Synchronizing Words for Probabilistic Automata

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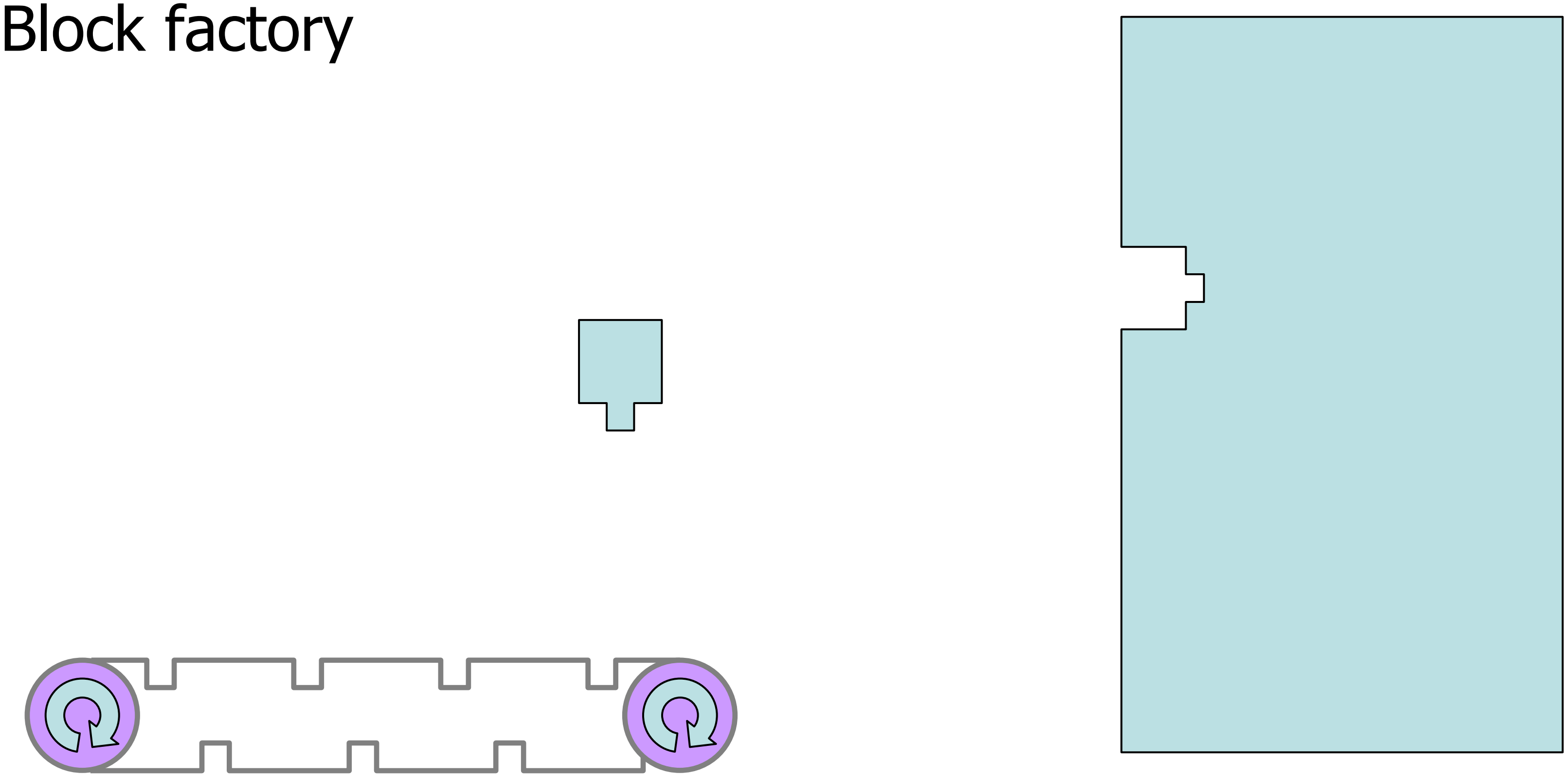
Université Libre de Bruxelles

5th Gasics meeting

Example [AV04]

Block factory

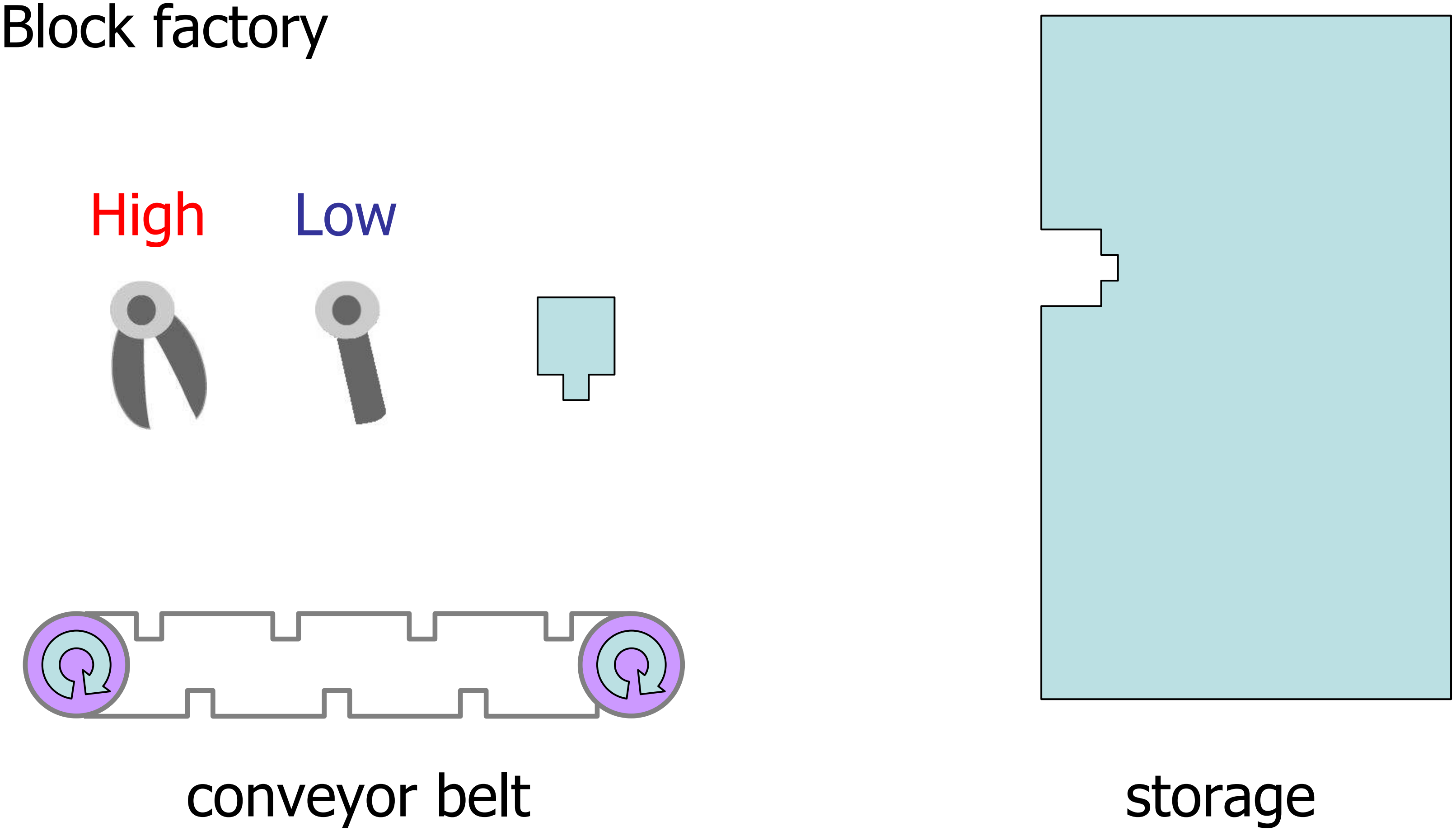


conveyor belt

storage

Example [AV04]

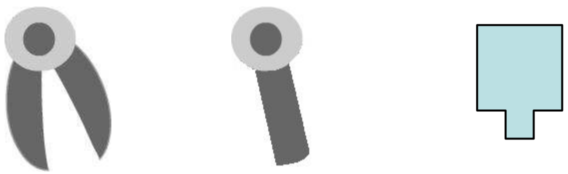
Block factory

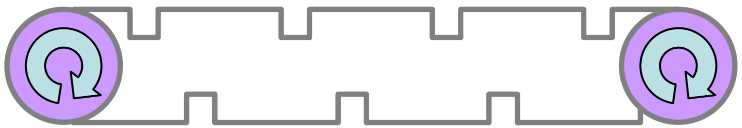


Example

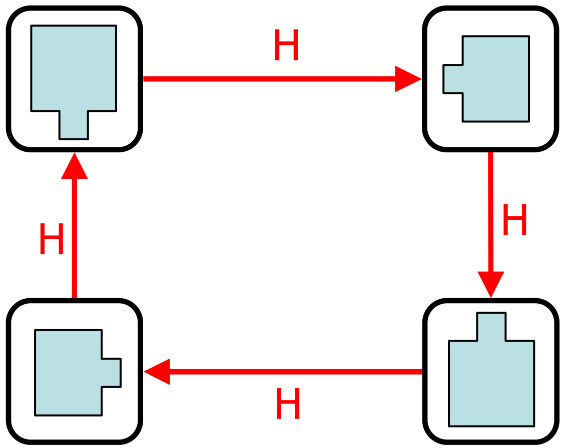
Block factory

High   Low





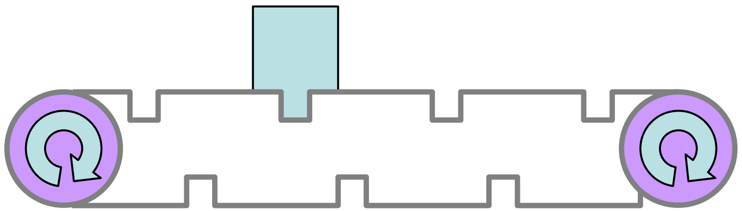
conveyor belt



Example

Block factory

High Low



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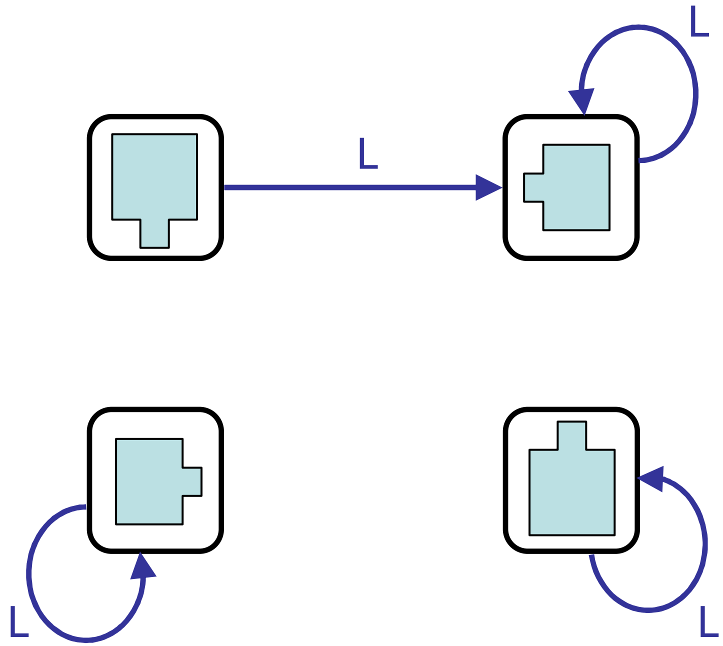
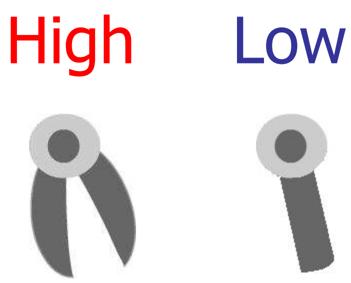
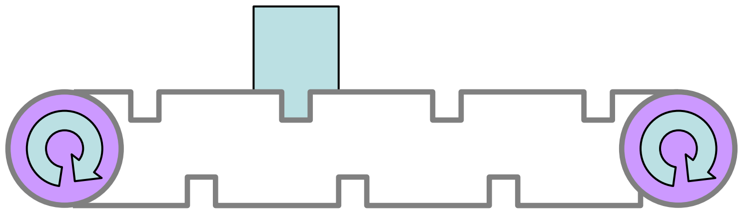


Example

Block factory





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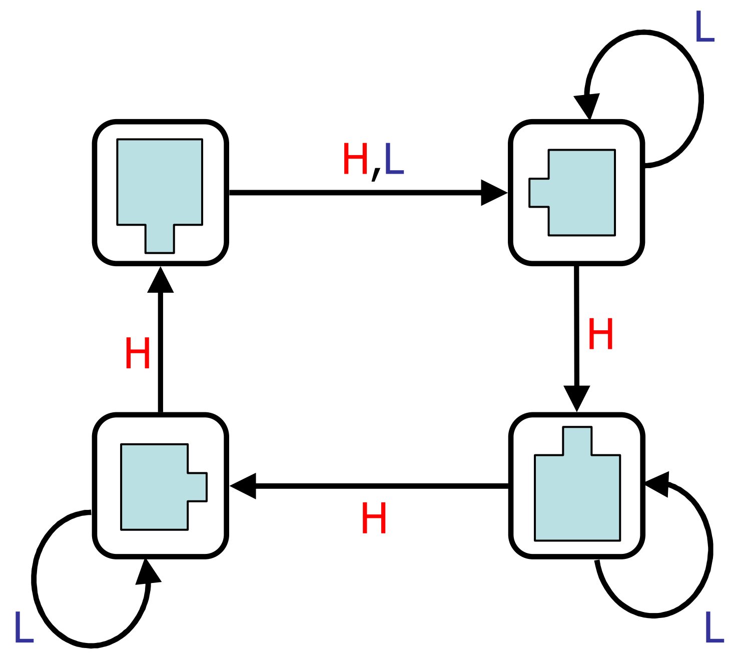


