Robust Synchronization in Markov Decision Processes

Laurent Doyen

LSV, ENS Cachan & CNRS

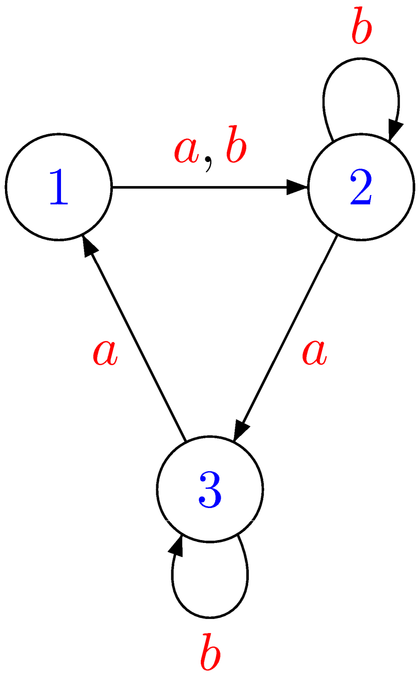
Joint work with
Thierry Massart, Mahsa Shirmohammadi

Concur 2014

Outline

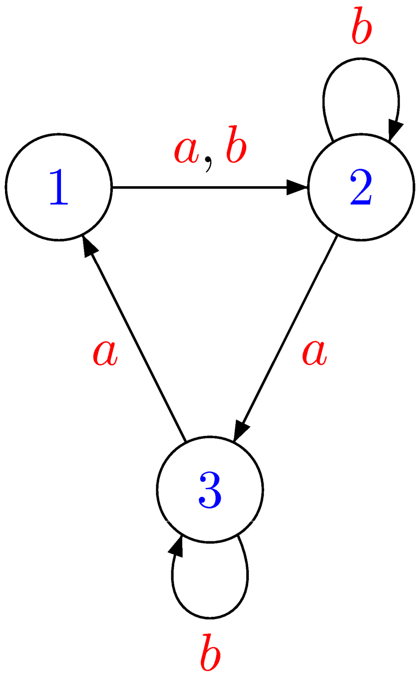
1. Synchronization (in finite-state automata)
2. Extension to Markov Decision Processes
3. Results

Synchronization in DFA



Synchronizing word brings the automaton from all states to the same state

Synchronization in DFA



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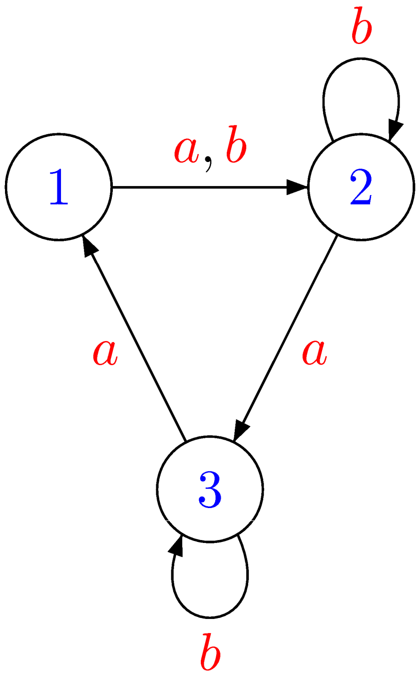


$$\delta : Q \times A \to Q$$

Synchronizing word
in DFA

1

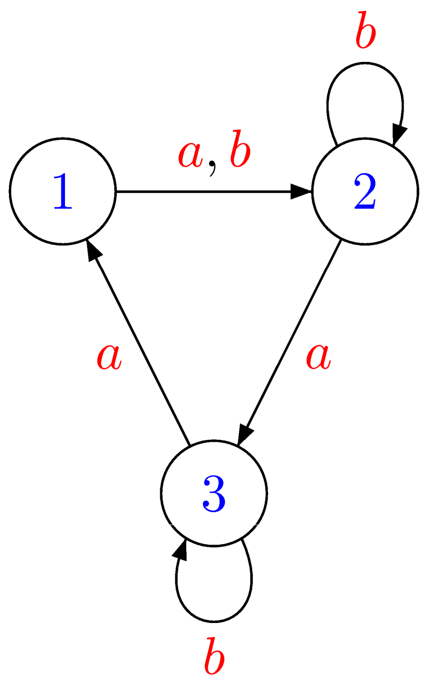
2

3

$b \quad a \quad a \quad b$

brings the automaton from all states
to the same state

Synchronization in DFA



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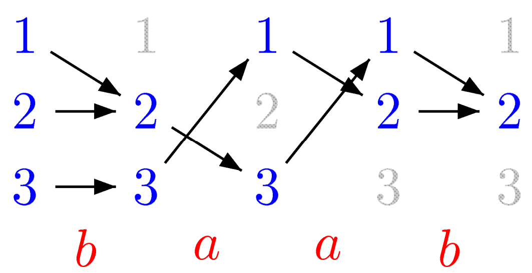

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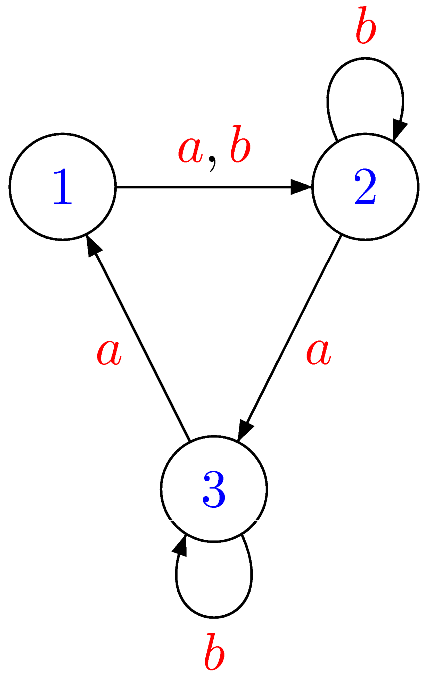

