On the Well-Foundedness of Adequate Orders Used for Construction of Complete Unfolding Prefixes

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Abstract

Petri net unfolding prefixes are an important technique for formal verification and synthesis of concurrent systems. In this paper we show that the requirement that the *adequate order* used for truncating a Petri net unfolding must be well-founded is superfluous in many important cases, i.e., it logically follows from other requirements. We give a complete analysis when this is the case. These results concern the very 'core' of the unfolding theory.

Key words: adequate order, unfolding prefix, Petri net, well-foundedness, concurrency

1. Introduction

McMillan's finite and complete prefixes of Petri net unfoldings [2,4,6] are a prominent technique for analysing the behaviour of reactive systems modelled by Petri nets. It alleviates the *state space explosion* problem, i.e., the problem that even a relatively small system specification can (and often does) have so many reachable states that the straightforward enumeration of them is infeasible. This technique relies on the partial order view of a concurrent computation.

A *finite and complete unfolding prefix* of a Petri net Ω is a finite acyclic net which implicitly represents all the reachable states of Ω together with transitions enabled at those states. Intuitively, it can be obtained through *unfolding* Ω, by successive firing of transitions, under the following assumptions: (i) for each new firing a fresh transition (called an *event*) is generated; (ii) for each newly produced token a fresh place (called a *condition*) is generated. Throughout the paper we will denote by h the labelling function mapping the conditions and events to the corresponding places and transitions, respectively, of the original Petri net.

Due to its structural properties (such as acyclicity), the reachable states of Ω can be represented using *configurations* of its unfolding. A configuration C is a finite downward-closed set of events (being downward-closed means that if $e \in C$ and f is a causal predecessor of e, denoted $f \prec e$, then $f \in C$) without *choices* (i.e., for all distinct events $e, f \in C$, there is no condition c in the unfolding such that the arcs (c, e) and (c, f) are in the unfolding). Intuitively, a configuration is a partially ordered execution, i.e., an execution where the order of firing of some of its events (viz. concurrent ones) is not important. We will denote by $[e]$ the *local* configuration of an event e, i.e., the smallest (w.r.t. set inclusion) configuration containing e (it is comprised of e and its causal predecessors). A finite set of events E is an *extension* of a configuration C if $C \cap E = \emptyset$ and $C \cup E$ is a configuration; in such a case the notation $C \oplus E$ will be used to denote the latter configuration.

The unfolding is infinite whenever the original Petri net has an infinite run; however, if the Petri net has finitely many reachable states then the unfolding

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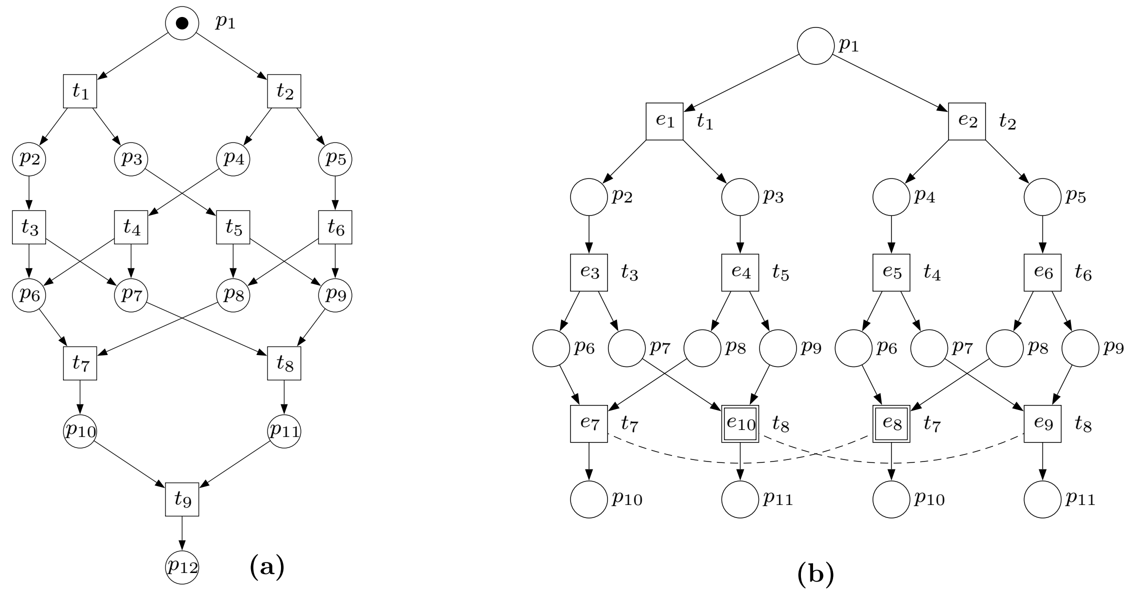