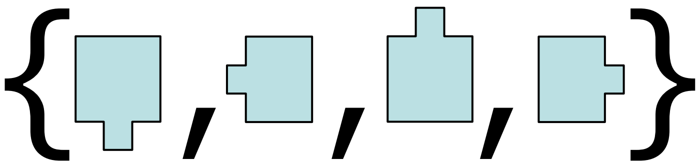
Deterministic finite automaton

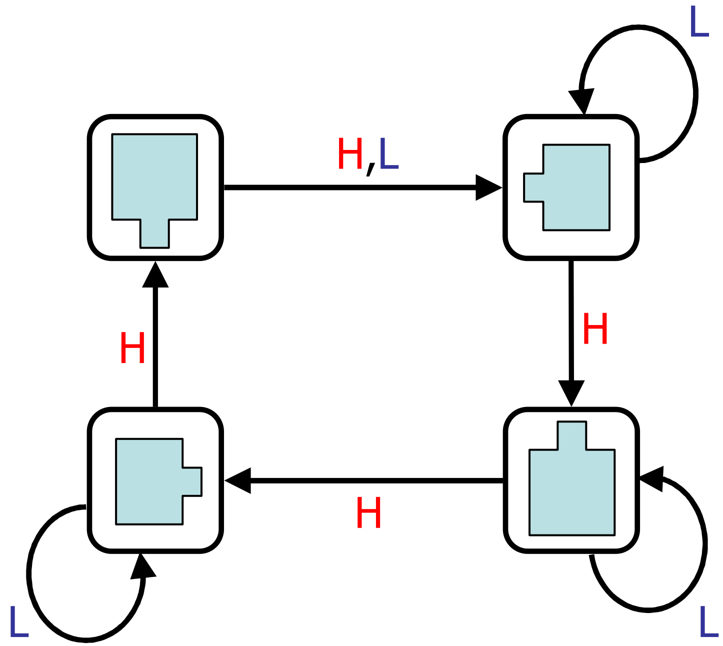
$$\delta : Q \times A \to Q$$

Example

Block factory

- no sensor

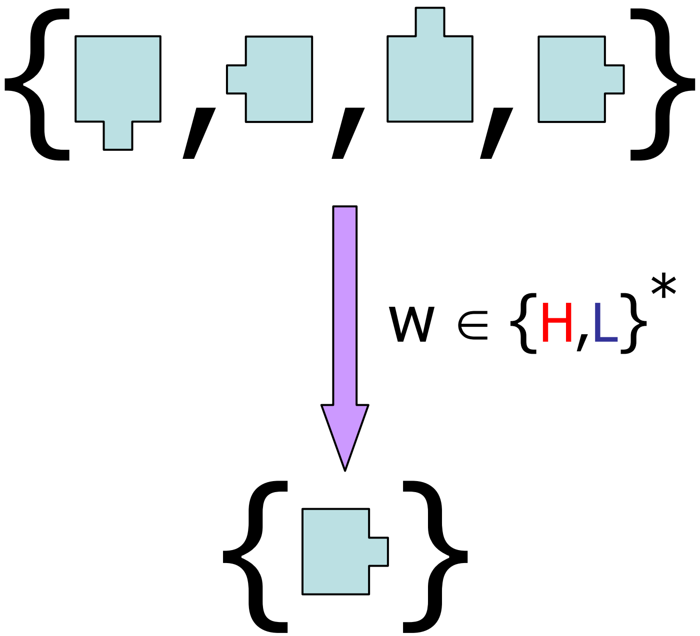


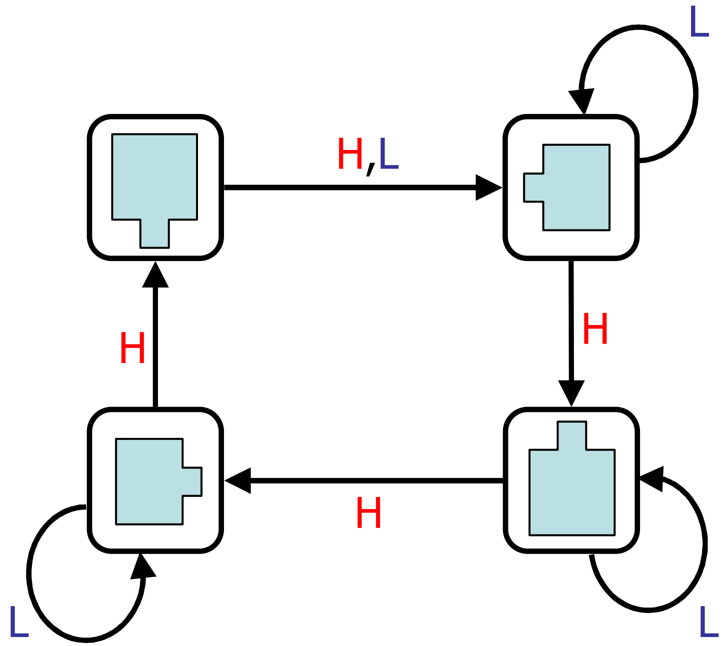


Example

Block factory

- no sensor
- robust control: $w \in \{H,L\}^*$





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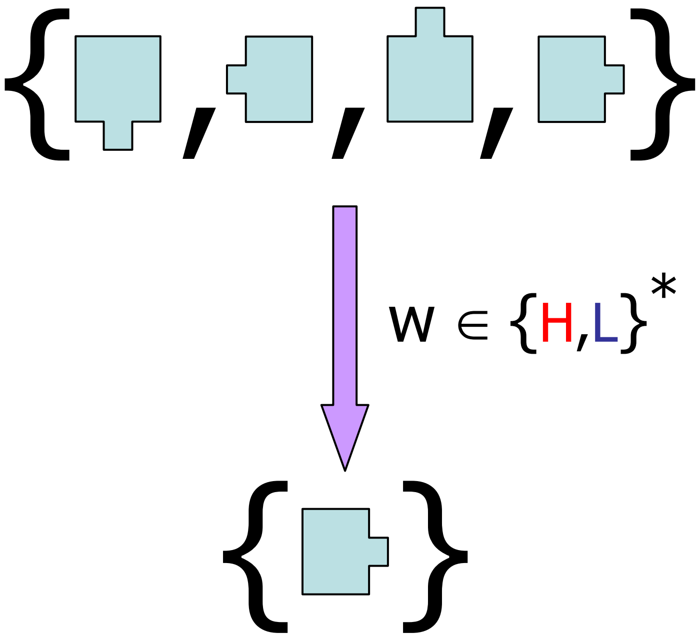

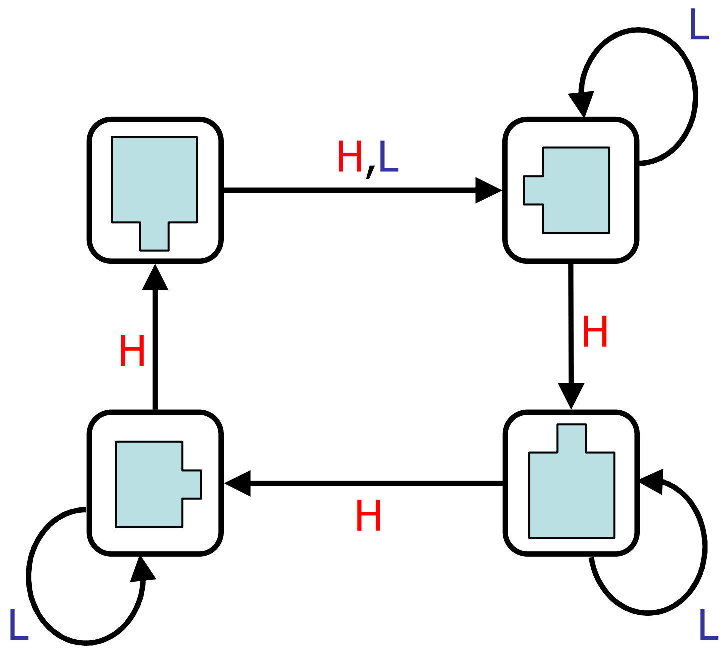

The word w is <u>synchronizing</u>:
no matter the initial state, the automaton ends up in a singleton

Reachability in subset construction

Example

Block factory

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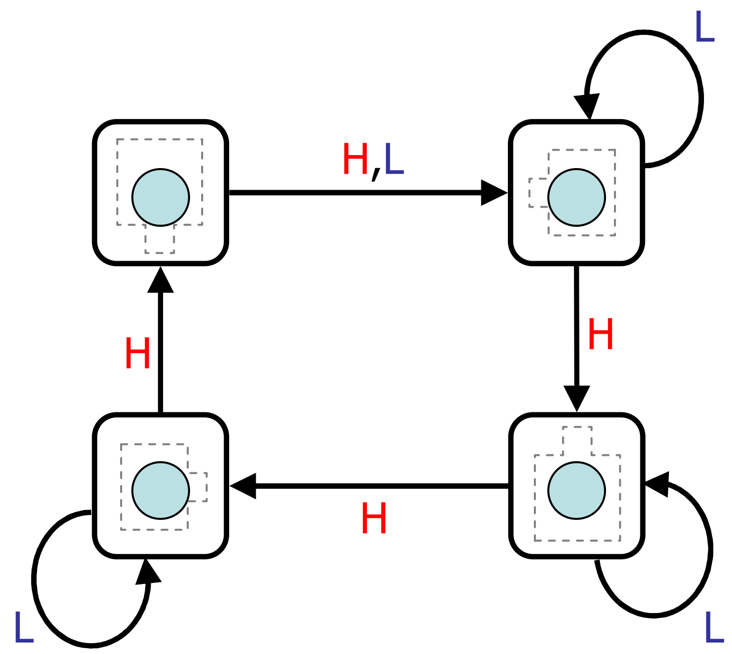
- robust control: w



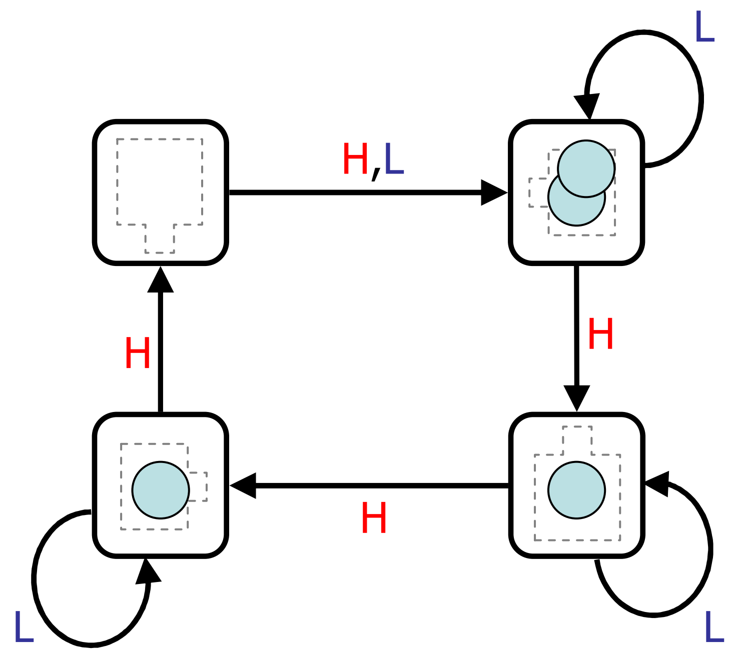
w = L H H H L H H H L

synchronizing word

Example

Block factory

- no sensor
- robust control: w



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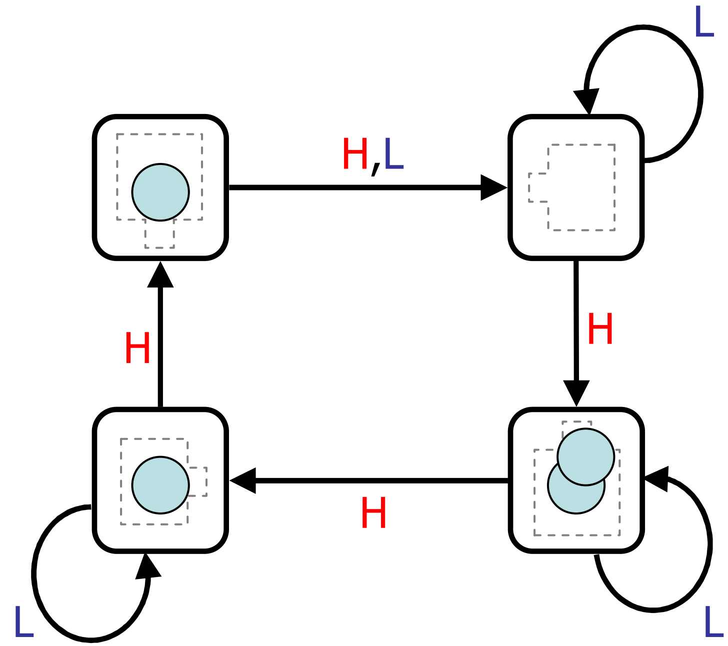

