One-Counter Automata with Counter Visibility

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In a one-counter automaton (OCA), one can read a letter from some finite alphabet, increment and decrement the counter by one, or test it for zero. It is well-known that universality and language inclusion for OCAs are undecidable. We consider here OCAs with counter visibility: Whenever the automaton produces a letter, it outputs the current counter value along with it. Hence, its language is now a set of words over an infinite alphabet. We show that universality and inclusion for that model are in PSPACE, thus no harder than the corresponding problems for finite automata, which can actually be considered as a special case. In fact, we show that OCAs with counter visibility are effectively determinizable and closed under all boolean operations. As a strict generalization, we subsequently extend our model by registers. The general nonemptiness problem being undecidable, we impose a bound on the number of register comparisons and show that the corresponding nonemptiness problem is NP-complete.

1 Introduction

One-counter automata (OCAs) are a fundamental model of infinite-state systems. Their expressive power resides between finite automata and pushdown automata. Unlike finite automata, however, OCAs are not robust: They lack closure under complementation and have an undecidable universality, equivalence, and inclusion problem [15]. Several directions to overcome this drawback have been taken. One may underapproximate the above decision problems in terms of bisimilarity [16] or overapproximate the system behavior by a finite-state abstraction, e.g., in terms of the downward closure or preserving the Parikh image [22, 29].

We will consider here a new and simple way of obtaining a robust model of one-counter systems. Whenever the automaton produces a letter from a finite alphabet Σ, it will also output the *current* counter value along with it (transitions that manipulate the counter are not concerned). Hence, its language is henceforth a subset of $(\Sigma \times \mathbb{N})^*$. For obvious reasons, we call this model *one-counter automata with counter visibility (CAVs)*. We will show that CAVs form a robust automata model: They are closed under all boolean operations. Moreover, their universality and inclusion problem are in PSPACE and, as a simple reduction from universality for finite automata shows, PSPACE-complete.

These results may come as a surprise given that universality for OCAs is undecidable and introducing counter visibility seems like an extension of OCAs. But, actually, the problem becomes quite different. The fact that a priori hidden details from a run (in terms of the counter values) are revealed makes the model more robust and the decision problems easier. Note that this is also what happens in input-driven/visibly pushdown automata or their restriction of visibly one-counter automata, where the stack/counter operation can be deduced from the letter that is read [3, 19]. It is worth noting that revealing details from a system configuration does not always help, quite the contrary: The universality problem of timed automata is decidable only if time stamps are excluded from the semantics [2].

But it is not only for the pure fact that we obtain a robust model that we consider counter visibility. Counter values usually have a *meaning*, such as energy level, value of a variable, or number of jobs to be accomplished (cf. Example 3). In those contexts, it is natural to include them in the semantics, just like including time stamps in timed automata.

Our constructions are purely automata-theoretic and conceptually simple so that possible extensions or variants of CAVs can build on an elementary theory as is the case with finite automata. More precisely, we show that CAVs are determinizable (and, therefore, complementable) using a powerset construction that relies only on a couple of basic lemmas concerning arithmetic progressions. The powerset construction can be performed on-the-fly so that a PSPACE upper bound for the universality problem follows. Note that this constitutes a sound approximation of the corresponding decision problem for classical one-counter automata, since universality of a CAV implies universality of the underlying ordinary one-counter automaton. Actually, the powerset automaton falls into a more general class that has the same expressive power but allows for a simple product construction. The latter is then used to show that the inclusion problem for CAVs is in PSPACE, too.

In the context of infinite alphabets, the usage of registers is natural [17]. Extending CAVs by registers will indeed allow us to formalize requirements such as "a service is only provided if the energy level is above a certain threshold", the threshold being stored in a register. Unfortunately, the resulting model is no longer closed under complementation and has an undecidable nonemptiness problem. We will impose a bound on the number of times the current counter value can be compared to a register. This allows us to reduce nonemptiness, in polynomial time, to satisfiability of an existential Presburger formula, which is in NP [26].

Related Work. Though our work shares its general idea with visibly pushdown automata, which were also shown to be determinizable and complementable [3], the closest model to ours is probably that of *strong automata* [6]. Strong automata operate on infinite alphabets and were introduced as an extension of *symbolic automata* [4,7]. A transition of a strong automaton is a formula $\varphi_{p,q}(x, x')$ from monadic second-order logic over some infinite structure, say $(\mathbb{N}, +1)$. If the last letter read so far is $n \in \mathbb{N}$ and the current state is p, the transition allows us to go to q and read $n' \in \mathbb{N}$, provided $\varphi_{p,q}(n, n')$ is evaluated to true. It follows from our constructions that CAVs are in fact strong automata over $(\mathbb{N}, +1)$ (extended by a component for Σ), whose universality problem is decidable [6]. However, strong automata do not allow us to derive any elementary complexity upper bounds. In a sense, our model is an operational version of strong automata. As such, it is well suited for extensions in terms of "operational" storing mechanisms like registers.

As far as CAVs with registers are concerned, our model is a strict extension of ordinary register automata (over $\mathbb{N}$), which are not equipped with a counter and where natural numbers can only be compared for equality [17, 27]. Registers were also used in *freeze LTL*, notably for specification of counter systems [8, 9, 24]. The main difference here is that we consider CAVs (with registers) as language acceptors and include registers in the system itself rather than in the specification language. This also yields different natural notions of underapproximation to get decidability of the nonemptiness problem.

Finally, note that there have been other quantitative settings such as availability languages and weighted automata where powerset constructions were applicable [14, 21]. Furthermore, the reduction to existential Presburger formulas is a common technique to solve reachability questions in counter systems and related quantitative settings (e.g., [1, 12, 13]).

Outline. In Section 2, we introduce one-counter automata with counter visibility. Section 3 presents a determinizable variant of CAVs and a corresponding powerset construction. The latter will allow us to conclude that CAVs are closed under boolean operations and have a PSPACE-complete universality problem. Finally, Section 4 considers an extension of CAVs by registers. It is shown that, under a suitable restriction, their nonemptiness problem is NP-complete. Missing proofs can be found in the appendix.

2 One-Counter Automata with Counter Visibility

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we set $[n] := \{1, \ldots, n\}$ and $[n]_0 := \{0, 1, \ldots, n\}$.

The Model. We consider ordinary one-counter automata over some finite alphabet Σ. In addition to a finite-state control and transitions that produce a letter from Σ, they have a counter that can be incremented, decremented, or tested for (non-)zero. Accordingly, the set of *counter operations* is $\mathsf{Op} = \{1, -1\}$, and a *counter guard* $\varphi \in \mathsf{Guards}$ is either tt (true) or a (non-)zero test:

$$\mathsf{Guards}: \quad \varphi ::= \mathsf{tt} \mid \mathsf{cnt} = 0 \mid \neg(\mathsf{cnt} = 0)$$

Here, cnt refers to the current counter value. Suppose the current counter value is $x \in \mathbb{N}$. The relation $x \models (\mathsf{cnt} = 0)$ (to be read as "x satisfies $\mathsf{cnt} = 0$") holds if $x = 0$, and $x \models \neg(\mathsf{cnt} = 0)$ holds if $x > 0$. Moreover, $x \models \mathsf{tt}$ holds for every $x \in \mathbb{N}$.

Syntactically, our model is just a one-counter automaton (with unary updates). Visibility of counter values becomes apparent in the semantics defined below.

Definition 1. A *one-counter automaton with counter visibility (CAV)* is a tuple $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ where Q is the finite set of *states*, Σ is a nonempty finite alphabet (disjoint from Op), $\iota \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, and $\longrightarrow \subseteq Q \times \mathsf{Guards} \times (\mathsf{Op} \cup \Sigma) \times Q$ is the *transition relation.*

We write a transition $(p, \varphi, \sigma, q) \in \longrightarrow$ as $p \xrightarrow{\varphi|\sigma} q$ and call it a σ-transition. When $\varphi = \mathsf{tt}$, we may just write $p \xrightarrow{\sigma} q$.

Semantics. The set of *configurations* of $\mathcal{A}$ is $\mathit{Conf}_{\mathcal{A}} := Q \times \mathbb{N}$. In a configuration (q, x), q is the current state and x is the current counter value. The *initial* configuration is $(\iota, 0)$, and a configuration (q, x) is *final* if $q \in F$.

Towards the language of $\mathcal{A}$, we determine, for each $t \in \longrightarrow$, a global transition relation $\Longrightarrow_{\mathcal{A},t} \subseteq \mathit{Conf}_{\mathcal{A}} \times (\{\varepsilon\} \cup (\Sigma \times \mathbb{N})) \times \mathit{Conf}_{\mathcal{A}}$. We keep t in the index as it will be convenient to extract, from a run, the transitions that were actually used. For two configurations (q, x) and (q', x') and a transition $t = (q, \varphi, \sigma, q') \in \longrightarrow$ such that $x \models \varphi$, we have

- $(q, x) \xRightarrow{\varepsilon}_{\mathcal{A},t} (q', x')$ if $\sigma \in \mathsf{Op}$ and $x' = x + \sigma$,
- $(q, x) \xRightarrow{(\sigma,x)}_{\mathcal{A},t} (q', x')$ if $\sigma \in \Sigma$ and $x' = x$.

