantics of *until* allows a finite or an infinite prefix in the factorization.

## 2. Mazurkiewicz traces

We recall some standard notations from trace theory which will be used in the paper. A *dependence alphabet* is a pair  $(\Sigma, D)$ , where the alphabet  $\Sigma$  is a finite set and the *dependence relation*  $D \subseteq \Sigma \times \Sigma$  is reflexive and symmetric. The *independence relation*  $I$  is the complement of  $D$ . For  $A \subseteq \Sigma$ , the set of letters dependent on  $A$  is denoted by  $D(A) = \{b \in \Sigma \mid (a, b) \in D \text{ for some } a \in A\}$ .

A *Mazurkiewicz trace* is an equivalence class of a labelled partial order  $t = [V, \leq, \lambda]$ , where  $V$  is a set of vertices labelled by  $\lambda : V \rightarrow \Sigma$  and  $\leq$  is a partial order over  $V$  satisfying the following conditions: For all  $x \in V$ , the downward set  $\downarrow x = \{y \in V \mid y \leq x\}$  is finite, and for all  $x, y \in V$  we have that  $(\lambda(x), \lambda(y)) \in D$  implies  $x \leq y$  or  $y \leq x$ , and that  $x < y$  implies  $(\lambda(x), \lambda(y)) \in D$ , where  $< = < \setminus <^2$  is the immediate successor relation in  $t$ . For  $x \in V$ , we also define  $\uparrow x = \{y \in V \mid x \leq y\}$  and  $\uparrow\uparrow x = \{y \in V \mid x < y\}$ .

The trace  $t$  is finite if  $V$  is finite. We denote the set of finite traces by  $\mathbb{M}(\Sigma, D)$  (or simply  $\mathbb{M}$ ). By  $\mathbb{R}(\Sigma, D)$  (or simply  $\mathbb{R}$ ), we denote the set of finite or infinite traces (also called *real traces*). We write  $\text{alph}(t) = \lambda(V)$  for the alphabet of  $t$  and we let  $\text{alphinf}(t) = \{a \in \Sigma \mid \lambda^{-1}(a) \text{ is infinite}\}$  be the set of letters occurring infinitely often in  $t$ .

We define the concatenation for traces  $t_1 = [V_1, \leq_1, \lambda_1]$  and  $t_2 = [V_2, \leq_2, \lambda_2]$ , provided  $\text{alphinf}(t_1) \times \text{alph}(t_2) \subseteq I$ . It is given by  $t_1 \cdot t_2 = [V, \leq, \lambda]$ , where  $V$  is the disjoint union of  $V_1$  and  $V_2$ ,  $\lambda = \lambda_1 \cup \lambda_2$ , and  $\leq$  is the transitive closure of the relation  $\leq_1 \cup \leq_2 \cup (V_1 \times V_2 \cap \lambda^{-1}(D))$ . The set  $\mathbb{M}$  of finite traces is then a monoid with the empty trace  $1 = (\emptyset, \emptyset, \emptyset)$  as unit. If we can write  $t = rs$ , then  $r$  is a prefix of  $t$  and  $s$  is a suffix of  $t$ .

We denote by  $\min(t)$  the set of minimal vertices of  $t$ . We let  $\mathbb{R}^1 = \{t \in \mathbb{R} \mid |\min(t)| = 1\}$  be the set of traces with exactly one minimal vertex.

A trace  $p$  is called a *prime*, if it is finite and has a unique maximal element. The set of all primes in  $\mathbb{R}$  is denoted by  $\mathbb{P}$ . We have  $\mathbb{P} \subseteq \mathbb{M}$ , whereas  $\mathbb{R}^1$  contains infinite traces (if  $\Sigma \neq \emptyset$ ).

We also use  $\min(t)$  for the set  $\lambda(\min(t))$  of labels of the minimal vertices of  $t$ , and similarly for  $\max(t)$ . What we actually mean is always clear from the context. If  $t = [V, \leq, \lambda] \in \mathbb{R}$  is a real trace and  $x \in V$  is a vertex then we also write  $x \in t$  instead of  $x \in V$ .

## 3. Local temporal logics

The syntax of the basic linear temporal logic  $\text{LTL}_\Sigma$  is given by

$$\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \text{EX } \varphi \mid \varphi \text{ U } \varphi,$$

where  $a$  ranges over  $\Sigma$  and  $\top$  denotes *true*. As usual, we use  $\text{F } \varphi$  (*future* or *eventually*  $\varphi$ ) as an abbreviation for  $\top \text{ U } \varphi$  and  $\text{G } \varphi = \neg \text{F } \neg\varphi$  (*globally* in the sense of *always*  $\varphi$ ).

Here,  $\text{EX}$  denotes the usual (existential) *next*-operator and  $\text{U}$  means *until*. For (non-empty) finite or infinite words there is a standard semantics, and we have the following classical results:

The notions of first-order definability, star-freeness, aperiodicity, and  $\text{LTL}_\Sigma$ -definability lead to the same class of formal languages.

These results have been generalized in a sequence of papers to traces, [9,11–13,15,25]. However, the situation for traces is more complex. Traces are labelled partial orders and hence there are (at least) two natural semantics for  $\text{LTL}_\Sigma$ . We can define a *local* and a *global* semantics of linear temporal logics, and these semantics are quite different. In the following, we review some of our results and the reference logic for us is always first-order logic. We do not repeat its definition (which is the usual one and can be found e.g., in [13]), because we never work explicitly with it. Our results rather relate expressive powers of different temporal logics to each other.

### 3.1. Local semantics

Let  $t = [V, \leq, \lambda] \in \mathbb{R}$  be a non-empty real trace and  $x \in t$  be a vertex. The local semantics of  $\text{LTL}_\Sigma$  is defined by

$$\begin{aligned} t, x \models \top, \\ t, x \models a & \quad \text{if } \lambda(x) = a, \\ t, x \models \neg\varphi & \quad \text{if } t, x \not\models \varphi, \\ t, x \models \varphi \vee \psi & \quad \text{if } t, x \models \varphi \text{ or } t, x \models \psi, \\ t, x \models \text{EX } \varphi & \quad \text{if } \exists y (x < y \text{ and } t, y \models \varphi), \\ t, x \models \varphi \cup \psi & \quad \text{if } \exists z (x \leq z \text{ and } t, z \models \psi \text{ and } \forall y (x \leq y < z) \Rightarrow t, y \models \varphi). \end{aligned}$$

Together with the local semantics  $\text{LTL}_\Sigma$  is denoted by  $\text{LocTL}_\Sigma[\text{EX}, \cup]$  henceforth. This semantics is called *local* since a formula is evaluated at some vertex  $x$  of  $t$  which corresponds to the occurrence of a local event of the concurrent behavior represented by  $t$ .

