Wadge Games
Between 1-Counter Automata
and Models of Set Theory

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Acceptance of infinite words

- **In the sixties**,
  Acceptance of infinite words by finite automata was firstly considered by **Büchi** in order to study the decidability of the monadic second order theory S1S of one successor over the integers.
- Since then ω-regular languages accepted by Büchi automata and their extensions have been much studied and used for **specification and verification of non terminating systems**.

Büchi acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of final states $F \subseteq K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff there is some state $q_f \in F$ appearing infinitely often during the reading of σ.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

An ω-language $L \subseteq \Sigma^\omega$ is accepted by $\mathcal{A}$ if it is the set of infinite words $\sigma \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Muller acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of accepting sets of states $\mathcal{F} \subseteq 2^K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff the set of states appearing infinitely often during this run is an accepting set $F \in \mathcal{F}$.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

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Context free or regular ω-languages

(Cohen and Gold 1977; Linna 1976)

Let $L \subseteq \Sigma^\omega$. Then the following propositions are equivalent :

- L is accepted by a Büchi pushdown automaton.
- L is accepted by a Muller pushdown automaton.
- $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$,
  for some context free finitary languages U_i and V_i.
- L is a context free ω-language.

A similar theorem holds if we:

- omit the pushdown stack and replace context free by regular,
- or replace pushdown and context-free by 1-counter.

Languages of infinite words

An ω-language over the alphabet Σ is a subset of Σ^ω.

An ω-language is regular iff it is accepted by a Büchi automaton.

An ω-language is context free iff it is accepted by a Büchi pushdown automaton.

A 1-counter ω-language is an ω-language which is accepted by a 1-counter Büchi automaton.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Wadge Reducibility

Definition (Wadge 1972)

For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, $L \leq_W L'$ iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.

L and L' are Wadge equivalent ($L \equiv_W L'$) iff $L \leq_W L'$ and $L' \leq_W L$. .

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called Wadge degrees.

Intuitively $L \leq_W L'$ means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function.

Wadge Degrees

Hence the Wadge degree of an ω-language is a measure of its topological complexity.

Wadge degrees were firstly studied by Wadge for Borel sets using Wadge games.

There is a close relationship between Wadge reducibility and games:

Wadge Games

Definition (Wadge 1972)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'.

The two players alternatively write letters a_n of X for Player 1 and b_n of Y for player 2.
Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps.

After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written $b \in Y^\omega$.
Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff :

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L')].$$

Wadge Games

Theorem (Wadge)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. Then $L \leq_W L'$ iff Player 2 has a winning strategy in the game $W(L, L')$.

By Martin's Theorem, the Wadge game $W(L, L')$, for Borel sets L and L', is determined: One of the two players has a winning strategy.

$\longrightarrow$ Study of the Wadge hierarchy on Borel sets.

Borel Hierarchy

Σ^0_1 is the class of open subsets of Σ^ω,

Π^0_1 is the class of closed subsets of Σ^ω,

for any integer $n \geq 1$:

Σ^0_{n+1} is the class of countable unions of Π^0_n-subsets of Σ^ω.

Π^0_{n+1} is the class of countable intersections of Σ^0_n-subsets of Σ^ω.

Π^0_{n+1} is also the class of complements of Σ^0_{n+1}-subsets of Σ^ω.

Borel Hierarchy

The Borel hierarchy is also defined for levels indexed by countable ordinals.

For any countable ordinal $\alpha \geq 2$:

Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

Borel Hierarchy

Below an arrow → represents a strict inclusion between Borel classes.

$$\begin{tikzcd}
 & \Pi^0_1 \arrow[dr] & & & & \Pi^0_\alpha \arrow[dr] & & \Pi^0_{\alpha+1} & \\
\Delta^0_1 \arrow[ur] \arrow[dr] & & \Delta^0_2 \arrow[ur] \arrow[dr] & \cdots \; \cdots & \Delta^0_\alpha \arrow[ur] \arrow[dr] & & \Delta^0_{\alpha+1} \arrow[ur] \arrow[dr] & & \cdots \\
 & \Sigma^0_1 \arrow[ur] & & & & \Sigma^0_\alpha \arrow[ur] & & \Sigma^0_{\alpha+1} &
\end{tikzcd}$$

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ^1_1 of analytic sets which are obtained by projection of Borel sets.