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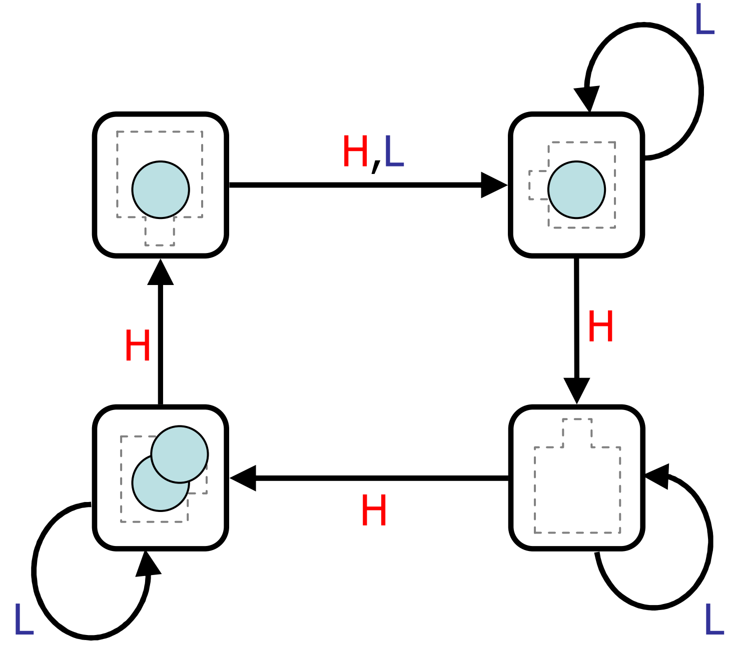

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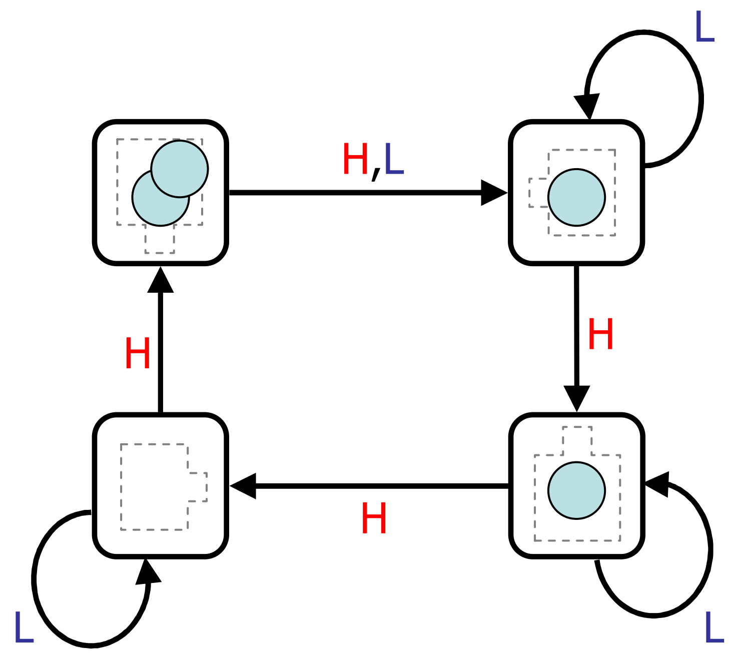

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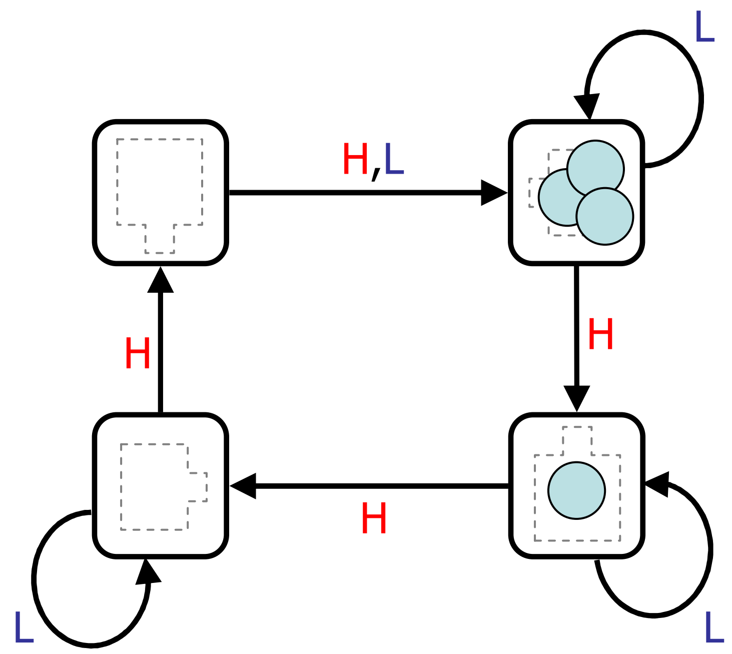

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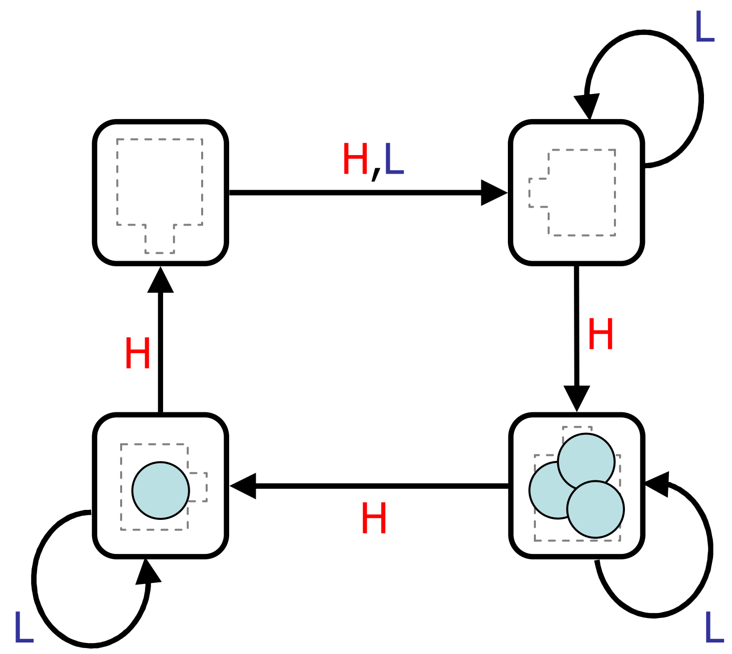

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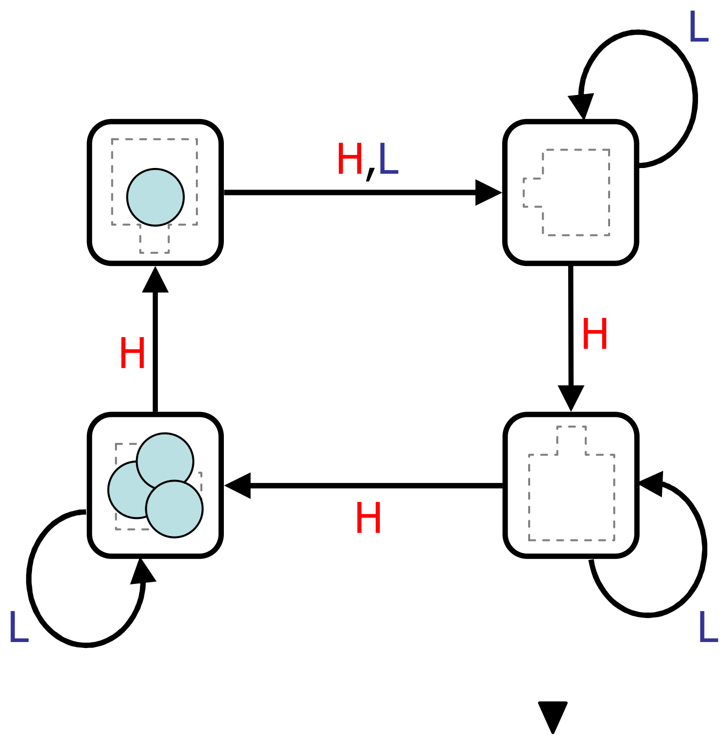

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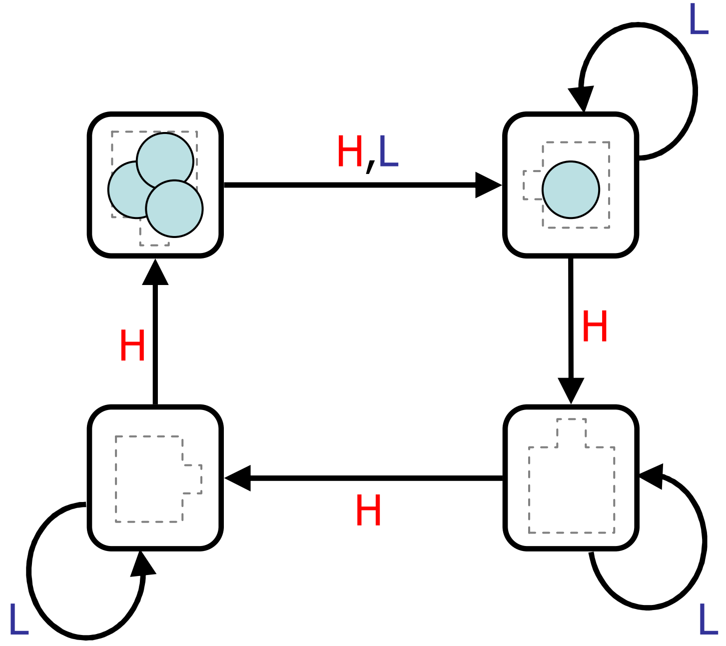

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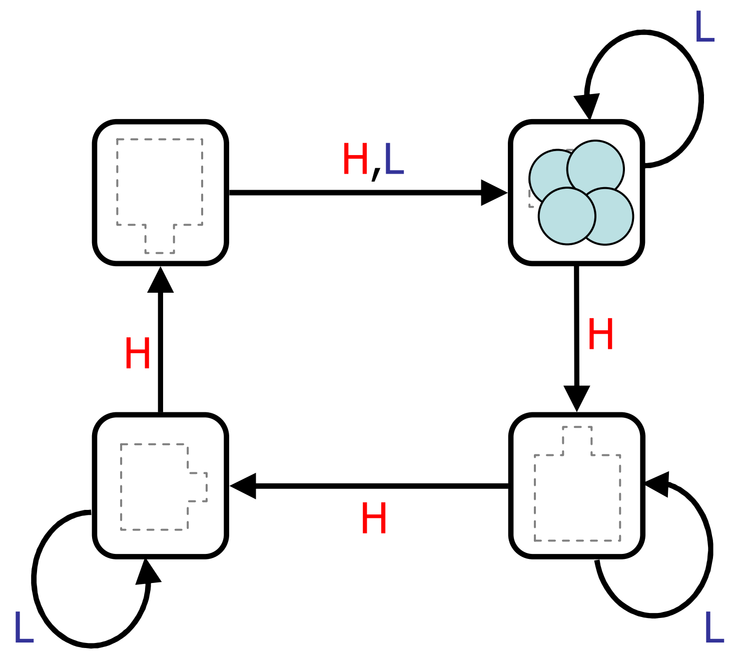