Note that the temporal logics  $\text{LocTL}_\Sigma[\text{EX}, \cup]$  is pure future, i.e., whether  $t, x \models \varphi$  holds or not only depends on the suffix of  $t$  defined by  $\uparrow x$  consisting of the events in  $t$  which are in the future of  $x$ . Formally, we have  $t, x \models \varphi$  if and only if  $\uparrow x, x \models \varphi$ . Therefore, we could also define the semantics  $t \models' \varphi$  for traces  $t \in \mathbb{R}^1$  only omitting the vertex  $x$  which is implicitly the minimal vertex of  $t$ , i.e.,  $t \models' \varphi$  if and only if  $t, \min(t) \models \varphi$  for  $t \in \mathbb{R}^1$ . For instance, we would have  $t \models' \varphi \cup \psi$  if there exists  $z \in t$  such that  $\uparrow z \models' \psi$  and for all  $y \in t$ ,  $y < z$  implies  $\uparrow y \models' \varphi$ . We draw the attention to this alternative definition because the corresponding one will be more convenient for the global semantics. Hence, this remark should help linking the two definitions.

In the following proofs, we will use mainly some fragments of  $\text{LocTL}_\Sigma[\text{EX}, \cup]$ . To introduce these fragments, we first need some notations. For  $x \in t$  and  $c \in \Sigma$ , we denote by  $x_c$  the unique minimal vertex of  $\uparrow x \cap \lambda^{-1}(c)$  if it exists, i.e., if  $\uparrow x \cap \lambda^{-1}(c) \neq \emptyset$ . Note that  $x < x_c$  if  $x_c$  exists.

We consider the local temporal logic  $\text{LocTL}_\Sigma[(X_a \leq X_b), \text{XU}_a]$  the syntax of which is given by

$$\varphi ::= \top \mid a \mid (X_a \leq X_b) \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \text{XU}_a \varphi,$$

where  $a, b$  range over  $\Sigma$ . The semantics of  $\text{LocTL}_\Sigma[(X_a \leq X_b), \text{XU}_a]$  is defined as follows:

$$\begin{aligned} t, x \models (X_a \leq X_b) & \quad \text{if } x_a, x_b \text{ exist and } x_a \leq x_b \\ t, x \models \varphi \text{XU}_a \psi & \quad \text{if } \exists z (x < z \text{ and } \lambda(z) = a \text{ and } t, z \models \psi \text{ and } \forall y (x < y < z \text{ and } \lambda(y) = a) \Rightarrow t, y \models \varphi). \end{aligned}$$

It is shown in [11] that  $\text{LocTL}_\Sigma[(X_a \leq X_b), \text{XU}_a]$  is indeed a fragment of  $\text{LocTL}_\Sigma[\text{EX}, \cup]$ . The main results of [11] can be stated now as follows:

**Theorem 1** (Diekert and Gastin [11]). *Let  $L \subseteq \mathbb{R}$  be a first-order definable real trace language. Then there are formulae  $\varphi \in \text{LocTL}_\Sigma[\text{EX}, \cup]$  and  $\psi \in \text{LocTL}_\Sigma[(X_a \leq X_b), \text{XU}_a]$  such that*

$$L \cap \mathbb{R}^1 = \{t \in \mathbb{R}^1 \mid t, \min(t) \models \varphi\} = \{t \in \mathbb{R}^1 \mid t, \min(t) \models \psi\}.$$

**Theorem 2** (Diekert and Gastin [11]). *Let  $L \subseteq \mathbb{R}$  be a real trace language and let  $\#$  be a new symbol ( $\# \notin \Sigma$ ) which depends on all letters of  $\Sigma$ . Then the following assertions are equivalent:*

- (1) *The language  $L$  is first-order definable.*
- (2) *There is a formula  $\varphi \in \text{LocTL}_\Sigma[\text{EX}, \cup]$  such that*

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

- (3) *There is a formula  $\varphi \in \text{LocTL}_\Sigma[(X_a \leq X_b), \text{XU}_a]$  such that*

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

### 3.2. Global semantics

Let  $t \in \mathbb{R}$  be a possibly empty real trace. The global semantics of the linear temporal logic  $\text{LTL}_\Sigma$  is defined by

$$\begin{aligned} t \models_g \top, \\ t \models_g a & \quad \text{if } t \in a\mathbb{R}, \\ t \models_g \neg \varphi & \quad \text{if } t \not\models_g \varphi, \\ t \models_g \varphi \vee \psi & \quad \text{if } t \models_g \varphi \text{ or } t \models_g \psi, \\ t \models_g \text{EX } \varphi & \quad \text{if } \exists a \in \Sigma, s \in \mathbb{R} : t = as \text{ and } s \models_g \varphi, \\ t \models_g \varphi \cup \psi & \quad \text{if } \exists r \in \mathbb{M}, s \in \mathbb{R} : t = rs \text{ and } s \models_g \psi \text{ and } \forall r', r'' \in \mathbb{R}, (r = r'r'' \text{ and } r'' \neq 1) \Rightarrow r''s \models_g \varphi. \end{aligned}$$

Analogously to the above, together with the global semantics we denote this logic by  $\text{GlobTL}_\Sigma[\text{EX}, \cup]$  henceforth.

Alternatively, the global semantics could define more generally when  $t, r \models_g \varphi$ , where  $t \in \mathbb{R}$  is a (possibly empty) real trace and  $r$  is a prefix of  $t$  corresponding to a partial execution. Note that  $r$  may have several maximal events and the corresponding suffix  $s$  (defined by  $t = rs$ ) may have several minimal events. Hence, the factorization  $t = rs$  defines a *global cut* in the behavior  $t$ . This is why we call this semantics *global*.

Since our logic  $\text{LTL}_\Sigma$  is *pure future*, the truth value of  $t, r \models_g \varphi$  only depends on the suffix  $s$ : formally, we have  $t, r \models_g \varphi$  if and only if  $s \models_g \varphi$ . Since we only deal with pure future logics, in this paper, we have chosen to omit the prefix  $r$  simplifying the definition of the semantics.