A set $E \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff :

$$\exists F \subseteq (\Sigma \times \{0, 1\})^\omega \text{ such that } F \text{ is } \Pi^0_2 \text{ and}$$

$$E \text{ is the projection of } F \text{ onto } \Sigma^\omega$$

A set $E \subseteq \Sigma^\omega$ is in the class Π^1_1 iff $\Sigma^\omega - E$ is in Σ^1_1.

Suslin's Theorem states that : Borel sets $= \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Complete Sets

A set $E \subseteq \Sigma^\omega$ **is $\mathcal{C}$-complete**, where $\mathcal{C}$ is a Borel class Σ^0_α or Π^0_α

or the class Σ^1_1, **for reduction by continuous functions** iff :

$\forall F \subseteq \Gamma^\omega \quad F \in \mathcal{C}$ **iff** $F \leq_W E$.

Example : $\{\sigma \in \{0,1\}^\omega \mid \exists^\infty i \;\; \sigma(i) = 1\}$ is a Π^0_2-complete-set and it is accepted by a deterministic Büchi automaton.

More Examples of Complete Sets

Examples :

$\{\sigma \in \{0,1\}^\omega \mid \exists i \;\; \sigma(i) = 1\}$ is a Σ^0_1**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \forall i \;\; \sigma(i) = 1\} = \{1^\omega\}$ is a Π^0_1**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \exists^{<\infty} i \;\; \sigma(i) = 1\}$ is a Σ^0_2**-complete-set**.

All these ω-languages are ω-regular.

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- ω-**regular languages** accepted by **deterministic Büchi automata** are Π^0_2-sets.
- ω-**regular languages** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

deterministic Turing machines

- ω-**languages** accepted by **deterministic Büchi Turing machines** are Π^0_2-sets.
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Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi or Muller Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ_1^1** of **effective analytic sets** is obtained as the class of **projections of arithmetical sets** and **Effective-$\Sigma_1^1 \subsetneq \Sigma_1^1$**.

Let ω_1^{CK} be the first non recursive ordinal.

Topological Complexity of Effective Analytic Sets

- There are some Σ_1^1-complete sets in **Effective-Σ_1^1.**
- For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ_α^0-complete and some Π_α^0-complete ω-languages in the class **Effective-Σ_1^1**.
- **(Kechris, Marker and Sami 1989)**
  The supremum of the set of Borel ranks of **Effective-Σ_1^1-sets** is a countable ordinal $\gamma_2^1 > \omega_1^{\mathrm{CK}}$.

Topological complexity of 1-counter or context free ω-languages

Let $1 - CL_\omega$ be the class of real-time 1-counter ω-languages.

Let $\mathcal{C}$ be a class of ω-languages such that:

$$1 - CL_\omega \subseteq \mathcal{C} \subseteq \textbf{ Effective-}\Sigma^1_1.$$

(a) **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class $\mathcal{C}$.**

(b) **(F. 2005)** The Borel hierarchy of **the class $\mathcal{C}$** is equal to the Borel hierarchy of **the class Effective-Σ^1_1.**

(c) γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class $\mathcal{C}$.**

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class $\mathcal{C}$.**

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- **(d)** For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class $\mathcal{C}$.**

Sketch of the proof

It is well known that every Turing machine can be simulated by a (non real time) 2-counter automaton.

We denote **BCL**$(2)_\omega$ the class of ω-languages accepted by Büchi 2-counter automata.

Thus the topological complexity of ω-languages in the class **BCL**$(2)_\omega$ is equal to the topological complexity of ω-languages accepted by Büchi Turing machines.

Sketch of the proof

First, from a 2-counter automaton A accepting an ω-language $L \subseteq X^\omega$, we construct a real-time 8-counter Büchi automaton B accepting an ω-language of the same topological complexity.

