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About the author

Thomas Chatain received his PhD in computer science from University of Rennes 1 in 2006. He is currently doing a post-doc in the Department of Computer Science of the University of Aalborg, Denmark. He is interested in the use of formal models for the supervision, verification and control of distributed systems. In particular he studies true concurrency models (including timed models), partial order semantics, unfoldings and timed games.

Victor Khomenko Obtained MSc with distinction in Computer Science, Applied Mathematics and Teaching of Mathematics and Computer Science in 1998 from Kiev Taras Shevchenko University, and PhD in Computing Science in 2003 from University of Newcastle upon Tyne. From September 2005 Victor is a Royal Academy of Engineering/EPSRC Post-Doctoral Research Fellow, working on the DAVAC project. His interests include model checking of Petri nets, Petri net unfolding techniques, self-timed (asynchronous) circuits.

Suggested keywords

ADEQUATE ORDER,
WELL-FOUNDEDNESS,
UNFOLDING PREFIX,
PETRI NET

A Note on the Well-Foundedness of Adequate Orders Used for Truncating Unfoldings

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Abstract. Petri net unfolding prefixes are an important technique for formal verification and synthesis. In this paper we show that the requirement that the *adequate order* used for truncating a Petri net unfolding must be well-founded is superfluous in many important cases, i.e., it logically follows from other requirements. We give a complete analysis when this is the case. These results concern the very 'core' of the unfolding theory.

1 Introduction and basic notions

McMillan's finite and complete prefixes of Petri net unfoldings [3, 8] are a prominent technique for analysing the behaviour of reactive systems modelled by Petri nets. It alleviates the *state space explosion* problem, i.e., the problem that even a relatively small system specification can (and often does) have so many reachable states that the straightforward enumeration of them is infeasible. This technique relies on the partial order view of concurrent computation.

A *finite and complete unfolding prefix* of a Petri net Ω is a finite acyclic net which implicitly represents all the reachable states of Ω together with transitions enabled at those states. Intuitively, it can be obtained through *unfolding* Ω, by successive firing of transitions, under the following assumptions: (i) for each new firing a fresh transition (called an *event*) is generated; (ii) for each newly produced token a fresh place (called a *condition*) is generated.

Due to its structural properties (such as acyclicity), the reachable states of Ω can be represented using *configurations* of its unfolding. A configuration C is a finite downward-closed set of events (being downward-closed means that if $e \in C$ and f is a causal predecessor of e, denoted $f \prec e$, then $f \in C$) without *choices* (i.e., for all distinct events $e, f \in C$, there is no condition c in the unfolding such that the arcs (c, e) and (c, f) are in the unfolding). Intuitively, a configuration is a partially ordered execution, i.e., an execution where the order of firing of some of its events (viz. concurrent ones) is not important. We will denote by $[e]$ the *local* configuration of an event e, i.e., the smallest (w.r.t. $\subset$) configuration containing e (it is comprised of e and its causal predecessors). A finite set of events E is a *suffix* of a configuration C if $C \cap E = \emptyset$ and $C \cup E$ is configuration; in such a case the notation $C \oplus E$ will be used to denote the latter configuration, called an *extension* of C.

The unfolding is infinite whenever the original Petri net has an infinite run; however, if the Petri net has finitely many reachable states then the unfolding eventually starts to repeat itself and can be truncated (by identifying a set of *cut-off* events) without loss of information, yielding a finite and complete prefix. Intuitively, an event e can be declared cut-off if the already build part of the prefix contains a configuration C^e (called the *corresponding* configuration of e) such that $Mark(C^e) = Mark([e])$ (where $Mark(C)$ denotes the final marking of a configuration C) and C^e is smaller than $[e]$ w.r.t. some well-founded partial order $\lhd$ on the configurations of the unfolding, called an *adequate order* [3, 6]. The importance of the latter condition is illustrated by the example in Figure 1, which is taken from [3]. The marking $\{p_{12}\}$ is reachable