w = L H H H L H H H L

synchronizing word

$$|\delta(Q, w)| = 1$$

Existence of a synchronising word can be decided in PTIME

Cerny'64

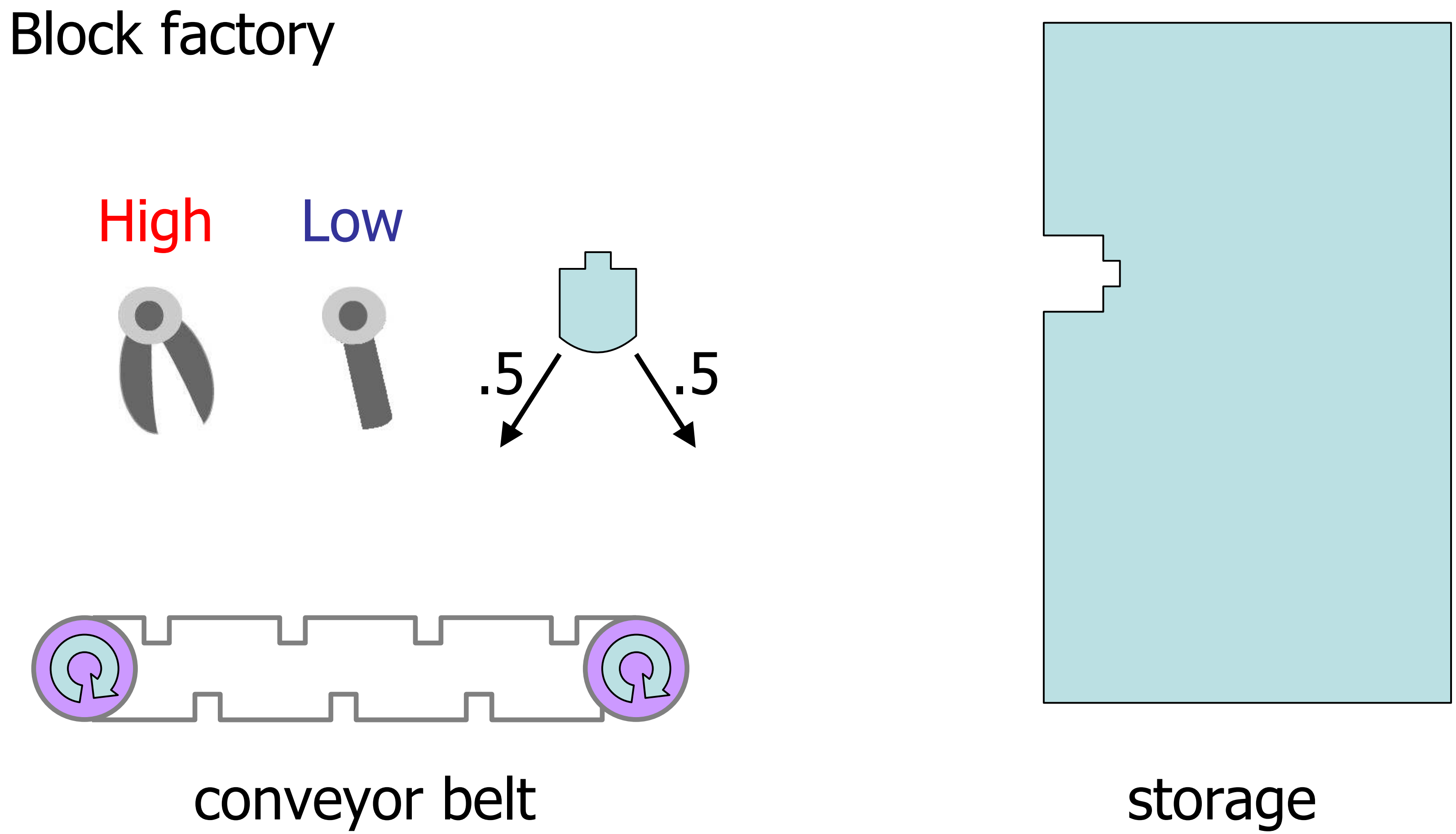
Applications

Robust control, reset from unknown state

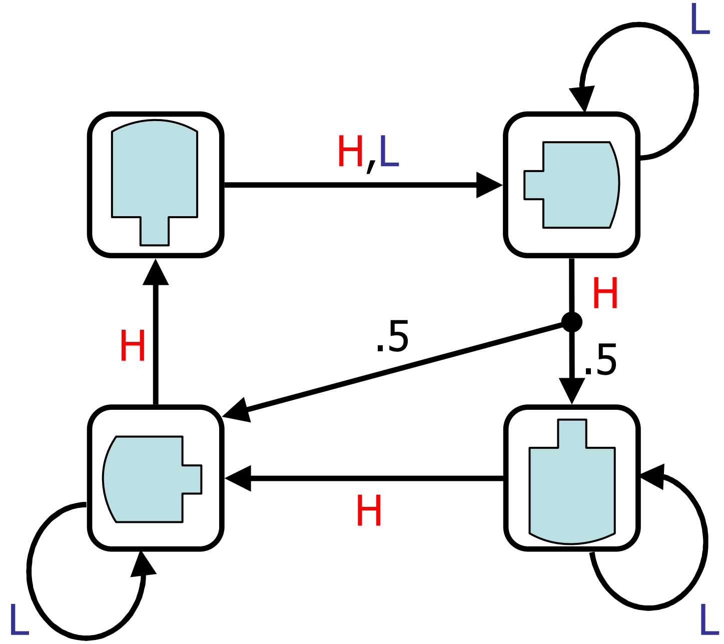
- Discrete-event systems
- Planning
- Biocomputing
- Robotics

See [Vol08]

Probabilistic systems



Probabilistic automata

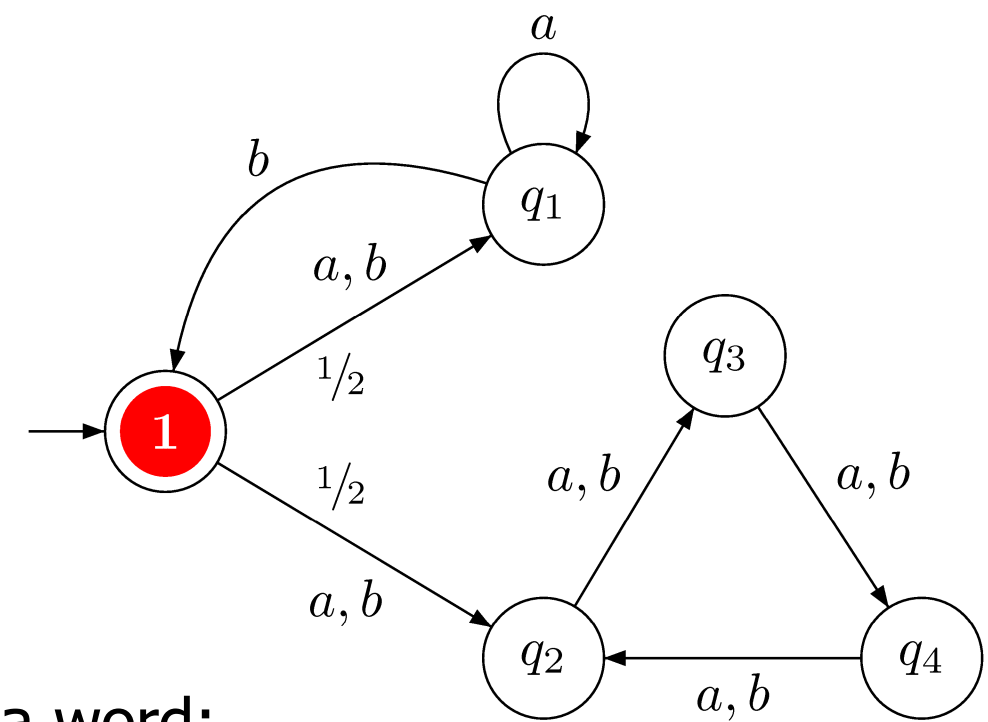


Probabilistic automaton

$$\delta : Q \times A \to \mathcal{D}(Q)$$

What is a synchronizing word for probabilistic automata ?

Probabilistic automata

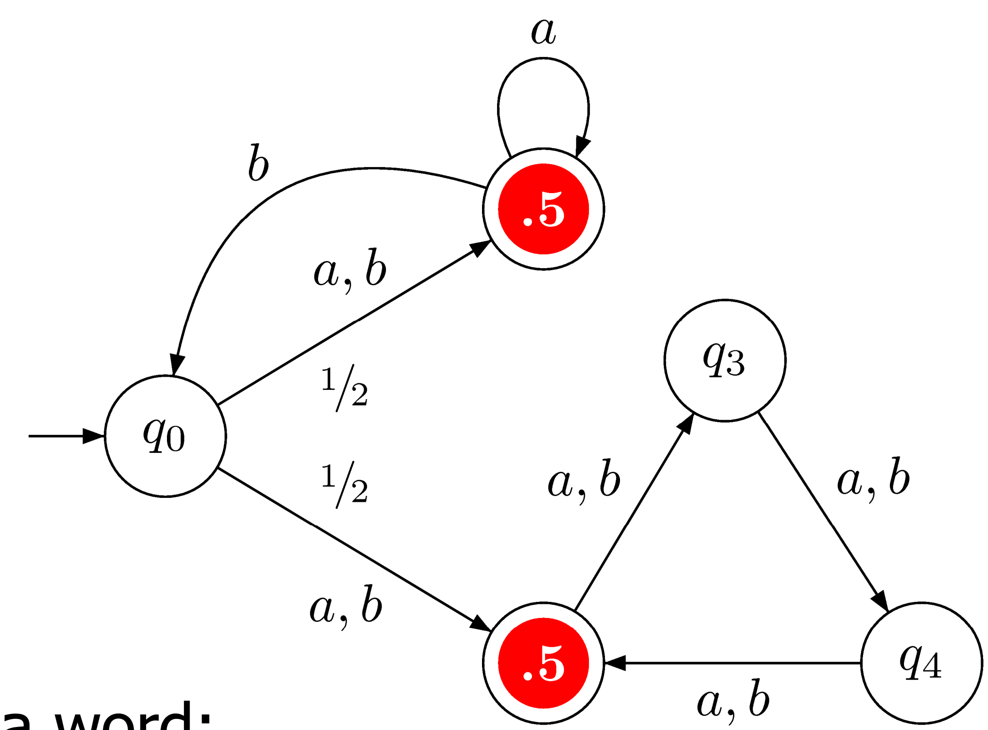


Outcome of a word:

$$\begin{matrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

w = aaba …

Probabilistic automata

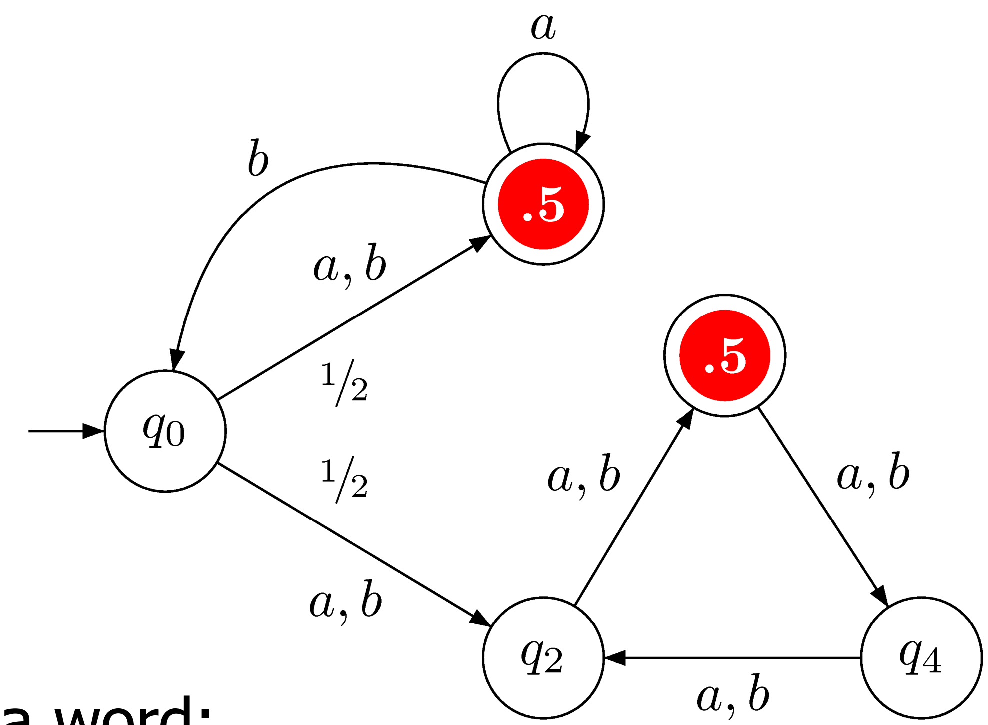


Outcome of a word:

$$
\begin{matrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix}
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\xrightarrow{a}
\begin{pmatrix} 0 \\ .5 \\ .5 \\ 0 \\ 0 \end{pmatrix}
$$

w = aaba ...

Probabilistic automata



Outcome of a word:

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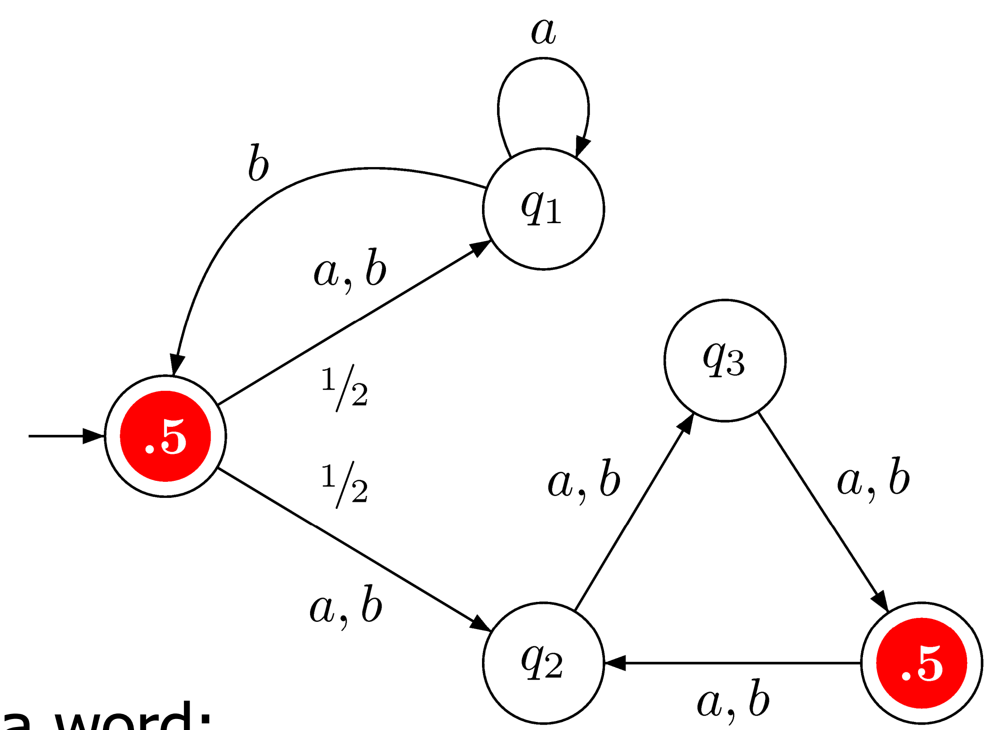
Probabilistic automata