Fig. 1. A safe Petri net **(a)** and a prefix of its unfolding **(b)**.

eventually starts to repeat itself and can be truncated (by identifying a set of *cut-off* events) without loss of information, yielding a finite and complete prefix. Intuitively, an event e can be declared cut-off if the already built part of the prefix contains a configuration C^e (called the *corresponding* configuration of e) such that $Mark(C^e) = Mark([e])$ (where $Mark(C)$ denotes the final marking of a configuration C) and C^e is smaller than $[e]$ w.r.t. some well-founded partial order $\lhd$ on the configurations of the unfolding, called an *adequate order* [2,4]. The importance of the latter condition is illustrated by the example in Figure 1, which is taken from [2]. The marking $\{p_{12}\}$ is reachable in the Petri net in Figure 1(a). However, without using an adequate order, one can generate the prefix shown in Figure 1(b), in which this marking is not represented. (The numbers of the events indicate the order in which they were added to the prefix.) The events e_8 and e_{10} are marked as cut-offs, because the final markings of the corresponding local configurations are $\{p_7, p_9, p_{10}\}$ and $\{p_6, p_8, p_{11}\}$, which are also the final markings of $[e_7]$ and $[e_9]$, respectively. Although no events can now be added, the prefix is not complete, because $\{p_{12}\}$ is not represented in it.

Practical algorithms exist for building complete prefixes [2,4], which ensure that the number of non-cut-off events in such a prefix can never exceed the number of reachable states of the Petri net. However, complete prefixes are often exponentially smaller than the corresponding state graphs, especially for highly concurrent Petri nets, because they represent concurrency directly rather than by multidimensional 'diamonds' as it is done in state graphs. For example, if the original Petri net consists of 100 transitions which can fire once in parallel, the state graph will be a 100-dimensional hypercube with 2^{100} vertices, whereas the complete prefix will coincide with the net itself. In many applications, e.g., in asynchronous circuit design, the Petri net models usually exhibit a lot of concurrency, but have rather few choice points, and so their unfolding prefixes are often exponentially smaller than the corresponding state graphs; in fact, in many of the experiments conducted in [4] they are just slightly bigger then the original Petri nets themselves. Therefore, unfolding prefixes are well-suited for alleviating the state space explosion problem.

Well-foundedness of the adequate order used to truncate the unfolding is an important part of the completeness proof of [2,5]. In this paper, we show that *the requirement of well-foundedness is superfluous in many important cases.* More precisely, we show that in many cases the well-foundedness of the adequate order is implied by other requirements the adequate order must satisfy.

2. Basic Notions

First, we introduce several important definitions related to adequate orders. For convenience, their form has been slightly changed compared with [2,4], but they are easily seen to be equivalent.

Definition 1 (Structural isomorphism). *Two finite sets of events of the unfolding of a Petri net* Ω, E *and* E', *are* structurally isomorphic,[1] *denoted* $E \sim_s E'$, *if the labelled (by* h*) digraphs induced by these two sets of events and their adjacent conditions are isomorphic.*

Note that this definition essentially compares two subnets of the unfolding, induced by the sets of events E and E' and their adjacent conditions, taking also the labels into account.

Definition 2 (Preservation by finite extensions). *A strict partial order* $\lhd$ *on the finite configurations of the unfolding of a Petri net is* strongly (resp. weakly) preserved by finite extensions *if for every pair of configurations* C', C'' *such that* $Mark(C') = Mark(C'')$ *and* $C' \lhd C''$, *and for every finite extension* E'' *of* C'' *and every (resp. there exists a) finite extension* E' *of* C' *such that* $E' \sim_s E''$, *it holds that (resp. and)* $C' \oplus E' \lhd C'' \oplus E''$.