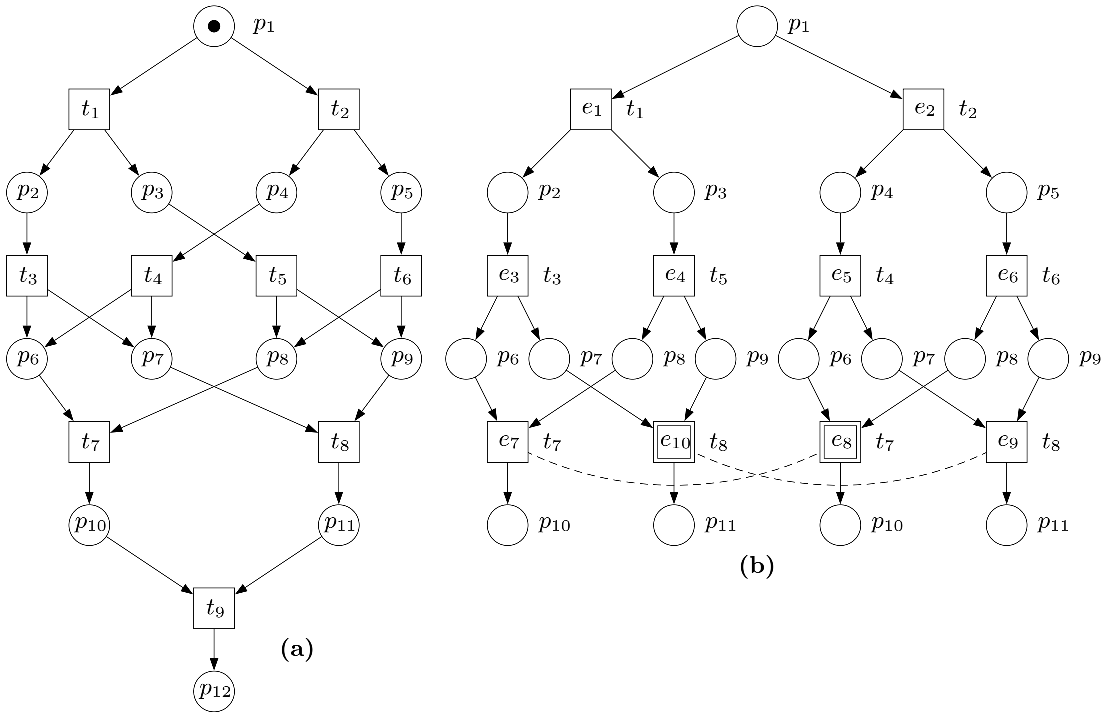


Fig. 1. A safe Petri net **(a)** and a prefix of its unfolding **(b)**

in the Petri net in Figure 1(a). However, one can generate the prefix shown in Figure 1(b), in which this marking is not represented. (The numbers of the events indicate the order in which they were added to the prefix.) The events e_8 and e_{10} are marked as cut-off, because the final markings of the corresponding local configurations are $\{p_7, p_9, p_{10}\}$ and $\{p_6, p_8, p_{11}\}$, which are also the final markings of $[e_7]$ and $[e_9]$, respectively. Although no events can now be added, the prefix is not complete, because $\{p_{12}\}$ is not represented in it.

Efficient algorithms exist for building such prefixes [3, 6], which ensure that the number of non-cut-off events in a complete prefix can never exceed the number of reachable states of the Petri net. However, complete prefixes are often exponentially smaller than the corresponding state graphs, especially for highly concurrent Petri nets, because they represent concurrency directly rather than by multidimensional 'diamonds' as it is done in state graphs. For example, if the original Petri net consists of 100 transitions which can fire once in parallel, the state graph will be a 100-dimensional hypercube with 2^{100} vertices, whereas the complete prefix will coincide with the net itself. In many applications, e.g., in asynchronous circuit design, the Petri net models usually exhibit a lot of concurrency, but have rather few choice points, and so their unfolding prefixes are often exponentially smaller than the corresponding state graphs; in fact, in many of the experiments conducted in [6] they are just slightly bigger then the original Petri nets themselves. Therefore, unfolding prefixes are well-suited for alleviating the state space explosion problem.

Well-foundedness of the adequate order used to truncate the unfolding is an important part of the completeness proof of [3, 7]. In this paper, we show that *the requirement of well-foundedness is superfluous in many important cases.* More precisely, we show that in many cases the well-foundedness of the adequate order is implied by other requirements the adequate order must satisfy.

First, we introduce several important definitions related to adequate orders. For convenience, their form has been slightly changed compared with [3, 6], but they are easily seen to be equivalent.