For two configurations $\gamma, \gamma' \in \mathit{Conf}_{\mathcal{A}}$ and $\tau \in \{\varepsilon\} \cup (\Sigma \times \mathbb{N})$, we write $\gamma \xRightarrow{\tau}_{\mathcal{A}} \gamma$ if there is a transition $t \in \longrightarrow$ such that $\gamma \xRightarrow{\tau}_{\mathcal{A},t} \gamma'$.

A *partial run* of $\mathcal{A}$ is a sequence $\rho = (q_0, x_0) \xRightarrow{\tau_1}_{\mathcal{A},t_1} (q_1, x_1) \xRightarrow{\tau_2}_{\mathcal{A},t_2} \ldots \xRightarrow{\tau_n}_{\mathcal{A},t_n} (q_n, x_n)$ with $n \geq 0$. If, in addition, (q_0, x_0) is the initial configuration, then we say that ρ is a *run.* We call ρ *accepting* if its last configuration (q_n, x_n) is final. What we actually observe during ρ is $\mathit{obs}(\rho) := \tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_n$, i.e., the concatenation of $\tau_1, \ldots, \tau_n$, which is thus a word in $(\Sigma \times \mathbb{N})^*$. We also say that ρ is a *(partial) run on obs*(ρ).

The language of $\mathcal{A}$ is now defined as $L(\mathcal{A}) = \{\mathit{obs}(\rho) \mid \rho \text{ is an accepting run of } \mathcal{A}\}$. Actually, $L(\mathcal{A})$ can be seen as the language of an infinite automaton using ε-transitions, with set of states $\mathit{Conf}_{\mathcal{A}}$, transition relation $\Longrightarrow_{\mathcal{A}} \subseteq \mathit{Conf}_{\mathcal{A}} \times (\{\varepsilon\} \cup (\Sigma \times \mathbb{N})) \times \mathit{Conf}_{\mathcal{A}}$, initial state $(\iota, 0)$, and set of final states $F \times \mathbb{N}$.

To summarize, a CAV is simply an ordinary one-counter automaton, but with the semantic variation that the current counter value is output whenever a letter from the alphabet Σ is produced.

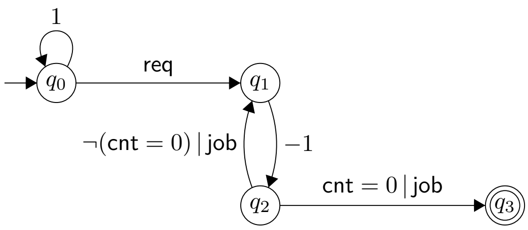

Figure 1 A one-counter automaton with counter visibility

The *nonemptiness problem for CAVs* is defined as follows: Given a CAV $\mathcal{A}$, do we have $L(\mathcal{A}) \neq \emptyset$? Of course, this problem reduces to reachability in ordinary one-counter automata:

Theorem 2 ([28]). *The nonemptiness problem for CAVs is NL-complete.*

Example 3. Consider the CAV $\mathcal{A}$ from Figure 1 over the alphabet $\Sigma = \{\mathsf{req}, \mathsf{job}\}$. We can think of a client that executes (req, n) to signalize that it requires the completion of $n \geq 1$ jobs. Performing job, the number of remaining jobs is then decreased by one and output along with the letter. In fact, the language of $\mathcal{A}$ is given as $L(\mathcal{A}) = \{(\mathsf{req}, n)(\mathsf{job}, n-1)(\mathsf{job}, n-2)\dots(\mathsf{job}, 0) \mid n \geq 1\}$. Note that, when seen as an ordinary one-counter automaton, the language of $\mathcal{A}$ is the projection of $L(\mathcal{A})$ to Σ, i.e., $(\mathsf{req})(\mathsf{job})^+$.

3 Determinizing and Complementing CAVs

We will show that CAVs are effectively closed under all boolean operations. Closure under complement follows from a determinization procedure, which we illustrate next.

Let $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ be the CAV to be determinized and $w = (a_1, x_1)\dots(a_n, x_n) \in (\Sigma \times \mathbb{N})^*$. Every run on w has to have reached the counter value x_1 by the time it reads the first letter a_1. In particular, it has to perform at least x_1 counter increments. In other words, we can identify x_1 transitions that lift the counter value from 0 to 1, from 1 to 2, and, finally, from $x_1 - 1$ to x_1, respectively, and that are separated by partial runs that "oscillate" around the current counter value but, at the end, return to their original level. Similarly, before reading the second letter a_2, $\mathcal{A}$ will perform $|x_2 - x_1|$-many counter operations $\mathrm{sign}(x_2 - x_1) \in \mathsf{Op}$ to reach x_2, again separated by some oscillation phases, and so on. This is illustrated on the left hand side of Figure 2.

Now, there may be several runs on a given word w, and even one given run may admit several decompositions into oscillations and counter increments/decrements (cf. Figure 2). To gather all possible runs of $\mathcal{A}$ on w and their decompositions deterministically in a single one, we will perform a powerset construction that keeps track of the states of $\mathcal{A}$ that can be reached by an oscillation followed by an increment/decrement. This is illustrated on the right hand side of Figure 2. The automaton starts in an increasing mode, goes straight to the value x_1, while maintaining the set of states $\mathcal{A}$ may be in at that stage. Once it reads letter a_1, it may go into an increasing or decreasing mode, and so on.

There is a little issue here, since the possibility of performing an oscillation leading from p_2 to p_2' (cf. Figure 2) depends on the current counter value. However, it was shown in [11] that the set of counter values allowing for such a shortcut can be described as a boolean combination of arithmetic progressions that can be computed in polynomial time. To perform the powerset construction, we will, therefore, work with an extended version of CAVs that includes arithmetic-progression tests. This extension is introduced in the next subsection.

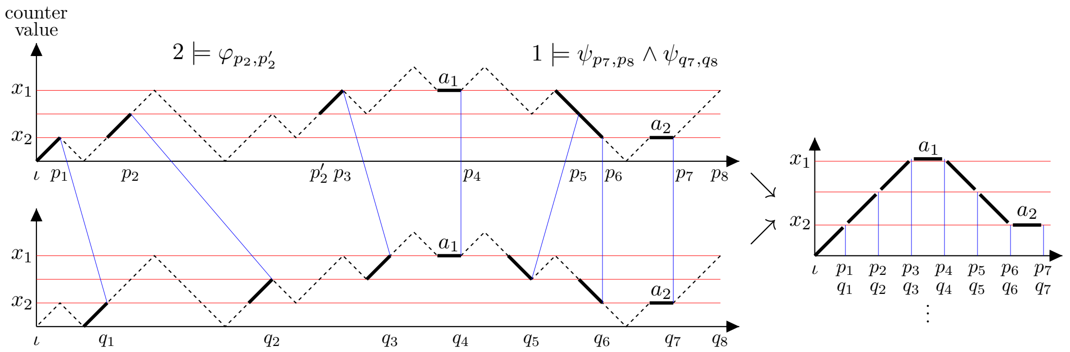

Figure 2 Two decompositions of a run on $(a_1, 3)(a_2, 1)$, and a run of the powerset automaton

3.1 Extended CAVs

Extended CAVs have access to a more expressive supply of guards, denoted $\mathsf{Guards}^{\mathrm{mod}}$:

$$\mathsf{Guards}^{\mathrm{mod}}: \qquad \varphi \ ::= \ \mathsf{cnt} \in c + d\mathbb{N} \ \mid \ \neg\varphi \ \mid \ \varphi \wedge \varphi$$

where $c, d \in \mathbb{N}$. We call $\mathsf{cnt} \in c + d\mathbb{N}$ an *arithmetic-progression formula*. We consider that constants are given in unary, as extended counter guards are an intermediate formalism and already constitute a succinct encoding of their "unfoldings" in terms of CAVs (cf. Lemma 6 below).

For $x \in \mathbb{N}$, we define $x \models (\mathsf{cnt} \in c + d\mathbb{N})$ if $x = c + d \cdot i$ for some $i \in \mathbb{N}$. Thus, we may use $\mathsf{cnt} = 0$ as an abbreviation for $\mathsf{cnt} \in 0 + 0\mathbb{N}$, and tt as an abbreviation for $\mathsf{cnt} \in 0 + 1\mathbb{N}$. The other formulas from $\mathsf{Guards}^{\mathrm{mod}}$ are interpreted as expected. Moreover, given $\varphi \in \mathsf{Guards}^{\mathrm{mod}}$, we set $[\![\varphi]\!] := \{x \in \mathbb{N} \mid x \models \varphi\}$.

Before we introduce extended CAVs, we will state the lemma saying that the "possibility" of a shortcut is definable in $\mathsf{Guards}^{\mathrm{mod}}$. Let $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ be a CAV and $p, q \in Q$. By $X^{\mathcal{A}}_{p,q}$, we denote be the set of natural numbers $x \in \mathbb{N}$ such that $(p, x) \xRightarrow{\varepsilon}{}^{*}_{\mathcal{A}} (q, x)$, i.e., there is a partial run from (p, x) to (q, x) that does not produce any pair from $\Sigma \times \mathbb{N}$. Moreover, we define $Y^{\mathcal{A}}_{p,q}$ to be the set of natural numbers $x \in \mathbb{N}$ such that $(p, x) \xRightarrow{\varepsilon}{}^{*}_{\mathcal{A}} (q, x')$ for some $x' \in \mathbb{N}$. Note that $X^{\mathcal{A}}_{p,q} \subseteq Y^{\mathcal{A}}_{p,q}$. The following result is due to [11, Lemmas 6–9]:

Lemma 4 ([11]). *Let $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ be a CAV and $p, q \in Q$. We can compute, in polynomial time, guards $\varphi_{p,q}, \psi_{p,q} \in \mathsf{Guards}^{\mathrm{mod}}$ such that $[\![\varphi_{p,q}]\!] = X^{\mathcal{A}}_{p,q}$ and $[\![\psi_{p,q}]\!] = Y^{\mathcal{A}}_{p,q}$. In particular, the constants in $\varphi_{p,q}$ and $\psi_{p,q}$ are all polynomially bounded.*

See Figure 2 for an illustration of the formulas. Note that disjunctions of formulas of type $\psi_{p,q}$ will represent the acceptance condition of the deterministic powerset automaton.