Definition 3 ((Pre-)adequate orders). *A strict partial order* $\lhd$ *on the finite configurations of the unfolding of a Petri net* Ω *is called* pre-adequate *if:*

- *it refines (strict) set inclusion* $\subset$, *i.e.,* $C' \subset C''$ *implies* $C' \lhd C''$;
- *it is weakly preserved by finite extensions.*

A pre-adequate order is called adequate *if it is well-founded.*

Note that the strong preservation by finite extensions is not required for the order to be adequate, since weak preservation is sufficient for the proof of completeness in [2,5]. However, the adequate orders used in practice are often strongly preserved by finite extensions, in particular this is the case for all the concrete adequate orders considered in [2].[2] We introduce the notion of strong preservation because some of the positive results in this paper hold only for strongly preserved adequate orders.

3. Well-foundedness of Pre-adequate Orders

We now proceed by describing in which cases the requirement of well-foundedness of the adequate order is superfluous, i.e., pre-adequate orders are automatically adequate. We consider, in turn, safe (1-bounded), bounded and general (unbounded) finite Petri nets. The proofs of all the results are postponed until Section 5.

3.1. *The Case of Safe Petri Nets*

The proposition below states that the well-foundedness requirement is superfluous for safe Petri nets.

Proposition 4 (The requirement of well-foundedness is superfluous for unfoldings of safe Petri nets). *A pre-adequate order on the configurations of the unfolding of a safe Petri net is adequate.*

3.2. *The Case of Bounded Petri Nets*

The case of bounded Petri nets differs from the previous case since the weak and the strong preservations by finite extensions no longer coincide, as illustrated by the following counterexample.

Counterexample 5 (The requirement of well–foundedness is not superfluous for unfoldings of bounded Petri nets in the case of weak preservation by finite extensions). *The pre-adequate order shown in Figure 2(c,d) is not a well-founded order on the configurations of the unfolding shown in Figure 2(b). Indeed, any finite execution starts by a series of firings of* a, *and then, optionally,* b *fires. When* b *fires,* p_2 *contains two tokens, and* b *can consume either of them; in the unfolding, the corresponding conditions and the instances of* b *can be easily distinguished. We denote for all* $n \geq 0$ *the finite configurations as* a^n, $a^n b$ *and* $a^n b'$.

Note that only the configurations of the form a^n *can be extended, either to* a^{n+k+1} *or* $a^{n+k}b$ *or* $a^{n+k}b'$, $k \geq 0$. *Suppose* $a^m \lhd a^n$ *(i.e.,* $m < n$*). If* a^n *is extended to* a^{n+k+1} *then we can extend (in a structurally isomorphic way)* a^m *to* a^{m+k+1}. *If* a^n *is extended to* $a^{n+k}b$, $k \geq 0$, *then we can extend (in a structurally isomorphic way)* a^m *to* $a^{m+k}b$, *and, by the definition of* $\lhd$, $a^{m+k}b \lhd a^{n+k}b$. *If* a^n *is extended to* $a^{n+k}b'$, $k \geq 0$, *then we can extend (in a structurally isomorphic way)* a^m *to* $a^{m+k}b$, *and, by the definition of* $\lhd$, $a^{m+k}b \lhd a^{n+k}b'$. *Hence,* $\lhd$ *is weakly preserved by finite extensions. However,* $\lhd$ *is not well-founded due to* $a^0b' \rhd a^1b' \rhd a^2b' \rhd \cdots$.

Remark 6. *In several seminal papers on unfoldings, like [1], the initial marking is assumed to be safe, i.e., it should contain at most one token on each place. The net of Figure 2(***a***) does not satisfy this*

[1] Such an isomorphism is used in [2] without a formal definition. It turns out that there are several alternative 'natural' isomorphisms which can be used; we discuss some of them in Section 4.

[2] One of those orders is defined only on configurations of unfoldings of safe Petri nets, for which the strong and weak preservation by finite extensions coincide.