For sake of completeness, let us discuss a subtle point. It is not clear that the choice to define  $t \models_g \varphi \cup \psi$  as done above is the only natural one for traces. We required the first factor  $r$  to be finite, and we did it this way in order to find the usual semantics in case of infinite words, but as an alternative one could consider a semantics defined as follows:

$$t \models_g \varphi \cup \psi \text{ if } \exists r \in \mathbb{R}, s \in \mathbb{R} : t = rs \text{ and } s \models'_g \psi \text{ and } \forall r', r'' \in \mathbb{R}, (r = r'r'' \text{ and } r'' \neq 1) \Rightarrow r''s \models'_g \varphi.$$

We get stronger and more convenient results, if we do not need to pay attention what choice has been taken. To make this formal let us inductively define *robust* formulae

$$\begin{aligned} \top & \quad \text{is robust,} \\ a & \quad \text{is robust (for } a \in \Sigma), \\ \neg \varphi & \quad \text{is robust if } \varphi \text{ is robust,} \\ \varphi \vee \psi & \quad \text{is robust if } \varphi \text{ and } \psi \text{ are robust,} \\ \text{EX } \varphi & \quad \text{is robust if } \varphi \text{ is robust,} \\ \varphi \cup \psi & \quad \text{is robust if } \varphi \text{ and } \psi \text{ are robust, and } \forall t \in \mathbb{R}, t \models_g \varphi \cup \psi \iff t \models'_g \varphi \cup \psi. \end{aligned}$$

Note that  $Fa$  is robust, whereas  $F\neg a$  is not robust since  $a^\omega \models'_g F\neg a$  but  $a^\omega \not\models_g F\neg a$ . Actually,  $t \models'_g F\neg a$  for all trace  $t \in \mathbb{R}$ , whereas  $t \models_g F\neg a$  if and only if  $t = a^k s$  for some  $k \geq 0$  and  $s \in \mathbb{R}$  with  $a \notin \min(s)$ .

Clearly, if  $\varphi$  is robust then  $t \models_g \varphi$  if and only if  $t \models'_g \varphi$ .

In [9], another type of global formulae has been used. For  $a \in \Sigma$  and a formula  $\varphi$  the global semantics of  $\langle a \rangle \varphi$  (read *next- $a$ - $\varphi$* ) is defined by

$$t \models_g \langle a \rangle \varphi \quad \text{if } \exists s \in \mathbb{R} : t = as \text{ and } s \models_g \varphi.$$

It was not noticed in [9] that the modality  $\langle a \rangle (-)$  can be expressed in  $\text{GlobTL}_\Sigma[\text{EX}, \cup]$ . So this is our first contribution.

**Lemma 3.** *A language  $L \subseteq \mathbb{R}$  is expressible in  $\text{GlobTL}_\Sigma[\text{EX}, \cup]$  if and only if it is expressible in  $\text{GlobTL}_\Sigma[\langle a \rangle, \cup]$ .*

*Moreover, if  $\varphi$  is robust, then  $\langle a \rangle \varphi$  can be expressed by some robust formula in  $\text{GlobTL}_\Sigma[\text{EX}, \cup]$ .*

**Proof.** Obviously, EX can be defined by  $\text{EX } \varphi = \bigvee_{a \in \Sigma} \langle a \rangle \varphi$ . For the other direction, we show that the formula  $\langle a \rangle \varphi$  can be expressed using EX and  $\varphi$ . In particular,  $\langle a \rangle \varphi$  is replaced by a robust formula, if  $\varphi$  is robust.

First, for a given  $\varphi \in \text{GlobTL}_\Sigma$  let  $m \in \mathbb{N}$  be such that for all  $t \in \mathbb{R}$  and  $a \in \Sigma$  we have  $a^m t \models_g \varphi$  if and only if  $a^{m+1} t \models_g \varphi$ . The existence of  $m$  is ensured by the fact that  $\{t \in \mathbb{R} \mid t \models_g \varphi\}$  is aperiodic by [13] (see also [10]). Here, we give a direct proof of this fact.

We define inductively  $m(\varphi)$  by  $m(\top) = 0$ ,  $m(b) = 1$  for  $b \in \Sigma$ ,  $m(\neg\varphi) = m(\varphi)$ ,  $m(\text{EX } \varphi) = 1 + m(\varphi)$  and  $m(\varphi \vee \psi) = m(\varphi \cup \psi) = \max(m(\varphi), m(\psi))$ . We show by structural induction on  $\varphi$  that for all  $n \geq m(\varphi)$  and all  $t \in \mathbb{R}$  and  $a \in \Sigma$ , we have  $a^n t \models_g \varphi$  if and only if  $a^{n+1} t \models_g \varphi$ .

The result is clear for  $\varphi = \top$ . For  $\varphi = b$ , we use the fact that for  $n \geq 1$ ,  $a^n t \models_g b$  if and only if  $a = b$  or  $b \in \min(t)$  and  $(a, b) \in I$ . The induction is trivial for negation and disjunction. Assume now that  $a^n t \models_g \text{EX } \varphi$  with  $n \geq 1 + m(\varphi)$ . We have  $a^n t = bs$  with  $b \in \Sigma$  and  $s \models_g \varphi$ . If  $a = b$  then  $s = a^{n-1} t \models_g \varphi$  and we get  $a^n t \models_g \varphi$  by induction. It follows that  $a^{n+1} t \models_g \text{EX } \varphi$ . Now, if  $a \neq b$  then  $(a, b) \in I$  and we have  $s = a^n r$  and  $t = br$ . By induction, we get  $a^{n+1} r \models_g \varphi$  and therefore  $a^{n+1} t = ba^{n+1} r \models_g \text{EX } \varphi$ . Hence, we have shown that  $a^n t \models_g \text{EX } \varphi$  implies  $a^{n+1} t \models_g \text{EX } \varphi$ . The converse implication can be shown similarly.