Definition 5. An *extended CAV (eCAV)* is a tuple $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ where all components are like in a CAV apart from $\longrightarrow$, which is a *finite* subset of $Q \times \mathsf{Guards}^{\mathrm{mod}} \times (\mathsf{Op} \cup \Sigma) \times Q$, and $F : Q \to \mathsf{Guards}^{\mathrm{mod}}$.

Runs and the language $L(\mathcal{A})$ of an eCAV $\mathcal{A}$ are defined in exactly the same way as for CAVs, with one exception: A run is now *accepting* if its last configuration (q, x) is such that $x \models F(q)$. Of course, every CAV is (almost) an eCAV. Conversely, every eCAV can be translated into an equivalent CAV, as stated in the next lemma.

Lemma 6. *For every eCAV $\mathcal{A}$, we can construct a CAV $\mathcal{A}'$ of exponential size such that $L(\mathcal{A}') = L(\mathcal{A})$.*

Proof. For every arithmetic-progression formula $\mathsf{cnt} \in c + d\mathbb{N}$ that occurs in $\mathcal{A}$ (for simplicity, let us assume $c, d > 0$), we will enrich the state space by an additional component $\{0, 1, \ldots, c\} \times \{0, 1, \ldots, d\}$. Its current state (i, j) will then tell us whether $\mathsf{cnt} \in c + d\mathbb{N}$ holds, namely iff $i = c$ and $j \in \{0, d\}$. Obviously, we start in $(0, 0)$. Whenever a transition performs a counter operation $op \in \mathsf{Op}$, the state component (i, j) is updated: We first determine, using a linear number of additional transitions decrementing and incrementing the counter at most c times, respectively, the relation between the current counter value x and c. According to the outcome of that test, we proceed as follows:

- If $x < c$, then we go to $(\max\{0, i + op\}, 0)$.
- If $x = c$, then we go to $(\min\{c, c + op\}, \max\{0, op\})$.
- If $x > c$, then we go to $(i, (j + op) \bmod (d + 1))$ (with $-1 \bmod (d + 1) = d$).

Thus, the size of $\mathcal{A}'$ is exponential in the number of arithmetic-progression formulas. $\square$

3.2 Deterministic Extended CAVs

It is not obvious to come up with a notion of a *deterministic* eCAV. Apparently, adapting the definition of determinism for ordinary one-counter automata is too restrictive: Two distinct words $w \in (\Sigma \times \mathbb{N})^*$ may have the same projection to Σ. We rather follow the scheme outlined at the beginning of this section. The states of a deterministic eCAV $\mathcal{A}$ are partitioned into Q_1, Q_0, and Q_{-1}. Being in Q_1, $\mathcal{A}$ *cannot* decrease its counter, and being in Q_{-1}, it *cannot* increase it. A direction change can only be effectuated by first reading a letter from Σ, which leads to a "neutral" state from Q_0. This is ensured by Condition [D1] below. In addition, $\mathcal{A}$ should behave deterministically on every possible instruction from $\mathsf{Op} \cup \Sigma$ (cf. Condition [D4]). Now, every word $w \in (\Sigma \times \mathbb{N})^*$ will have a unique run stopping in some neutral state from Q_0 (cf. Lemma 8 below). When we require that only states from Q_0 are "accepting", complementation is achieved by negating the final-state function (Proposition 9).

Definition 7. An eCAV $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ is called *deterministic* (deCAV) if there is a partition $Q = Q_1 \uplus Q_0 \uplus Q_{-1}$ such that the following hold:

[D1] For all $(p, \varphi, \sigma, q) \in \longrightarrow$, $\sigma \in \mathsf{Op}$ implies $q \in Q_\sigma$, and $\sigma \in \Sigma$ implies $q \in Q_0$.

[D2] The initial state ι is contained in Q_0 and has no incoming transitions.

[D3] We have $F(q) = \neg\mathsf{tt}$ for all $q \in Q \setminus Q_0$.

[D4] Let $p \in Q$ and $\sigma \in \mathsf{Op} \cup \Sigma$. Suppose $\{p \xrightarrow{\varphi_i | \sigma} q_i \mid i \in [k]\}$, with $k \in \mathbb{N}$, is the set of σ-transitions outgoing from p.

(a) If $(p, \sigma) \in Q_1 \times \{-1\}$ or $(p, \sigma) \in Q_{-1} \times \{1\}$, then $k = 0$.

(b) Otherwise, we have $k \geq 1$, $\bigcup_{i \in [k]} [\![\varphi_i]\!] = \mathbb{N}$, and $[\![\varphi_i]\!] \cap [\![\varphi_j]\!] = \emptyset$ for all $i, j \in [k]$ such that $i \neq j$.

Note that one can decide whether a given eCAV is deterministic: The semantic property [D4b] is decidable, as each $[\![\varphi_i]\!]$ is a (one-dimensional) semilinear set.

We call a run of an (e)CAV *minimal* if consists only of the initial configuration, or its last transition label is a letter from Σ (rather than a counter operation). By the following lemma, it is indeed justified to call a deCAV deterministic:

Lemma 8. *Let $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ be a deCAV and $w \in (\Sigma \times \mathbb{N})^*$. There is a unique minimal run ρ such that $obs(\rho) = w$. We denote this run by $\rho_{\mathcal{A}}(w)$. We have that $w \in L(\mathcal{A})$ iff $\rho_{\mathcal{A}}(w)$ is accepting.*

Proposition 9. *For every deCAV $\mathcal{A}$ over some alphabet Σ, there is a deCAV $\mathcal{A}'$ of the same size as $\mathcal{A}$ such that $L(\mathcal{A}') = (\Sigma \times \mathbb{N})^* \setminus L(\mathcal{A})$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ be the given deCAV. By Lemma 8, we obtain a deCAV recognizing $(\Sigma \times \mathbb{N})^* \setminus L(\mathcal{A})$ by replacing F with F' where $F'(q) = \neg F(q)$ for all $q \in Q_0$, and $F'(q) = \neg\mathsf{tt}$ for all $q \in Q \setminus Q_0$. $\square$

Another useful property of deCAVs is that of a simple product construction, which will be used later to solve the inclusion problem for CAVs in PSPACE.

Proposition 10. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be deCAVs over the same alphabet Σ. There is a deCAV $\mathcal{A}$ of size $O(|\mathcal{A}_1| \cdot |\mathcal{A}_2|)$ such that $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.*

3.3 The Powerset Construction

Next, we show that CAVs are determinizable in terms of deCAVs. As mentioned at the beginning of this section, we perform a powerset construction. At each transition, we collect the states that can be reached after an oscillation phase followed by reading a letter from Σ. In order to compute these states deterministically, we need to identify the exact set K of pairs $(p, q) \in Q \times Q$ such that the current counter value satisfies $\varphi_{p,q}$. To do so, we first "guess" K and then apply, as a transition guard, the *minterm* of the formulas $\varphi_{p,q}$ where $\varphi_{p,q}$ occurs positively iff $(p, q) \in K$.

Theorem 11. *Let $\mathcal{A}$ be a CAV. We can construct, in exponential time, a deCAV $\tilde{\mathcal{A}}$ such that $L(\tilde{\mathcal{A}}) = L(\mathcal{A})$.*

Proof. Fix a CAV $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$. Without loss of generality, we assume that there is a partition $Q = Q_1 \uplus Q_0 \uplus Q_{-1}$ satisfying Conditions [D1] and [D2] from Definition 7. We also assume that every transition guard is either $\mathsf{cnt} = 0$ or $\neg(\mathsf{cnt} = 0)$. A CAV can be brought into that form involving a linear blow-up. For every $p, q \in Q$, we compute, according to Lemma 4, $\varphi_{p,q}, \psi_{p,q} \in \mathsf{Guards}^{\mathrm{mod}}$ such that $[\![\varphi_{p,q}]\!] = X^{\mathcal{A}}_{p,q}$ and $[\![\psi_{p,q}]\!] = Y^{\mathcal{A}}_{p,q}$.

Given $P \subseteq Q$ (representing the set of possible "current" states), $\sigma \in \mathsf{Op} \cup \Sigma$, $K \subseteq Q \times Q$, and $\psi \in \mathsf{Guards}_0 := \{\mathsf{cnt} = 0\,, \neg(\mathsf{cnt} = 0)\}$, we let

- $\varphi_{K,\psi} = \psi \wedge \bigwedge_{(p,q) \in K} \varphi_{p,q} \;\wedge \bigwedge_{(p,q) \in Q^2 \setminus K} \neg\varphi_{p,q}$ (the minterm associated with K and ψ), and
- $\langle\!\langle P, K, \psi, \sigma \rangle\!\rangle = \{q \in Q \mid$ there is $(p, p') \in K \cap (P \times Q)$ such that $p' \xrightarrow{\psi|\sigma} q\}$ (which is the set of successor states when the counter value satisfies $\varphi_{K,\psi}$ and σ is executed).