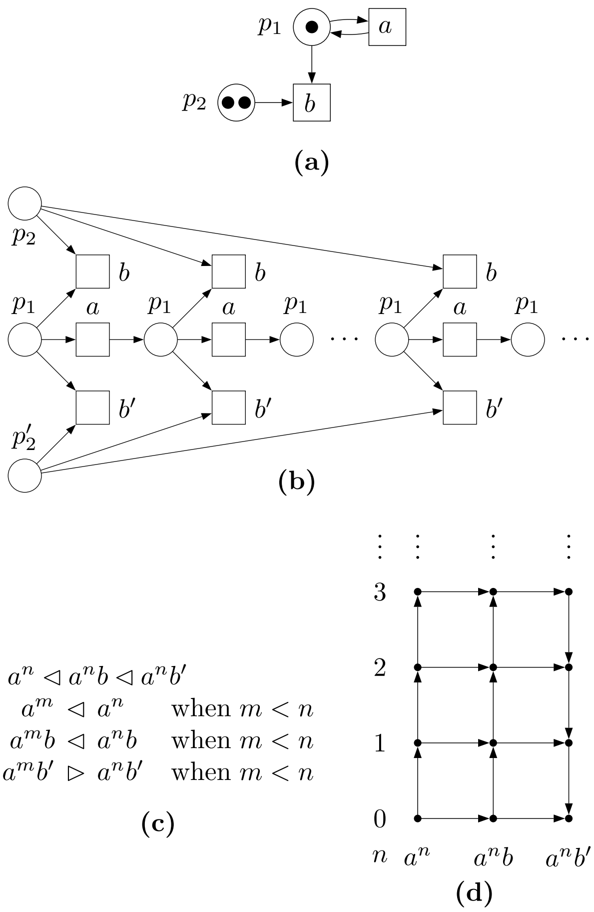
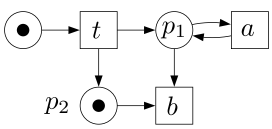


Fig. 2. A 2-bounded Petri net **(a)**, its unfolding **(b)** and an order on its configurations **(c, d)**.

requirement. Nevertheless, it is easy to adapt this counterexample as follows:



This net starts by firing t*, which leads to the same marking as in Figure 2***(a)***. Denote* C_0 *the initial configuration. The configuration that is reached after firing* t *corresponds to* a^0 *in Figure 2. We still denote it* a^0*, and re-use the notations* a^n*,* $a^n b$ *and* $a^n b'$ *as before. The order* $\lhd$ *is also re-used, and extended with* $C_0 \lhd a^0$*.*

Proposition 7 (The requirement of well-foundedness is superfluous for unfoldings of bounded Petri nets in the case of strong preservation by finite extensions). *If a pre-adequate order* $\lhd$ *on the finite configurations of the unfolding of a bounded Petri net is strongly preserved by finite extensions then* $\lhd$ *is adequate.*

3.3. *The Case of Unbounded Petri Nets*

We complete our analysis by considering the case of general (unbounded) Petri nets. This case might be less interesting in practice, since the complete prefixes of unbounded nets are infinite. However, this case is interesting from the theoretical point of view. Moreover, [4] shows that a finite and complete prefix of an unbounded nets can be obtained if instead of the equivalence of final markings a coarser equivalence is used to compare the configurations in the cut-off criterion.

The definition of the preservation by finite extensions (Definition 2) requires that $\lhd$ is only preserved by extensions of configurations reaching the same markings. The counterexample below shows that in this case the requirement of well-foundedness is not superfluous.

Counterexample 8 (The requirement of well–foundedness is not superfluous for unfoldings of unbounded Petri nets). *Consider Figure 3. The finite configurations of the unfolding have the form either* a^n *or* $a^n b$*, where* n *ranges over the set of integers. The shown order is pre-adequate, as it refines the set inclusion and it is trivially preserved by finite extensions of configurations reaching the same marking, since* no two configurations reach the same marking. *However,* $\lhd$ *is not well-founded due to* $a^0 b \rhd a^1 b \rhd a^2 b \rhd \cdots$*.*

Technically, this counterexample settles the case of unbounded Petri nets. However, one can observe that this negative result holds due to the trivial reason that it is possible to construct an unbounded Petri net such that in its unfolding no two configurations have the same final marking. Hence, it seems reasonable to strengthen the assumptions about the pre-adequate order in the unbounded case, by requiring that $\lhd$ is preserved not only by configurations that reach the same marking, but also each time isomorphic finite extensions can be added to two comparable configurations.