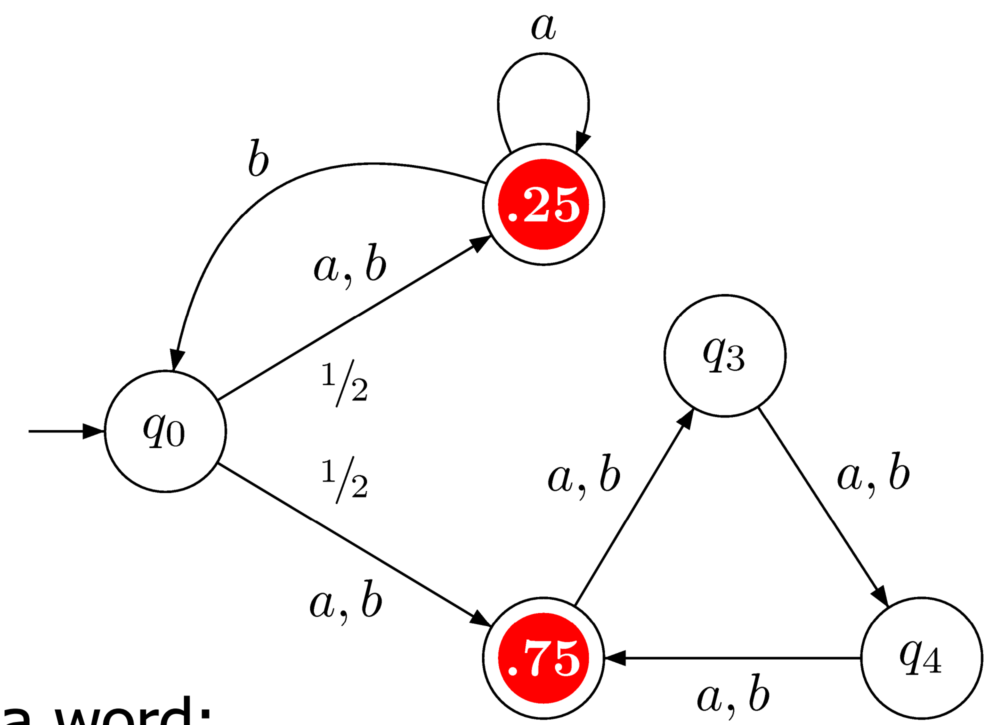


Outcome of a word:

$$\begin{matrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ .5 \\ .5 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ .5 \\ 0 \\ .5 \\ 0 \end{pmatrix} \xrightarrow{b} \begin{pmatrix} \color{red}{.5} \\ 0 \\ 0 \\ 0 \\ \color{red}{.5} \end{pmatrix}$$

w = aaba …

Probabilistic automata



Outcome of a word:

$$\begin{matrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ .5 \\ .5 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ .5 \\ 0 \\ .5 \\ 0 \end{pmatrix} \xrightarrow{b} \begin{pmatrix} .5 \\ 0 \\ 0 \\ 0 \\ .5 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ .25 \\ .75 \\ 0 \\ 0 \end{pmatrix} \cdots$$

w = aaba …

Probabilistic automata

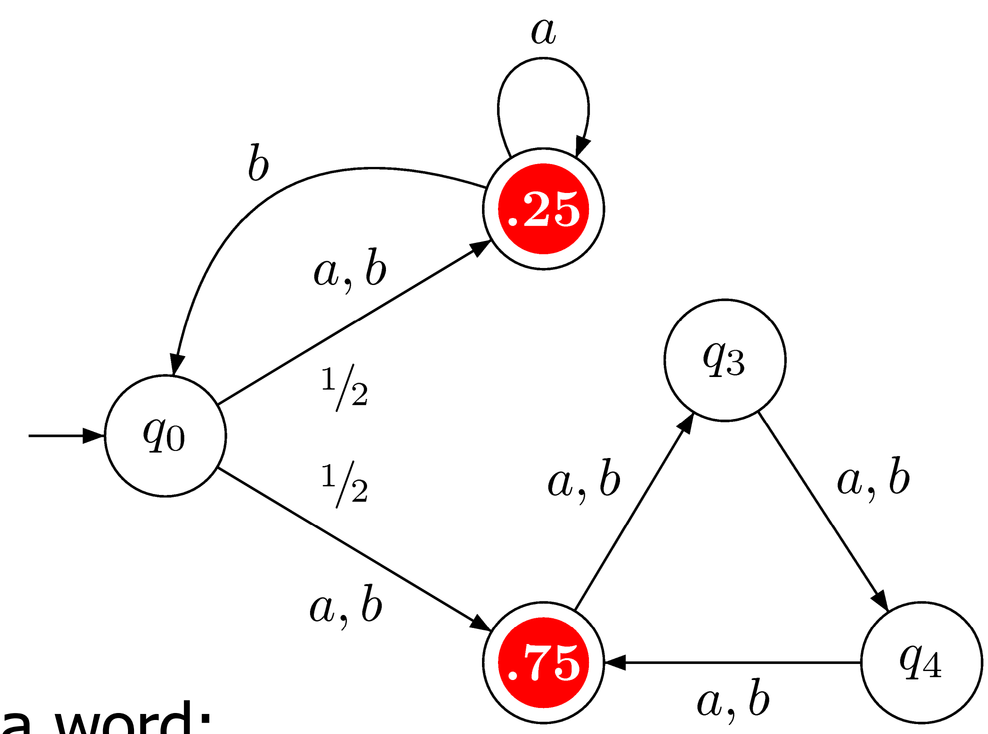
What is a synchronizing word for probabilistic automata ?

An infinite word $w \in \Sigma^\omega$

$$\begin{pmatrix} 0 \\ .5 \\ .5 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\quad w \in \Sigma^\omega \quad} \begin{pmatrix} . \\ 1-\epsilon \\ . \\ . \\ . \end{pmatrix} \longrightarrow$$

The probability mass tends to accumulate in a single state.

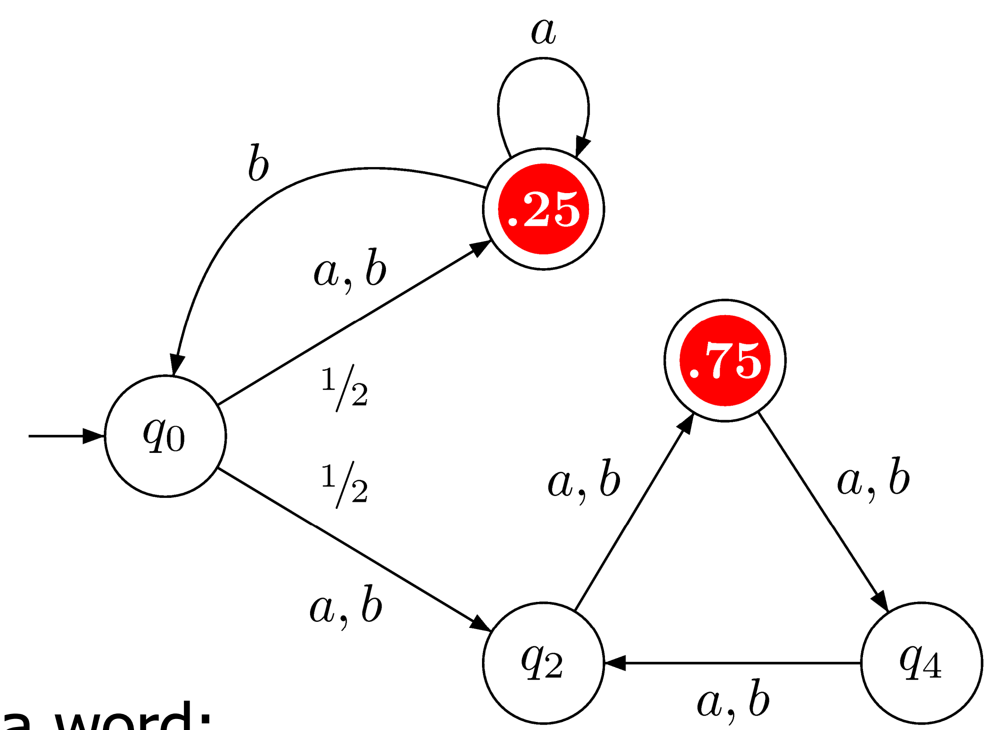
Probabilistic automata



Outcome of a word:

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.5\\.5\\0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.5\\0\\.5\\0\end{pmatrix} \xrightarrow{b} \begin{pmatrix}.5\\0\\0\\0\\.5\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.25\\.75\\0\\0\end{pmatrix} \ldots$$

Probabilistic automata



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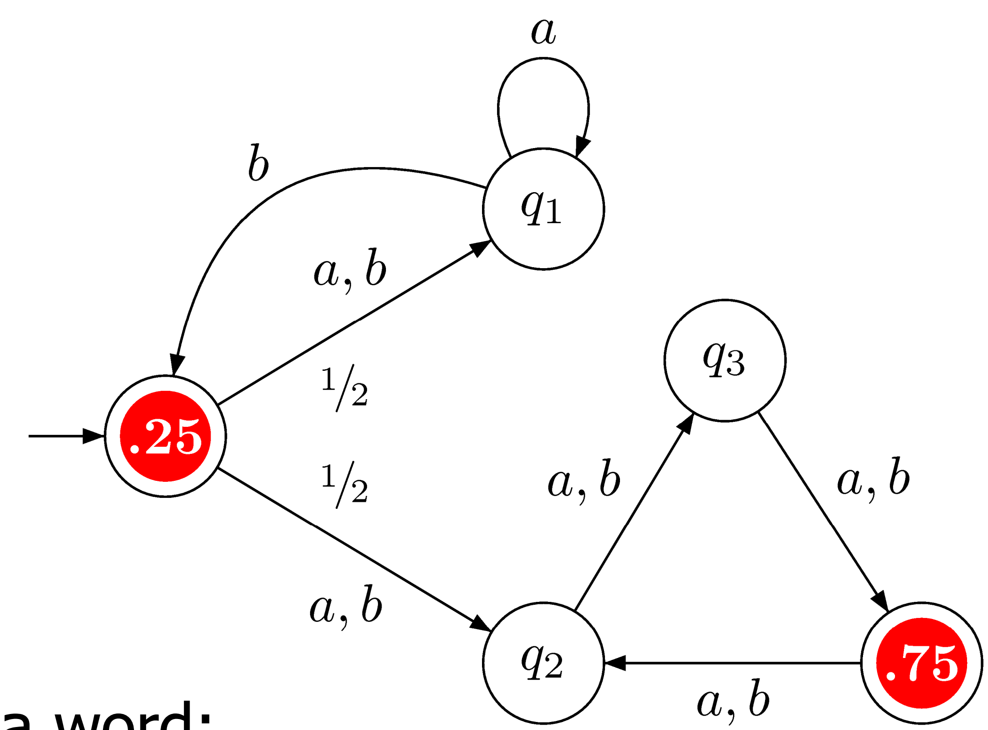
Probabilistic automata



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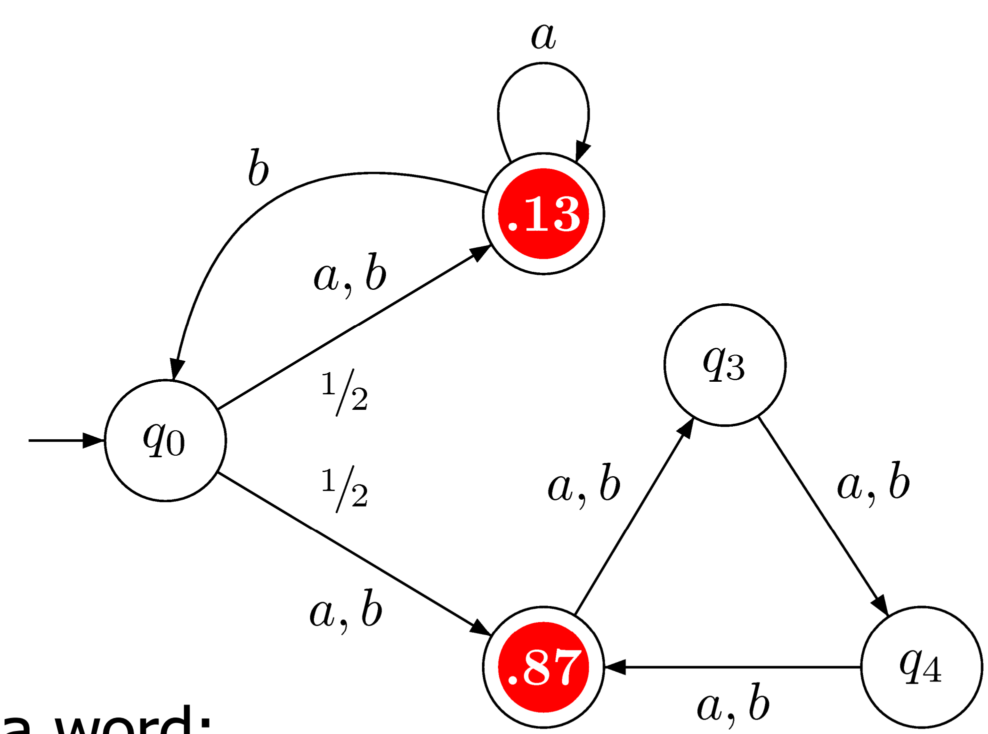
Probabilistic automata



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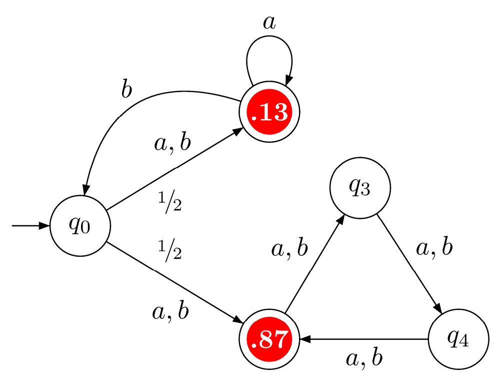
Probabilistic automata

What is a synchronizing word for probabilistic automata ?