Finally, assume that  $a^n t \models_g \varphi \cup \psi$  and write  $a^n t = rs$  with  $s \models_g \psi$  and  $r'' s \models_g \varphi$  for all  $r' r'' = r$  with  $r'' \neq 1$ . If  $a$  is independent of  $r$  then  $s = a^n s'$  and we get by induction that  $as = a^{n+1} s' \models_g \psi$  and  $r'' as = a^{n+1} r'' s' \models_g \varphi$  for all  $r' r'' = r$  with  $r'' \neq 1$ . Hence in this case  $a^{n+1} t \models_g \varphi \cup \psi$ . Now, if  $a$  depends on  $r$  then  $a \leq r$  and  $a^{n+1} t = (ar)s$ . Let  $r' r'' = ar$  with  $r'' \neq 1$ . If  $r' = ar_1$  then  $r = r_1 r''$  and we get  $r'' s \models_g \varphi$ . Otherwise  $a$  is independent of  $r'$  and  $r'' = ar_2$ . We obtain  $r = r' r_2$  and  $a \leq r_2 \neq 1$ . Therefore,  $r_2 s \models_g \varphi$  and since  $a^n \leq r_2 s$  we get also  $r'' s = ar_2 s \models_g \varphi$ . Finally, we have shown  $a^{n+1} t \models_g \varphi \cup \psi$ . The converse can be shown similarly.

Now, for each  $k \geq 1$  we define a macro  $a^k$  by

$$a^k = a \wedge \bigwedge_{1 \leq \ell < k} \neg \text{EX}^\ell \neg a.$$

Note that  $a^k$  is a robust formula. We show that  $t \models_g a^k$  if and only if  $t \in a^k \mathbb{R}$  by induction on  $k$ . If  $k = 1$  then the result is clear. Let now  $k > 1$  and assume that  $t \models_g a^k$ . Note that  $a^k = a^{k-1} \wedge \neg \text{EX}^{k-1} \neg a$ . By induction, we deduce that  $t = a^{k-1} s$  with  $s \in \mathbb{R}$ . If  $s \notin a \mathbb{R}$  then  $s \models_g \neg a$  and  $t = a^{k-1} s \models_g \text{EX}^{k-1} \neg a$ , a contradiction. Hence  $s \in a \mathbb{R}$  and  $t \in a^k \mathbb{R}$ .

Conversely, if  $t \in a^k \mathbb{R}$  then we have  $t \models_g a^{k-1}$  by induction and it remains to show that  $t \models_g \neg \text{EX}^{k-1} \neg a$ . We have  $t \models_g \text{EX}^{k-1} \neg a$  if and only if there is a factorization  $t = rs$  with  $|r| = k - 1$  and  $s \notin a \mathbb{R}$ . This is indeed impossible if  $t \in a^k \mathbb{R}$ .

To conclude the proof of the lemma, we show that  $\langle a \rangle \varphi$  is equivalent to the robust formula

$$(a^{m+1} \wedge \varphi) \vee \bigvee_{1 \leq k \leq m} (a^k \wedge \text{EX}(\neg a^k \wedge \varphi)).$$

Assume first that  $t \models_g \langle a \rangle \varphi$ . If  $t = a^{m+1} s \in a^{m+1} \mathbb{R}$  then we have  $a^m s \models_g \varphi$  and we get  $t = a^{m+1} s \models_g \varphi$  by definition of  $m$ . In this case,  $t \models_g a^{m+1} \wedge \varphi$ . Now, assume that  $t = a^k s \in a^k \mathbb{R}$  for some  $k \leq m$  and  $s \in \mathbb{R} \setminus a \mathbb{R}$ . Then  $t \models_g a^k$  and  $a^{k-1} s \models_g \varphi \wedge \neg a^k$ .

Conversely, assume that  $t \models_g a^{m+1} \wedge \varphi$  then  $t = a^{m+1} s \models_g \varphi$  and by definition of  $m$  we get  $a^m s \models_g \varphi$  and  $t \models_g \langle a \rangle \varphi$ . Finally, assume that  $t \models_g a^k \wedge \text{EX}(\varphi \wedge \neg a^k)$  for some  $1 \leq k \leq m$ . We have  $t \in a^k \mathbb{R}$  and  $t = bs$  with  $b \in \Sigma$ ,  $s \models_g \varphi$  and  $s \notin a^k \mathbb{R}$ . We deduce that  $b = a$  and  $t \models_g \langle a \rangle \varphi$ .  $\square$

With the help of Lemma 3 the main result of [9] can be stated as follows:

**Theorem 4** (Diekert and Gastin [9]). *A language  $L \subseteq \mathbb{R}$  is first-order definable if and only if it is expressible in  $\text{GlobTL}_\Sigma[\text{EX}, \text{U}]$ .*

#### 4. From local to global logic

The proofs in [9,11] are both very complex and technical. They are independent of each other, but one has the impression that the local result Theorem 2 is stronger than the global one in Theorem 4. We can confirm this impression by the following proposition. In the following,  $\#$  denotes again a new symbol (not in  $\Sigma$ ) which depends on all letters of  $\Sigma$ .

**Proposition 5.** Let  $c \in \Sigma' = \Sigma \cup \{\#\}$  and  $\varphi \in \text{LocTL}_\Sigma[(X_a \leq X_b), XU_a]$ . Then we can effectively construct a robust formula  $\overline{\varphi}^c \in \text{GlobTL}_\Sigma[\langle a \rangle, U]$  such that for all  $t \in \mathbb{R}$  we have

$$ct, \min_c(ct) \models \varphi \quad \text{if and only if} \quad t \models_g \overline{\varphi}^c$$

where  $\min_c(ct)$  is the minimal vertex of  $ct$  which is labelled  $c$ .

So, if we are willing to use Theorem 2, we get as a corollary a strengthening of Theorem 4 since the expressibility is obtained with *robust* formulae.

**Corollary 6.** A language  $L \subseteq \mathbb{R}$  is first-order definable if and only if it is expressible by some robust formula in  $\text{GlobTL}_\Sigma[EX, U]$ .

**Proof.** Actually, we get as a corollary only the difficult part of the equivalence. So let  $L \subseteq \mathbb{R}$  be first-order definable. By Theorem 2, we find a local formula  $\varphi \in \text{LocTL}_\Sigma[(X_a \leq X_b), XU_a]$  such that

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

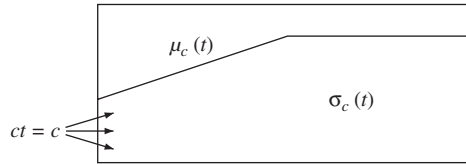
Let  $\overline{\varphi}^\# \in \text{GlobTL}_\Sigma[\langle a \rangle, U]$  be the *robust* formula given by Proposition 5. We get

$$L = \{t \in \mathbb{R} \mid t \models_g \overline{\varphi}^\#\}.$$

It remains to apply Lemma 3 to get a *robust* formula from  $\text{GlobTL}_\Sigma[EX, U]$  defining  $L$ .  $\square$

The remaining of this section is devoted to the proof of Proposition 5. The construction of  $\overline{\varphi}^c$  is done by structural induction. Clearly, we have  $\overline{\top}^c = \perp$ ,  $\overline{\neg\varphi}^c = \neg\overline{\varphi}^c$ , and  $\overline{\varphi \vee \psi}^c = \overline{\varphi}^c \vee \overline{\psi}^c$ . For  $a \in \Sigma$ , we define  $\overline{a}^c = \top$ , if  $a = c$ , and  $\overline{a}^c = \perp$ , if  $a \neq c$ . The translation for  $(X_a \leq X_b)$  and  $XU_a$  is much more involved and will use some auxiliary macros.