First, we add a storage type called a queue to a 2-counter Büchi automaton in order to read ω-words in real-time.

Then the queue can be simulated by

- two pushdown stacks or
- four counters,
  because each pushdown stack may be simulated by two counters.

Sketch of the proof

This simulation is not done in real-time but one can bound the number of transitions needed to simulate the queue. This allows to pad the strings in L with enough extra letters so that the new language $\theta_S(L)$ will be read in real-time by a 8-counter Büchi automaton.

The padding is obtained via the function $\theta_S : X^\omega \to (X \cup \{E\})^\omega$, where $S = (3k)^3$, with $k = card(X) + 2$, and for all $x \in X^\omega$:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

The ω-language $\theta_S(L)$ is accepted in real time by a Büchi automaton with $2 + 4 + 2 = 8$ counters.

Sketch of the proof

The next step is to simulate a *real-time* 8-counter Büchi automaton $\mathcal{A}$, by a *real-time* 1-counter Büchi automaton $\mathcal{B}$.

The eight first prime numbers are $2; 3; 5; 7; 11; 13; 17; 19$.

We code the content $(c_1, c_2, \ldots, c_8)$ of eight counters by the product $2^{c_1} \times 3^{c_2} \times \ldots \times (17)^{c_7} \times (19)^{c_8}$.

Then we code ω-words in $Y = X \cup \{E\}$ by ω-words in $Z = Y \cup \{A, B, 0\}$.

The new ω-words will have a **special shape** which will allow the propagation of the values of the counters of $\mathcal{A}$.

Sketch of the proof

The product of the eight first prime numbers is:

$$K = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9699690$$

An ω-word $x \in Y^\omega$ is coded by the ω-word

$$h(x) = A.0^K.x(1).B.0^{K^2}.A.0^{K^2}.x(2).B.\ldots B.0^{K^n}.A.0^{K^n}.x(n).B\ldots$$

If $L(\mathcal{A}) \subseteq Y^\omega$ is accepted by a real time 8-counter Büchi automaton $\mathcal{A}$, then one can construct from $\mathcal{A}$ a 1-counter Büchi automaton $\mathcal{B}$, reading words over $Y \cup \{A, B, 0\}$, such that:

$$\forall x \in Y^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A})$$

Sketch of the proof

The mapping $h : Y^\omega \to (Y \cup \{A, B, 0\})^\omega$ is continuous.

The complement $h(Y^\omega)^-$ of the ω-language $h(Y^\omega)$ is an open subset of $(Y \cup \{A, B, 0\})^\omega$ and is accepted by a real time 1-counter automaton.

Thus the ω-language

$$h(L(\mathcal{A})) \cup h(Y^\omega)^- = L(\mathcal{B}) \cup h(Y^\omega)^-$$

is in the class $\mathbf{BCL}(1)_\omega$ and it has the same topological complexity as the ω-language $L(\mathcal{A})$.

Independence from the Axiomatic System ZFC of Set Theory

ZFC : Zermelo-Fraenkel Axiomatic System ZF + Axiom of Choice AC.

ZFC : commonly accepted Axiomatic System for Set Theory in which all usual mathematics can be developped.

Using some notions of set theory we show:

The topological complexity of a 1-counter ω-language may depend on the models of ZFC.

The Axiomatic System ZFC of Set Theory

The axioms of ZFC express some natural facts that we consider to hold in the universe of sets.

These axioms are first-order sentences in the logical language of set theory whose only non logical symbol is the membership binary relation symbol $\in$.

The Axiomatic System ZFC of Set Theory

The *Axiom of Extensionality* states that two sets x and y are equal iff they have the same elements:

$$\forall x \forall y \, [\, x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y) \,].$$

The *Pairing Axiom* states that for all sets x and y there exists a set $z = \{x, y\}$ whose elements are x and y:

$$\forall x \forall y \, [\, \exists z (\forall w (w \in z \leftrightarrow (w = x \vee w = y)))]$$

Similarly the *Powerset Axiom* states the existence of the set of subsets of a set x.