Outcome of a word:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{X_0} \xrightarrow{a} \underbrace{\begin{pmatrix} 0 \\ .5 \\ .5 \\ 0 \\ 0 \end{pmatrix}}_{X_1} \xrightarrow{a} \underbrace{\begin{pmatrix} 0 \\ .5 \\ 0 \\ .5 \\ 0 \end{pmatrix}}_{X_2} \xrightarrow{b} \underbrace{\begin{pmatrix} .5 \\ 0 \\ 0 \\ 0 \\ .5 \end{pmatrix}}_{X_3} \xrightarrow{a} \underbrace{\begin{pmatrix} 0 \\ .25 \\ .75 \\ 0 \\ 0 \end{pmatrix}}_{X_4} \xrightarrow{a} \underbrace{\begin{pmatrix} 0 \\ .25 \\ 0 \\ .75 \\ 0 \end{pmatrix}}_{X_5} \xrightarrow{b} \underbrace{\begin{pmatrix} .25 \\ 0 \\ 0 \\ 0 \\ .75 \end{pmatrix}}_{X_6} \xrightarrow{a} \underbrace{\begin{pmatrix} 0 \\ .125 \\ .875 \\ 0 \\ 0 \end{pmatrix}}_{X_7} \cdots$$

where $\|X_i\| = \max_{q \in Q} X_i(q)$



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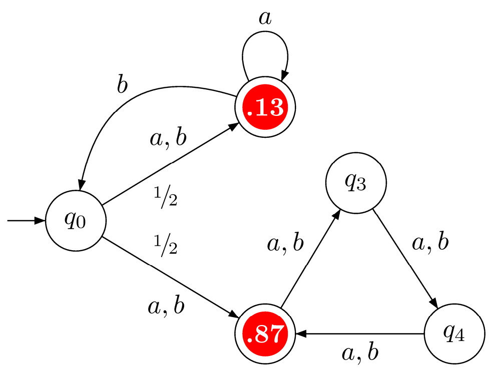
is synchronizing if $\lim_{n \to \infty} \|X_n\| = 1$

where $\|X_i\| = \max_{q \in Q} X_i(q)$

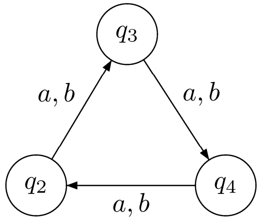


Synchronizing words

Two variants:

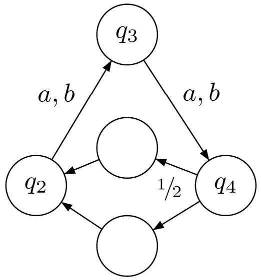
$$\liminf_{n\to\infty} \|X_n\| = 1$$

strongly synchronizing



$$\limsup_{n\to\infty} \|X_n\| = 1$$

weakly synchronizing



$$\|X_i\| = \max_{q\in Q} X_i(q)$$

Decision Problems

Decision problems

- Emptiness

  Does there exist a synchronizing word ?

- Universality

  Are all words synchronizing ?

Note: we consider randomized words $w \in \mathcal{D}(\Sigma)^\omega$

Words $w \in \Sigma^\omega$ are called pure words.

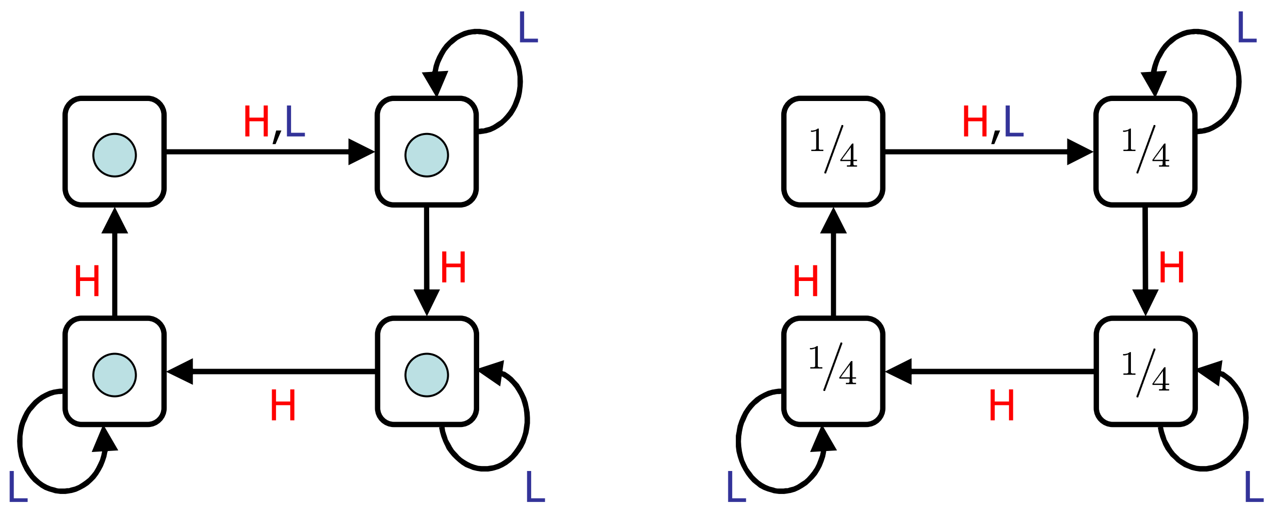
Synchronizing words for DFA

If we view DFA as special case of probabilistic automata:

there exists a synchronizing (finite) word for DFA A

iff

there exists a synchronizing (infinite) word for A with uniform initial distribution



Emptiness problem

Does there exist a synchronizing word ?

- Pure words are sufficient
- The emptiness problem is PSPACE-complete

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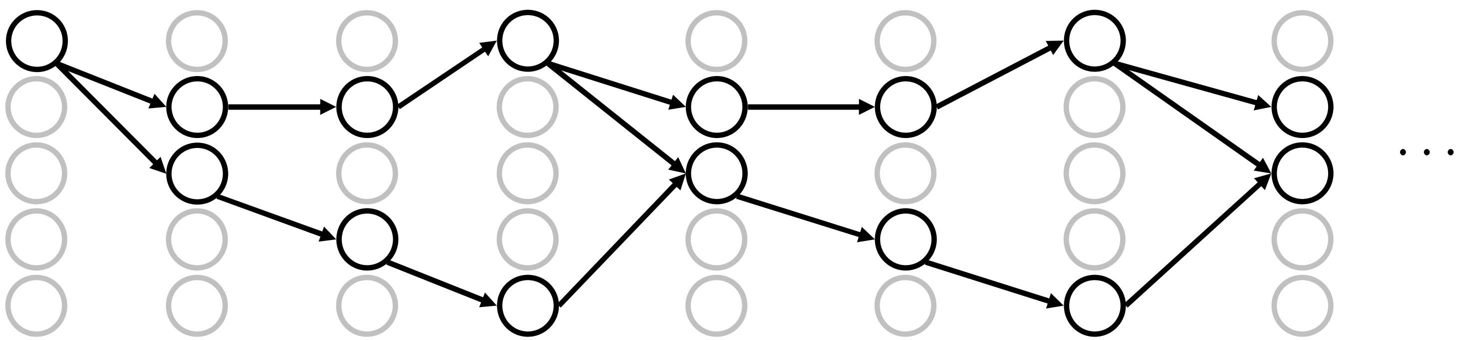

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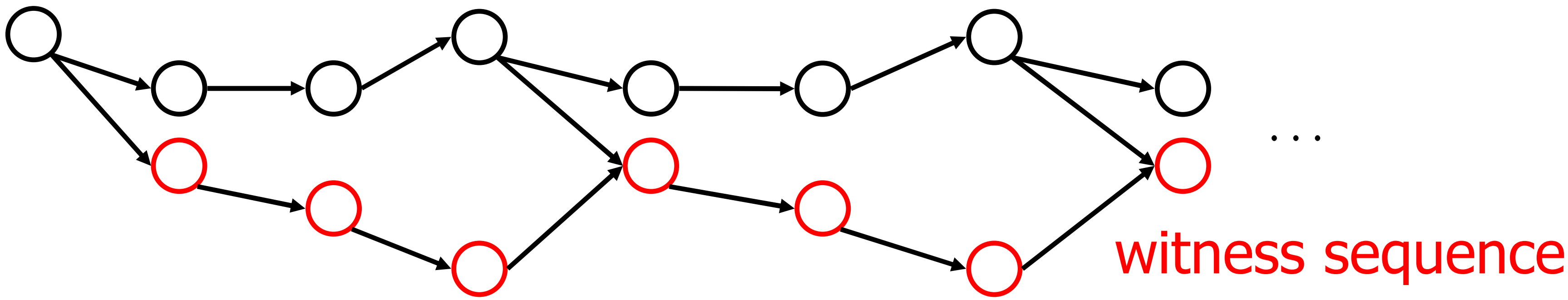
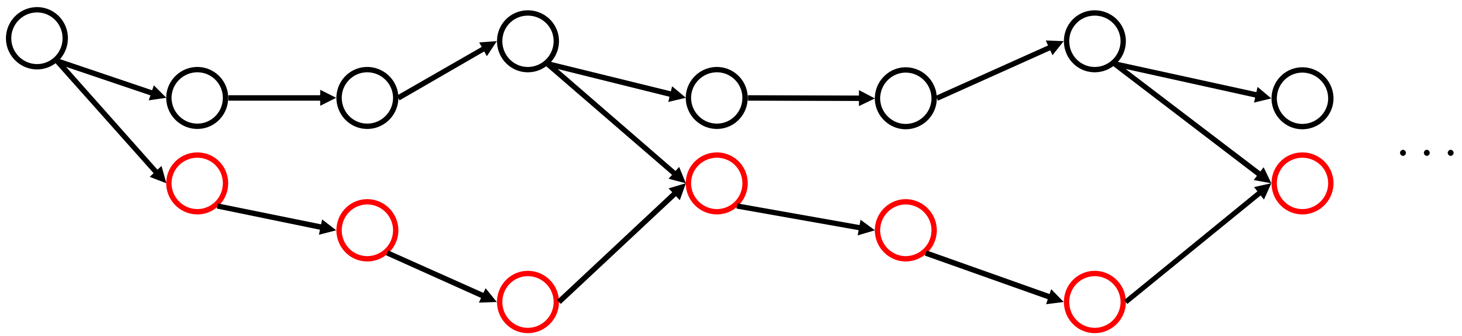


Emptiness problem

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.5\\.5\\0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.5\\0\\.5\\0\end{pmatrix} \xrightarrow{b} \begin{pmatrix}.5\\0\\0\\0\\.5\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.25\\.75\\0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.25\\0\\.75\\0\end{pmatrix} \xrightarrow{b} \begin{pmatrix}.25\\0\\0\\0\\.75\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\.125\\.875\\0\\0\end{pmatrix} \ldots$$



States in the witness sequence have exactly one successor

Emptiness problem

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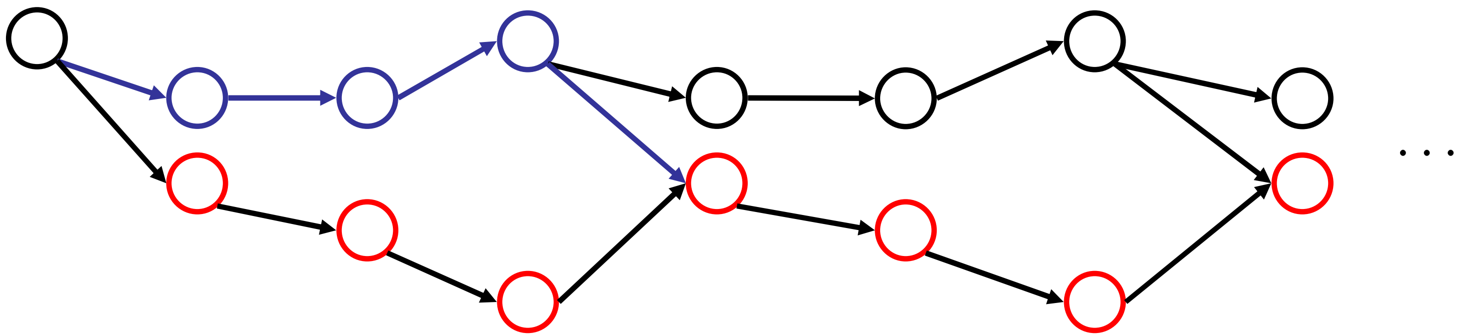


States in the witness sequence have exactly one successor

All other states have to inject some probability in the witness sequence

Emptiness problem

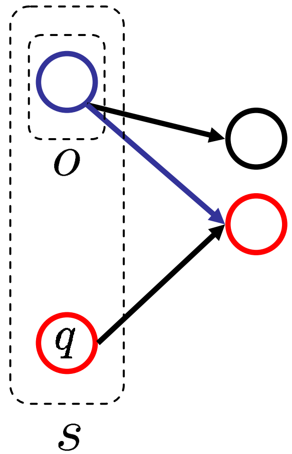
Does there exist a synchronizing word ?