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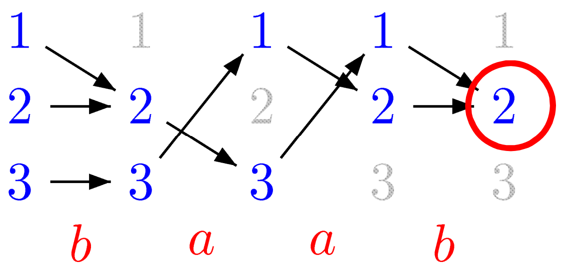

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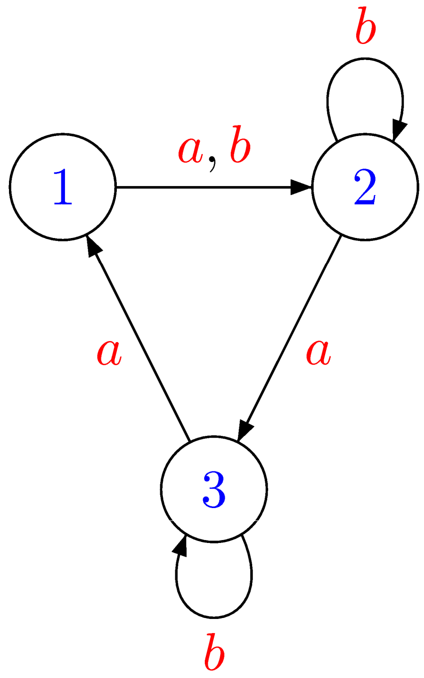

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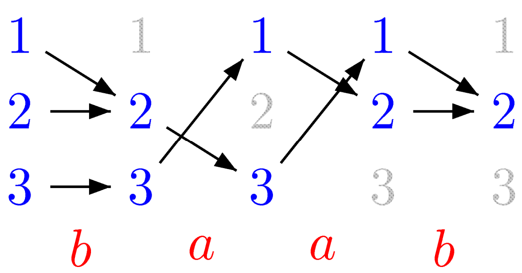