Definition 9 (Extendible pre-adequate order). *A pre-adequate order* $\lhd$ *on the finite configurations of the unfolding of a Petri net* Ω *is called* extendible *if for all configurations* C' *and* C'' *such that* $C' \lhd C''$*, and for all finite extensions* E' *and* E'' *of* C' *and* C''*, respectively, such that* $E' \sim_s E''$*, it holds that* $C' \oplus E' \lhd C'' \oplus E''$*.*

Note that extendible pre-adequate orders are

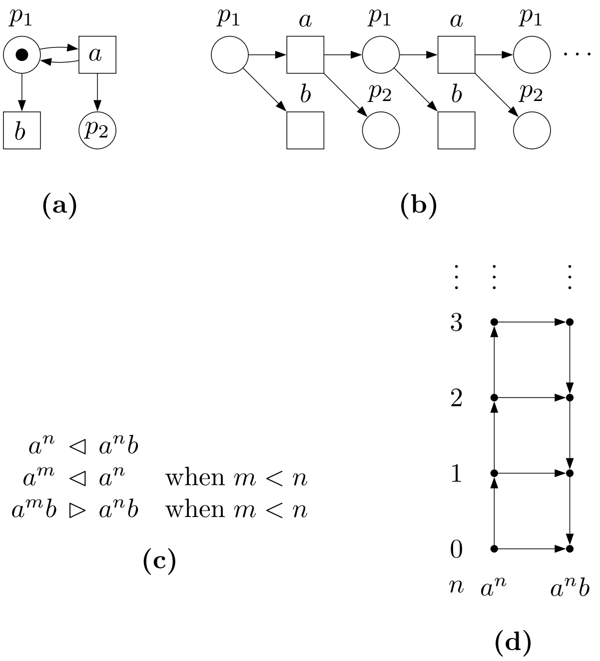


Fig. 3. An unbounded Petri net **(a)**, its unfolding **(b)** and an order on its configurations **(c, d)**.

strongly preserved by finite extensions. The proposition below shows that a positive result can be obtained in the case of an extendible pre-adequate order.

Proposition 10 (The requirement of well-foundedness is superfluous for unfoldings of unbounded Petri nets in the case of an extendible order). *An extendible pre-adequate order $\lhd$ on the finite configurations of the unfolding of a (possibly unbounded) Petri net is adequate.*

4. Summary and Further Considerations

Our results are summarised in Table 1, where ✓ means that the requirement of well-foundedness is superfluous, and × means that it is not superfluous. Moreover, we now show that these results are *robust*, i.e., they are not affected if an alternative notion of preservation of $\lhd$ by extensions is used, or if $\sim_s$ is replaced by a different isomorphism.

4.1. *Single-Event Extensions*

In practice it is often convenient to consider single-event extensions, i.e., extensions of cardinality one. One can easily show by induction on the size of the extensions that strong preservation by single-event

	weak preservation	strong preservation	extendible order
safe	✓ (Prop. 4)		
bounded	× (Counterex. 5)	✓ (Prop. 7)	
unbounded	× (Counterex. 8)		✓ (Prop. 10)

Table 1
Summary of results.

extension coincides with strong preservation by finite extensions, and so Propositions 7 and 10 still hold for single-event extensions. On the other hand, weak preservation by single event extensions is even weaker than weak preservation by finite extensions, and so Counterexamples 5 and 8 also hold for weak preservation by single-event extensions.

Moreover, one can easily show that for safe Petri nets, weak preservation by single-event extensions is equivalent to strong preservation by single-event extensions (which, in turn, is equivalent to weak or strong preservation by finite extensions), and so Proposition 4 holds for single-event extensions as well.

To summarise, using single-event extensions instead of finite ones does not change our results.

4.2. *Other Isomorphisms*

So far, we considered the structural isomorphism, $\sim_s$, which is in a sense strongest possible, as it takes the full structure of the net into account. Below we consider other natural isomorphisms, which are coarser then $\sim_s$.

Definition 11 (Pomset-isomorphism and Parikh-isomorphism). *Let E and E' be two finite sets of events of the unfolding of a Petri net Ω.*

- *E and E' are* pomset-isomorphic*, denoted $E \sim_p E'$, if the labelled (by h) digraphs induced by these two sets of events in the digraph corresponding to the causality relation on the events of the unfolding are isomorphic.*
- *E and E' are* Parikh-isomorphic*, denoted $E \sim_\# E'$, if for every transition t of Ω, $\#_t E = \#_t E'$, where $\#_t E$ denotes the number of instances of t in E.*