Since we deal with pure future logics, we have already noticed that whether  $ct, \min_c(ct) \models \varphi$  only depends on the future of  $\min_c(ct)$  in the trace  $ct$ . Hence, we define  $\mu_c(t)$  and  $\sigma_c(t)$  by the equation  $t = \mu_c(t)\sigma_c(t)$  with  $c$  independent of  $\mu_c(t)$  and  $\min(\sigma_c(t)) \subseteq D(c)$ . Note that  $c\sigma_c(t) \in \mathbb{R}^1$  is the future of  $\min_c(ct)$ . Therefore,  $ct, \min_c(ct) \models \varphi$  if and only if  $c\sigma_c(t), \min(c\sigma_c(t)) \models \varphi$ .



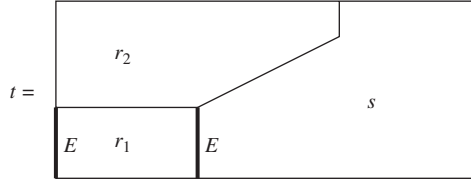
It is easier to give a formula for  $(X_a \leq X_b)$  if the first  $a$  in  $t$  coincide with the first  $a$  in  $ct$  which is above  $x = \min_c(ct)$  and similarly for  $b$ . This is the case if and only if  $a, b \notin \text{alph}(\mu_c(t))$ . In order to reduce the general case to this simpler case, we will introduce a macro  $(\text{shift}_{c,E}(-))$  which allows to skip an arbitrary finite prefix of  $\mu_c(t)$ . In the special case of  $(X_a \leq X_b)$ , we will use this macro to skip all  $a$ 's and  $b$ 's contained in  $\mu_c(t)$ . Skipping a prefix that might be arbitrarily long requires some *until* formula. Since we would like not to skip any vertex from  $\sigma_c(t)$  we insist that the set  $E$  of minimal letters depending on  $c$  remains constant along the until move. This requirement does not ensure that no vertex from  $\sigma_c(t)$  will be skipped but it is powerful enough for our purposes.

We start by defining some abbreviations. For  $A \subseteq \Sigma$ , we define the macro  $(\min = A) \in \text{GlobTL}_\Sigma$  by  $\bigwedge_{a \in A} a \wedge \bigwedge_{a \notin A} \neg a$  so that for all  $t \in \mathbb{R}$  we have  $t \models_g (\min = A)$  if and only if  $\min(t) = A$ . We will also use macros  $(\min \cap D(a) = A)$  for  $a \in \Sigma$  and  $A \subseteq \Sigma$  with the obvious meanings and definitions. Note that all these macros are robust. For a formula  $\alpha \in \text{GlobTL}_\Sigma$  and a subalphabet  $E \subseteq \Sigma$ , we define the global formula

$$\text{shift}_{c,E}(\alpha) = (\min \cap D(c) = E) \cup (\alpha \wedge (\min \cap D(c) = E)).$$

**Lemma 7.** For all  $t \in \mathbb{R}$ , we have  $t \models_g \text{shift}_{c,E}(\alpha)$  if and only if  $t = r_1 r_2 s$  with  $r_1 r_2$  finite,  $s \models_g \alpha$ ,  $\text{alph}(r_1) \subseteq E$ ,  $r_2$  independent of  $E \cup \{c\}$  and  $E = \min(s) \cap D(c)$ .

Moreover,  $\text{shift}_{c,E}(\alpha)$  is robust if the following two conditions hold:  $\alpha$  is robust and  $s \models_g \alpha$  implies  $rs \models_g \alpha$  for  $r, s \in \mathbb{R}$  with  $r$  independent of  $s$  and  $c$ .



**Proof.** Assume first that  $t \models_g \text{shift}_{c,E}(\alpha)$  then we write  $t = rs$  with  $r$  finite,  $s \models_g \alpha$  and  $\min(r''s) \cap D(c) = E$  for all  $r'r'' = r$ . Using  $r'' = 1$ , we get  $E = \min(s) \cap D(c)$ . There is a unique factorization  $r = r_1r_2$  with  $\text{alph}(r_1) \subseteq E$  and  $\min(r_2) \cap E = \emptyset$ . It remains to show that  $r_2$  is independent of  $E$  and  $c$ . Assume that  $r_2 = r'_2br''_2$  with  $r'_2$  independent of  $E \cup \{c\}$  and  $r'_2b$  prime. We first show that  $b \notin E$ . If  $r'_2 = 1$  then  $b \in \min(r_2)$  hence  $b \notin E$ . Otherwise, let  $a \in \max(r'_2)$ . Since  $(a, b) \in D$ , we have  $b \notin \min(abr''_2s)$  and  $E \subseteq \min(abr''_2s)$ , hence again  $b \notin E$ . Now, since  $b \notin E = \min(br''_2s) \cap D(c)$ , we deduce that  $b$  is independent of  $E \cup \{c\}$ .

Conversely, let  $t = r_1r_2s$  with  $r_1r_2$  finite,  $s \models_g \alpha$ ,  $\text{alph}(r_1) \subseteq E$ ,  $r_2$  independent of  $E \cup \{c\}$  and  $E = \min(s) \cap D(c)$ . We have to show that if  $r'r'' = r_1r_2$  then  $E = \min(r''s) \cap D(c)$ . By Levi's lemma,  $r'r'' = r_1r_2$  implies  $r' = r'_1r'_2$ ,  $r'' = r''_1r''_2$ ,  $r_1 = r'_1r''_1$  and  $r_2 = r'_2r''_2$ . Since  $E \subseteq \min(s)$ , the letters in  $E$  are pairwise independent and using  $\text{alph}(r_1) \subseteq E$ , we get  $E = \min(r'_1s) \cap D(c)$ . Using in addition  $r_2$  independent of  $E \cup \{c\}$  we obtain  $E = \min(r''s) \cap D(c)$ . This proves the first part of the claim.