We are now ready to define the deCAV $\tilde{\mathcal{A}} = (\tilde{Q}, \Sigma, \longrightarrow\!\!\!\!\!\rightarrow, \tilde{\iota}, \tilde{F})$ satisfying $L(\tilde{\mathcal{A}}) = L(\mathcal{A})$:

[P1] The set of states is $\tilde{Q} = (2^{Q_1} \times \{1\}) \cup (2^{Q_0} \times \{0\}) \cup (2^{Q_{-1}} \times \{-1\})$. It induces a partitioning $\tilde{Q} = \tilde{Q}_1 \uplus \tilde{Q}_0 \uplus \tilde{Q}_{-1}$ in the obvious way. In fact, we include the second component $\{1, 0, -1\}$ to be able to distinguish three different "versions" of the empty set so that we meet the requirements from Definition 7.

[P2] The initial state is $\tilde{\iota} = (\{\iota\}, 0) \in \tilde{Q}_0$.

[P3] For $(P, m) \in \tilde{Q}$, we set $\tilde{F}((P, m)) = \begin{cases} \bigvee_{(p,q) \in P \times F} \psi_{p,q} & \text{if } m = 0\,, \\ \neg\mathsf{tt} & \text{if } m \neq 0\,. \end{cases}$

[P4] For all $(P, m) \in \tilde{Q}$, $a \in \Sigma$, $K \subseteq Q \times Q$, and $\psi \in \mathsf{Guards}_0$, we have a transition

$$(P, m) \xrightarrow{\varphi_{K,\psi}|a}\!\!\!\!\!\rightarrow (\langle\!\langle P, K, \psi, a \rangle\!\rangle, 0)\,.$$

[P5] For all $(P, m) \in \tilde{Q}$, $op \in \mathsf{Op} \setminus \{-m\}$, $K \subseteq Q \times Q$, and $\psi \in \mathsf{Guards}_0$, we have a transition

$$(P, m) \xrightarrow{\varphi_{K,\psi} \mid op} (\langle\!\langle P, K, \psi, op \rangle\!\rangle, op) \,.$$

This concludes the construction of $\tilde{\mathcal{A}}$. Note that $\tilde{\mathcal{A}}$ has exponential size and can be computed in exponential time. In the appendix, we show that $\tilde{\mathcal{A}}$ is indeed deterministic and that $L(\tilde{\mathcal{A}}) = L(\mathcal{A})$. ☐

Corollary 12. *CAVs are effectively closed under all boolean operations.*

The *universality problem for CAVs* is defined as follows: Given a CAV $\mathcal{A}$ over some alphabet Σ, do we have $L(\mathcal{A}) = (\Sigma \times \mathbb{N})^*$? The *inclusion problem* is defined as expected.

Theorem 13. *The universality problem and the inclusion problem for CAVs are PSPACE-complete. In both cases, PSPACE-hardness already holds when $|\Sigma| = 1$.*

Proof. To solve the universality problem for a given CAV $\mathcal{A}$ in (nondeterministic) polynomial space, one constructs (the complement of) $\tilde{\mathcal{A}}$ *on-the-fly*, applying Theorem 2, Lemma 6, Proposition 9, and Theorem 11. We start in the initial state $(\{\iota\}, 0)$. In addition to the current state of $\tilde{\mathcal{A}}$, we will maintain a component for the current counter value. In fact, the latter can be supposed to be polynomially bounded (cf. [5] for a tight upper bound) in the size of $\tilde{\mathcal{A}}'$, which is the CAV obtained from $\tilde{\mathcal{A}}$ according to Lemma 6. The size of $\tilde{\mathcal{A}}'$ is exponential in the size of $\mathcal{A}$ (the construction from Lemma 6 does not add an extra blow-up), and so the required information can be stored in polynomial space.

At each new step, we guess $\sigma \in \mathsf{Op} \cup \Sigma$, $K \subseteq Q \times Q$, and $\psi \in \mathsf{Guards}_0$ and compute, according to [P4] or [P5], the set $\langle\!\langle P, K, \psi, \sigma \rangle\!\rangle$. While this computation is immediate, we still have to ensure that the transition guard $\varphi_{K,\psi}$ is satisfied. But this can be derived from the (bounded) current counter value by an easy membership test. To check whether a state $(P, m) \in \tilde{Q}_0$ is "final" (for the complement of $\tilde{\mathcal{A}}$), we evaluate the *negation* of $\tilde{F}((P, m))$ in the very same way. Clearly, all this takes no more than polynomial space. For the inclusion problem, we rely on Proposition 10 and perform the determinization procedure on-the-fly for *both* of the given CAVs.

For the lower bound, we restrict to universality. We reduce from the universality problem for ordinary finite automata, which is known to be PSPACE-complete [20]. Let $\mathcal{A}$ be a finite automaton over some alphabet $\Gamma = \{a_0, \ldots, a_{n-1}\}$. We construct a CAV $\mathcal{A}'$ over a singleton alphabet Σ such that $L(\mathcal{A}) = \Gamma^*$ iff $L(\mathcal{A}') = (\Sigma \times \mathbb{N})^*$. The idea is to represent letter a_i by (counter) value i. To obtain $\mathcal{A}'$, an a_i-transition in $\mathcal{A}$ is replaced with a gadget that nondeterministically outputs i or any other natural number strictly greater than $n - 1$. ☐

4 Register One-Counter Automata with Counter Visibility

Having established that CAVs constitute a robust automata model, we now include registers. Registers are a natural storing mechanism when dealing with infinite domains [17, 27]. They will allow us to model requirements such as "a service is only provided if the energy level is above a certain threshold". Unfortunately, even the nonemptiness problem becomes undecidable for that extension so that we will have to impose some restriction.

4.1 CAVs with Registers

Let R be a finite set (of registers). The set $\mathsf{Guards}^{\mathrm{store}}(R)$ of *guards over R* is generated by the following grammar:

$$\mathsf{Guards}^{\mathrm{store}}(R): \quad \varphi \ ::= \ \mathsf{tt} \ \mid \ \mathsf{cnt} = 0 \ \mid \ \neg(\mathsf{cnt} = 0) \ \mid \ \mathsf{cnt} \bowtie r$$

where $\bowtie \in \{<, =, >\}$ and $r \in R$. Satisfaction of a guard $\varphi \in \mathsf{Guards}^{\mathrm{store}}(R)$ is now defined wrt. a pair (x, ν) where $x \in \mathbb{N}$ is a counter value and $\nu \in \mathbb{N}^R$ a *register valuation*. We write $(x, \nu) \models \varphi$ if φ is evaluated to true when cnt is interpreted as x and register $r \in R$ as $\nu(r) \in \mathbb{N}$. In particular, $(x, \nu) \models \mathsf{cnt} \bowtie r$ if $x \bowtie \nu(r)$.

Our extended model is essentially a CAV, but there are additional transitions of the form $p \xrightarrow{\varphi|r} q$, which assign to register $r \in R$ the current counter value. A guard of the form $\mathsf{cnt} \bowtie r$ then allows us to compare the stored value with a counter value obtained later in a run.

Definition 14. A *register one-counter automaton with counter visibility* (RCAV) is a tuple $\mathcal{A} = (Q, \Sigma, R, \longrightarrow, \iota, F)$ where Q is the finite set of *states*, Σ is a nonempty finite alphabet, R is the finite set of *registers*, $\iota \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, and $\longrightarrow \subseteq Q \times \mathsf{Guards}^{\mathrm{store}}(R) \times (\mathsf{Op} \cup \Sigma \cup R) \times Q$ is the *transition relation*. We assume that the sets Op, Σ, and R are pairwise disjoint.

Note that an RCAV with empty register set is simply a CAV.

The set of configurations of $\mathcal{A}$ is $\mathit{Conf}_{\mathcal{A}} := Q \times \mathbb{N} \times \mathbb{N}^R$, the third component taking into account the register contents. The *initial configuration* is $(\iota, 0, \nu)$ where $\nu(r) = 0$ for all $r \in R$. The set of *final configurations* is $F \times \mathbb{N} \times \mathbb{N}^R$.

For a transition $t \in \longrightarrow$, we again define a global transition relation $\Longrightarrow_{\mathcal{A},t} \subseteq \mathit{Conf}_{\mathcal{A}} \times (\{\varepsilon\} \cup (\Sigma \times \mathbb{N})) \times \mathit{Conf}_{\mathcal{A}}$. For configurations $(q, x, \nu), (q', x', \nu') \in \mathit{Conf}_{\mathcal{A}}$ and a transition $t = (q, \varphi, \sigma, q') \in \longrightarrow$ such that $(x, \nu) \models \varphi$, we have

- $(q, x, \nu) \xRightarrow{\varepsilon}_{\mathcal{A},t} (q', x', \nu')$ provided one of the following holds:
  - $\sigma \in \mathsf{Op}$, $x' = x + \sigma$, and $\nu' = \nu$, or
  - $\sigma \in R$, $x' = x$, and $\nu' = \nu[\sigma \mapsto x]$ (i.e., ν' coincides with ν except that it maps σ to x),
- $(q, x, \nu) \xRightarrow{(\sigma,x)}_{\mathcal{A},t} (q', x', \nu')$ if $\sigma \in \Sigma$, $x' = x$, and $\nu' = \nu$.