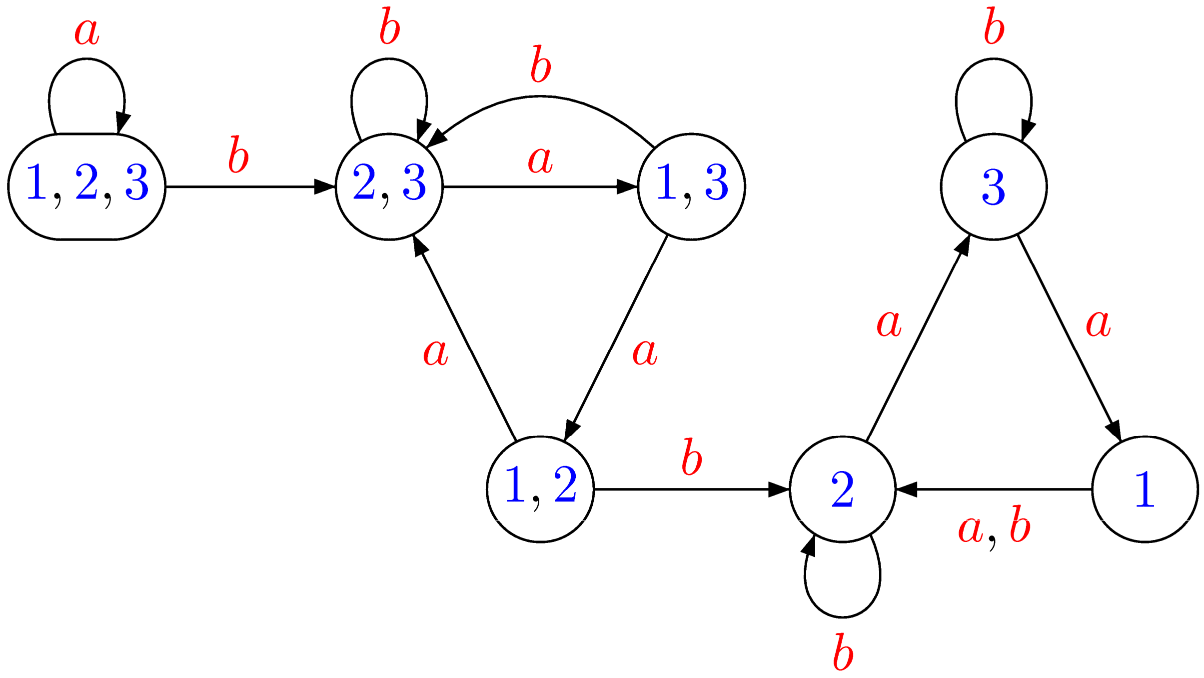

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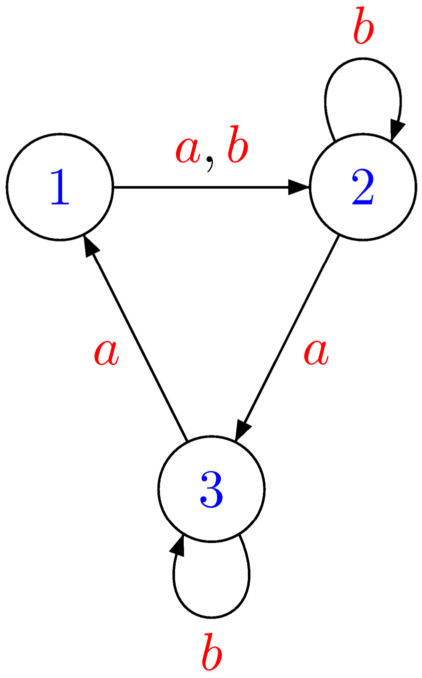
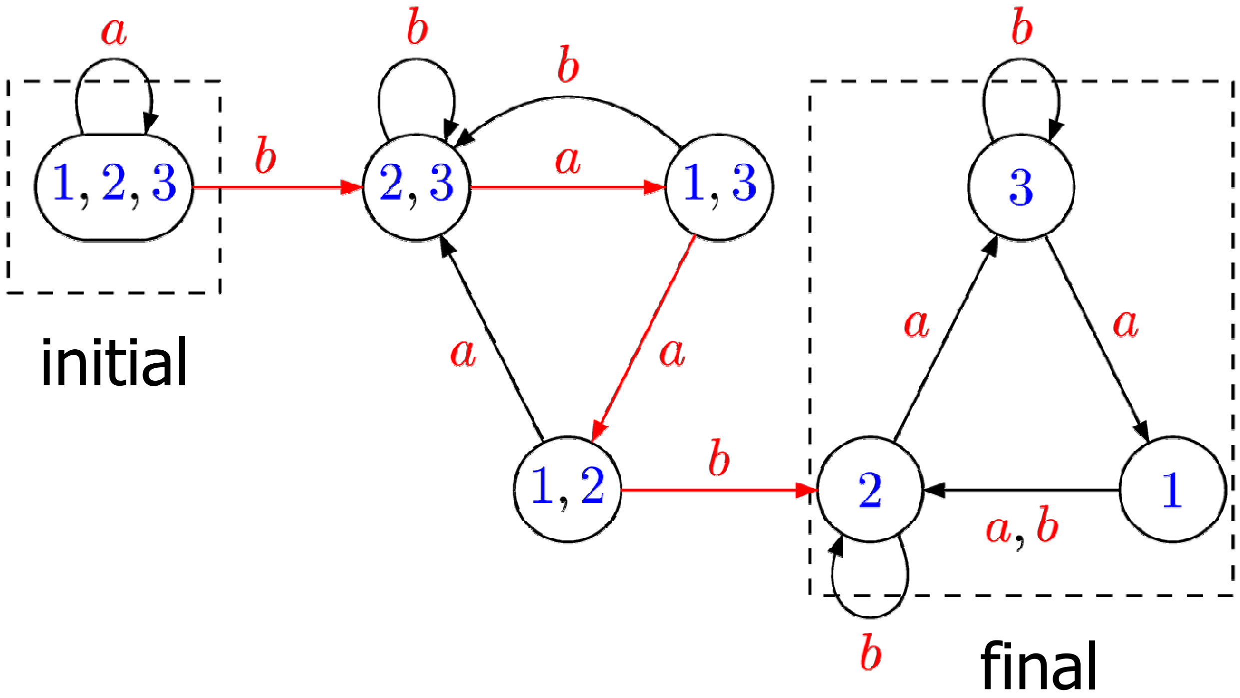
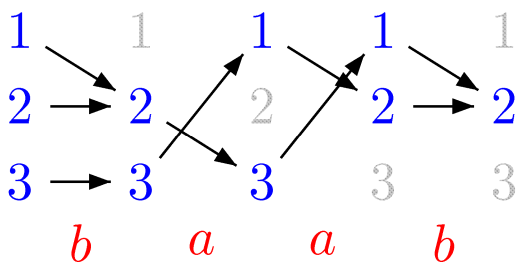
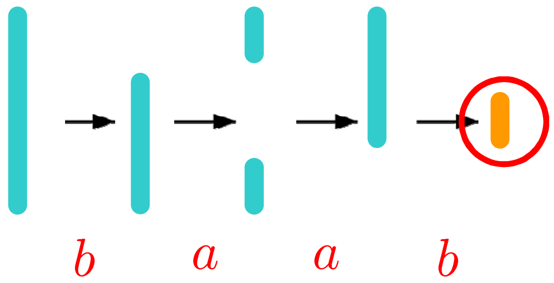




Synchronizing word in DFA $\Leftrightarrow$ Reachability question in powerset graph