Assume now that  $\alpha$  is robust and  $s \models_g \alpha$  implies  $rs \models_g \alpha$  for  $r, s \in \mathbb{R}$  with  $r$  independent of  $s$  and  $c$ . Let  $t \in \mathbb{R}$  be such that  $t \models_g \text{shift}_{c,E}(\alpha)$  and write  $t = rs$  with  $s \models_g \alpha$  and  $\min(r''s) \cap D(c) = E$  for all  $r'r'' = r$ . Since  $\alpha$  is robust, we get  $s \models_g \alpha$ . Now, write  $r = r'r''$  with  $r'$  finite and  $\text{alph}(r'') = \text{alphinf}(r)$ . Then,  $r''$  is independent of  $s$ . Now, if  $b \in \text{alph}(r'')$  then there is a suffix  $br'''$  of  $r$  and we obtain  $\{b\} \cap D(c) \subseteq \min(br'''s) \cap D(c) = E$ . Since  $b$  is independent of  $s$  and  $E \subseteq \min(s)$ , we deduce that  $b \notin D(c)$ . Therefore,  $r''s \models_g \alpha$  and obtain easily  $t = r'(r''s) \models_g \text{shift}_{c,E}(\alpha)$ .  $\square$

Now, for  $a \in \Sigma$ , we let

$$(a \in \mu_c) = \bigvee_{E \mid a \notin E} \text{shift}_{c,E}(a),$$

where the disjunction ranges over all subsets  $E \subseteq \Sigma$  such that  $a \notin E$ .

**Lemma 8.** For all  $t \in \mathbb{R}$ , we have  $t \models_g (a \in \mu_c)$  if and only if  $a \in \text{alph}(\mu_c(t))$ . Moreover, the formula  $(a \in \mu_c)$  is robust.

**Proof.** Let  $t \in \mathbb{R}$  with  $a \in \text{alph}(\mu_c(t))$  and let  $E = \min(t) \cap D(c) = \min(\sigma_c(t))$ . We have  $\mu_c(t)$  independent of  $E \cup \{c\}$ . In particular,  $a \notin E$ . We can write  $\mu_c(t) = r_2r'$  with  $r_2$  finite and  $a \in \min(r')$ . Then, with  $s = r'\sigma_c(t)$  we have  $a \in \min(s)$  and  $\min(s) \cap D(c) = E$ . Hence,  $t = r_2s \models_g \text{shift}_{c,E}(a)$  by Lemma 7 with  $r_1 = 1$ .

Conversely, assume that  $t \models_g \text{shift}_{c,E}(a)$  for some  $E$  with  $a \notin E$ . Let  $t = r_1r_2s$  be the factorization given by Lemma 7. From  $E = \min(s) \cap D(c)$ ,  $a \in \min(s)$  and  $a \notin E$ , we deduce that  $a$  is independent of  $E \cup \{c\}$ . Using in addition that  $r_2$  is independent of  $E \cup \{c\}$  and  $\text{alph}(r_1) \subseteq E$  we infer that the trace  $r_2a$  is a prefix of  $\mu_c(t)$  and we get  $a \in \text{alph}(\mu_c(t))$  as desired.

Finally,  $\text{shift}_{c,E}(a)$  is robust since  $\alpha = a$  satisfies the additional requirement of Lemma 7 for robustness.  $\square$

We deal now with  $(X_a \leq X_b)$ . If  $a = b \in D(c)$  then we let  $\overline{(X_a \leq X_b)}^c = F a$  and if  $a \neq b$  or if  $a = b \notin D(c)$  then we let

$$\overline{(X_a \leq X_b)}^c = \bigvee_{E \mid b \notin E} \text{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c)),$$

where  $\zeta = (\neg b) \cup (a \wedge (b \notin \mu_a) \wedge F b)$ .

**Lemma 9.** *The formula  $\overline{(X_a \leq X_b)^c}$  satisfies the requirement of Proposition 5.*

**Proof.** In all this proof, we let  $x = \min_c(ct)$ .

If  $a = b \in D(c)$  then  $a \notin \text{alph}(\mu_c(t))$ , hence  $a \in \text{alph}(t)$  if and only if  $ct, \min_c(ct) \models (X_a \leq X_a)$ . Moreover, the formula  $F_a$  is robust.

Next, we consider the special case where  $a, b \notin \text{alph}(\mu_c(t))$ . We show that for all  $t \in \mathbb{R}$  such that  $a, b \notin \text{alph}(\mu_c(t))$ , we have  $t \models_g \zeta$  if and only if  $ct, x \models (X_a \leq X_b)$ .

Assume first that  $a, b \notin \text{alph}(\mu_c(t))$  and  $t \models_g \zeta$ . We can write  $t = rs$  with  $a \in \min(s)$  and  $b \in \text{alph}(s) \setminus \text{alph}(\mu_a(s))$  and  $b \notin \text{alph}(r)$ . We have  $a, b \in \text{alph}(t) \setminus \text{alph}(\mu_c(t))$ . We deduce that  $x_a$  and  $x_b$  exist. Let  $y \in \min(s)$  with  $\lambda(y) = a$ . Since  $a \notin \text{alph}(\mu_c(t))$  we deduce that  $x < y$  and therefore  $x_a \leq y$ . Now,  $b \notin \text{alph}(r) \cup \text{alph}(\mu_a(s))$ , hence  $x_b$  is in  $\sigma_a(s)$  and we get  $y = \min(\sigma_a(s)) \leq x_b$ .

Conversely, note that if  $a, b \notin \text{alph}(\mu_c(t))$  then  $x_a$  and  $x_b$  are the first vertices labelled  $a$  and  $b$ , respectively, in the trace  $t$ . So assume that  $a, b \notin \text{alph}(\mu_c(t))$  and  $x_a \leq x_b$  exist. We can write  $t = rs$  with  $ra$  prime,  $a \notin \text{alph}(r)$  and  $a \in \min(s)$ . Since  $a \notin \text{alph}(\mu_c(t))$ , we deduce that  $x_a \in \min(s)$  and therefore  $\{x_a\} = \min(\sigma_a(s))$ . From  $x_a \leq x_b$  we deduce that  $x_b$  is in  $\sigma_a(s)$  and therefore  $b \in \text{alph}(s)$ . Also,  $b \notin \text{alph}(\mu_a(s))$  since  $x_b$  is the first vertex labelled  $b$  in  $t$ . We have thus shown  $s \models_g a \wedge (b \notin \mu_a) \wedge F_b$ . Finally, let  $r = r'r''$  with  $r'' \neq 1$ . We have  $x_a \in r''s$  and the vertex  $x_a$  is not minimal in  $r''s$  since  $ra$  is prime and  $r'' \neq 1$ . Then  $b \notin \min(r''s)$  since  $x_a \leq x_b$  and  $x_b$  is the first vertex of  $t$  which is labelled  $b$ . Therefore,  $t \models_g \zeta$  as desired.