Note that $\sim_s$ refines $\sim_p$, which in turn refines $\sim_\#$, i.e., $E' \sim_s E'' \Rightarrow E' \sim_p E'' \Rightarrow E' \sim_\# E''$. Moreover, one can observe that if $\sim_1$ and $\sim_2$ are two isomorphisms such that $\sim_2$ refines $\sim_1$ then:

- weak preservation w.r.t. $\sim_1$ is even weaker than weak preservation w.r.t. $\sim_2$ (i.e., there exists an E' such that $E' \sim_1 E''$ and $C' \oplus E' \lhd C'' \oplus E''$, but maybe $E' \not\sim_2 E''$);

– strong preservation w.r.t. $\sim_1$ is even stronger than strong preservation w.r.t. $\sim_2$ (for all E' such that $E' \sim_1 E''$, $C' \oplus E' \lhd C'' \oplus E''$, even for such E' that $E' \not\sim_2 E''$).

Consequently, Counterexample 5, as well as Propositions 7 and 10, still hold for $\sim_p$ and $\sim_\#$. Moreover, since in the case of safe Petri nets it is enough to consider only single-event extensions, and $\sim_s$, $\sim_p$ and $\sim_\#$ coincide on such extensions, Proposition 4 holds for either of these isomorphisms. Finally, one can observe that Counterexample 8 still holds for $\sim_p$ and $\sim_\#$.

To summarise, using $\sim_p$ or $\sim_\#$ (or any other isomorphism refining $\sim_\#$ and refined by $\sim_s$) instead of $\sim_s$ does not change our results.

5. Proofs

The main result of the paper is stated in Proposition 14; all the propositions in Section 3 easily follow from it. Its proof is organised in two steps.

– First, we show that if there is an infinite descending sequence of configurations $C_1 \rhd C_2 \rhd \cdots$, then there are two indices $i < j$ such that C_i and C_j can be decomposed as $D \oplus E$ and $D' \oplus E'$, respectively, with $D \lhd D'$ and $E \leftharpoondown E'$, where $\leftharpoondown$ is a relation defined below. For this we use Lemma 12 and Higman's lemma (explained below).
– Then Lemma 13 states that $D \oplus E \lhd D' \oplus E'$, leading to a contradiction.

The following lemma states basically that from an infinite sequence of distinct configurations one can always extract an infinite subsequence of configurations 'stringed' on some infinite causal chain p of events $e_1 \prec e_2 \prec \cdots$ of the unfolding, in the sense that the configurations of this subsequence have increasing intersections with p.

Lemma 12 (Stringing lemma). *For every infinite sequence $C_1, C_2, \ldots$ of distinct configurations of the unfolding of a finite Petri net, there exists an infinite causal chain of events $e_1 \prec e_2 \prec \cdots$ and extensions $E_1, E_2, \ldots$ of $[e_1], [e_2], \ldots$ respectively, such that for every $n > 0$, $[e_n] \oplus E_n = C_{k_n}$ for some integers $0 < k_1 < k_2 < \cdots$. Moreover one can impose that $e_{n+1} \notin E_n$ and $|E_1| \le |E_2| \le \cdots$.*

Proof. The events in the union of the configurations $C_1, C_2, \ldots$ together with their adjacent conditions induce (in the graph-theoretical sense) an infinite branching process, and the analog of König's lemma for branching processes [5] states that it has an infinite causal chain of events $e''_1 \prec e''_2 \prec \cdots$.

For each $i > 0$, let $x_i \stackrel{\mathrm{df}}{=} \max\{n > 0 \mid e''_n \in C_i\}$ be the index of the last event in the chain belonging to C_i. The sequence $x_1, x_2, \ldots$ is unbounded because for every n, e''_n belongs to some configuration C_i, which implies that $x_i \ge n$. Therefore, one can extract an infinite increasing subsequence $x_{i_1} < x_{i_2} < \cdots$, where $0 < i_1 < i_2 < \cdots$. Let $e'_n \stackrel{\mathrm{df}}{=} e''_{x_{i_n}}$; then $e'_n \in C_{i_n}$ (and thus $[e'_n] \subseteq C_{i_n}$) and $e'_{n+1} \notin C_{i_n}$. Let $E'_n \stackrel{\mathrm{df}}{=} C_{i_n} \setminus [e'_n]$, i.e., $e'_{n+1} \notin E'_n$ and $[e'_n] \oplus E'_n = C_{i_n}$.