Extensions

Basic model

DFA	$\delta : Q \times A \longrightarrow Q$
word	$\mathbb{N} \longrightarrow A$

Extensions

Basic model

DFA	$\delta : Q \times A \to Q$
word	$\mathbb{N} \to A$

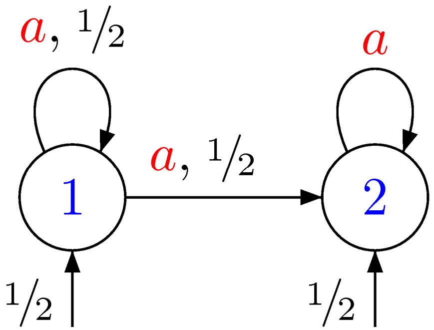
$\downarrow$ stochastic transitions

MDP – Markov decision process

$\delta : Q \times A \to \mathcal{D}(Q)$

$d_0 \in \mathcal{D}(Q)$ initial distribution

Synchronization in MDP



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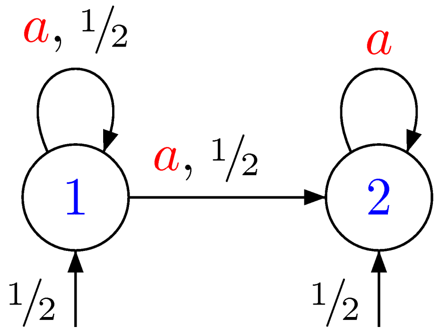


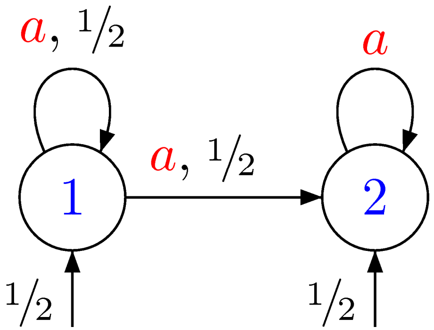
Finite state space

Stochastic transitions

Probability measure over events, i.e. sets of state sequences

$\mathbb{P}(1a1(a2)^{\omega}) = \frac{1}{8}$

Synchronization in MDP



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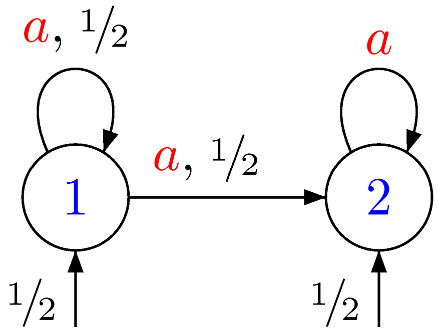
Traditional semantics

Synchronization in MDP



$$\begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} .5 \\ .5 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} .25 \\ .75 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} .125 \\ .875 \end{pmatrix} \xrightarrow{a} \cdots$$

Finite state space

Stochastic transitions

Probability measure over events, i.e. sets of state sequences

$$\mathbb{P}(1a1(a2)^\omega) = \frac{1}{8}$$

Traditional semantics

Infinite state space

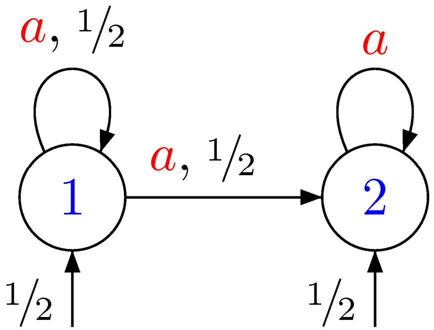
Deterministic transitions

Set of distribution sequences

$$\Omega \subseteq \mathcal{D}(Q)^\omega$$

Distribution-based semantics

Synchronization in MDP



Finite state space

Stochastic transitions

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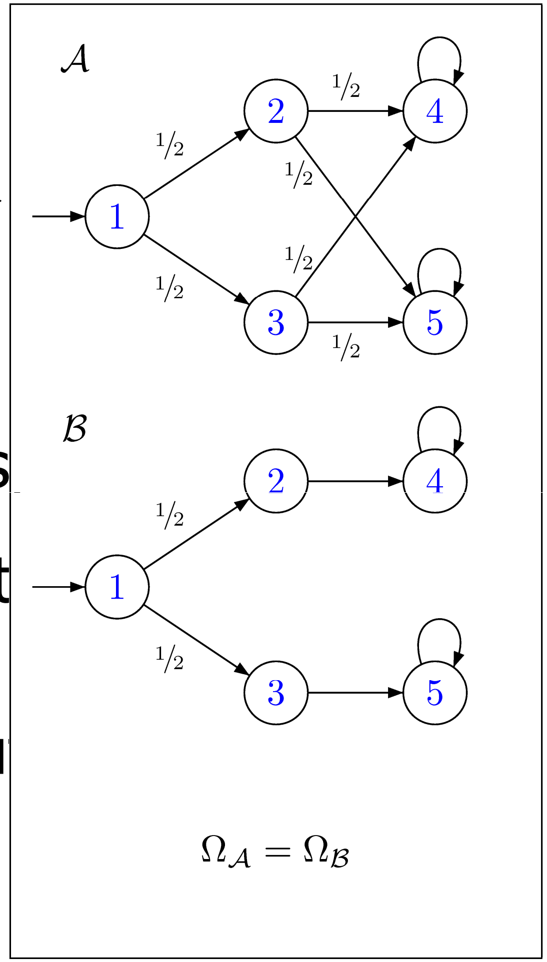

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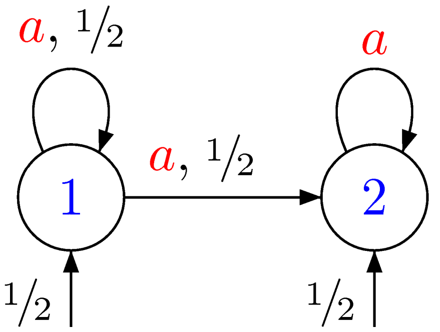