Definition 1.1 (Structural isomorphism). Two finite sets of events of the unfolding of a Petri net Ω, E and E', are *structurally isomorphic*,[3] denoted $E \sim_s E'$, if the labelled digraphs induced by these two sets of events and their adjacent conditions are isomorphic. $\Diamond$

Definition 1.2 (Preservation by finite extensions). A strict partial order $\lhd$ on the finite configurations of the unfolding of a Petri net is *strongly (resp. weakly) preserved by finite extensions* if for every pair of configurations C', C'' such that $Mark(C') = Mark(C'')$ and $C' \lhd C''$, and for every finite suffix E'' of C'' and every (resp. there exists a) finite suffix E' of C' such that $E' \sim_s E''$, it holds that $C' \oplus E' \lhd C'' \oplus E''$. $\Diamond$

Definition 1.3 ((Pre-)adequate orders). A strict partial order $\lhd$ on the finite configurations of the unfolding of a Petri net Ω is called *pre-adequate* if:
- it refines $\subset$, i.e., $C' \subset C''$ implies $C' \lhd C''$;
- it is weakly preserved by finite extensions.

A pre-adequate order is called *adequate* if it is well-founded. $\Diamond$

We now proceed by showing that in many cases the requirement of well-foundedness of the adequate order is superfluous, i.e., that pre-adequate orders are automatically adequate. We consider, in turn, several classes of Petri nets.

2 The case of safe Petri nets

The proposition below states that the well-foundedness requirement is superfluous for safe Petri nets.

Proposition 2.1 (The requirement of well-foundedness is superfluous for unfoldings of safe Petri nets). *A pre-adequate order on the finite configurations of the unfolding of a safe Petri net is adequate.*

Proof. Since for safe Petri nets, weak preservation by finite extensions implies strong preservation by finite extensions, this is a special case of Proposition 3.3 below. $\Box$

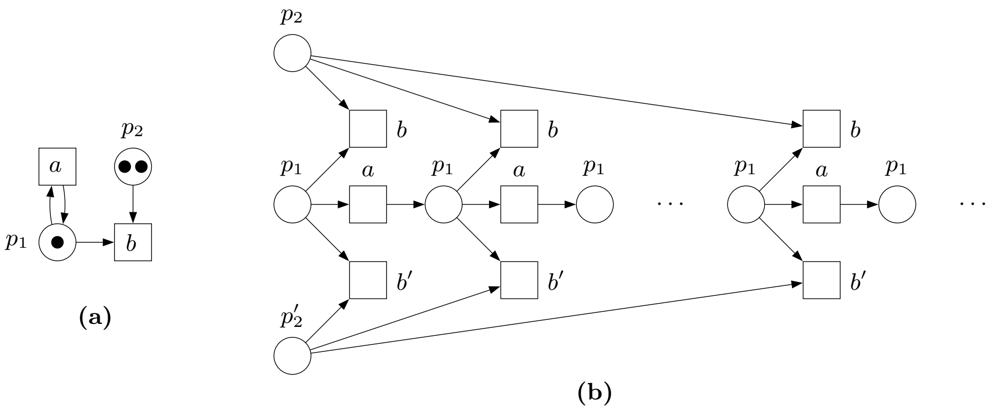
3 The case of bounded Petri nets

The case of bounded Petri nets differs from the previous case since the weak and the strong preservations by finite extensions no longer coincide, as illustrated by the following counterexample.

Counterexample 3.1 (The requirement of well-foundedness is not superfluous for unfoldings of bounded Petri nets in the case of weak preservation by finite extensions). The pre-adequate order shown in Figure 2(c,d) is not a well-founded order on the configurations of the unfolding shown in Figure 2(b). Indeed, any finite execution starts by a series of firings of a, and then, optionally, b fires. When b fires, p_2 contains two tokens, and b can consume either of them; in the unfolding, the corresponding conditions and the instances of b can be easily distinguished. We denote for all $n \geq 0$ the finite configurations as a^n, $a^n b$ and $a^n b'$.