(Accepting) runs and the language $L(\mathcal{A}) \subseteq (\Sigma \times \mathbb{N})^*$ of $\mathcal{A}$ are then defined similarly to CAVs. A *run* of $\mathcal{A}$ is a sequence

$$\rho = (q_0, x_0, \nu_0) \xRightarrow{\tau_1}_{\mathcal{A},t_1} (q_1, x_1, \nu_1) \xRightarrow{\tau_2}_{\mathcal{A},t_2} \dots \xRightarrow{\tau_n}_{\mathcal{A},t_n} (q_n, x_n, \nu_n) \tag{1}$$

(where $n \geq 0$ and $\tau_i \in \{\varepsilon\} \cup (\Sigma \times \mathbb{N})$) such that (q_0, x_0, ν_0) is the initial configuration. We say that ρ is *accepting* if its last configuration is final. Again, what we observe during ρ is $\mathit{obs}(\rho) := \tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_n \in (\Sigma \times \mathbb{N})^*$. The language of $\mathcal{A}$ is then $L(\mathcal{A}) = \{\mathit{obs}(\rho) \mid \rho \text{ is an accepting run of } \mathcal{A}\}$.

Example 15. Figure 3 depicts an RCAV $\mathcal{A}$ over $\Sigma = \{a, b\}$ and set of registers $R = \{r\}$. Its language is $L(\mathcal{A}) = \{u(b, i)v(a, j)w \mid i, j \in \mathbb{N} \text{ with } i < j \text{ and } u, v, w \in (\{a\} \times \mathbb{N})^*\}$. In fact, after reading the unique b, the automaton makes sure that the counter level eventually exceeds the value that has been registered when b was read. Similarly, we can construct an RCAV $\mathcal{B}$ over $\Sigma = \{a\}$ such that $L(\mathcal{B}) = \{u(a, i)v(a, i)w \mid i \in \mathbb{N} \text{ and } u, v, w \in (\{a\} \times \mathbb{N})^*\}$. Though both languages look similar, the complement of $L(\mathcal{A})$ is recognized by some RCAV (we can ensure that no output value exceeds the registered counter value), while the complement of $L(\mathcal{B})$ is not (like for register automata over infinite alphabets [17]). Intuitively, one would need an unbounded number of registers to ensure that no counter value is output twice.

Note that RCAVs can easily simulate a two-counter Minsky machine: One register stores the current value of counter one, and one register stores the value of the second counter. Thus, we obtain another negative result concerning RCAVs:

Theorem 16. *Nonemptiness for RCAVs is undecidable.*

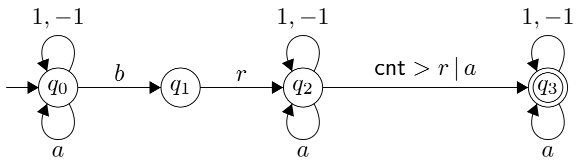

Figure 3 A register CAV

4.2 Test-Bounded RCAVs

Given the above undecidability result, we are in the quest for suitable modifications of the model. In several settings of infinite-state systems, a fruitful approach has been to underapproximate the actual system behavior (e.g., [1, 13, 18, 23]). A natural restriction in our case is to impose a bound on the number of register tests that are allowed during a run. Note that this still generalizes the basic model of CAVs.

Let $k \in \mathbb{N}$. Consider the run ρ as in (1) above and suppose $t_i = (q_{i-1}, \varphi_i, \sigma_i, q_i)$. We say that ρ is *k-test-bounded* if the number of steps $i \in [n]$ such that guard φ_i is of the form $\mathsf{cnt} \bowtie r$ for some $r \in R$, is at most k. With this, we set $L_k(\mathcal{A}) = \{obs(\rho) \mid \rho$ is a k-test-bounded accepting run of $\mathcal{A}\}$.

Example 17. Let $\mathcal{A}$ be the RCAV from Figure 3. We have $L(\mathcal{A}) = L_1(\mathcal{A})$.

We define *nonemptiness for test-bounded RCAVs* as the following problem: Given an RCAV $\mathcal{A}$ and $k \in \mathbb{N}$ (encoded in unary), do we have $L_k(\mathcal{A}) \neq \emptyset$?

Theorem 18. *Nonemptiness for test-bounded RCAVs is NP-complete.*

Proof. The proof structure is inspired by [13], where pushdown systems with several reversal-bounded counters are translated into existential Presburger formulas. The concrete steps are different, though. *Existential Presburger formulas* Φ over the natural numbers are built according to the grammar $e ::= 0 \mid 1 \mid \mathsf{x} \mid e + e \quad \Phi ::= e = e \mid e > e \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \exists \mathsf{x} \Phi$ where x is a variable. The *solution space* of a formula Φ is the set of variable assignments from $\mathbb{N}^{\mathit{Free}(\Phi)}$ that make the formula true, $\mathit{Free}(\Phi)$ being the set of free variables of Φ. Checking whether the solution space of a given formula is nonempty is NP-complete [26].

Let $\mathcal{A} = (Q, \Sigma, R, \longrightarrow, \iota, F)$ be the given RCAV with $|R| \geq 1$, and let $k \geq 1$ be the bound on the number of register tests (when $|R| = 0$ or $k = 0$, the problem reduces to nonemptinesss for CAVs). To facilitate the proof, we will make a couple of assumptions/modifications on $\mathcal{A}$. First, we suppose that $\mathsf{Op} = \{1, 0, -1\}$, with the expected meaning of 0-transitions. As we are interested in nonemptiness, we require, without loss of generality, that there are no Σ-transitions in $\mathcal{A}$. Moreover, we will assume that each run of $\mathcal{A}$ starts with a sequence of transitions $(t_r)_{r \in R}$ where the label of t_r is $\mathsf{tt}|r$. Finally, we suppose that register tests are only applied on 0-transitions and that, conversely, all 0-transitions carry register tests. In the following, *write* will refer to a register update, and *read* to a register test.

Let us outline the proof. With every k-test-bounded run ρ of $\mathcal{A}$, we associate an abstraction $\mathit{abstr}(\rho) \in \Gamma^*$, where Γ is a new finite alphabet. A letter from Γ gathers some (global) information on the run that a single transition does not provide, such as "write on r that will be followed by another write on r before r is read". We will construct a (small) existential Presburger formula $\Phi_{\mathcal{A}}$ with free variables $(\mathsf{x}_\alpha)_{\alpha \in \Gamma}$, whose solution space is precisely the Parikh image of $\mathfrak{L}_k(\mathcal{A}) := \{\mathit{abstr}(\rho) \mid \rho$ is a k-test-bounded accepting run of $\mathcal{A}\}$. Thus, nonemptiness of $L_k(\mathcal{A})$ is reduced to nonemptiness of the solution space of $\Phi_{\mathcal{A}}$.

ρ	r_1	r_2	1	1	r_1	1	r_2	r_1	-1	r_1	1	$(\mathsf{cnt} = r_2)$	r_2	-1	$(\mathsf{cnt} = r_1)$
	$\overline{r_1}$	$\overline{r_2}$	1	1	$\overline{r_1}$	1	r_2	$\overline{r_1}$	-1	r_1	1	$(\mathsf{cnt} = r_2, 1, 0)$	$\overline{r_2}$	-1	$(\mathsf{cnt} = r_1, 2, 0)$
$abstr(\rho)$	0	0	0	0	0	0	1	1	1	2	2	2	2	2	2
	0	0	0	0	0	0	0	0	0	0	0	1	1	1	2

Figure 4 Abstracting a run

The first step is to define an abstraction of $\mathcal{A}$ in terms of an ordinary one-counter automaton $\mathcal{B}$ over Γ. By [30], the Parikh image of $L(\mathcal{B}) \subseteq \Gamma^*$ can be described by an existential Presburger formula $\Phi_{\mathcal{B}}$, and this formula can be computed in polynomial time. As $L(\mathcal{B})$ is a one-counter language, it will already take into account zero tests in $\mathcal{A}$. However, we still have to check whether the (at most k-many) register tests are satisfied. Thus, we only have $\mathfrak{L}_k(\mathcal{A}) \subseteq L(\mathcal{B})$. However, using the information provided by Γ, we are able to enforce the missing requirements in terms of another existential Presburger formula Ψ, so that the Parikh image of $\mathfrak{L}_k(\mathcal{A})$ is precisely the solution space of $\Phi_{\mathcal{A}} = \Phi_{\mathcal{B}} \wedge \Psi$.

We will now define the alphabet Γ. Let $\overline{R} = \{\overline{r} \mid r \in R\}$, i.e., each register gets a (distinct) copy of r. In the following, $\overline{r}$ will denote a write (register update) that is never read (tested), i.e., r will be overwritten again before it is read. We let $\text{Tests} = \{\mathsf{cnt} \bowtie r \mid {\bowtie} \in \{<, =, >\}$ and $r \in R\}$ and, for a given $r \in R$, $\text{Tests}_r = \{\mathsf{cnt} \bowtie r \mid {\bowtie} \in \{<, =, >\}\}$. Finally, we set $\text{Write} = R \cup \overline{R}$ and $\text{Read} = \text{Tests} \times [k] \times [k]_0$. Here, $(\mathsf{cnt} \bowtie r, i, j) \in \text{Read}$ denotes a read whose unique corresponding write is in *phase* (i, j) (the meaning of (i, j) is defined below). With this, our new alphabet is $\Gamma = (\mathsf{Op} \cup \text{Write} \cup \text{Read}) \times [k]_0 \times [k]_0$.