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Not synchronizing

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ not reachable

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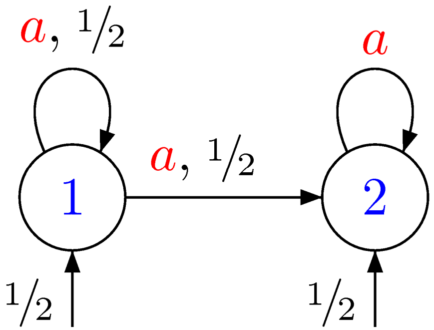

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Not synchronizing

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ not reachable

...but almost-sure synchronizing:

$\text{Final}^{\epsilon} = \{d \in \mathcal{D}(Q) \mid d(2) > 1 - \epsilon\}$

is reachable for all $\epsilon > 0$

$\begin{pmatrix} \epsilon \\ 1-\epsilon \end{pmatrix}$ reachable for arbitrarily small ε

Extensions

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$d_0 \in \mathcal{D}(Q)$ initial distribution

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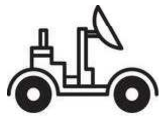
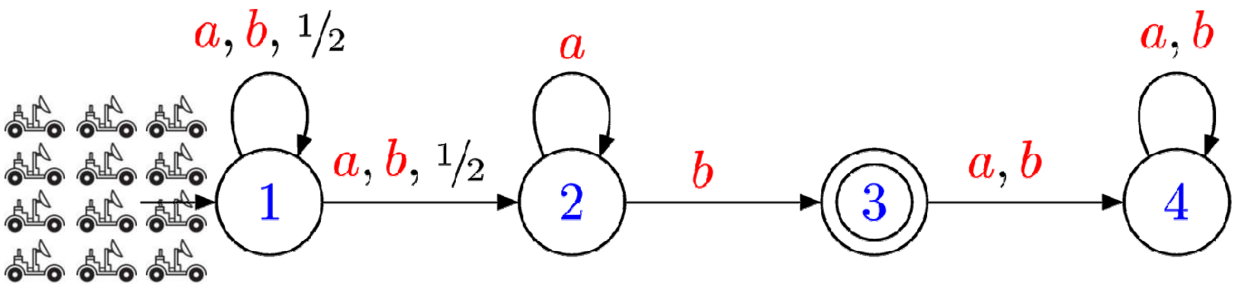
MDP: model of a robot crossing a bridge





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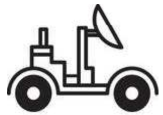

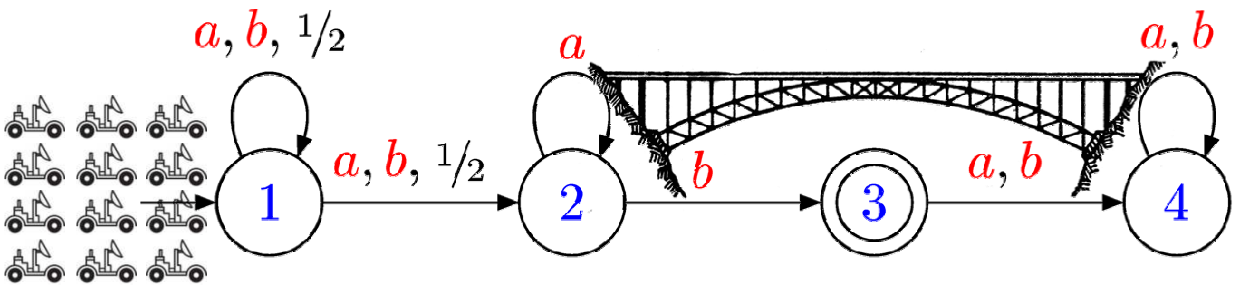

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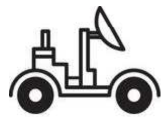
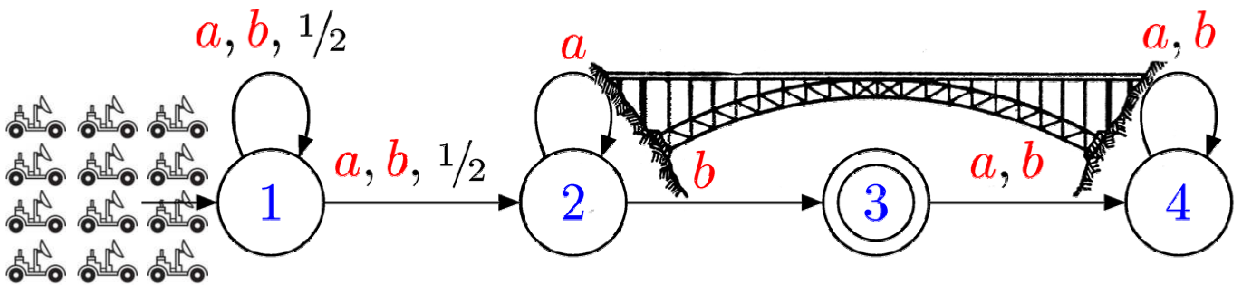


Embed a program α in each robot to ensure a group eventually meet on the bridge



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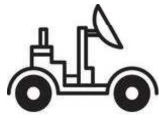