Note that only the configurations of the form a^n can be extended, either to a^{n+k+1} or $a^{n+k}b$ or $a^{n+k}b'$, $k \geq 0$. Suppose $a^m \lhd a^n$ (i.e., $m < n$). If a^n is extended to a^{n+k+1} then we can extend (in a structurally isomorphic way) a^m to a^{m+k+1}. If a^n is extended to $a^{n+k}b$, $k \geq 0$, then we can extend (in a structurally isomorphic way) a^m to $a^{m+k}b$, and, by the definition of $\lhd$, $a^{m+k}b \lhd a^{n+k}b$. If a^n is extended to $a^{n+k}b'$, $k \geq 0$, then we can extend (in a structurally isomorphic way) a^m to $a^{m+k}b$, and, by the definition of $\lhd$, $a^{m+k}b \lhd a^{n+k}b'$. Hence, $\lhd$ is weakly preserved by finite extensions. However, $\lhd$ is not well-founded due to $a^1b' \rhd a^2b' \rhd a^3b' \rhd \ldots \Diamond$

[3] [3] used such an isomorphism without formally defining it. It turns out that there are several alternative 'natural' isomorphisms which can be used; we discuss some of them in Section 5.



$$
\begin{array}{lll}
a^n \lhd a^n b \lhd a^n b' & & \\
a^m \lhd a^n & & \text{when } m < n \\
a^m b \lhd a^n b & & \text{when } m < n \\
a^m b' \rhd a^n b' & & \text{when } m < n
\end{array}
$$

(c)

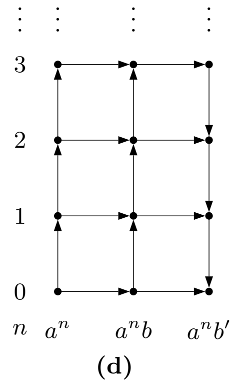
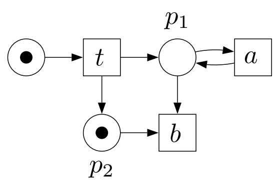


Fig. 2. A 2-bounded Petri net **(a)**, its unfolding **(b)** and an order on its configurations **(c,d)**.

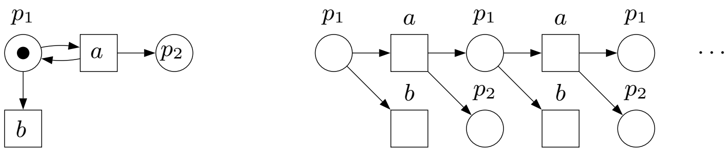
Remark 3.2. In several seminal papers on unfoldings, like [2], the initial marking is assumed to be safe, i.e., it should contain at most one token on each place. The net of Figure 2**(a)** does not satisfy this requirement. Nevertheless, it is easy to adapt this counterexample as follows:



This net starts by firing t, which leads to the same marking as in Figure 2**(a)**. Denote C_0 the initial configuration. The configuration that is reached after firing t corresponds to a^0 in Figure 2. We still denote it a^0, and re-use the notations a^n, $a^n b$ and $a^n b'$ as before. The order $\lhd$ is also re-used, and extended with $C_0 \lhd a^0$. $\Diamond$

Proposition 3.3 (The requirement of well-foundedness is superfluous for unfoldings of bounded Petri nets in the case of strong preservation by finite extensions). *If a pre-adequate order $\lhd$ on the finite configurations of the unfolding of a bounded Petri net is strongly preserved by finite extensions then $\lhd$ is adequate.*

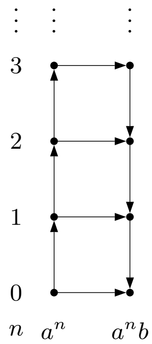
Proof. Follows from Proposition A.1 with $\Sigma = RM \times T$ and $\sigma(C, e) = (Mark(C), h(e))$. $\square$



(a) **(b)**

$$
\begin{aligned}
a^n &\lhd a^n b \\
a^n &\lhd a^m \quad \text{when } n < m \\
a^n b &\lhd a^m b \text{ when } n > m
\end{aligned}
$$

(c)



(d)

Fig. 3. An unbounded Petri net **(a)**, its unfolding **(b)** and an order on its configurations **(c,d)**.

4 The case of unbounded Petri nets

We complete our analysis by considering the case of general (unbounded) Petri nets. This case might be less interesting in practice, since the complete prefixes of unbounded nets are infinite. However, this case is interesting from the theoretical point of view. Moreover, [6] shows that a finite and complete prefix of an unbounded nets can be obtained if instead of the equivalence of final markings a coarser equivalence is used to compare the configurations in the cut-off criterion.