With a k-test-bounded run ρ employing transitions $t_1 \dots t_n$, we can now associate the word $abstr(\rho) = (\sigma_1, i_1, j_1) \dots (\sigma_n, i_n, j_n) \in \Gamma^*$. If transition t_i (i) executes a counter operation $op \in \{1, -1\}$, then $\sigma_i = op$, (ii) performs a write $r \in R$ that is read later in the run, then $\sigma_i = r$, (iii) performs a write that is not read at all or overwritten before it is read, then $\sigma_i = \overline{r}$, and (iv) performs a read $\mathsf{cnt} \bowtie r$ that refers to the (most recent) write to r at position $\ell \in [n]$, then $\sigma_i = (\mathsf{cnt} \bowtie r, i_\ell, j_\ell)$ where i_ℓ, j_ℓ are given as follows. In fact, for each $\ell \in [n]$, we let $i_\ell = |\{i \in [\ell] \mid \sigma_i \in R\}|$ and $j_\ell = |\{j \in [\ell] \mid t_j \text{ performs a register test}\}|$. The translation is illustrated in Figure 4.

From $\mathcal{A}$, we now construct the (ordinary) one-counter automaton $\mathcal{B} = (\tilde{Q}, \Gamma, \twoheadrightarrow, \tilde{\iota}, \tilde{F})$ whose language is an overapproximation of $\mathfrak{L}_k(\mathcal{A})$. Its transition relation is of the form $\twoheadrightarrow \subseteq \tilde{Q} \times \mathsf{Guards} \times (\mathsf{Op} \cup \Gamma) \times \tilde{Q}$, and its language $L(\mathcal{B}) \subseteq \Gamma^*$ is defined as expected. We let $\tilde{Q} = Q \times [k]_0 \times [k]_0$. Moreover, $\tilde{\iota} = (\iota, 0, 0)$ and $\tilde{F} = F \times [k]_0 \times [k]_0$. Let us turn to the transitions. For $(q, i, j), (q', i', j') \in \tilde{Q}$, a transition $(q, \varphi, \sigma, q') \in \longrightarrow$ of $\mathcal{A}$, $\sigma' \in \mathsf{Op} \cup \text{Write} \cup \text{Read}$, and $\varphi' \in \mathsf{Guards}$, we introduce a transition

$$(q, i, j) \xrightarrow{\varphi' | (\sigma', i', j')} \!\!\!\!\rightarrow (q', i', j')$$

if one of the following holds:

- $\sigma \in \{1, -1\}$, $\varphi' = \varphi$, and $(\sigma', i', j') = (\sigma, i, j)$ (counter operation),
- $\sigma \in R$, $\varphi' = \varphi$, and $(\sigma', i', j') = (\sigma, i + 1, j)$ (write that will be read),
- $\sigma \in R$, $\varphi' = \varphi$, and $(\sigma', i', j') = (\overline{\sigma}, i, j)$ (write that will be dismissed),
- $\sigma = 0$, $\varphi' = \mathsf{tt}$, $\sigma' \in \{\varphi\} \times [i] \times [j]_0$ and $(i', j') = (i, j + 1)$ (read).

Recall that, due to [30], we get a small existential Presburger formula $\Phi_{\mathcal{B}}$ for the Parikh image of $L(\mathcal{B})$. Next, we define an existential Presburger formula Ψ such that $\Phi_{\mathcal{B}} \wedge \Psi$ defines the Parikh image of $\mathfrak{L}_k(\mathcal{A})$. Recall that we have one (free) variable x_α for every $\alpha \in \Gamma$. For $A \subseteq \mathsf{Op} \cup \mathrm{Write} \cup \mathrm{Read}$ and sets $I, J \subseteq [k]_0$, we use $\sum\big(A, I, J\big)$ as a macro for the frequently used expression $\sum_{\alpha \in A \times I \times J} \mathsf{x}_\alpha$. For readability, when one component describes the whole domain (e..g., $I = [k]_0$), we will just write _ as a placeholder. Moreover, in a singleton set such as $\{i\}$, we omit brackets and simply write i. Finally, we use $[i, j]$ and $[i, j)$ for $\{i, i+1, \ldots, j\}$ and $\{i, i+1, \ldots, j-1\}$, respectively.

The following formula says that every write should have a corresponding read:

$$\bigwedge_{r \in R} \; \bigwedge_{(i,j) \in [k] \times [k]_0} \Big(\mathsf{x}_{(r,i,j)} \geq 1 \implies \sum\big(\mathrm{Tests}_r \times \{i\} \times \{j\}, \_, \_\big) \geq 1 \Big) \tag{2}$$

Conversely, every read has a unique corresponding (i.e., most recent) write (this is why we assumed that $\mathcal{A}$ performs some redundant register initializations):

$$\bigwedge_{r \in R} \; \bigwedge_{(i,j) \in [k] \times [k]_0} \Big(\sum\big(\mathrm{Tests}_r \times \{i\} \times \{j\}, \_, \_\big) \geq 1 \implies \mathsf{x}_{(r,i,j)} = 1 \Big) \tag{3}$$

Let us formalize that a "real" write is not overwritten by another write before it is read:

$$\bigwedge_{\substack{(\mathsf{cnt} \bowtie r) \\ \in \mathrm{Tests}}} \; \bigwedge_{\substack{(i,j) \in [k] \times [k]_0 \\ (i',j') \in [k] \times [k]}} \Big(\mathsf{x}_{((\mathsf{cnt} \bowtie r, i, j), i', j')} = 1 \implies \sum\big(\{r, \overline{r}\}, [i, i'], [j, j')\big) = 1 \Big) \tag{4}$$

Finally, all register tests should be satisfied (subtraction is used here as a macro):

$$\bigwedge_{\substack{(\mathsf{cnt} \bowtie r) \\ \in \mathrm{Tests}}} \; \bigwedge_{\substack{(i,j) \in [k] \times [k]_0 \\ (i',j') \in [k] \times [k]}} \left(\begin{array}{l} \qquad \mathsf{x}_{((\mathsf{cnt} \bowtie r, i, j), i', j')} = 1 \\ \implies \left(\begin{array}{r} \sum\big(1, [0, i'], [0, j')\big) \\ - \; \sum\big(-1, [0, i'], [0, j')\big) \end{array} \right) \bowtie \left(\begin{array}{r} \sum\big(1, [0, i), [0, j]\big) \\ - \; \sum\big(-1, [0, i), [0, j]\big) \end{array} \right) \end{array} \right) \tag{5}$$

We let Ψ be the conjunction of the formulas (2)–(5), which concludes the construction. Clearly, Ψ has polynomial size so that checking nonemptiness of $L_k(\mathcal{A})$ is in NP. The NP-hardness proof is similar to that for classical register automata [25] (cf. appendix). $\square$

5 Conclusion

Our new semantics opens several directions for follow-up work. We may carry it over to other classes of infinite-state systems, e.g., to Petri nets as a model of concurrency. Our model would have a concrete interpretation in that context: In some states, we are allowed to observe the global system state, i.e., the token marking, while in others we are not (or we may only observe a projection to certain places). Are there infinite-state restrictions of Petri nets whose visibility semantics is robust?

Other natural questions arise: To which extent can we relax the requirement that the counter value be output with every letter $a \in \Sigma$? It may indeed be possible to deal with a bounded number of Σ-transitions between any two counter outputs. In the case of registers, one can imagine alternative restrictions of RCAVs such as imposing a bound on the number of reversals that the counter may effectuate. Finally, our automata are particularly suited to model the development of an energy level over time. In this realm, extensions to parametric one-counter automata [10] and infinite words would be worthwhile.

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A Proof of Lemma 8

Let us show that there is a unique minimal run ρ such that $obs(\rho) = w$. By [D2], the statement is obvious for $w = \varepsilon$. So assume $w \neq \varepsilon$ starts with $(a, x) \in \Sigma \times \mathbb{N}$. By [D1] and [D4a], when performing a counter increment $op = 1$ in some configuration (q, z), the next counter value that is output is necessarily greater than z. Similarly, we cannot return to the original value after a decrement $op = -1$, unless we perform some output action. Thus, a run outputting w will necessarily have a prefix of the form

$$\rho = (\iota, 0) \xRightarrow{\varepsilon}_{\mathcal{A},t_1} (p_1, 1) \xRightarrow{\varepsilon}_{\mathcal{A},t_2} (p_2, 2) \ldots \xRightarrow{\varepsilon}_{\mathcal{A},t_x} (p_x, x) \xRightarrow{(a,x)}_{\mathcal{A},t_{x+1}} (p_{x+1}, x)$$

such that $p_1, \ldots, p_x \in Q_1$ and $p_{x+1} \in Q_0$. By [D4b], for each $j \in [x + 1]$, the applicable transition t_j indeed exists and is uniquely determined.