We show now that  $\zeta$  is robust. Let  $t \in \mathbb{R}$  with  $t \models'_g \zeta$  and write  $t = rs$  the associated factorization. Since  $a \in \min(s)$ , we can write  $r = r_1r_2$  with  $r_1$  finite and  $r_2$  independent of  $a$ . We have  $\mu_a(r_2s) = r_2\mu_a(s)$  and  $b \notin \text{alph}(r)$ , hence  $b \in \text{alph}(r_2s) \setminus \text{alph}(\mu_a(r_2s))$ . Also,  $a \in \min(r_2s)$  and we deduce easily that  $t = r_1(r_2s) \models_g \zeta$ .

For the general case, note that if  $a \neq b$  or  $a = b \notin D(c)$  then  $ct, x \models (X_a \leq X_b)$  implies  $b \notin \min(t) \cap D(c)$ . Hence, it is enough to show that for all  $t \in \mathbb{R}$  and  $E \subseteq \Sigma$  with  $b \notin E$ , we have  $E = \min(t) \cap D(c)$  and  $ct, x \models (X_a \leq X_b)$  if and only if  $t \models_g \text{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$ .

Let  $t \in \mathbb{R}$  and  $E \subseteq \Sigma$  with  $b \notin E$ . Assume that  $E = \min(t) \cap D(c)$  and that  $x_a, x_b$  exist and  $x_a \leq x_b$ . We can write  $\mu_c(t) = r_2r'$  with  $r_2$  finite and  $\text{alph}(r') = \text{alphinf}(\mu_c(t))$ . Since  $x_a, x_b$  exist, we have  $a, b \in \text{alph}(\sigma_c(t))$ . Using  $\sigma_c(t)$  independent of  $\text{alphinf}(\mu_c(t))$  we deduce that  $a, b \notin \text{alph}(r')$ . With  $s = r'\sigma_c(t)$ , we have  $\mu_c(s) = r'$  and  $\sigma_c(s) = \sigma_c(t)$ . Hence,  $x = \min_c(cs)$  and  $cs, x \models (X_a \leq X_b)$ . Moreover,  $a, b \notin \text{alph}(\mu_c(s))$  and we deduce that  $s \models_g \zeta$  from the special case above. Hence, we have shown  $s \models_g \zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c)$ . Now, we have  $\mu_c(t)$  (hence also  $r_2$ ) independent of  $E \cup \{c\}$ . Also,  $E = \min(t) \cap D(c) = \min(\sigma_c(t)) = \min(\sigma_c(s)) = \min(s) \cap D(c)$ . Using Lemma 7 with  $r_1 = 1$ , we obtain as desired  $t = r_2s \models_g \text{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$ .

Conversely, let  $t \in \mathbb{R}$  be such that  $t \models_g \text{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$  for some  $E$  with  $b \notin E$ . Let  $t = r_1r_2s$  be the factorization given by Lemma 7. Note that  $E = \min(t) \cap D(c)$ . Let  $y = \min_c(cs)$ . Since  $a, b \notin \text{alph}(\mu_c(s))$ , we deduce from the special case above that  $y_a, y_b$  exist and  $y_a \leq y_b$ . We have  $\sigma_c(t) = r_1\sigma_c(s)$ , hence  $x_a, x_b$  exist and  $x_a \leq y_a$ . Using  $b \notin E$  and  $\text{alph}(r_1) \subseteq E$ , we obtain  $x_b = y_b$  and therefore,  $x_a \leq x_b$ .

Finally, the formula  $\alpha = \zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c)$  is robust. Let  $r, s \in \mathbb{R}$  with  $s \models_g \alpha$  and  $r$  independent of  $s$  and  $c$ . We have  $\mu_c(rs) = r\mu_c(s)$  and  $\sigma_c(rs) = \sigma_c(s)$ . Since  $s \models_g \zeta$  we have  $a, b \in \text{alph}(s)$  and using  $r$  independent of  $s$  and  $a, b \notin \text{alph}(\mu_c(s))$  we deduce that  $a, b \notin \text{alph}(\mu_c(rs))$ . Now, using the special case we have  $s \models_g \zeta$  if and only if  $\sigma_c(s) \models_g \zeta$  and  $rs \models_g \zeta$  if and only if  $\sigma_c(rs) \models_g \zeta$ . Since  $\sigma_c(s) = \sigma_c(rs)$  we deduce that  $rs \models_g \zeta$ . Therefore,  $\alpha$  satisfies the additional requirement of Lemma 7 for robustness and we deduce that  $\text{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$  is robust.  $\square$

It remains to deal with  $\varphi XU_a \psi$ . We define

$$\overline{\varphi XU_a \psi^c} = (\xi \wedge (a \notin \mu_c)) \vee \bigvee_{E|a \notin E} \text{shift}_{c,E}(\xi \wedge (a \notin \mu_c)),$$

where  $\xi = (\neg a \vee \langle a \rangle \overline{\varphi^a}) \cup \langle a \rangle \overline{\psi^a}$ .

**Lemma 10.** *The formula  $\overline{\varphi XU_a \psi^c}$  satisfies the requirement of Proposition 5.*

**Proof.** We first show that for all  $t \in \mathbb{R}$  such that  $a \notin \text{alph}(\mu_c(t))$ , we have  $t \models_g \zeta$  if and only if  $ct, x \models \varphi XU_a \psi$ , where  $x = \min_c(ct)$  is the minimal vertex of  $ct$  which is labelled  $c$ .