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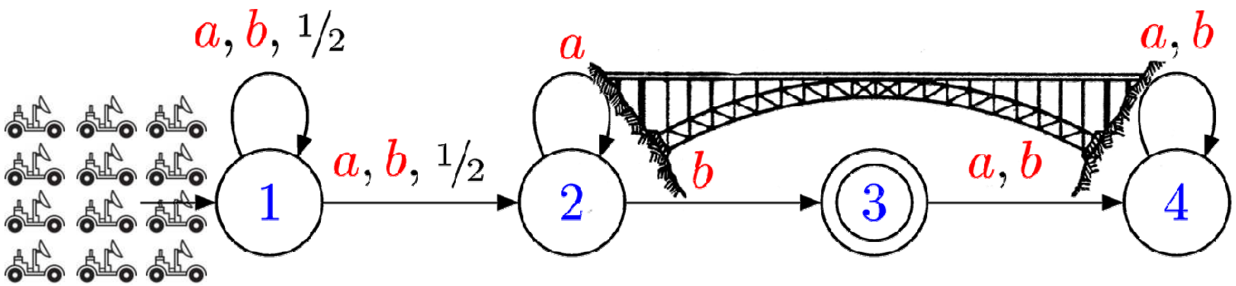

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MDP – Markov decision process

$\delta : Q \times A \to \mathcal{D}(Q)$

$d_0 \in \mathcal{D}(Q)$ initial distribution

produces a <u>single deterministic</u> trace (of distributions)

$d_0, d_1, d_2, d_3, \ldots$

Extensions

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$\alpha : Q^* \to A$

How to specify distribution traces ?

MDP – Ma... process

$\delta : Q \times A \to$ ...

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produces a <u>single deterministic</u> trace (of distributions)

$d_0, d_1, d_2, d_3, \ldots$

Distribution traces

MDP + strategy $\alpha \longrightarrow d_0^\alpha, d_1^\alpha, d_2^\alpha, d_3^\alpha, \ldots$

- reach a given set D of distributions

  $\exists \alpha \cdot \exists n : d_n^\alpha \in D$ (related to Skolem problem)

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  $\exists \alpha \cdot \exists n : d_n^\alpha(T) = 1$ sure eventually synchronizing

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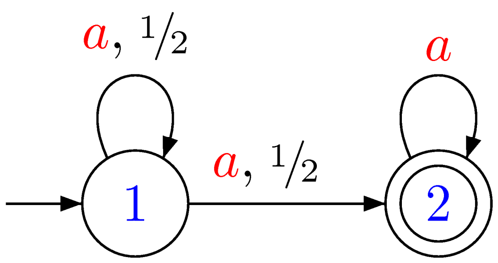

almost-sure

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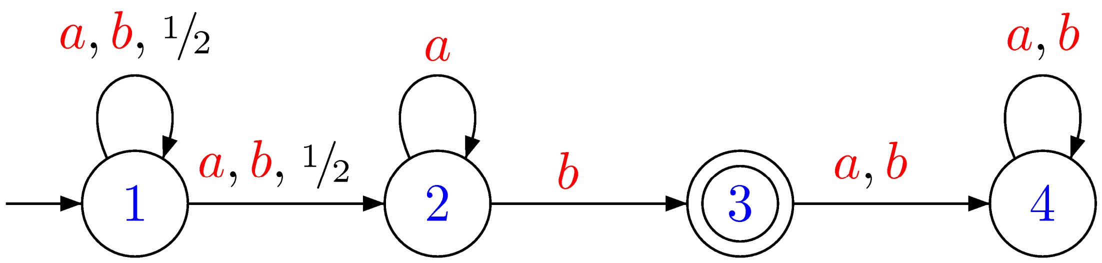

limit-sure

Distribution traces

MDP + strategy $\alpha \longrightarrow d_0^\alpha, d_1^\alpha, d_2^\alpha, d_3^\alpha, \ldots$

- reach a given set D of distributions

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- reach a distribution with support in $T \subseteq Q$

  $\exists \alpha \cdot \exists n : d_n^\alpha(T) = 1$ sure eventually synchronizing

  $\exists \alpha : \sup_n d_n^\alpha(T) = 1$ almost-sure eventually synchronizing

  $\sup_\alpha \sup_n d_n^\alpha(T) = 1$ limit-sure eventually synchronizing

- visit infinitely often a distribution with support in $T \subseteq Q$

- eventually visit only distributions with support in $T \subseteq Q$

Distribution traces

MDP + strategy $\alpha \longrightarrow d_0^{\alpha}, d_1^{\alpha}, d_2^{\alpha}, d_3^{\alpha}, \ldots$

	Eventually	Weakly	Strongly
Sure	$\exists \alpha \quad \exists n \quad d_n^{\alpha}(T) = 1$		
Almost-sure	$\exists \alpha \quad \sup_n d_n^{\alpha}(T) = 1$		
Limit-sure	$\sup_{\alpha} \sup_n d_n^{\alpha}(T) = 1$		

eventually

Distribution traces

MDP + strategy $\alpha \longrightarrow d_0^\alpha, d_1^\alpha, d_2^\alpha, d_3^\alpha, \ldots$

	Eventually	Weakly	Strongly
Sure	$\exists \alpha \quad \exists n \quad d_n^\alpha(T) = 1$	$\exists \alpha \quad \forall N \, \exists n \geq N \; d_n^\alpha(T) = 1$	
Almost-sure	$\exists \alpha \quad \sup_n \, d_n^\alpha(T) = 1$	$\exists \alpha \quad \limsup_{n \to \infty} \, d_n^\alpha(T) = 1$	
Limit-sure	$\sup_\alpha \, \sup_n \, d_n^\alpha(T) = 1$	$\sup_\alpha \, \limsup_{n \to \infty} \, d_n^\alpha(T) = 1$	

eventually

weakly

Distribution traces

MDP + strategy $\alpha \longrightarrow d_0^{\alpha}, d_1^{\alpha}, d_2^{\alpha}, d_3^{\alpha}, \ldots$