Now, suppose the second letter of w is (b, y) and let $n = |y - x|$ and $d = \mathrm{sign}(y - x)$. Then, a run of w will extend ρ by

$$(p_{x+1}, x) \xRightarrow{\varepsilon}_{\mathcal{A},s_1} (q_1, x + d) \xRightarrow{\varepsilon}_{\mathcal{A},s_2} (q_2, x + 2d) \ldots \xRightarrow{\varepsilon}_{\mathcal{A},s_n} (q_n, y) \xRightarrow{(b,y)}_{\mathcal{A},s_{n+1}} (q_{n+1}, y)$$

such that $q_1, \ldots, q_n \in Q_d$ and $q_{n+1} \in Q_0$. Again, [D4b] makes sure that the transitions s_j exist and are uniquely determined. Continuing this scheme, we obtain a unique *minimal* run $\rho_{\mathcal{A}}(w)$ of $\mathcal{A}$ on w.

Of course, if $\rho_{\mathcal{A}}(w)$ is accepting, then $w \in L(\mathcal{A})$ by definition. Conversely, suppose $w \in L(\mathcal{A})$. As $\rho_{\mathcal{A}}(w)$ is uniquely determined, any accepting run of $\mathcal{A}$ on w must have $\rho_{\mathcal{A}}(w)$ as prefix. Towards a contradiction, suppose $\hat{\rho}$ is an accepting run on w and a proper extension of $\rho_{\mathcal{A}}(w)$. As $\hat{\rho}$ is accepting, it ends in a configuration (q, x) such that $x \models F(q)$. By [D3], $q \in Q_0$. However, due to [D1], the last transition of $\hat{\rho}$ produces a letter from $\Sigma \times \mathbb{N}$ so that $\hat{\rho}$ is no longer a run on w, a contradiction. Since $w \in L(\mathcal{A})$, $\rho_{\mathcal{A}}(w)$ itself has to be accepting.

B Proof of Proposition 10

For $i \in \{1, 2\}$, let $\mathcal{A}_i = (Q_i, \Sigma, \longrightarrow_i, \iota_i, F_i)$. Suppose that the induced partition of Q_i is $Q_i^1 \uplus Q_i^0 \uplus Q_i^{-1}$. We define $\mathcal{A} = (Q, \Sigma, \longrightarrow, \iota, F)$ by

- $Q = (Q_1^1 \times Q_2^1) \cup (Q_1^0 \times Q_2^0) \cup (Q_1^{-1} \times Q_2^{-1})$ (inducing the obvious partition),
- $\iota = (\iota_1, \iota_2)$,
- $F((q_1, q_2)) = \begin{cases} F_1(q_1) \wedge F_2(q_2) & \text{if } (q_1, q_2) \in Q_0\,, \\ \neg\mathsf{tt} & \text{otherwise.} \end{cases}$

Moreover, for transitions $q_1 \xrightarrow{\varphi_1|\sigma}_1 q_1'$ and $q_2 \xrightarrow{\varphi_2|\sigma}_2 q_2'$ such that $(q_1, q_2) \in Q$, we introduce a transition

$$(q_1, q_2) \xrightarrow{\varphi_1 \wedge \varphi_2|\sigma} (q_1', q_2')\,.$$

Note that, as $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic, we have $(q_1', q_2') \in Q$. We easily check that $\mathcal{A}$ is deterministic, too, and that we have $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.

C Correctness of $\tilde{\mathcal{A}}$ (from Theorem 11)

C.1 $\tilde{\mathcal{A}}$ is Deterministic

We show that $\tilde{\mathcal{A}}$ is deterministic according to Definition 7. We use the partition of $\tilde{Q}$ into $\tilde{Q}_1 \uplus \tilde{Q}_0 \uplus \tilde{Q}_{-1}$ where $\tilde{Q}_i = 2^{Q_i} \times \{i\}$.

From [P4] and [P5], we can deduce that $\mathcal{A}$ satisfies [D1] and [D4a]. The initial state $\tilde{\iota} = (\iota, 0) \in \tilde{Q}_0$ does not have any incoming transitions, as we assumed that ι has no incoming transitions in $\mathcal{A}$. This shows [D2]. By the definition of $\tilde{F}$, [D3] is satisfied, too. Towards showing [D4b], let $(P, m) \in \tilde{Q}$ and $\sigma \in \mathsf{Op} \cup \Sigma$ such that neither $m = 1$ and $\sigma = -1$ nor $m = -1$ and $\sigma = 1$. By [P4] and [P5], every $K \subseteq Q \times Q$ and $\psi \in \mathsf{Guards}_0$ determine a unique transition of the form

$$(P, m) \xrightarrow{\varphi_{K,\psi}|\sigma}\!\!\!\!\twoheadrightarrow (\langle\!\langle P, K, \psi, \sigma \rangle\!\rangle, m')$$

and all these transitions are in fact the unique σ-transitions outgoing from (P, m).

Clearly, for $K, K' \subseteq Q \times Q$ and $\psi, \psi' \in \mathsf{Guards}_0$, $(K, \psi) \neq (K', \psi')$, implies $[\![\varphi_{K,\psi}]\!] \cap [\![\varphi_{K',\psi'}]\!] = \emptyset$. Moreover, for all $x \in \mathbb{N}$, there are $K \subseteq Q \times Q$ and $\psi \in \mathsf{Guards}_0$ such that $x \in [\![\varphi_{K,\psi}]\!]$: We will choose $\psi = (\mathsf{cnt} = 0)$ if $x = 0$ and $\psi = \neg(\mathsf{cnt} = 0)$ if $x > 0$. With this, we have $x \models \psi$. Furthermore, let K be the set of pairs $(p, q) \in Q \times Q$ such that $x \models \varphi_{p,q}$. Clearly, $x \models \varphi_{K,\psi}$.

We conclude that $\tilde{\mathcal{A}}$ is deterministic according to Definition 7.

C.2 Proof of $L(\mathcal{A}) = L(\tilde{\mathcal{A}})$

Let $w = (a_1 x_1) \dots (a_n, x_n) \in (\Sigma \times \mathbb{N})^*$. We have seen in the proof of Lemma 8 that $\rho_{\tilde{\mathcal{A}}}(w)$ is uniquely determined and is of the form:

$$\begin{aligned}
(\tilde{\iota}, 0) = ((P_0, 0), 0) \quad & \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{1,1}} \dots \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{1,k_1}} \xRightarrow{(a_1, x_1)}_{\tilde{\mathcal{A}}, t_1} ((P_1, 0), x_1) \\
& \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{2,1}} \dots \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{2,k_2}} \xRightarrow{(a_2, x_2)}_{\tilde{\mathcal{A}}, t_2} ((P_2, 0), x_2) \\
& \qquad \vdots \\
& \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{n,1}} \dots \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{n,k_n}} \xRightarrow{(a_n, x_n)}_{\tilde{\mathcal{A}}, t_n} ((P_n, 0), x_n)
\end{aligned}$$

where, assuming $x_0 = 0$, we have $k_i = |x_i - x_{i-1}|$ for all $i \in [n]$, and $t_{i,j} = (\dots, op_i, \dots)$ with $op_i = \mathrm{sign}(x_i - x_{i-1}) \in \mathsf{Op}$.

By Lemma 8, we also have $w \in L(\tilde{\mathcal{A}})$ iff $\rho_{\tilde{\mathcal{A}}}(w)$ is accepting, i.e., $x_n \models \tilde{F}((P_n, 0))$. We will show that $\rho_{\tilde{\mathcal{A}}}(w)$ is accepting iff $w \in L(\mathcal{A})$. By the definition of $\tilde{F}$, it is sufficient to prove that

$$P_n = \mathit{Reach}_{\mathcal{A}}(w)$$

where $\mathit{Reach}_{\mathcal{A}}(w)$ is defined to be the set of states $q \in Q$ for which there is a *minimal* run ρ of $\mathcal{A}$ on w that ends in q. We proceed by induction on the length of w.

Base case. Suppose $w = \varepsilon$, i.e., $n = 0$. The unique minimal run of $\mathcal{A}$ on w is $(\iota, 0)$, and that of $\tilde{\mathcal{A}}$ is $(\tilde{\iota}, 0) = ((P_0, 0), 0)$ where $P_0 = \{\iota\}$. Thus, $P_0 = \mathit{Reach}_{\mathcal{A}}(w)$.

Inductive step. Suppose $w = (a_1, x_1) \dots (a_n, x_n) \in (\Sigma \times \mathbb{N})^*$ with $n \geq 1$ and let $u = (a_1, x_1) \dots (a_{n-1}, x_{n-1})$. Recall that we set $x_0 = 0$. We have

$$((P_{n-1}, 0), x_{n-1}) = \gamma_0 \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{n,1}} \gamma_1 \dots \xRightarrow{\varepsilon}_{\tilde{\mathcal{A}}, t_{n,k_n}} \gamma_{k_n} \xRightarrow{(a_n, x_n)}_{\tilde{\mathcal{A}}, t_n} ((P_n, 0), x_n)$$

where, for $i \in \{0, 1, \dots, k_n\}$, we assume

$$\gamma_i = ((P^{(i)}, op_n), x_{n-1} + i \cdot op_n)\,.$$

Starting from the induction hypothesis $P_{n-1} = \mathit{Reach}_{\mathcal{A}}(u)$, we will show $P_n = \mathit{Reach}_{\mathcal{A}}(w)$.