Finally, from the sequence of non-negative integers $|E'_1|, |E'_2|, \ldots$ an infinite non-decreasing subsequence can always be extracted, i.e., there exist integers $0 < j_1 < j_2 < \cdots$ such that $|E'_{j_1}| \le |E'_{j_2}| \le \cdots$.

Now for every $n > 0$ we take $k_n \stackrel{\mathrm{df}}{=} i_{j_n}$, $e_n \stackrel{\mathrm{df}}{=} e'_{j_n}$ and $E_n \stackrel{\mathrm{df}}{=} E'_{j_n}$. We have $0 < k_1 < k_2 < \cdots$, $e_1 \prec e_2 \prec \cdots$, $|E_1| \le |E_2| \le \cdots$, and for every $n > 0$, $e_{n+1} \notin E_n$ and $[e_n] \oplus E_n = C_{k_n}$. □

From now on we assume that $\lhd$ is a pre-adequate order on the configurations of the unfolding of a finite Petri net. Let Σ be some finite alphabet and $\sigma(C, e)$ be a mapping that assigns a letter from Σ to each pair (C, e), where C is a configuration and e is an event that extends C, satisfying: $\forall C_1, e_1, C_2, e_2$:

$$\left.\begin{matrix} C_1 \lhd C_2 \\ \sigma(C_1, e_1) = \sigma(C_2, e_2) \end{matrix}\right\} \Rightarrow C_1 \oplus \{e_1\} \lhd C_2 \oplus \{e_2\} \quad (*)$$

Let E be a set of events of the unfolding of a finite Petri net. A *linearisation* of E is a sequence of events $u = e_1, \ldots, e_{|E|}$ such that $\{e_1, \ldots, e_{|E|}\} = E$ and for all $i, j \in \{1, \ldots, |E|\}$, if $e_i \prec e_j$ then $i < j$. Note that any linearisation of a configuration C is an execution of the unfolding; moreover, a linearisation of $C \oplus E$ can be obtained by concatenating any linearisations of C and E.

For a linearisation $u = e_1, \ldots, e_n$ of a set of events, we define $W_\sigma(u)$ as the word $a_1 \ldots a_n$ with $a_i \stackrel{\mathrm{df}}{=} \sigma(\{e_1, \ldots, e_{i-1}\}, e_i)$.

Given two words $u = u_1 \ldots u_m$ and $v = v_1 \ldots v_n$ over the same alphabet, u is a *subword* of v if there exist integers $0 < j_1 < \cdots < j_m \le n$ such that for every $i \in \{1, \ldots, m\}$, $u_i = v_{j_i}$.

Given two sets of events E and E', we write $E \leftharpoondown E'$ if there are linearisations u and u' of E and E', respectively, such that $W_\sigma(u)$ is a subword of $W_\sigma(u')$. Note that in general $\leftharpoondown$ is *not* an order (it is not transitive).

The following lemma generalises the preservation of $\lhd$ by finite extensions. It applies to extensions that are not necessarily isomorphic.

Lemma 13. *Let D and D' be two configurations, and E and E' be extensions of D and D', respectively. If $D \lhd D'$ and $E \leftharpoondown E'$, then $D \oplus E \lhd D' \oplus E'$.*

Proof. Let $u = e_1 \ldots e_{|u|}$ and $u' = e'_1 \ldots e'_{|u'|}$ be linearisations of E and E', respectively, such that $W_\sigma(u) = a_1 \ldots a_{|u|}$ is a subword of $W_\sigma(u') = a'_1 \ldots a'_{|u'|}$. Then there exist $0 = j_0 < j_1 < \cdots < j_{|u|} \leq |u'|$ such that for all $i \in \{1, \ldots, |u|\}$, $a_i = a'_{j_i}$, i.e.,

$$\begin{aligned} W_\sigma(u') &= \ldots\ a'_{j_1}\ \ldots\ a'_{j_2}\ \ldots\ a'_{j_{|u|}}\ \ldots \\ &= \ldots\ a_1\ \ldots\ a_2\ \ldots\ a_{|u|}\ \ldots \end{aligned}$$

