Perebor: Brute-Force Search

- **1950s** – Early computer science in Soviet Union: some problems seem to be solvable only via brute-force search – research program ultimately unsuccessful.


- **1980s** – Levin, Gurevich: explanation via *average-case* NP-completeness – not too successful (few problems amenable)
“Where The Hard Problems Are”

Cheeseman-Kanefsky-Taylor, 1991:

- Worst-case complexity not always useful – not a good guide for real-world performance

- Instances of NP-complete problems can be quite easy in practice, e.g., industrial SAT instances with millions of clauses!

- Decision problems can often be characterized in terms of “constrainedness”

- The hard problems are those that lie in the transition from under-constrained to over-constrained.
Example: 3-SAT

**Motivation:**
- NP-complete
- Basic reasoning task
- Useful algorithmic abstraction

- **Literal:** $\pm x$
- **Clause:** $\pm x_{i_1} \lor \pm x_{i_2} \lor \pm x_{i_3}$
- **Instance:** $\bigwedge_{i=1}^{m} C_i$
- **Order:** number of variables
- **Density:** ratio between number of clauses and number of variables
- **Random 3-SAT:** fix order and density, choose clauses uniformly
- **Asymptotics:** Fix density, take order to infinity
Phase Transition of Random 3-SAT
Asymptotic Random 3-SAT

Intuition

• Easy satisfiability for underconstrained problems

• Hard satisfiability for overconstrained problems

Known:

• Kaporis et al., 2006 Density < 3.52: asymptotic probability 1

• Diaz et. al, 2009: Density > 4.4898: asymptotic probability 0

Conjecture: sharp phase transition!
Phase Transition and Solver Time for Random 3-SAT
“Easy, Hard, Easy”

Mitchell–Selman–Levesque, 1992

- Constrainedness = Density!

- *Empirically*: phase transition from probability 1 to probability 0 at density 4.26.


**Bottom Line:**
- Low density: easy
- High density: easy
- Crossover: *hard*
Phase Transitions and Computational Complexity

**Conjecture:** There is a profound connection between computational complexity and phase transitions!

**Evidence:** Random-Graph Colorability

- **Order:** Number of nodes
- **Density:** Ratio between number of edges and number of nodes
- **Random Graphs:** Fix order and density, and choose edges uniformly
- **Asymptotics:** Fix density and take order to infinity

**Known:**
- 2-COLOR *does not* have a phase transition (provably)
  - 2-COLOR is in PTIME.
- 3-COLOR *does* have a phase transition (empirically)
  - 3-COLOR is NP-complete.
Phase Transitions and Computational Complexity

1990s: *lots of excitement*!!!

- Physicists get involved
- Articles in *Science*
- Many invited talks
- Microsoft Research forms a Theory Group, planning to use Statistical Physics to solve P vs. NP!

**Difficulties:**
- Nice results about probability, but no concrete results about complexity.
- 2-SAT has a phase transition at density 1 [*Chvatal–Reed, Goerdt, 1992*]
- XOR-SAT has a phase transition at density 1 [*Creignu–Daube, 2000*]
HornSAT: at most one positive literal in a clause


Demopolous-V., 2003: Random 1-3-HornSAT

- $n$ variables
- 1 negative clause
- $d_1 n$ positive (unit) clauses
- $d_3 n$ implications ($x_i \land x_j \rightarrow x_k$)
Phase Transition for HornSAT?

Satisfiability probability of a 1-3-Horn formula of size 20,000

Istrate-Moore-Demopolous-V., 2003:

**Analytical Results** – \((d_1, d_3)\)-plane

- Phase transition for asymptotic probability
- Phase transition between region with phase transition and region without phase transition

**Bottom Line:** Phase transitions are *not* intrinsically related to computational complexity!
What is “Easy, Hard, Easy”?  

Coarfa–Demopolous–San Miguel Aguirre–Subramanian–V., 2000:

- What is meant by “easy” and “hard”?
- Where are the transitions between “easy” and “hard”?

**Note**: MSL’s “Easy-Hard-Easy” refers to *fixed order* and *varying density*.

**Proposed Approach**:

- Focus on *scalability*:
  - *Easy*: polynomial scalability
  - *Hard*: exponential scalability

- *Fix density, scale order*
  - Corresponds to analytical studies
  - Corresponds to applications
“Easy” ≠ Easy

- **Very low density problems are indeed easy**: Algorithms with linear median time below density 1.63 [Broder–Frieze–Upfal, 1993] and below density 3 [Frieze–Suen, 1996]

- **High density problems may be hard**: Exponential lower bound for resolution-proof length at density above 5.2 [Chvatal–Szemeredi, 1988]
  - SAT solvers are resolution based!

- **Ordered-DPLL is exponential with constant probability at density 3.81** [Achlioptas–Beame–Molloy, 2001].