We first show the inclusion $\underline{P_n \subseteq \mathit{Reach}_{\mathcal{A}}(w)}$. Let $q \in P_n$ and set $k = k_n$. We will determine states $p_0 \in P^{(0)}, \dots, p_k \in P^{(k)}$ and partial runs $\rho_1, \dots, \rho_k, \rho$ of $\mathcal{A}$ such that, for all $i \in [k]$,

- ρ_i is a partial run on ε from $(p_{i-1}, x_{n-1} + (i-1) \cdot \mathit{op}_n)$ to $(p_i, x_{n-1} + i \cdot \mathit{op}_n)$, and
- ρ is a partial run on (a_n, x_n) from $(p_k, x_{n-1} + k \cdot \mathit{op}_n) = (p_k, x_n)$ to (q, x_n).

Suppose, for the moment, that we already have these states and partial runs. By the induction hypothesis $P_{n-1} = \mathit{Reach}_{\mathcal{A}}(u)$, there is a minimal run ρ_0 of $\mathcal{A}$ on u ending in (p_0, x_{n-1}). Thus, concatenating $\rho_0, \rho_1, \dots, \rho_k, \rho$, we obtain a minimal run of $\mathcal{A}$ on w ending in (q, x_n). We deduce that $q \in \mathit{Reach}_{\mathcal{A}}(w)$.

It remains to determine $p_0, p_1, \dots, p_k$ as well as $\rho_1, \dots, \rho_k, \rho$. We start with p_k and ρ. By [P4], there are a unique set $K \subseteq Q \times Q$ and a unique $\psi \in \mathsf{Guards}_0$ such that $x_n \models \varphi_{K,\psi}$. The latter implies $x_n \models \varphi_{\hat{p},\hat{q}}$ for all $(\hat{p}, \hat{q}) \in K$ and $x_n \models \psi$. By the definition of $P_n = \langle\!\langle P^{(k)}, K, \psi, a_n \rangle\!\rangle$, there is $(p, p') \in K \cap (P^{(k)} \times Q)$ such that $p' \xrightarrow{\psi | a_n} q$. Denote the latter transition by t. Due to $(p, p') \in K$, we have $x_n \models \varphi_{p,p'}$ and, therefore,

$$(p, x_n) \overset{\varepsilon}{\Longrightarrow}{}^{*}_{\mathcal{A}} (p', x_n) \,.$$

Moreover, since $x_n \models \psi$, we get

$$(p', x_n) \xRightarrow{(a_n, x_n)}_{\mathcal{A},t} (q, x_n) \,.$$

We set $p_k = p$ and, altogether, obtain a run ρ from (p_k, x_n) to (q, x_n) as required.

Now, suppose that we have already determined $p_i, \dots, p_k$ and $\rho_{i+1}, \dots, \rho_k$ for some $i \in \{1, \dots, k\}$. Similarly to the previous case, but now relying on [P5], we will construct p_{i-1} and ρ_i with the required properties.

Again, there are a unique set $K \subseteq Q \times Q$ and a unique $\psi \in \mathsf{Guards}_0$ such that $y_{i-1} = x_{n-1} + (i-1) \cdot \mathit{op}_n \models \varphi_{K,\psi}$. This implies $y_{i-1} \models \varphi_{\hat{p},\hat{q}}$ for all $(\hat{p}, \hat{q}) \in K$ and $y_{i-1} \models \psi$. By the definition of $P^{(i)} = \langle\!\langle P^{(i-1)}, K, \psi, \mathit{op}_n \rangle\!\rangle$, there is $(p, p') \in K \cap (P^{(i-1)} \times Q)$ such that $p' \xrightarrow{\psi | \mathit{op}_n} p_i$. Again, denote the latter transition by t. By $(p, p') \in K$, we have $y_{i-1} \models \varphi_{p,p'}$, which implies

$$(p, y_{i-1}) \overset{\varepsilon}{\Longrightarrow}{}^{*}_{\mathcal{A}} (p', y_{i-1}) \,.$$

As we also have $y_{i-1} \models \psi$, we get

$$(p', y_{i-1}) \overset{\varepsilon}{\Longrightarrow}_{\mathcal{A},t} (p_i, y_{i-1} + \mathit{op}_n) \,.$$

We set $p_{i-1} = p$ and, altogether, obtain a run ρ_i from (p_{i-1}, y_{i-1}) to $(p_i, y_{i-1} + \mathit{op}_n)$ as required.

Towards showing $\underline{\mathit{Reach}_{\mathcal{A}}(w) \subseteq P_n}$, let $q \in \mathit{Reach}_{\mathcal{A}}(w)$. By the definition of $\mathit{Reach}_{\mathcal{A}}(w)$, there exists a minimal run of $\mathcal{A}$ on w ending in (q, x_n). That run can be decomposed into a run ρ and a partial run ρ' such that ρ is a minimal run on $u = a_1 \dots a_{n-1}$.

Suppose the last configuration of ρ (and, therefore, the first configuration of ρ') is (q_0, y_0) and that

$$\rho' = (q_0, y_0) \overset{\tau_1}{\Longrightarrow}_{\mathcal{A},s_1} (q_1, y_1) \overset{\tau_2}{\Longrightarrow}_{\mathcal{A},s_2} \dots \overset{\tau_\ell}{\Longrightarrow}_{\mathcal{A},s_\ell} (q_\ell, y_\ell) = (q, x_n)$$

Note that, as ρ' is a minimal partial run (the definition of *minimal* carries over to partial runs as expected) on (a_n, x_n), we have $\tau_1 = \ldots = \tau_{\ell-1} = \varepsilon$ and $\tau_\ell = (a_n, x_n)$.

By the induction hypothesis, we have $q_0 \in \mathit{Reach}_{\mathcal{A}}(u) = P_{n-1}$. We will now decompose ρ' to fit it into a normal form accepted by $\tilde{\mathcal{A}}$. In fact, letting $k = k_n = |x_n - x_{n-1}|$, we can find indices

$$0 = i_0 \leq i_0^* < i_1 \leq i_1^* < \ldots < i_k \leq i_k^* < i_{k+1} = \ell$$

such that

(i) $y_{i_j} = y_{i_j^*}$ for all $j \in \{0, \ldots, k\}$,
(ii) $i_j = i_{j-1}^* + 1$ for all $j \in \{1, \ldots, k+1\}$, and
(iii) $y_{i_j} = y_{i_{j-1}^*} + \mathit{op}_n$ for all $j \in \{1, \ldots, k\}$.

(recall that $\mathit{op}_n = \mathrm{sign}(x_n - x_{n-1})$). By (i), we have $y_{i_0} \models \varphi_{q_{i_0}, q_{i_0^*}}$. Choose $K_1 \subseteq Q \times Q$ and $\psi_1 \in \mathsf{Guards}_0$ such that $y_{i_0} \models \varphi_{K_1, \psi_1}$. Recall that K_1 and ψ_1 are uniquely determined. We will show that

$$q_{i_1} \in P^{(1)} = \langle\!\langle P^{(0)}, K_1, \psi_1, \mathit{op}_n \rangle\!\rangle\,,$$

which reduces to showing

- $(q_{i_0}, q_{i_0^*}) \in K_1$ and
- $q_{i_0^*} \xrightarrow{\psi_1 | \mathit{op}_n} q_{i_1}$.

The first statement is true due to $y_{i_0} \models \varphi_{q_{i_0}, q_{i_0^*}}$. The second statement follows from (ii) and (iii) and the fact that ψ_1 is the only guard from Guards_0 that is satisfied by $y_{i_0} = y_{i_0^*}$ (recall that we assumed that all transition guards employed by $\mathcal{A}$ are from Guards_0).

In exactly the same way, we then successively prove $q_{i_2} \in P^{(2)}, \ldots, q_{i_k} \in P^{(k)}$ and, finally, $q_{i_{k+1}} = q \in P_n$ so that we are done.

We showed $L(\mathcal{A}) = L(\tilde{\mathcal{A}})$, which concludes proof of Theorem 11.

D NP Lower Bound for Theorem 18

There is a simple reduction from 3-CNF-SAT, which is an adaptation of a reduction for ordinary register automata on infinite alphabets [25]. Interestingly, the counter does not even need to exceed 1 to get NP-hardness. For every variable v that occurs in the given 3-CNF-formula Φ, we introduce a register r_v. The RCAV $\mathcal{A}_\Phi$ that we are going to construct proceeds in two phases. In the initialization phase, $\mathcal{A}_\Phi$ chooses, nondeterministically, among two possibilities for every variable v. It sets the counter value to 1 or to 0 and stores that value in register r_v. The contents of r_v is to be interpreted as the truth value of v. In the evaluation phase, we first set the counter value to 0. For every clause $l_1 \vee l_2 \vee l_3$ of Φ, $\mathcal{A}_\Phi$ then chooses nondeterministically among three transitions, which correspond to the literals l_i (each of which is a variable or its negation). If $l_i = v$, then the transition will check $\mathsf{cnt} < r_v$ (i.e., v has been set to true). If $l_i = \neg v$, then it will check $\mathsf{cnt} = r_v$ (i.e., v has been set to false). After processing the last clause, $\mathcal{A}_\Phi$ goes into a final state. The size of $\mathcal{A}_\Phi$ is linear in the size of Φ. Moreover, Φ is satisfiable iff $L_k(\mathcal{A}_\Phi)$ is nonempty, where k is the number of clauses in Φ.