The definition of the preservation by finite extensions (Definition 1.2) requires that $\lhd$ is only preserved by extensions of configurations reaching the same markings. The counterexample below shows that in this case the requirement of well-foundedness is not superfluous.

Counterexample 4.1 (The requirement of well-foundedness is not superfluous for unfoldings of unbounded Petri nets). Consider Figure 3. The finite configurations of the unfolding have the form either a^n or $a^n b$, where n ranges over the set of integers. The shown order is pre-adequate, as it refines the set inclusion and it is trivially preserved by finite extensions of configurations reaching the same marking, since *no two configurations reach the same marking.* However, $\lhd$ is not well-founded due to $b \rhd ab \rhd aab \rhd \ldots$ $\lozenge$

Technically, this counterexample settles the case of unbounded Petri nets. However, one can observe that this negative result holds due to the trivial reason that it is possible to construct an unbounded Petri net such that in its unfolding no two configurations have the same final marking. Hence, it seems reasonable to strengthen the assumptions about the pre-adequate order in the unbounded case, by requiring that $\lhd$ is preserved not only by configurations that reach the same marking, but also each time isomorphic finite suffixes can be added to two comparable configurations.

Definition 4.2 (Extendible pre-adequate order). A pre-adequate order $\lhd$ on the finite configurations of the unfolding of a Petri net Ω is called *extendible* if for all configurations C' and C'' such that $C' \lhd C''$, and for all finite suffixes E' and E'' of C' and C'', resp., such that $E' \sim_s E''$, it holds that $C' \oplus E' \lhd C' \oplus E'$. $\lozenge$

	weak preservation	strong preservation
safe nets	✓ (Proposition 2.1)	
bounded nets	× (Counterexample 3.1)	✓ (Proposition 3.3)
unbounded nets	× (Counterexample 4.1)	
unbounded nets (extendible order)	× (Counterexample 3.1)	✓ (Proposition 4.3)

Table 1. Summary of results.

Note that extendible pre-adequate orders are strongly preserved by finite extensions. The proposition below shows that a positive result can be obtained in the case of an extendible pre-adequate order.

Proposition 4.3 (The requirement of well-foundedness is superfluous for unfoldings of unbounded Petri nets in the case of an extendible order). *An extendible pre-adequate order $\lhd$ on the finite configurations of the unfolding of a (possibly unbounded) Petri net is adequate.*

Proof. Follows from Proposition A.1 with $\Sigma = T$ and $\sigma(C, e) = h(e)$. □

5 Summary and further considerations

Our results are summarised in Table 1, where ✓ means that the requirement of well-foundedness is superfluous, and × means that it is not superfluous. Moreover, we now show that these results are *robust*, i.e., they are not affected if an alternative notion of preservation of $\lhd$ by extensions is used, or if $\sim_s$ is replaced by a different isomorphism.

5.1 Single-Event Extensions

Definition 5.1. $\lhd$ is *weakly (resp. strongly) preserved by single-event extensions* if it is weakly (resp. strongly) preserved by finite extensions with singleton suffixes. ◊

One can easily show by induction on the size of the configuration suffixes that strong preservation by single-event extension coincides with strong preservation by finite extensions, and so Propositions 3.3 and 4.3 still hold for single-event extensions. On the other hand, weak preservation by single event extensions is even weaker than weak preservation by finite extensions, and so Counterexample 3.1 also holds for weak preservation by single-event extensions.

Moreover, one can easily show that for safe Petri nets, weak preservation by single-event extensions is equivalent to strong preservation by single-event extensions (which is in turn equivalent to weak or strong preservation by finite extensions), and so Proposition 2.1 holds for single-event extensions as well.

To summarise, using single-event extensions instead of finite ones does not change our results.

5.2 Other Isomorphisms

So far, we considered the structural isomorphism, $\sim_s$, which is in a sense strongest possible, as it takes the full structure of the net into account. Below we consider other natural isomorphisms, which are coarser then $\sim_s$.

Definition 5.2 (Pomset-isomorphism and Parikh-isomorphism). Let E and E' be two finite sets of events of the unfolding of a Petri net Ω.