We define $C_i \stackrel{\text{df}}{=} D \oplus \{e_1, \ldots, e_i\}$ and $C'_i \stackrel{\text{df}}{=} D' \oplus \{e'_1, \ldots, e'_i\}$. Starting from $C_0 = D \lhd D' = C'_{j_0}$, we show by induction on i that $C_i \lhd C'_{j_i}$ for all $i \leq |u|$, which gives $C_{|u|} \lhd C'_{j_{|u|}} \subseteq C'_{|u'|}$, i.e., $D \oplus E \lhd D' \oplus E'$. We get the inductive step as follows: if $C_{i-1} \lhd C'_{j_{i-1}}$ then $C_{i-1} \lhd C'_{j_i-1}$ due to $C'_{j_{i-1}} \subseteq C'_{j_i-1}$; moreover $\sigma(C_{i-1}, e_i) = a_i = a'_{j_i} = \sigma(C'_{j_i-1}, e'_{j_i})$, so $C_{i-1} \oplus \{e_i\} \lhd C'_{j_i-1} \oplus \{e'_{j_i}\}$, i.e., $C_i \lhd C'_{j_i}$. □

The proposition below is the central result of this paper. Its proof makes use of Higman's lemma [3,7], stating that for every infinite sequence $W_1, W_2, \ldots$ of finite words over a finite alphabet, there are indices $i \neq j$ such that W_i is a subword of W_j.

Proposition 14 (Well-foundedness of $\lhd$). *Let $\lhd$ be a pre-adequate order on configurations of the unfolding of a finite Petri net, Σ be a finite alphabet and σ be a mapping satisfying $(*)$. Then $\lhd$ is well-founded (and hence adequate).*

Proof. For the sake of contradiction, suppose that $\lhd$ is not well-founded, i.e., there is an infinite descending sequence $C_1 \rhd C_2 \rhd \cdots$. Since $\lhd$ is a strict order, for all $i \neq j$, $C_i \neq C_j$. By Lemma 12, there exists an infinite causal chain of events $e_1 \prec e_2 \prec \cdots$ and extensions $E_1, E_2, \ldots$ with non-decreasing cardinalities, such that for every $n > 0$, $[e_n] \oplus E_n = C_{k_n}$ for some integers $0 < k_1 < k_2 < \cdots$.

For each $n > 0$, let u_n be an arbitrary linearisation of the events of E_n. The $W_\sigma(u_n)$'s form an infinite set of finite words over the finite alphabet Σ. By Higman's lemma, there are integers $i \neq j$ such that $W_\sigma(u_i)$ is a subword of $W_\sigma(u_j)$. As the $W_\sigma(u_n)$'s have non-decreasing length, we can assume that $i < j$. Then we have $e_i \prec e_j$, which implies $[e_i] \subset [e_j]$ and $[e_i] \lhd [e_j]$. By Lemma 13, $[e_i] \oplus E_i \lhd [e_j] \oplus E_j$, i.e., $k_i < k_j$ and $C_{k_i} \lhd C_{k_j}$, a contradiction. □

We are now in a position to prove all the results announced in Section 3. One can observe that Proposition 7, stating that the requirement of well-foundedness is superfluous for unfoldings of bounded Petri nets in the case of strong preservation by finite extensions, follows from Proposition 14 if we take $\Sigma \stackrel{\text{df}}{=} \mathcal{RM} \times T$ and $\sigma(C, e) \stackrel{\text{df}}{=} (\mathit{Mark}(C), h(e))$, where $\mathcal{RM}$ is the set of reachable markings of the Petri net and T is the set of its transitions. Moreover, since for safe Petri nets weak preservation by finite extensions implies strong preservation by finite extensions, Proposition 4, stating that the requirement of well-foundedness is superfluous for unfoldings of safe Petri nets, is simply a special case of Proposition 7.

Proposition 10, stating that the requirement of well-foundedness is superfluous for unfoldings of unbounded Petri nets in the case of an extendible order, follows from Proposition 14 if we take $\Sigma \stackrel{\text{df}}{=} T$ and $\sigma(C, e) \stackrel{\text{df}}{=} h(e)$.

6. Conclusions

In this paper we have demonstrated that the requirement that the adequate order must be well-founded is superfluous in many important cases, i.e., it follows from other requirements. We have produced a complete analysis when this is the case, by providing either a proof or a counterexample in each situation.

It is noteworthy that even though the unfolding technique has been around for more than a decade, these results concerning the very 'core' of the unfolding theory have been obtained only now. We hope that these results contribute to a conceptual clarification of the basic theory of complete prefixes.

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