	Eventually	Weakly	Strongly
Sure	$\exists \alpha \quad \exists n \quad d_n^{\alpha}(T) = 1$	$\exists \alpha \quad \forall N\, \exists n \geq N \; d_n^{\alpha}(T) = 1$	$\exists \alpha \quad \exists N\, \forall n \geq N \; d_n^{\alpha}(T) = 1$
Almost-sure	$\exists \alpha \quad \sup_n \; d_n^{\alpha}(T) = 1$	$\exists \alpha \quad \limsup_{n \to \infty} \; d_n^{\alpha}(T) = 1$	$\exists \alpha \quad \liminf_{n \to \infty} \; d_n^{\alpha}(T) = 1$
Limit-sure	$\sup_{\alpha} \; \sup_n \; d_n^{\alpha}(T) = 1$	$\sup_{\alpha} \; \limsup_{n \to \infty} \; d_n^{\alpha}(T) = 1$	$\sup_{\alpha} \; \liminf_{n \to \infty} \; d_n^{\alpha}(T) = 1$

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	Eventually	Weakly	Strongly
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eventually
[FoSSaCS'14]

weakly
[this paper]

strongly
[this paper]

Results

Robustness

Complexity

Results

Robustness

Almost-sure and limit-sure coincide for weakly and strongly synchronizing

Complexity

Robustness

Almost-sure and limit-sure coincide for weakly and strongly synchronizing

	Eventually	Weakly	Strongly
Sure	$\exists \alpha \quad \exists n \quad d_n^{\alpha}(T) = 1$	$\exists \alpha\ \forall N\ \exists n \geq N\ d_n^{\alpha}(T) = 1$	$\exists \alpha\ \exists N\ \forall n \geq N\ d_n^{\alpha}(T) = 1$
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Limit-sure	$\sup_{\alpha}\ \sup_n\ d_n^{\alpha}(T) = 1$		

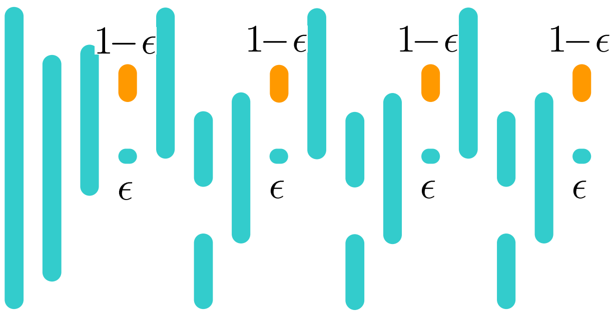
Robustness

Almost-sure and limit-sure coincide for weakly and strongly synchronizing

	Eventually	Weakly	Strongly
Sure	$\exists\alpha \quad \exists n \quad d_n^\alpha(T) = 1$	$\exists\alpha\ \forall N\ \exists n \geq N\ \ d_n^\alpha(T) = 1$	$\exists\alpha\ \exists N\ \forall n \geq N\ \ d_n^\alpha(T) = 1$
Almost-sure	$\exists\alpha \quad \sup_n\ d_n^\alpha(T) = 1$	$\exists\alpha\ \limsup_{n\to\infty}\ d_n^\alpha(T) = 1$	$\exists\alpha\ \liminf_{n\to\infty}\ d_n^\alpha(T) = 1$
Limit-sure	$\sup_\alpha\ \sup_n\ d_n^\alpha(T) = 1$		

Proof sketch (for Weakly)

Limit-sure $\quad \forall\epsilon > 0 \cdot \exists\alpha : \limsup_{n\to\infty} d_n^\alpha(T) \geq 1 - \epsilon$



implies almost-sure by "partly" switching to ε-strategies for smaller and smaller ε

$$\exists\alpha \cdot \forall\epsilon > 0 : \limsup_{n\to\infty} d_n^\alpha(T) \geq 1 - \epsilon$$

Results

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Almost-sure and limit-sure coincide for
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Results

Robustness

Almost-sure and limit-sure coincide for weakly and strongly synchronizing

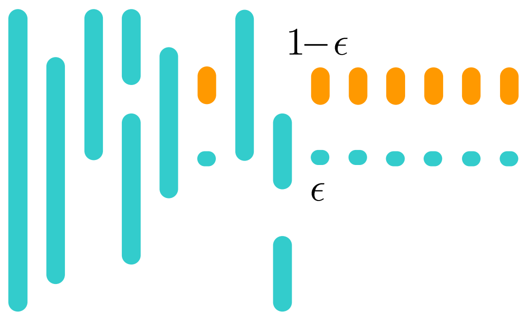
Complexity

Deciding Weakly synchronization is PSPACE-complete

Deciding Strongly synchronization is PTIME-complete

Complexity

Deciding Strongly synchronization is PTIME-complete

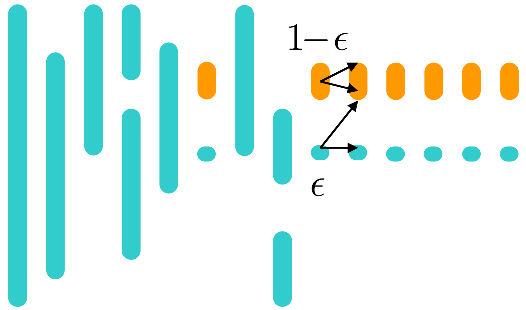


	Strongly
Sure	$\exists\alpha\ \exists N\ \forall n \geq N\ d_n^\alpha(T) = 1$
Almost-sure	$\exists\alpha\ \liminf_{n\to\infty}\ d_n^\alpha(T) = 1$
Limit-sure	