Models of the Axiomatic System ZFC

A model $(\mathbf{V}, \in)$ of the axiomatic system **ZFC** is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of **ZFC**.

Perfect Sets, Thin Sets

Definition

Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is a perfect subset of Σ^ω iff it is a non-empty closed set which has no isolated points.

A perfect subset of Σ^ω has cardinality $2^{\aleph_0}$.

Definition

A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.

Theorem (Souslin)

*(**ZFC**) An analytic set $X \subseteq \Sigma^\omega$ is either countable or contains a perfect subset. Thus every thin analytic set is countable.*

This result is not true for co-analytic sets in **ZFC**. We need additional axioms like analytic determinacy.

The Largest Thin Effective Coanalytic Set

Theorem (Kechris 1975; Guaspari, Sacks)

(**ZFC**) *Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

Theorem (Kechris 1975; Guaspari, Sacks)

1. *There is a model V_1 of* **ZFC** *in which $\mathcal{C}_1(\Sigma^\omega)$ is countable.*
2. *There is a model V_2 of* **ZFC** *in which $\mathcal{C}_1(\Sigma^\omega)$ is uncountable.*

The Largest Thin Effective Coanalytic Set

Theorem

1. *There is a model V_1 of* **ZFC** *in which the largest thin Π_1^1-set in Σ^ω is countable, hence a Σ_2^0-set.*
2. *There is a model V_2 of* **ZFC** *in which the largest thin Π_1^1-set in Σ^ω is not a Borel set.*

Proof. There is a model V_1 of **ZFC** in which the largest thin Π_1^1-set in Σ^ω is countable. It is a countable union of singletons, and each singleton is a closed set. Thus $\mathcal{C}_1(\Sigma^\omega)$ is a countable union of closed sets, i.e. a Σ_2^0-set.

There is a model V_2 of **ZFC** in which the largest thin Π_1^1-set in Σ^ω is uncountable. But it is thin, hence has no perfect subset. Thus it cannot be a Borel set because Borel sets have the perfect set property: a Borel set is either countable or contains a perfect subset.

From effective coanalytic sets to 1-counter automata

The complement of $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ is an effective analytic set accepted by a Büchi Turing machine $\mathcal{T}$.

We can now use previous constructions to obtain:

- A 2-counter Büchi automaton $\mathcal{A}_1$,
- A real time 8-counter Büchi automaton $\mathcal{A}_2$,
- A real time 1-counter Büchi automaton $\mathcal{A}_3$,

such that $L(\mathcal{T})$, $L(\mathcal{A}_1)$, $L(\mathcal{A}_2)$, and $L(\mathcal{A}_3)$, all have the same toplogical complexity.

The Topological complexity of a 1-counter ω-language depends on the models of ZFC

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system ***ZFC****.*

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is a Π^0_2-set.*

Wadge Games Between 1-Counter Automata

The ω-language $(0^\star \cdot 1)^\omega \subseteq \{0,1\}^\omega$ is ω-regular, accepted by a Büchi automaton $\mathcal{B}$, and is Π^0_2-complete in every model of **ZFC**. This implies:

Theorem

There exist two 1-counter Büchi automata $\mathcal{A}$ and $\mathcal{B}$ such that $L(\mathcal{A}) \leq_W L(\mathcal{B})$ is independent from **ZFC***:*

(1) There is a model V_1 of **ZFC** *in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.*

(2) There is a model V_2 of **ZFC** *in which Player 2 has no winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.*

A similar result holds for 2-tape Büchi automata.

Wadge Games Between 1-Counter Automata

Theorem

There exist two 1-counter Büchi automata $\mathcal{A}$ and $\mathcal{B}$ such that "$W(L(\mathcal{A}), L(\mathcal{B}))$ is determined" is independent from **ZFC***:*

(1) There is a model V_1 of **ZFC** *in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined.*

(2) There is a model V_2 of **ZFC** *in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.*

A similar result holds for 2-tape Büchi automata.