- Pure words are sufficient
- The emptiness problem is PSPACE-complete

PSPACE upper bound: emptiness of a Büchi automaton

$$\begin{pmatrix} s \subseteq Q \\ o \subseteq s \\ q \in Q \end{pmatrix}$$

subset sonctruction
obligation set
witness sequence



Büchi condition: o is empty infinitely often

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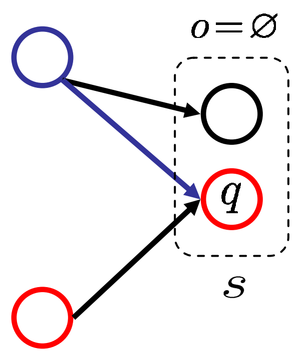

Büchi condition: o is empty infinitely often

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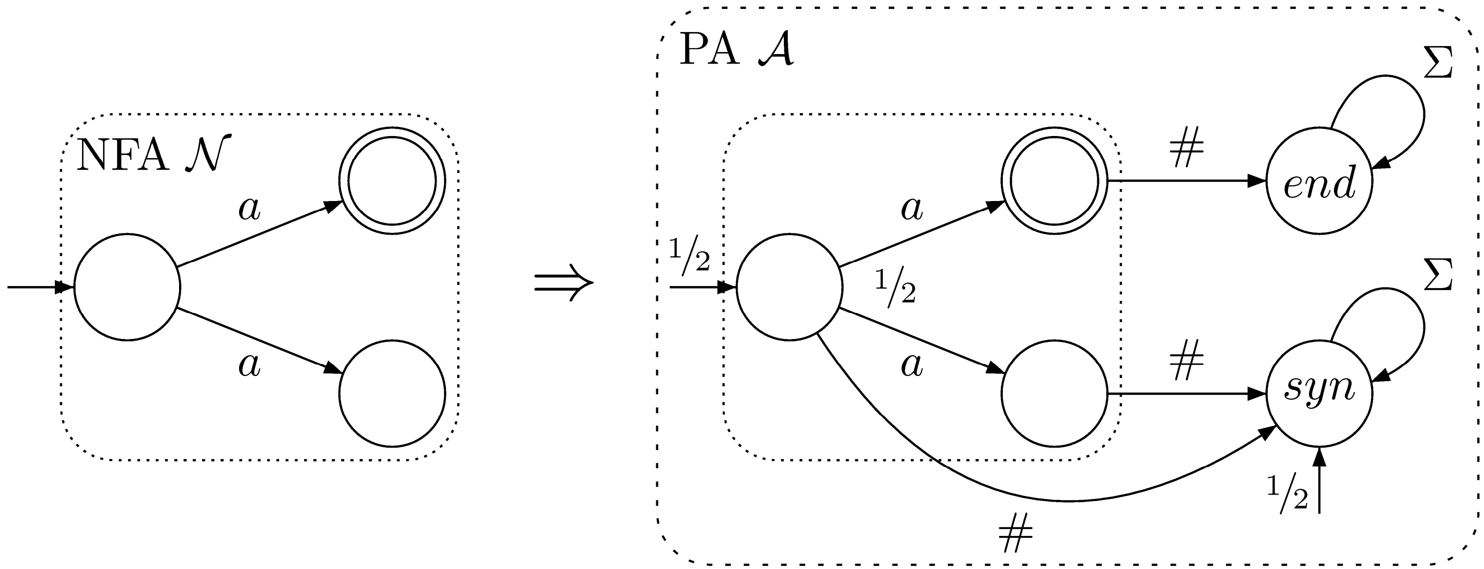
PSPACE lower bound: universality of NFA



Decision problems

- Emptiness

  Does there exist a synchronizing word ?

- Universality

  Are all words synchronizing ?

Note: we consider <u>randomized</u> words $w \in \mathcal{D}(\Sigma)^\omega$

Words $w \in \Sigma^\omega$ are called <u>pure</u> words.

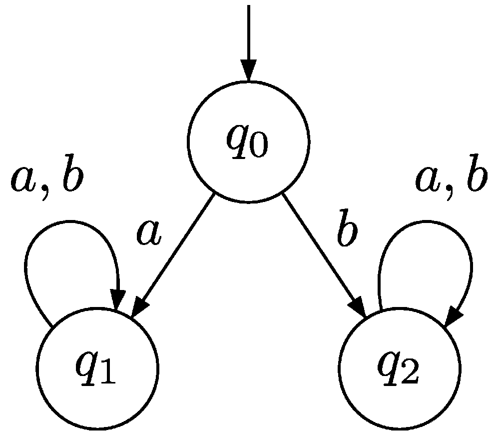
Universality problem

Are all words synchronizing ?

- Pure words are not sufficient

All pure words are synchronizing,

not all randomized words.



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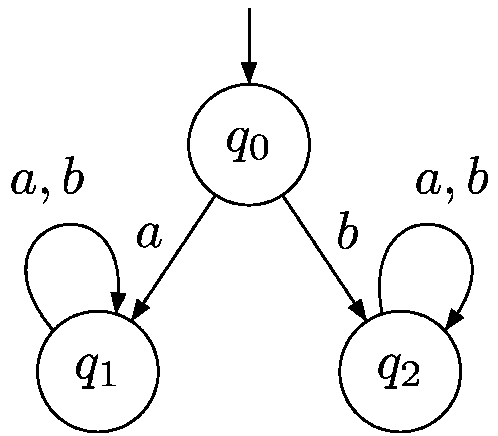
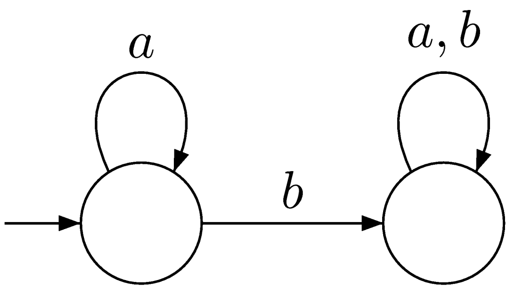


- The uniformly randomized word is not sufficient



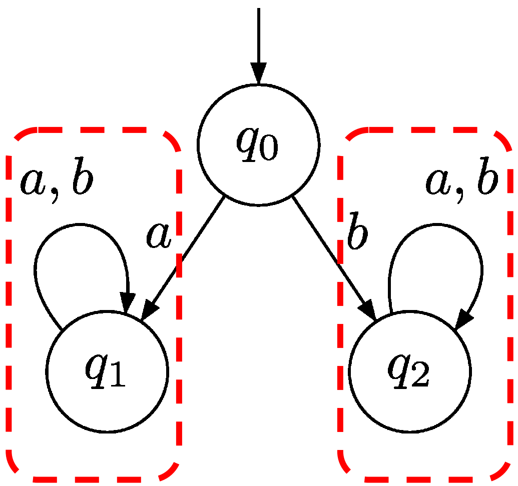
$(a \mapsto \frac{1}{2}, b \mapsto \frac{1}{2}) \cdot a^\omega$ is not synchronizing

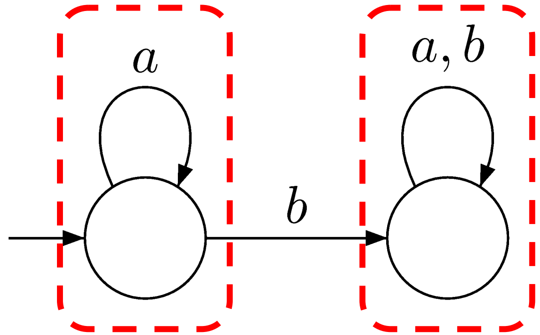
Universality problem

Are all words synchronizing ?

No, if there are two absorbing components.

(from which there is a word to stay inside).

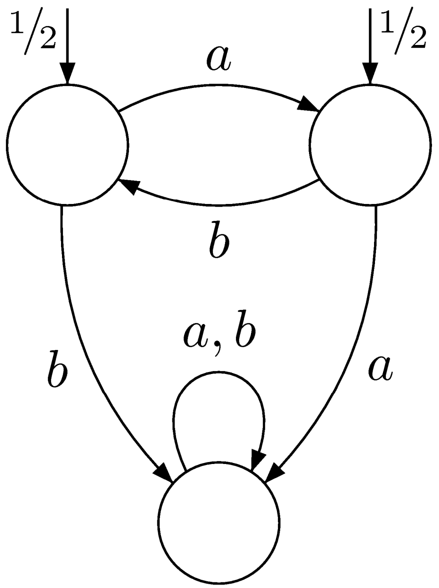


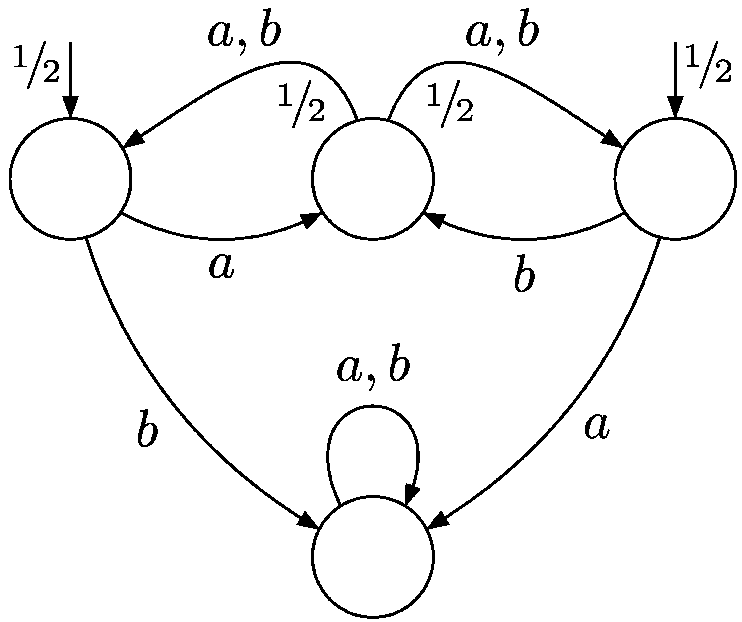


Universality problem

Are all words synchronizing ?

If there is only one absorbing component, then it is sufficient to check whether the uniformly randomized word is synchronizing.





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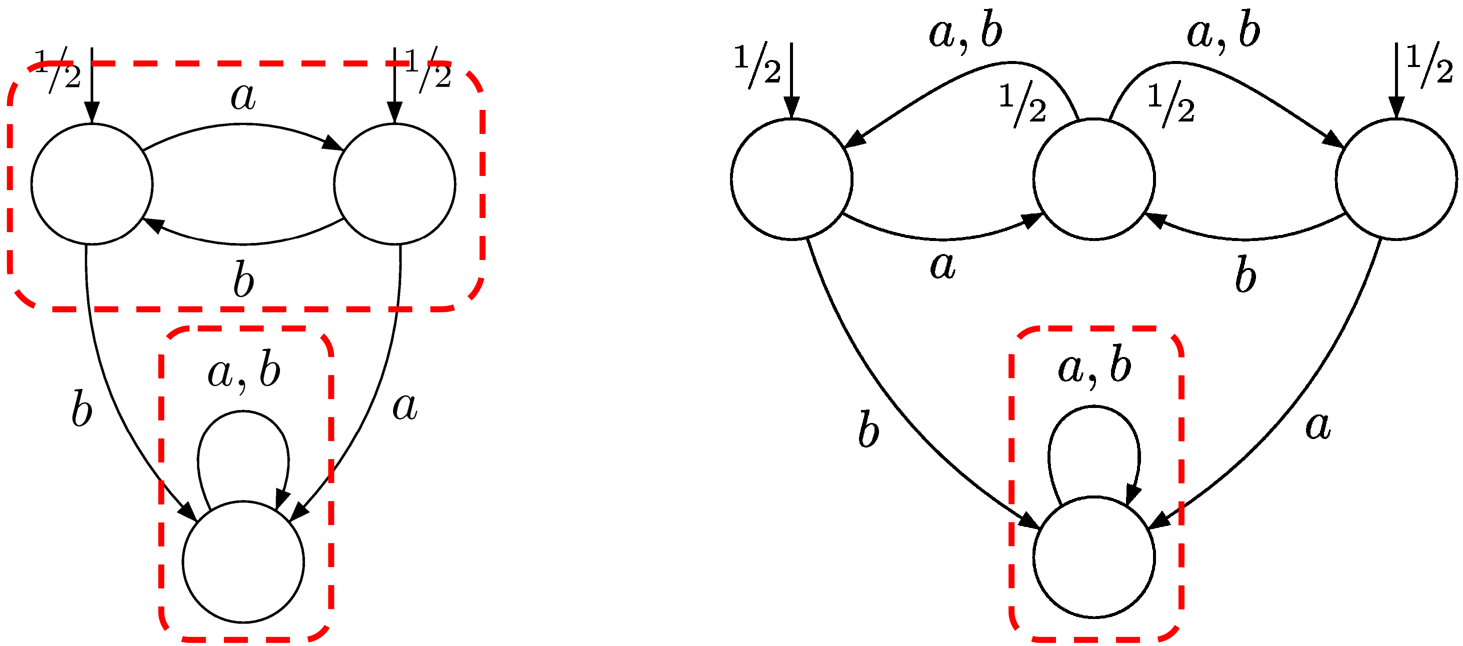

Uniformly randomized word is synchronizing.

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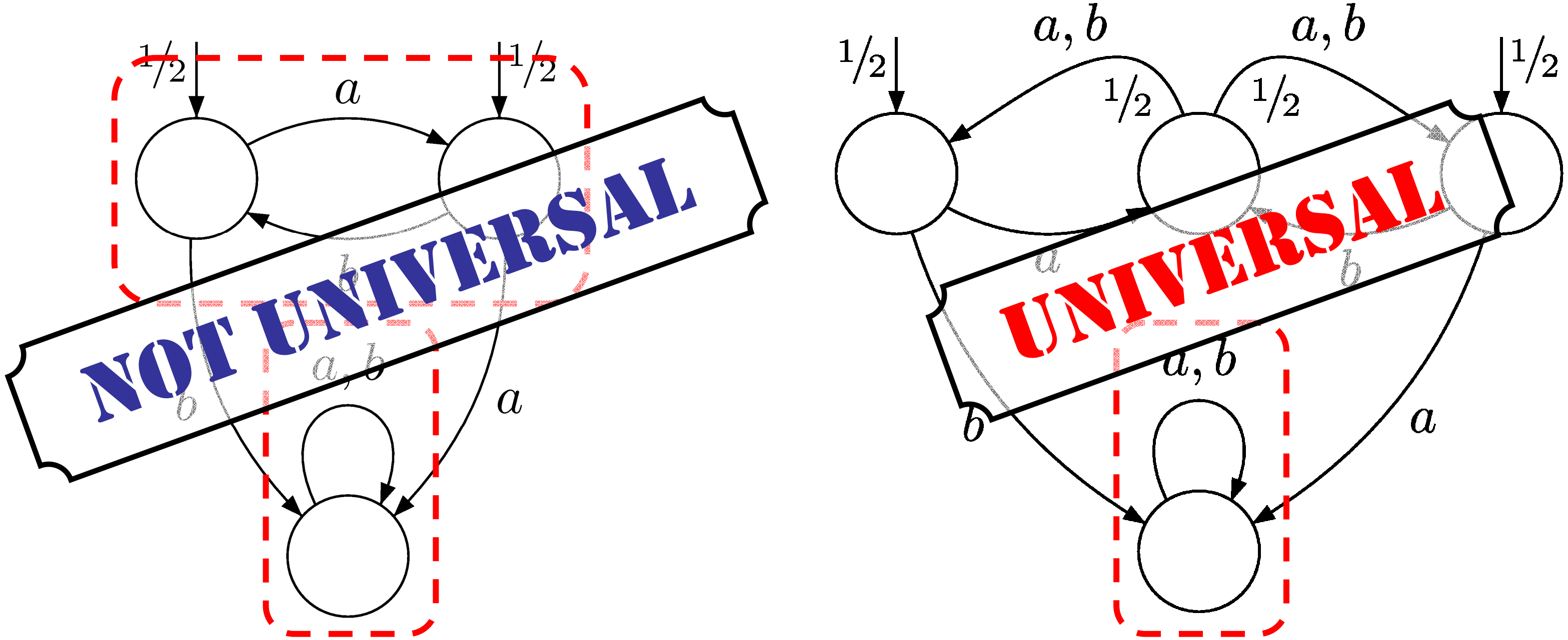

Uniformly randomized word is synchronizing.