– E and E' are *pomset-isomorphic*, denoted $E \sim_p E'$, if the labelled digraphs induced by these two sets of events in the digraph corresponding to the causality relation on the events on the unfolding are isomorphic.
– E and E' are *Parikh-isomorphic*, denoted $E \sim_\# E'$, if for every transition t of Ω, $\#_t E = \#_t E'$, where $\#_t E$ denotes the number of instances of t in E. $\Diamond$

Note that $\sim_s$ refines $\sim_p$, which in turn refines $\sim_\#$, i.e., $E' \sim_s E'' \Rightarrow E' \sim_p E'' \Rightarrow E' \sim_\# E''$. Moreover, one can observe that if $\sim_1$ and $\sim_2$ are two isomorphisms such that $\sim_2$ refines $\sim_1$ then:

– weak preservation w.r.t. $\sim_1$ is even weaker than weak preservation w.r.t. $\sim_2$ (i.e., there exists an E' such that $E' \sim_1 E''$ and $C' \oplus E' \lhd C'' \oplus E''$, but maybe $E' \not\sim_2 E''$);
– strong preservation w.r.t. $\sim_1$ is even stronger than strong preservation w.r.t. $\sim_2$ (for all E' such that $E' \sim_1 E''$, $C' \oplus E' \lhd C'' \oplus E''$, even for those E' such that $E' \not\sim_2 E''$).

Consequently, Counterexample 3.1 (for bounded or unbounded nets), as well as Propositions 3.3 and 4.3, still hold for $\sim_p$ and $\sim_\#$. Moreover, since in the case of safe Petri nets it is enough to consider only single-event extensions, and $\sim_s$, $\sim_p$ and $\sim_\#$ coincide on such extensions, Proposition 2.1 holds for either of these isomorphisms. Finally, one can observe that Counterexample 4.1 still holds for $\sim_p$ and $\sim_\#$.

To summarise, using $\sim_p$ or $\sim_\#$ (or any other isomorphism refining $\sim_\#$ and refined by $\sim_s$) instead of $\sim_s$ does not change our results.

6 Conclusions

In this paper we have demonstrated that the requirement that the adequate order must be well-founded is superfluous in many important cases, i.e., it logically follows from other requirements. We have produced a complete analysis when this is the case, by providing either a proof or a counterexample in each situation.

It is noteworthy that even though the unfolding technique has been around for more than a decade, these results concerning the very 'core' of the unfolding theory have been obtained only now.

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References

1. E. M. Clarke, O. Grumberg and D. Peled: *Model Checking.* MIT Press (1999).
2. J. Engelfriet: Branching Processes of Petri Nets. *Acta Informatica* 28 (1991) 575–591.
3. J. Esparza, S. Römer and W. Vogler: An Improvement of McMillan's Unfolding Algorithm. *Formal Methods in System Design* 20(3) (2002) 285–310.
4. K. Heljanko, V. Khomenko and M. Koutny: Parallelization of the Petri Net Unfolding Algorithm. Proc. of *TACAS'2002*, Springer-Verlag, Lecture Notes in Computer Science 2280 (2002) 371–385.
5. G. Higman: Ordering by Divisibility in Abstract Algebras. *Proc. London Math. Soc.* 2 (1952) 326–336.
6. V. Khomenko: *Model Checking Based on Prefixes of Petri Net Unfoldings.* PhD Thesis, School of Computing Science, Newcastle University (2003).
7. V. Khomenko, M. Koutny and V. Vogler: Canonical Prefixes of Petri Net Unfoldings. *Acta Informatica* 40(2) (2003) 95–118.
8. K. L. McMillan: Using Unfoldings to Avoid State Explosion Problem in the Verification of Asynchronous Circuits. Proc. of *CAV'1992*, Springer-Verlag, Lecture Notes in Computer Science 663 (1992) 164–174.

9. T. Murata: Petri Nets: Properties, Analysis and Applications. *Proceedings of the IEEE* 77(4) (1989) 541–580.
10. A. Valmari: The State Explosion Problem. In: *Lectures on Petri Nets I: Basic Models*, W. Reisig and G. Rozenberg (Eds.). Springer-Verlag, Lecture Notes in Computer Science 1491 (1998) 429–528.