**Conclusions**:  
- “Easy-Hard-Easy” picture is **patently wrong**.  
- Algorithmic behavior is algorithm dependent!

Coarfa et al., 2000: **empirical investigation**
3-D plot of median running time
Polynomial scalability at density 3.7

Median running time of GRASP as function of order n when $d=3.7$
Exponential scalability at density 3.8

Median run–time of GRASP as a function of order n when d=3.8

running time (in secs)

order n
New View:

- Density below 3.8: polynomial scalability – “easy”
- Density between 3.8 and 4.26: exponential scalability, increasing exponent
- Density larger than 4.26: exponential scalability, decreasing exponent

Single Transition $\rightarrow$ Multiple Transitions:

- Density 1.63: algorithmic correctness transition (analytical)
- Density 3.0: algorithmic correctness transition (analytical)
- Density 3.8: scalability transition (empirical)
- Density 4.26: probability transition (empirical)

Question: Any theory?
Algorithmic-Barrier Theory

Solution-Space Geometry

- **Solution Space**: \( sat(f) = \{ \alpha \in \{0, 1\}^n : \alpha \models f \} \)
- **Adjacency**: Hamming distance 1 between solutions
- **Cluster**: a connected component of \( sat(f) \)

Achlioptas–Coja-Oghlan, 2008:

- For density < 1.63: \( sat(f) \) has a single cluster.
- There is a density in the satisfiable region where \( sat(f) \) **shatters** into exponentially many small clusters.

**Intuition**: algorithms stop being effective after the shattering point.
The $P \neq NP$ Claim

On August 6, 2010, Vinay Deolalikar announced a proof (100-page manuscript) that $P \neq NP$.

- Aug. 6: Manuscript sent to 22 people and put on web page
- Aug. 7: First blog post [Greg Baker]
- Aug. 8: Second blog post [Richard Lipton], Slashdot, extensive commentary
- Aug. 9: Wikipedia article about V.D. (deleted later)
- Aug. 10: Wiki for technical discussion established
  - hundreds of edits
  - Fields medalists involved
- Aug. 15: CACM blogpost by Lipton
- Aug. 16: New York Times article
10 Days of Fame for
Finite-Model Theory

Richard Lipton, Blog, Aug. 8, 2010:

“At the highest level he is using the characterization of polynomial time via finite-model theory. His proof uses the beautiful result of Moshe Vardi (1982) and Neil Immerman (1986).”

Theorem: On ordered structures, a relation is defined by a first-order formula plus the Least Fixed Point (LFP) operator if and only if it is computable in polynomial time.
Crux: $9$-SAT can not be in $P$!

- If $9$-SAT is in $P$, then it can be expressed in $\text{FO+LFP}$, by the Immerman-V. Theorem.
- But, the $\text{FO+LFP}$ normal form is inconsistent with shattering.
**XOR-SAT**

**XOR-SAT:** “and of xors”

**Example:**

\[(-x_1 \oplus x_2 \oplus x_3) \land (-x_2 \oplus -x_3 \oplus x_4) \land (x_3 \oplus x_1 \oplus x_4)\]

**In essence:** Linear equations modulo 2

● Solve using Gaussian Elimination

**But:** The solution space of XOR-SAT shatters! [Mezard–Ricci-Tersenghi–Zecchina, 2003]

**Refuting Deolalikar’s Proof:** XOR-SAT is in \(P\) and it shatters!

**Conclusion:** Shattering is not a good algorithmic-barrier theory.
BDD-Based Algorithms

Coarfa et al., 2000, San Miguel Aguirre–V., 2001]: study BDD-based algorithms (can solve XOR-SAT efficiently [AKV’05]).

3-D Plot of median running time

Peak: Density 2.3!
BDDs: Polynomial to Exponential Transition

median running time for (left) density 0.2 and (right) density 1.0

Poly2Exp Phase Transition:

- Density 0.2: quadratic running time
- Density 1.0: exponential running time

Pattern: Easy-Harder-Less Hard
Conclusions

- “Easy-Hard-Easy” pattern – plain wrong
- “Easy-Harder-Less Hard” – more accurate pattern
  - fundamental pattern?
- Algorithm-dependent behavior
- Low-density problems are indeed easy.
- Where are the hard problems?
  - No evidence of intrinsic hardness!

**Bottom Line**: phase transition and computational complexity – it is a great story, but the evidence is simply not there!
Postscript: Counting Independent Sets

Weighted Independent Sets: parameter $\lambda$

- $wgt(I) = \lambda^{|I|}$

- **Input**: Undirected graph with maximum degree $d$
- **Output**: Sum of $wgt(I)$, over independent sets $I$

Results:

- Weitz, 2006: FPTAS for $\lambda < \lambda_c$
- Sly, 2012: No FPRAS for $\lambda$ just above $\lambda_c$ (assuming NP $\neq$ RP)

“Critical fugacity of the Hardcore Model”:

$\lambda_c = (d - 1)^{d-1}/(d - 2)^d$
Hardcore Model

**Hardcore Model:**
- Choose independent sets at random according to their (normalized) weights.
- Check if a given node $v$ is in selected independent set $I$.

**Question:** How does $\text{Prob}(v \in I)$ depends on some other selected nodes $V'$ belonging to $I$?

**Answer:** Influence of $V'$ declines as function of distance from $v$ iff $\lambda \leq \lambda_c$.

Atserias: It’s a connection between physics and complexity!

V.: So what? The setup is completely different! It says nothing about “constraindness” and complexity!