Universality problem

Are all words synchronizing ?

The universality problem is in PSPACE.

- existence of absorbing component, check in PSPACE
- whether unif. rand. word is synchronizing, check in PTIME

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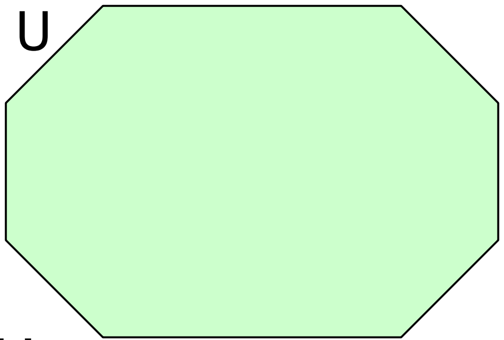

- Guess component $U \subseteq Q$
- Guess state $q \in U$ and finite word w
- Check that all runs from q on w stay in U

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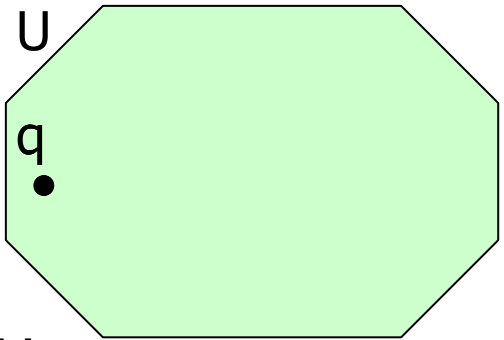

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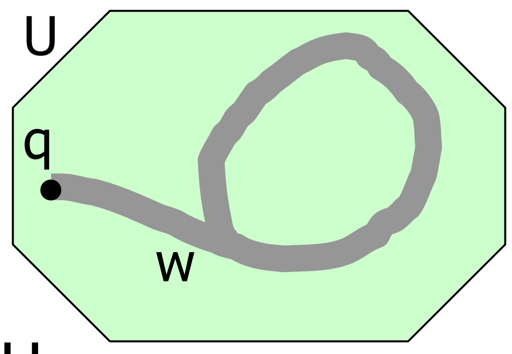

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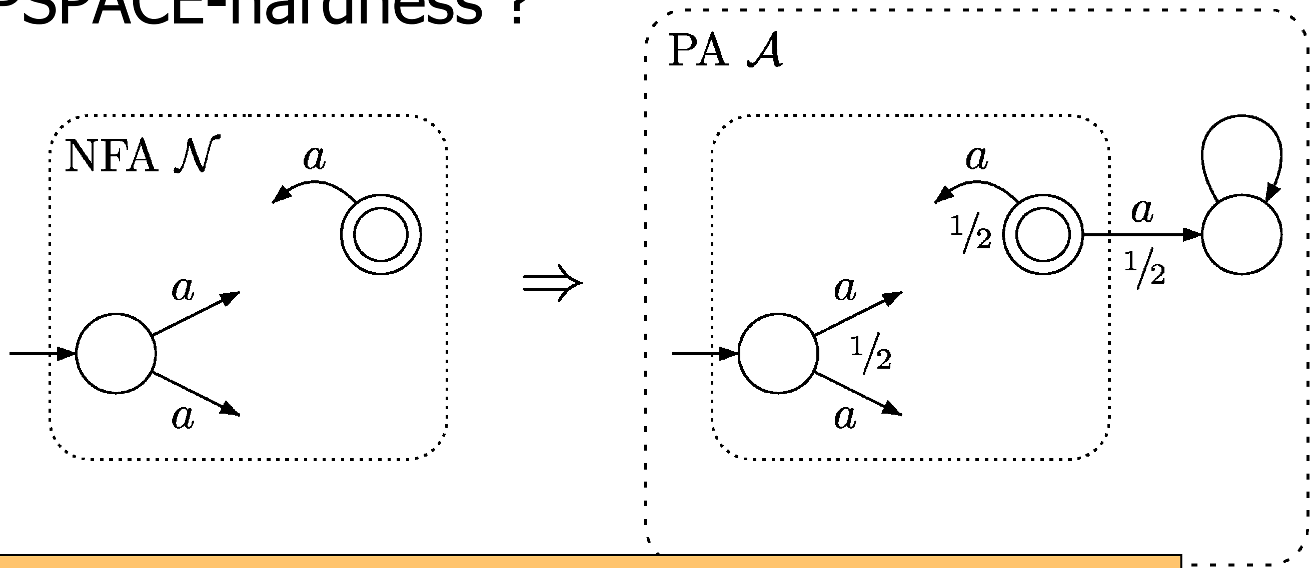
Towards PSPACE-hardness ?



There exists a state q and a word w such that all runs of $\mathcal{N}$ from q on w avoid accepting states iff $\mathcal{A}$ is not universal.

Universality problem

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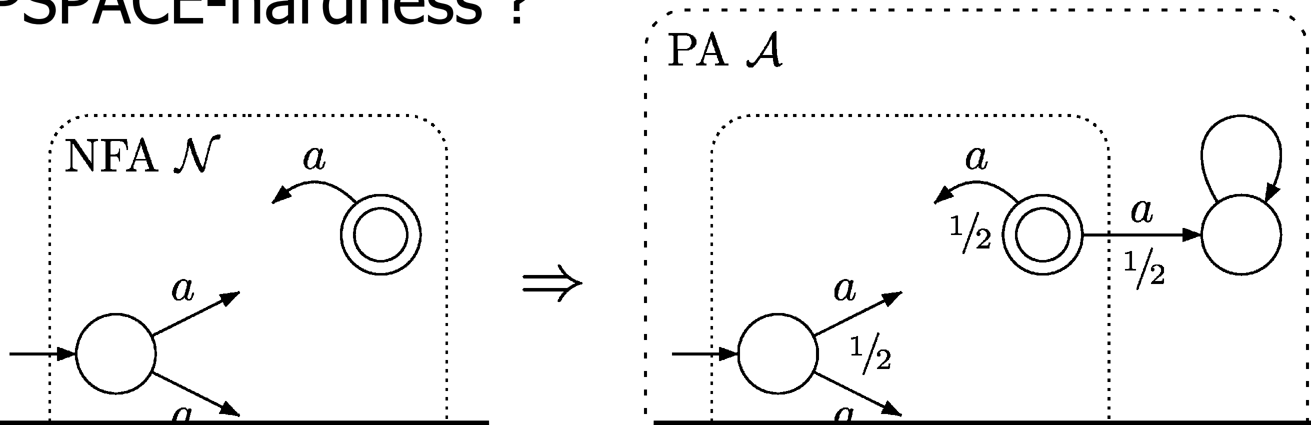
The universality problem is in PSPACE.

Towards PSPACE-hardness ?



Positive coBüchi automaton

Existential blind safety game

wo

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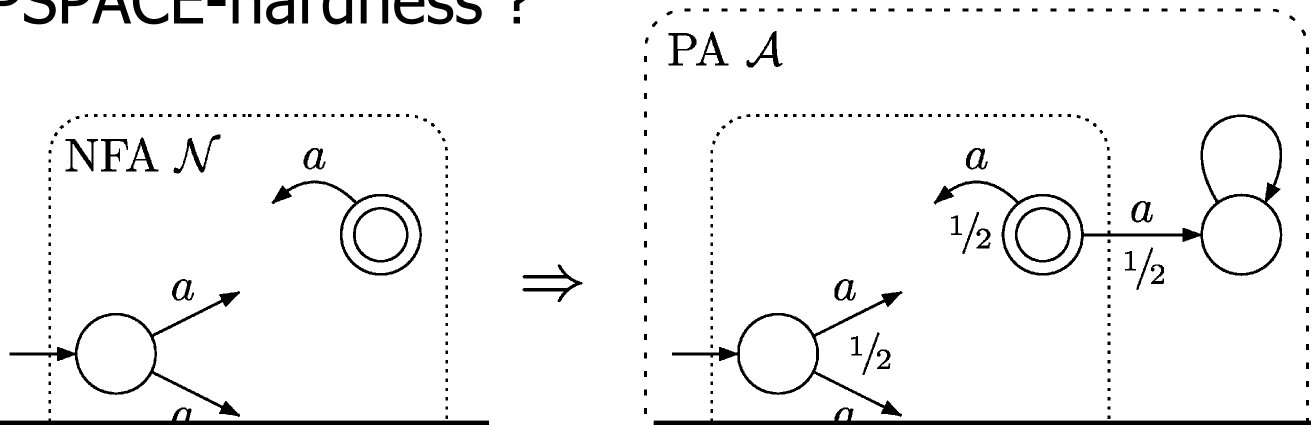

Positive coBüchi automaton

PSPACE-HARD ?

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PSPACE-HARD ?

... wo ...
runs of $\mathcal{N}$ from ... avoid accepting states
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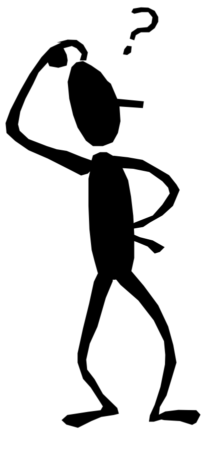
Summary

- Infinite synchronizing words for PA
- Generalizes finite sync. words for DFA
- Emptiness is PSPACE-complete
- Universality is in PSPACE – lower bound ?

Outlook

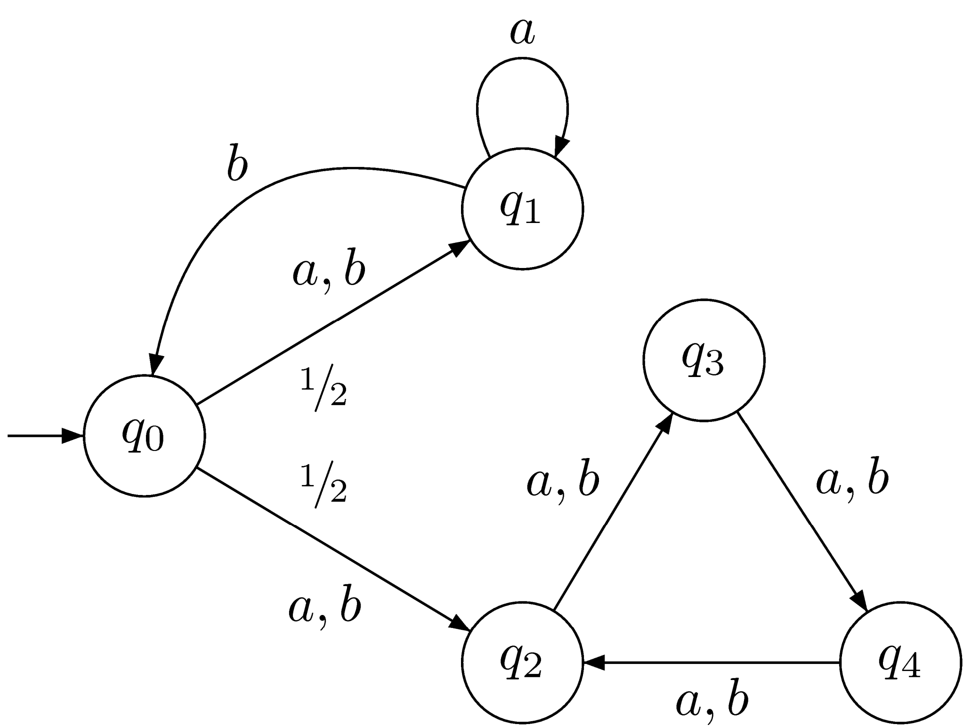
- Labeled automata, MDPs
- Universality in pure words
- Optimal synchronization
- Stochastic games

Thank you !



Questions ?

Probabilistic automata



$$\begin{matrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} \xrightarrow{b} \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1/4 \\ 3/4 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{aba} \begin{pmatrix} 0 \\ 1/8 \\ 7/8 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{(aba)^{n-3}} \begin{pmatrix} 0 \\ 1/2^n \\ 1 - 1/2^n \\ 0 \\ 0 \end{pmatrix}$$

References

[AV04] D. S. Ananichev and M. V. Volkov. Synchronizing Monotonic Automata. Theor. Comput. Sci. 327(3): 225-239 (2004)

[Vol08] M. V. Volkov. Synchronizing Automata and the Cerny Conjecture. LATA 2008: 11-27