Appendix

A The proof of the main result

Proposition A.1. *Let $\lhd$ be a strict partial order on configurations of the unfolding of a Petri net Ω, Σ be a finite alphabet and $\sigma(C,e)$ be a mapping that assigns a letter from Σ to each pair (C,e), where C is a configuration and e is an event that extends C, satisfying*

$$\forall C_1, e_1, C_2, e_2 : \quad \left.\begin{cases} C_1 \lhd C_2 \\ \sigma(C_1,e_1) = \sigma(C_2,e_2) \end{cases}\right\} \Rightarrow C_1 \oplus \{e_1\} \lhd C_2 \oplus \{e_2\}.$$

Then $\lhd$ is well-founded.

Proof. For the sake of contradiction, assume that $\lhd$ is not well-founded, i.e., there is an infinite descending sequence $C_1 \rhd \ldots \rhd C_n \rhd \ldots$ We assume that the configurations have strictly increasing sizes (an infinite subsequence of $C_1,\ldots,C_n,\ldots$ satisfying this property can always be extracted).

The union of the configurations $C_1,\ldots,C_n,\ldots$ induces an infinite branching process, and the analog of König's lemma for branching processes [7] states that it has an infinite causal chain of events p. Each configuration C_i has a finite intersection with p, since configurations are finite by definition. On the other hand, the union of all these configurations has an infinite intersection with p. Hence, infinitely many configurations have non-empty intersection with p, and these intersections can be arbitrarily large (since if some event e of p belongs to C_i then all the preceding events of p also belong to C_i). Therefore, without loss of generality, we can assume that $|p\cap C_1| < \ldots < |p\cap C_n| < \ldots$ (an infinite subsequence of $C_1,\ldots,C_n,\ldots$ satisfying this property can always be extracted).

Let $D_n \subseteq C_n$ be the configuration defined as the causal past of the events of p that are in C_n, and $E_n \stackrel{\mathrm{df}}{=} C_n \setminus D_n$, i.e., $C_n = D_n \oplus E_n$. We assume that the sizes of the E_n's are non-decreasing (an infinite subsequence of $C_1,\ldots,C_n,\ldots$ satisfying this property can always be extracted).

For each E_n, let $e_{n,1},\ldots,e_{n,s_n}$ be an arbitrary linearisation of the events of E_n consistent with the causal order, $s_n \stackrel{\mathrm{df}}{=} |E_n|$, $E_{n,k} \stackrel{\mathrm{df}}{=} \{e_{n,1},\ldots,e_{n,k}\}$ and $C_{n,k} \stackrel{\mathrm{df}}{=} D_n \oplus E_{n,k}$. We define the word $W_n \stackrel{\mathrm{df}}{=} a_{n,1}\ldots a_{n,s_n}$ with $a_k \stackrel{\mathrm{df}}{=} \sigma(C_{n,k-1},e_{n,k})$. Now we can apply Higman's lemma [5] to $W_1,\ldots,W_n,\ldots$, which are finite words over the finite alphabet Σ. This returns two integers $i<j$ such that W_i is a subword of W_j. Let $0 = l_0 < l_1 < \ldots < l_{s_i} \le s_j$ such that for all $k \in \{1,\ldots s_i\}$, $a_{i,k} = a_{j,l_k}$. We have $W_j = \ldots a_{j,l_1} \ldots a_{j,l_2} \ldots a_{j,l_{s_i}} \ldots = \ldots a_{i,1} \ldots a_{i,2} \ldots a_{i,s_i} \ldots$ Starting from $C_{i,0} = D_i \lhd D_j = C_{j,l_0}$, we show by induction on k that $C_{i,k} \lhd C_{j,l_k}$ for all $k \le s_i$, which gives $C_{i,s_i} \lhd C_{j,l_{s_i}} \subseteq C_{j,s_j}$, i.e., $C_i \lhd C_j$, which leads to contradiction. We get the inductive step as follows: if $C_{i,k-1} \lhd C_{j,l_{k-1}}$, then $C_{i,k-1} \lhd C_{j,l_k-1}$ because $C_{j,l_{k-1}} \subseteq C_{j,l_k-1}$; moreover $\sigma(C_{i,k-1},e_{i,k}) = a_{i,k} = a_{j,l_k} = \sigma(C_{j,l_k-1},e_{l_k})$, so $C_{i,k-1} \oplus \{e_{i,k}\} \lhd C_{j,l_k-1} \oplus \{e_{j,l_k}\}$, i.e., $C_{i,k} \lhd C_{j,l_k}$. □