Complexity

Deciding Strongly synchronization is PTIME-complete

	Strongly
Sure	$\exists \alpha\ \exists N\ \forall n \geq N\ d_n^{\alpha}(T) = 1$
Almost-sure	$\exists \alpha\ \liminf_{n \to \infty}\ d_n^{\alpha}(T) = 1$
Limit-sure	

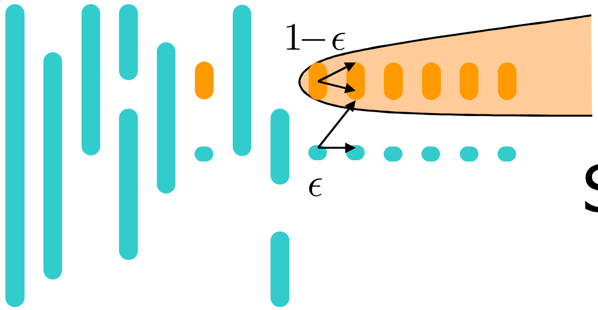


All transitions
from T stay in T

Complexity

Deciding Strongly synchronization is PTIME-complete

	Strongly
Sure	$\exists\alpha\ \exists N\ \forall n \geq N\ d_n^\alpha(T) = 1$
Almost-sure	$\exists\alpha\ \liminf_{n\to\infty}\ d_n^\alpha(T) = 1$
Limit-sure	



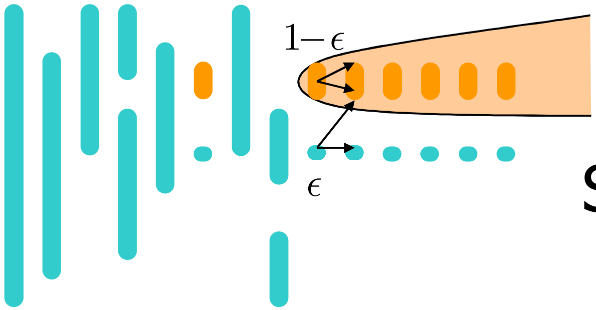
Safe region for $\square T$

$$\text{Reduces to} \left\{ \begin{array}{l} \text{sure} \\ \text{almost-sure} \\ \text{limit-sure} \end{array} \right\} \text{reachability to safe region for } \square T$$

Complexity

Deciding Strongly synchronization is PTIME-complete

	Strongly
Sure	$\exists \alpha\ \exists N\ \forall n \geq N\ d_n^{\alpha}(T) = 1$
Almost-sure	$\exists \alpha\ \liminf_{n \to \infty}\ d_n^{\alpha}(T) = 1$
Limit-sure	



Safe region for $\Box T$

$$\text{Reduces to } \underbrace{\left\{\begin{array}{l}\text{sure}\\ \text{almost-sure}\\ \text{limit-sure}\end{array}\right\} \text{ reachability to}}_{\text{in PTIME}}\ \underbrace{\text{safe region for } \Box T}_{\text{in PTIME}}$$

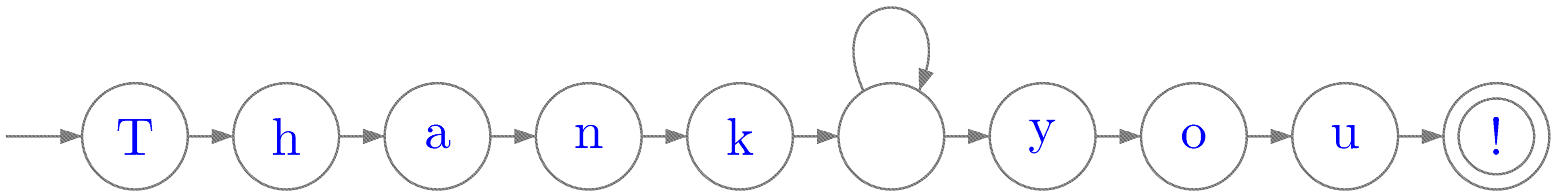
Corollary: almost-sure and limit-sure coincide

Results

	Eventually	Weakly	Strongly
Sure	PSPACE-C [FoSSaCS'14]	**PSPACE-C**	**PTIME-C**
Almost-sure	PSPACE-C [FoSSaCS'14]	**PSPACE-C**	**PTIME-C**
Limit-sure	PSPACE-C [FoSSaCS'14]		

Also in the paper:

- Variants with same complexity
- Memory requirement for synchronizing strategies



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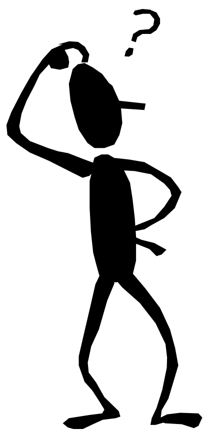
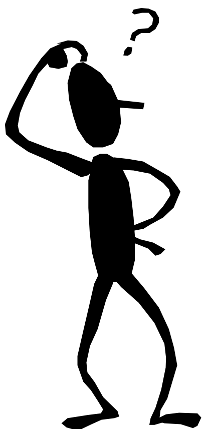


Thank you !



Questions ?