Assume first that  $a \notin \text{alph}(\mu_c(t))$  and  $t \models_g \xi$ . We can write  $t = ras$  with  $s \models_g \bar{\psi}^a$  and for all  $r'r'' = r$  with  $r'' \neq 1$ ,  $r''as \models_g (\neg a \vee \langle a \rangle \bar{\varphi}^a)$ . Let  $z = \min_a(as)$  be the minimal vertex of  $as$  which is labelled  $a$ . By induction, we get  $as, z \models \psi$ . Since  $a \notin \text{alph}(\mu_c(t))$ , we have  $z \in \sigma_c(t)$  and  $x < z$ . Therefore  $ct, z \models \psi$ . Now, let  $x < y < z$  with  $\lambda(y) = a$ . Then  $y \in r$  and we have a factorization  $r = r'ar''$  with  $y = \min_a(ar'') = \min_a(ar''as)$ . Since  $ar'' \neq 1$  and  $ar''as \models_g a$ , we get  $r''as \models_g \bar{\varphi}^a$ . By induction we obtain  $ar''as, y \models \varphi$  and therefore  $ct, y \models \varphi$ . We have thus shown that  $ct, x \models \varphi \text{XU}_a \psi$ .

Conversely, assume that  $a \notin \text{alph}(\mu_c(t))$  and  $ct, x \models \varphi \text{XU}_a \psi$ . We have to show that  $t \models_g \xi$ . Let  $z > x$  with  $\lambda(z) = a$  and  $ct, z \models \psi$  and for all  $x < y < z$  with  $\lambda(y) = a$  we have  $ct, y \models \varphi$ . We can write  $t = ras$  with  $ra$  prime and  $z$  being the maximal vertex of  $ra$ . Since  $ct, z \models \psi$  we also have  $as, z \models \psi$  and by induction we get  $s \models_g \bar{\psi}^a$ . Now, let  $r'r'' = r$  with  $r'' \neq 1$ . By definition of  $r$ ,  $z$  is not minimal in  $r''as$ . Assume that  $a \in \min(r''as)$  and let  $y = \min_a(r''as)$ . We have  $x < y < z$  since  $a \notin \text{alph}(\mu_c(t))$ . Therefore  $ct, y \models \varphi$  and also  $r''as, y \models \varphi$ . By induction, with  $r'' = ar'''$ , we obtain  $r'''as \models_g \bar{\varphi}^a$  and  $r''as \models_g \langle a \rangle \bar{\varphi}^a$ . Therefore,  $t \models_g \xi$ .

Next, we show that  $\xi$  is robust. Assume that  $t \models'_g \xi$  and write  $t = ras$  with  $s \models'_g \bar{\psi}^a$  and  $r''as \models'_g \neg a \vee \langle a \rangle \bar{\varphi}^a$  for all  $r'r'' = r$  with  $r \neq 1$ . Since  $\bar{\psi}^a$  is robust we also have  $s \models_g \bar{\psi}^a$ . We can write  $r = r_1r_2$  with  $r_1$  finite and  $r_2$  independent of  $a$ . We have  $s \models_g \bar{\psi}^a$  if and only if  $as, \min_a(as) \models \psi$  if and only if  $a\sigma_a(s), \min(a\sigma_a(s)) \models \psi$ . Since  $\sigma_a(s) = \sigma_a(r_2s)$  and  $s \models_g \bar{\psi}^a$ , we deduce  $r_2s \models_g \bar{\psi}^a$ . Also, for all  $r'r'' = r_1$  with  $r'' \neq 1$  we have  $r''ar_2s \models'_g \neg a \vee \langle a \rangle \bar{\varphi}^a$ , hence also  $r''ar_2s \models_g \neg a \vee \langle a \rangle \bar{\varphi}^a$  since  $\langle a \rangle \bar{\varphi}^a$  is robust. Therefore,  $t \models_g \xi$ .

Note that if  $a \in \text{alph}(\mu_c(t))$  then  $a \notin E = \min(t) \cap D(c)$ . Hence, it remains to show that for all  $t \in \mathbb{R}$ ,  $x = \min_c(ct)$  and  $E \subseteq \Sigma$  with  $a \notin E$ , we have  $E = \min(t) \cap D(c)$  and  $ct, x \models \varphi \text{XU}_a \psi$  if and only if  $t \models_g \text{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$ .

Let  $t \in \mathbb{R}$  and  $E \subseteq \Sigma$  with  $a \notin E$ . Assume that  $E = \min(t) \cap D(c)$  and that  $ct, x \models \varphi \text{XU}_a \psi$ . We can write  $\mu_c(t) = r_2r'$  with  $r_2$  finite and  $\text{alph}(r') = \text{alphinf}(\mu_c(t))$ . We have  $a \in \text{alph}(\sigma_c(t))$ . Using  $\sigma_c(t)$  independent of  $\text{alphinf}(\mu_c(t))$  we deduce that  $a \notin \text{alph}(r')$ . With  $s = r'\sigma_c(t)$ , we have  $\mu_c(s) = r'$  and  $\sigma_c(s) = \sigma_c(t)$ . Hence  $a \notin \text{alph}(\mu_c(s))$  and  $cs, x \models \varphi \text{XU}_a \psi$ . We deduce that  $s \models_g \xi$  from the special case above. We have thus shown  $s \models_g \xi \wedge (a \notin \mu_c)$ . Now, we have  $\mu_c(t)$  (hence also  $r_2$ ) independent of  $E \cup \{c\}$ . Also,  $E = \min(t) \cap D(c) = \min(\sigma_c(t)) = \min(\sigma_c(s)) = \min(s) \cap D(c)$ . Using Lemma 7 with  $r_1 = 1$ , we obtain as desired  $t = r_2s \models_g \text{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$ .

Conversely, let  $t \in \mathbb{R}$  be such that  $t \models_g \text{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$  for some  $E$  with  $a \notin E$ . Let  $t = r_1r_2s$  be the factorization given by Lemma 7. Note that  $E = \min(t) \cap D(c)$ . Since  $a \notin \text{alph}(\mu_c(s))$ , we deduce from the special case above that  $cs, \min_c(cs) \models \varphi \text{XU}_a \psi$ . We have  $\sigma_c(t) = r_1\sigma_c(s)$ . Using  $a \notin E$  and  $\text{alph}(r_1) \subseteq E$ , we obtain  $ct, x \models \varphi \text{XU}_a \psi$ .

Finally, we can show exactly as in the proof of Lemma 9 that  $\alpha = \xi \wedge (a \notin \mu_c)$  satisfies the additional requirement of Lemma 7 for robustness. Therefore, the formula  $\text{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$  is robust.  $\square$

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